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Monge's problem with a quadratic cost by the zero-noise limit of *h-pass* processes

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Abstract

We study the asymptotic behavior, in the zero noise limit, of solutions to Schrödinger's functional equations and that of *h-pass* processes, and give a new proof of the existence of the minimizer of Monge's problem with a quadratic cost.

1 Introduction.

Let $L : \mathbf{R}^d \mapsto [0, \infty)$ be convex, P_0 and P_1 be Borel probability measures on $\mathbf{R}^d$, and put

$$V(P_0, P_1) := \inf \left\{ \int_{\mathbf{R}^d} L(\psi(X) - X) P_0(dx) : P_0(\psi(X) \in dx) = P_1(dx) \right\}. \quad (1.1)$$

The study of the minimizer of (1.1) can be considered as a special case of Monge's problem.

Kantorovich's approach to Monge's problem is to study the minimizer of the following relaxed problem:

$$\begin{aligned}\tilde{V}(P_0, P_1) &:= \inf \left\{ \iint_{\mathbf{R}^d \times \mathbf{R}^d} L(y - x) \mu(dx dy) \right. \\ &\quad \left. : \mu(dx \times \mathbf{R}^d) = P_0(dx), \mu(\mathbf{R}^d \times dy) = P_1(dy) \right\}.\end{aligned}\tag{1.2}$$

If there exists a Borel measurable function ψ , on $\mathbf{R}^d$, such that the minimizer of (1.2) is $P_0(dx)\delta_{\psi(x)}(dy)$, then $V(P_0, P_1) = \tilde{V}(P_0, P_1)$ and ψ is a minimizer of (1.1).

This is called the Monge-Kantorovich problem and plays a crucial role in many fields and has been studied by many authors (see [8, 20, 25] and the references therein).

It is easy to see that the following holds:

$$\tilde{V}(P_0, P_1) = \inf \left\{ E \left[\int_0^1 L \left(\frac{d\phi(t)}{dt} \right) dt \right] \right\},\tag{1.3}$$

where the infimum is taken over all absolutely continuous stochastic processes $\{\phi(t)\}_{0 \leq t \leq 1}$ for which $P(\phi(t) \in dx) = P_t(dx)$ ($t = 0, 1$). (In this paper we use the same notation P for different probability measures for the sake of simplicity when it is not confusing.) Indeed, the minimizer of (1.3) is linear in t (see e.g. [5], [10, p. 35]).

This implies that Monge's problem with a quadratic cost $L(u) = |u|^2$ should be the zero noise limit of h -pass processes (see [7, p. 566]), which enables us not to use Kantorovich's approach to study (1.1).

To make the point clearer, we introduce Schrödinger's functional equation and then describe the h -pass process briefly. For $\varepsilon > 0$ and $x \in \mathbf{R}^d$, put

$$g_\varepsilon(x) := \frac{1}{\sqrt{2\pi\varepsilon}^d} \exp\left(-\frac{|x|^2}{2\varepsilon}\right),\tag{1.4}$$

$$P_{1,\varepsilon}(dy) := \left(\int_{\mathbf{R}^d} g_\varepsilon(z - y) P_1(dz) \right) dy.\tag{1.5}$$

The following is a special case of Schrödinger's functional equations: find nonnegative Borel measures $(\nu_{0,\varepsilon}, \nu_{1,\varepsilon})$ for which

$$\begin{aligned}P_0(dx) &= \left(\int_{\mathbf{R}^d} g_\varepsilon(x - y) \nu_{1,\varepsilon}(dy) \right) \nu_{0,\varepsilon}(dx), \\ P_{1,\varepsilon}(dy) &= \left(\int_{\mathbf{R}^d} g_\varepsilon(x - y) \nu_{0,\varepsilon}(dx) \right) \nu_{1,\varepsilon}(dy).\end{aligned}\tag{1.6}$$

It is known that there exists a unique solution $(\nu_{0,\varepsilon}, \nu_{1,\varepsilon})$ to (1.6) (see [13], and also [22] for the recent development).

Let $(\Omega, \mathbf{B}, P)$ be a probability space, $\{\mathbf{B}_t\}_{t \geq 0}$ be a right continuous, increasing family of sub σ -fields of $\mathbf{B}$, X_o be a $\mathbf{R}^d$ -valued, $\mathbf{B}_0$ -adapted random variable such that $P(X_o)^{-1} = P_0$, and $\{W(t)\}_{t \geq 0}$ denote a d -dimensional $(\mathbf{B}_t)$ -Wiener process (see e.g. [7], [10] or [12]).

The h -pass process on $[0, 1]$ with an initial distribution P_0 and a terminal one $P_{1,\varepsilon}$, and with the diffusion matrix $\sqrt{\varepsilon} \times (\text{Identity matrix})$ is the unique weak solution to the following (see [14]): for $t \in [0, 1]$,

$$X_\varepsilon(t) = X_o + \int_0^t b_\varepsilon(s, X_\varepsilon(s)) ds + \sqrt{\varepsilon} W(t), \quad (1.7)$$

where

$$b_\varepsilon(s, x) := \varepsilon D_x \log \left(\int_{\mathbf{R}^d} g_{\varepsilon(1-s)}(x - y) \nu_{1,\varepsilon}(dy) \right) \quad ((s, x) \in [0, 1] \times \mathbf{R}^d). \quad (1.8)$$

Here $D_x := (\partial/\partial x_i)_{i=1}^d$.

It is known that

$$P((X_\varepsilon(0), X_\varepsilon(1)) \in dxdy) = \mu_\varepsilon(dxdy) := \nu_{0,\varepsilon}(dx) g_\varepsilon(x - y) \nu_{1,\varepsilon}(dy). \quad (1.9)$$

It is also known that the minimizer of the following is the h -pass process in (1.7) (see [11]):

$$V_\varepsilon(P_0, P_{1,\varepsilon}) := \inf \left\{ E \left[\int_0^1 |u(t)|^2 dt \right] \right\}, \quad (1.10)$$

where the infimum is taken over all $\mathbf{R}^d$ -valued, $(\mathbf{B}_t)$ -progressively measurable $\{u(t)\}_{0 \leq t \leq 1}$ for which the distribution of $X_o + \int_0^1 u(s) ds + \sqrt{\varepsilon} W(1)$ is $P_{1,\varepsilon}$, provided that the right hand side of (1.10) is finite.

It seems likely that the limit of h -path processes as $\varepsilon \rightarrow 0$ is the minimizer of (1.3) with $L(u) = |u|^2$. But it is not trivial that this limit is a function of t and X_o since a continuous strong Markov process which is of bounded variation in time is not always a function of the initial point and time (see [23] and also [19]). Therefore we prove that the limit of $X_\varepsilon(1)$ as $\varepsilon \rightarrow 0$ is a function of X_o .

If $P_0(dx)$ is absolutely continuous with respect to dx and $L(u) = |u|^2$, then (1.1) and (1.2) have the unique minimizers $D\varphi(x)$ and $P_0(dx)\delta_{D\varphi(x)}(dy)$ respectively, where $\varphi : \mathbf{R}^d \mapsto (-\infty, \infty]$ is convex (see [3, 4], and also [8, 15, 16, 20, 21, 25] and the reference therein, and also [18, 19] for the continuum limit of (1.3)).

In this paper, independently of known results on the Monge-Kantorovich problem, we show that $V_\varepsilon(P_0, P_{1,\varepsilon})$ converges to $V(P_0, P_1)$ and $X_\varepsilon(1)$ converges, in L^2 , to the minimizer of (1.1) as $\varepsilon \rightarrow 0$, when $L(u) = |u|^2$. As a by-product, we give a new proof of the existence of the minimizer of (1.1) with $L(u) = |u|^2$.

From a probabilistic interest, replacing $P_{1,\varepsilon}$ by P_1 in (1.6)-(1.10), we also show the similar result to above.

When $L(u) = |u|$, in [9] they studied (1.2) by the “ $p \rightarrow \infty$ ” limit of the minimization problem for which the Euler-Lagrange equation is the p -Laplacian PDE under the assumption that P_0 and P_1 have disjoint compact supports, and in [6] and [24] they studied (1.2) by the “ $q \downarrow 1$ ” limit of (1.2) with $L(u) = |u|^q$ under the assumption that P_0 and P_1 have compact supports (see also [1]).

In future we would like to study the zero noise limit of the minimizer of (1.10) with a more general cost function $L(u)$, instead of $|u|^2$, and then apply the result to Monge’s problem.

In section 2 we give our main result which will be proved in section 3.

2 Main Result.

In this section we give our main result. We first state assumptions.

(A.0) P_0 and P_1 are Borel probability measures on $\mathbf{R}^d$ such that the following holds:

$$\int_{\mathbf{R}^d} |x|^2 (P_0(dx) + P_1(dx)) < \infty.$$

(A.1) $p_0(x) := P_0(dx)/dx$ exists.

Then the following holds.

Theorem 2.1 *Suppose that (A.0) holds. Then $\{\mu_\varepsilon\}_{\varepsilon>0}$ is tight, and any weak limit point of $\{\mu_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \rightarrow 0$ is supported on a cyclically monotone set.*

For the readers' convenience, we introduce the following.

Definition 2.1 *The set $A \in \mathbf{R}^d \times \mathbf{R}^d$ is called cyclically monotone if for any $n \geq 1$ and any $(x_i, y_i) \in A$ ($i = 1, \dots, n$),*

$$\sum_{i=1}^n \langle y_i, x_{i+1} - x_i \rangle \leq 0 \quad (2.1)$$

(see e. g. [25, p. 80]), where $x_{n+1} := x_1$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^d$.

Since a cyclically monotone set in $\mathbf{R}^d \times \mathbf{R}^d$ is contained in the subdifferential of a proper lower semicontinuous convex function on $\mathbf{R}^d$ and since a proper convex function is differentiable dx -a.e. in the interior of its domain (see [25, pp. 52, 82]), we obtain the following.

Corollary 2.1 *Suppose that (A.0) and (A.1) hold. Then for any weak limit point μ of $\{\mu_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \rightarrow 0$, there exists a proper lower semicontinuous convex function $\varphi : \mathbf{R}^d \mapsto (-\infty, \infty]$ such that*

$$\mu(dxdy) = P_0(dx)\delta_{D\varphi(x)}(dy). \quad (2.2)$$

Remark 2.1 *If (A.1) holds and $p_1(y) := P_1(dy)/dy$ exists, then Corollary 2.1 gives a new proof of the existence to the following Monge-Ampère equation:*

$$p_0(x) = p_1(D\varphi(x)) \det(D^2\varphi(x)) \quad (2.3)$$

in the sense that $P_0(D\varphi)^{-1} = P_1$, where $D^2 := (\partial^2/\partial x_i \partial x_j)_{i,j=1}^d$.

The following which can be proved from Theorem 2.1 and Corollary 2.1, independently of known results on the Monge-Kantorovich problem [1, 3, 4, 15, 16, 21], is our main result.

Theorem 2.2 *Suppose that (A.0) and (A.1) hold, and that $L(u) = |u|^2$. Then*

$$\lim_{\varepsilon \rightarrow 0} V_\varepsilon(P_0, P_{1,\varepsilon}) = V(P_0, P_1) < \infty. \quad (2.4)$$

In particular, $D\varphi$ in Corollary 2.1 is the unique minimizer of (1.1), and the following holds:

$$\lim_{\varepsilon \rightarrow 0} E\left[\int_0^1 |b_\varepsilon(t, X_\varepsilon(t)) - (D\varphi(X_o) - X_o)|^2 dt\right] = 0, \quad (2.5)$$

$$\lim_{\varepsilon \rightarrow 0} E\left[\sup_{0 \leq t \leq 1} |X_\varepsilon(t) - \{X_o + t(D\varphi(X_o) - X_o)\}|^2\right] = 0. \quad (2.6)$$

The following is known on (1.1)-(1.3) with $L(u) = |u|^2$.

(i) Suppose that (A.0) holds. Then a probability measure supported on a cyclically monotone set in $\mathbf{R}^d \times \mathbf{R}^d$ is a minimizer of (1.2) (see [15, 16] and also [25, pp. 66, 82], [1, Theorem 3.2]).

(ii) Suppose that (A.0) and (A.1) hold. Then there exists a convex function φ such that $P_0(dx)\delta_{D\varphi(x)}(dy)$ is the unique minimizer of (1.2) (see [3, 4]).

Using these facts, we have the following.

Corollary 2.2 (i) Suppose that (A.0) holds and that $L(u) = |u|^2$. Then any weak limit point of $\{\mu_\varepsilon\}_{\varepsilon > 0}$ as $\varepsilon \rightarrow 0$ is a minimizer of (1.2). (ii) Suppose in addition that (A.1) holds. Then μ_ε weakly converges to the unique minimizer of (1.2) as $\varepsilon \rightarrow 0$, and $X_o + t(D\varphi(X_o) - X_o)$ in (2.6) is the unique minimizer of (1.3).

To replace a terminal distribution $P_{1,\varepsilon}$ by P_1 in Theorems 2.1-2.2, we need extra assumptions.

(A.2) $p_1(x) := P_1(dx)/dx$ exists.

(A.3)

$$\int_{\mathbf{R}^d} \log\left(\frac{P_1(dx)}{dx}\right) P_1(dx) < \infty.$$

Replace $P_{1,\varepsilon}$ by P_1 in (1.6)-(1.7). Then there exists the unique solution $\tilde{X}_\varepsilon(t)$ to (1.7) from (A.2), and $V_\varepsilon(P_0, P_1)$ is finite from (A.3) (see Lemma 3.4). Besides, the following holds.

Proposition 2.1 Suppose that (A.0)-(A.3) hold, and replace X_ε by $\tilde{X}_\varepsilon$ in (2.6). Then (2.6) still holds.

3 Proof.

In this section we prove our results stated in section 2.

We first state and prove technical lemmas to prove Theorem 2.1. For $x, y \in \mathbf{R}^d$, $m \geq 1$ and $\varepsilon > 0$, put

$$H_{m,\varepsilon}(x, y) := \varepsilon \log \left\{ \iint_{U_m(o) \times U_m(o)} \exp \left(\frac{\langle x, z_2 \rangle + \langle y, z_1 \rangle}{\varepsilon} - \frac{\langle z_1, z_2 \rangle}{\varepsilon} \right) \mu_\varepsilon(dz_1 dz_2) \right\}, \quad (3.1)$$

$$H_{i,m,\varepsilon}(x) := \varepsilon \log \left(\int_{U_m(o)} g_\varepsilon(x - y) \nu_{j,\varepsilon}(dy) \right) + \frac{|x|^2}{2} \quad (i, j = 0, 1, i \neq j), \quad (3.2)$$

$$\mu_{0,m,\varepsilon}(dx) := \mu_\varepsilon(dx \times U_m(o)), \quad \mu_{1,m,\varepsilon}(dy) := \mu_\varepsilon(U_m(o) \times dy), \quad (3.3)$$

where $U_m(o) := \{x \in \mathbf{R}^d : |x| < m\}$. Then the following holds.

Lemma 3.1 (i) For $x, y \in \mathbf{R}^d$, $m \geq 1$ and $\varepsilon > 0$,

$$\begin{aligned} H_{m,\varepsilon}(x, y) &= H_{0,m,\varepsilon}(x) + H_{1,m,\varepsilon}(y) + \varepsilon \log \sqrt{2\pi\varepsilon}^d \\ &= \varepsilon \log \left\{ \iint_{U_m(o) \times U_m(o)} \exp \left(\frac{\langle x, z_2 \rangle + \langle y, z_1 \rangle}{\varepsilon} - \frac{H_{m,\varepsilon}(z_1, z_2)}{\varepsilon} \right) \mu_{0,m,\varepsilon}(dz_1) \mu_{1,m,\varepsilon}(dz_2) \right\}, \end{aligned} \quad (3.4)$$

$$\mu_\varepsilon(dxdy) = \exp \left(\frac{1}{\varepsilon} (\langle x, y \rangle - H_{m,\varepsilon}(x, y)) \right) \mu_{0,m,\varepsilon}(dx) \mu_{1,m,\varepsilon}(dy), \quad (3.5)$$

provided that $\mu_\varepsilon(U_m(o) \times U_m(o)) > 0$.

(ii) For $m \geq 1$ and $\varepsilon > 0$, $(x, y) \mapsto H_{m,\varepsilon}(x, y)$ is convex, and for any x and $y \in \mathbf{R}^d$,

$$|H_{m,\varepsilon}(x, y)| \leq (|x| + |y|)m + m^2 - \varepsilon \log \mu_\varepsilon(U_m(o) \times U_m(o)). \quad (3.6)$$

Proof. The first equality in (3.4) and (3.6) can be obtained from (3.1)-(3.2) easily. (3.5) holds from (1.9), the first equality in (3.4) and the following: for $i, j = 0, 1$ for which $i \neq j$,

$$\frac{\mu_{i,m,\varepsilon}(dx)}{\nu_{i,\varepsilon}(dx)} = \int_{U_m(o)} g_\varepsilon(x-y) \nu_{j,\varepsilon}(dy) = \exp\left(\frac{1}{\varepsilon} \left(H_{i,m,\varepsilon}(x) - \frac{|x|^2}{2}\right)\right). \quad (3.7)$$

The second equality in (3.4) can be obtained from (3.1) and (3.5).

Q. E. D.

Remark 3.1 For $x \in \mathbf{R}^d$, $m \geq 1$, $\varepsilon > 0$, and $i, j = 0, 1$ ($i \neq j$),

$$H_{i,m,\varepsilon}(x) = \varepsilon \log\left(\int_{U_m(o)} \frac{1}{\sqrt{2\pi\varepsilon}^d} \exp\left(\frac{1}{\varepsilon} \langle x, y \rangle - H_{j,m,\varepsilon}(y)\right) \mu_{j,m,\varepsilon}(dy)\right)$$

from (3.2) and (3.7).

Lemma 3.2 Suppose that (A.0) holds. Then for any sequence $\{\varepsilon_n\}_{n \geq 1}$ for which $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, there exist $m_0 \geq 1$ and subsequences $\{\{\varepsilon_{m,n}\}_{n \geq 1}\}_{m \geq m_0}$ such that $H_{m,\varepsilon_{m,n}}$ is convergent in $C(\mathbf{R}^d \times \mathbf{R}^d)$ as $n \rightarrow \infty$ for all $m \geq m_0$, and such that

$$\{\varepsilon_{m+1,n}\}_{n \geq 1} \subset \{\varepsilon_{m,n}\}_{n \geq 1} \quad (m \geq m_0). \quad (3.8)$$

In particular, $m \mapsto H_m := \lim_{n \rightarrow \infty} H_{m,\varepsilon_{m,n}}$ is nondecreasing on $\{m_0, m_0 + 1, \dots\}$,

$$(x, y) \mapsto H(x, y) := \lim_{m \rightarrow \infty} H_m(x, y) \in (-\infty, \infty] \quad (3.9)$$

is a lower semicontinuous convex function,

$$\langle x, y \rangle - H(x, y) \leq 0 \quad ((x, y) \in \text{supp}(P_0) \times \text{supp}(P_1)), \quad (3.10)$$

and the following set is cyclically monotone:

$$S := \{(x, y) \in \text{supp}(P_0) \times \text{supp}(P_1) \mid \langle x, y \rangle = H(x, y)\}. \quad (3.11)$$

Proof. There exist $m_0 \geq 1$ such that for any $m \geq m_0$, $\{H_{m,\varepsilon_n}\}_{n \geq 1}$ is bounded in $U_{\ell+1}(o) \times U_{\ell+1}(o)$ for any $\ell \geq 1$ from (3.6) and from the following:

$$\begin{aligned}
& 1 - \mu_\varepsilon(U_m(o) \times U_m(o)) \tag{3.12} \\
& \leq \frac{\iint_{\mathbf{R}^d \times \mathbf{R}^d} (|x|^2 + |y|^2) \mu_\varepsilon(dx dy)}{m^2} = \frac{\int_{\mathbf{R}^d} |x|^2 P_0(dx) + \int_{\mathbf{R}^d} |y|^2 P_{1,\varepsilon}(dy)}{m^2} \\
& = \frac{\int_{\mathbf{R}^d} |x|^2 P_0(dx) + \iint_{\mathbf{R}^d \times \mathbf{R}^d} |x+y|^2 g_\varepsilon(x) dx P_1(dy)}{m^2} \\
& \leq \frac{\int_{\mathbf{R}^d} |x|^2 P_0(dx) + 2(\varepsilon d + \int_{\mathbf{R}^d} |y|^2 P_1(dy))}{m^2} \rightarrow 0 \quad (\text{as } m \rightarrow \infty \text{ from (A.0)}).
\end{aligned}$$

Hence for any $m \geq m_0$, $\{H_{m,\varepsilon_n}\}_{n \geq 1}$ contains a uniformly convergent subsequence on $U_\ell(o) \times U_\ell(o)$ (see [2, p. 21, Theorem 3.2]). By the diagonal argument, $\{H_{m,\varepsilon_n}\}_{n \geq 1}$ contains a convergent subsequence $\{H_{m,\varepsilon_{m,n}}\}_{n \geq 1}$ in $C(\mathbf{R}^d \times \mathbf{R}^d)$. In particular, we can take $\{\varepsilon_{m,n}\}_{n \geq 1}$ so that (3.8) holds.

$m \mapsto H_m$ is nondecreasing on $\{m_0, m_0 + 1, \dots\}$ since

$$H_{m+1,\varepsilon_{m+1,n}} \geq H_{m,\varepsilon_{m+1,n}}$$

for all $m \geq m_0$ from (3.1), and since $H_{m,\varepsilon_{m+1,n}} \rightarrow H_m$ as $n \rightarrow \infty$ from (3.8). Hence for any $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$, $H_m(x, y)$ is convergent or diverges to ∞ as $m \rightarrow \infty$.

As the limit of convex functions, H in (3.9) is convex in $\mathbf{R}^d \times \mathbf{R}^d$. H is also lower semicontinuous. Indeed, if $(x_n, y_n) \rightarrow (x, y)$ as $n \rightarrow \infty$, then

$$\begin{aligned}
H(x_n, y_n) & \geq H_m(x_n, y_n) \rightarrow H_m(x, y) \quad (\text{as } n \rightarrow \infty, \text{ for all } m \geq m_0) \\
& \rightarrow H(x, y) \quad (\text{as } m \rightarrow \infty)
\end{aligned}$$

since $H_m \in C(\mathbf{R}^d \times \mathbf{R}^d)$ as a finite convex function (see (3.6)).

For any $(x, y) \in \text{supp}(P_0) \times \text{supp}(P_1)$, $r > 0$, $m \geq r + |x| + |y| + m_0$ and $n \geq 1$, from the second equality of (3.4),

$$\begin{aligned}
& H_{m,\varepsilon_{m,n}}(x, y) \tag{3.13} \\
& \geq \inf_{(z_1, z_2) \in U_r(x) \times U_r(y)} \{ \langle x, z_2 \rangle + \langle y, z_1 \rangle - H_{m,\varepsilon_{m,n}}(z_1, z_2) \} \\
& \quad + \varepsilon \log \{ \mu_{0,m,\varepsilon_{m,n}}(U_r(x)) \mu_{1,m,\varepsilon_{m,n}}(U_r(y)) \}.
\end{aligned}$$

Since $H_{m,\varepsilon_{m,n}}$ converges to H_m as $n \rightarrow \infty$, uniformly on every compact subset of $\mathbf{R}^d \times \mathbf{R}^d$,

$$\begin{aligned}
& \inf_{(z_1, z_2) \in U_r(x) \times U_r(y)} (< x, z_2 > + < y, z_1 > - H_{m,\varepsilon_{m,n}}(z_1, z_2)) \quad (3.14) \\
\rightarrow & \inf_{(z_1, z_2) \in U_r(x) \times U_r(y)} (< x, z_2 > + < y, z_1 > - H_m(z_1, z_2)) \quad (\text{as } n \rightarrow \infty) \\
\rightarrow & 2 < x, y > - H_m(x, y) \quad (\text{as } r \rightarrow 0) \\
\rightarrow & 2 < x, y > - H(x, y) \quad (\text{as } m \rightarrow \infty).
\end{aligned}$$

From (A.0), for sufficiently large $m \geq 1$,

$$\liminf_{\varepsilon \rightarrow 0} \{\mu_{0,m,\varepsilon}(U_r(x))\mu_{1,m,\varepsilon}(U_r(y))\} > 0. \quad (3.15)$$

Indeed,

$$\begin{aligned}
& \mu_{0,m,\varepsilon}(U_r(x))\mu_{1,m,\varepsilon}(U_r(y)) \\
= & \{P_0(U_r(x)) - \mu_\varepsilon(U_r(x) \times U_m(o)^c)\}\{P_{1,\varepsilon}(U_r(y)) - \mu_\varepsilon(U_m(o)^c \times U_r(y))\}.
\end{aligned}$$

$$\mu_\varepsilon(U_r(x) \times U_m(o)^c) \leq \frac{1}{m^2} \int_{\mathbf{R}^d} |z|^2 P_{1,\varepsilon}(dz) \leq \frac{2(\varepsilon + \int_{\mathbf{R}^d} |z|^2 P_1(dz))}{m^2}$$

as in (3.12), and

$$\mu_\varepsilon(U_m(o)^c \times U_r(y)) \leq \frac{1}{m^2} \int_{\mathbf{R}^d} |z|^2 P_0(dz).$$

$$\liminf_{\varepsilon \rightarrow 0} P_{1,\varepsilon}(U_r(y)) \geq P_1(U_r(y))$$

since $P_{1,\varepsilon}$ weakly converges to P_1 as $\varepsilon \rightarrow 0$, and

$$(P_0 \times P_1)(U_r(x) \times U_r(y)) > 0.$$

(3.13)-(3.15) implies (3.10).

The set S is cyclically monotone. Indeed, for any $k, n \geq 1$, $(x_1, y_1), \dots, (x_k, y_k) \in S$ and $m \geq m_0$, putting $x_{k+1} := x_1$,

$$\sum_{i=1}^k (H_{m,\varepsilon_{m,n}}(x_{i+1}, y_i) - H_{m,\varepsilon_{m,n}}(x_i, y_i)) = 0 \quad (3.16)$$

from the first equality of (3.4). Let $n \rightarrow \infty$ and then $m \rightarrow \infty$. Then from (3.10),

$$\sum_{i=1}^k \langle y_i, x_{i+1} - x_i \rangle \leq \sum_{i=1}^k (H(x_{i+1}, y_i) - H(x_i, y_i)) = 0. \quad (3.17)$$

(Notice that $H(x_i, y_i)$ is finite for all $i = 1, \dots, k$.)

Q. E. D.

Remark 3.2 *If $H(x, y)$ and $H(a, b)$ are finite, then $H(x, b)$ and $H(a, y)$ are also finite since for sufficiently large $m \geq 1$, from (3.9) and (3.16),*

$$-\infty < H_m(x, b) + H_m(a, y) \leq H(x, b) + H(a, y) = H(x, y) + H(a, b) < \infty.$$

In particular,

$$H(x, y) = H(x, b) + H(a, y) - H(a, b).$$

(Proof of Theorem 2.1.) $\{\mu_\varepsilon\}_{\varepsilon>0}$ is tight from (3.12) (see e.g. [12, p. 7]). Take a weakly convergent subsequence $\{\mu_{\varepsilon_n}\}_{n \geq 1}$ and denote by μ its weak limit, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

By taking $m_0 \geq 1$ and subsequences $\{\varepsilon_{m,n}\}_{n \geq 1}$ ($m \geq m_0$), construct a convex function H as in Lemma 3.2.

From (3.10)-(3.11), we only have to show the following to complete the proof:

$$\mu(\{(x, y) \mid \langle x, y \rangle - H(x, y) < 0\}) = 0. \quad (3.18)$$

By the monotone convergence theorem and Lemma 3.2,

$$\begin{aligned} & \mu(\{(x, y) \mid \langle x, y \rangle - H(x, y) < 0\}) \\ &= \lim_{r \downarrow 0} (\lim_{m \uparrow \infty} \mu(\{(x, y) \mid \langle x, y \rangle - H_m(x, y) < -r\})). \end{aligned} \quad (3.19)$$

For any $m \geq m_0$, $H_{m,\varepsilon_{m,n}}$ converges to H_m as $n \rightarrow \infty$, uniformly on every compact subset of $\mathbf{R}^d \times \mathbf{R}^d$. Therefore for any $R > 0$,

$$\begin{aligned}
& \mu(\{(x, y) \mid |x, y| < -H_m(x, y) < -r, |x|, |y| < R\}) \\
& \leq \liminf_{n \rightarrow \infty} \mu_{\varepsilon_{m,n}}(\{(x, y) \mid |x, y| < -H_m(x, y) < -r, |x|, |y| < R\}) \\
& \leq \liminf_{n \rightarrow \infty} \mu_{\varepsilon_{m,n}}(\{(x, y) \mid |x, y| < -H_{m,\varepsilon_{m,n}}(x, y) < -r/2, |x|, |y| < R\}) \\
& \leq \liminf_{n \rightarrow \infty} \exp\left(-\frac{r}{2\varepsilon_{m,n}}\right) = 0 \quad (\text{from (3.5)}).
\end{aligned} \tag{3.20}$$

Notice that the set $\{(x, y) \mid |x, y| < -H_m(x, y) < -r, |x|, |y| < R\}$ is open since $H_m \in C(\mathbf{R}^d \times \mathbf{R}^d)$ from Lemma 3.1, (ii).

Letting $R \rightarrow \infty$ in (3.20), we obtain (3.18) from (3.19).

Q. E. D.

Next we prove Theorem 2.2.

(Proof of Theorem 2.2). The proof of (2.4) is divided into the following:

$$\liminf_{\varepsilon \rightarrow 0} V_\varepsilon(P_0, P_{1,\varepsilon}) \geq V(P_0, P_1), \tag{3.21}$$

$$\limsup_{\varepsilon \rightarrow 0} V_\varepsilon(P_0, P_{1,\varepsilon}) \leq V(P_0, P_1) < \infty. \tag{3.22}$$

To prove (3.21), we only have to show that for any $\{\varepsilon_n\}_{n \geq 1}$ for which $\varepsilon_n \rightarrow 0$ and $E[\int_0^1 |b_{\varepsilon_n}(s, X_{\varepsilon_n}(s))|^2 ds]$ is convergent as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} E[\int_0^1 |b_{\varepsilon_n}(s, X_{\varepsilon_n}(s))|^2 ds] \geq V(P_0, P_1) \tag{3.23}$$

(see (1.7) for notation). (3.23) holds since $\{X_{\varepsilon_n}(\cdot)\}_{n \geq 1}$ is tight in $C([0, 1])$, any weak limit point $X(\cdot)$ of $\{X_{\varepsilon_n}(\cdot)\}_{n \geq 1}$ is an absolutely continuous stochastic process (see e.g. [19, Lemmas 2-3]), and

$$\begin{aligned}
& \lim_{n \rightarrow \infty} E[\int_0^1 |b_{\varepsilon_n}(s, X_{\varepsilon_n}(s))|^2 ds] \\
& \geq E[\int_0^1 \left|\frac{dX(s)}{ds}\right|^2 ds] \geq E[|X(1) - X(0)|^2] \geq V(P_0, P_1)
\end{aligned} \tag{3.24}$$

from (1.9) and (2.2) (see e.g. [19, the proof of (3.17)]).

Next we prove (3.22). Take ψ for which $P_0\psi^{-1} = P_1$, which is possible from (2.2). Then from (A.0),

$$V(P_0, P_1) \leq E[|\psi(X_o) - X_o|^2] \leq 2 \int_{\mathbf{R}^d} |x|^2 (P_0(dx) + P_1(dx)) < \infty. \quad (3.25)$$

Put

$$X_{\varepsilon, \psi}(t) := X_o + t(\psi(X_o) - X_o) + \sqrt{\varepsilon}W(t). \quad (3.26)$$

Then $P(X_{\varepsilon, \psi}(1))^{-1} = P_{1, \varepsilon}$, which implies (3.22).

By (2.2), (2.4) and (3.24), $D\varphi$ in Corollary 2.1 is a minimizer of (1.1) with $L(u) = |u|^2$. In particular, $D\varphi$ is the unique minimizer. Indeed, if ψ is a minimizer of (1.1) with $L(u) = |u|^2$, then

$$\begin{aligned} E[\langle X_o, \psi(X_o) \rangle] &= E[\langle X_o, D\varphi(X_o) \rangle] \\ &= E[\varphi(X_o) + \varphi^*(D\varphi(X_o))] = E[\varphi(X_o) + \varphi^*(\psi(X_o))], \end{aligned}$$

which implies that $\psi(X_o) \in \partial\varphi(X_o)$ a.s., where

$$\varphi^*(y) := \sup_{x \in \mathbf{R}^d} \{\langle x, y \rangle - \varphi(x)\},$$

$$\partial\varphi(x) := \{p \in \mathbf{R}^d | \varphi(y) \geq \varphi(x) + \langle p, y - x \rangle \text{ for all } y \in \mathbf{R}^d\}.$$

Here we used the fact that for any $(x, y) \in \mathbf{R}^d \times \mathbf{R}^d$,

$$\langle x, y \rangle \leq \varphi(x) + \varphi^*(y),$$

where the equality holds if and only if $y \in \partial\varphi(x)$ (see e.g. [25]). From (A.1), $\psi(X_o) = D\varphi(X_o)$ a.s. since a proper convex function is differentiable dx -a.e. in the interior of its domain (see [25, pp. 52]).

(2.5)-(2.6) is an easy consequence of (2.4). For $t \in [0, 1]$,

$$\begin{aligned} & |X_{\varepsilon}(t) - \{X_o + t(D\varphi(X_o) - X_o)\}| \\ & \leq \int_0^1 |b_{\varepsilon}(s, X_{\varepsilon}(s)) - (D\varphi(X_o) - X_o)| ds + \sqrt{\varepsilon} \sup_{0 \leq t \leq 1} |W(t)|. \end{aligned} \quad (3.27)$$

$$E\left[\sup_{0 \leq t \leq 1} |W(t)|^2\right] \leq 4d \quad (3.28)$$

(see e.g. [12, p. 34]), and from (2.4),

$$\begin{aligned} & E\left[\int_0^1 |b_\varepsilon(s, X_\varepsilon(s)) - (D\varphi(X_o) - X_o)|^2 ds\right] \\ = & E\left[\int_0^1 |b_\varepsilon(s, X_\varepsilon(s))|^2 ds + |D\varphi(X_o) - X_o|^2\right] \\ & - 2E[< X_\varepsilon(1) - X_o - \sqrt{\varepsilon}W(1), D\varphi(X_o) - X_o >] \\ \rightarrow & 2V(P_0, P_1) - 2E[< D\varphi(X_o) - X_o, D\varphi(X_o) - X_o >] = 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (3.29)$$

Indeed,

$$E[< W(1), D\varphi(X_o) - X_o >] = < E[W(1)], E[D\varphi(X_o) - X_o] > = 0.$$

For any $R > 0$, taking $f_R \in C(\mathbf{R}^d : [0, 1])$ for which $f_R(x) = 1$ ($|x| \leq R$) and $f_R(x) = 0$ ($|x| \geq R + 1$),

$$\begin{aligned} & E[< X_\varepsilon(1), D\varphi(X_o) - X_o >] \\ = & E[< X_\varepsilon(1), D\varphi(X_o) - X_o > (1 - f_R(X_\varepsilon(1))f_R(X_o))] \\ & + E[< X_\varepsilon(1), D\varphi(X_\varepsilon(0)) - X_\varepsilon(0) > f_R(X_\varepsilon(1))f_R(X_\varepsilon(0))]. \end{aligned}$$

$$\begin{aligned} & E[|< X_\varepsilon(1), D\varphi(X_o) - X_o > (1 - f_R(X_\varepsilon(1))f_R(X_o))|] \\ \leq & \sqrt{E[|D\varphi(X_o) - X_o|^2]E[|X_\varepsilon(1)|^2 : |X_\varepsilon(1)| \geq R]} \\ & + \sqrt{E[|X_\varepsilon(1)|^2]E[|D\varphi(X_o) - X_o|^2 : |X_o| \geq R]} \rightarrow 0 \quad \text{as } R \rightarrow 0 \end{aligned}$$

uniformly in $\varepsilon \in [0, 1]$. Since $(X_\varepsilon(0), X_\varepsilon(1))$ weakly converges to $(X_o, D\varphi(X_o))$ as $\varepsilon \rightarrow 0$ by the uniqueness of the minimizer of $V(P_0, P_1)$, one can assume, by taking a new probability space $(\tilde{\Omega}, \tilde{\mathbf{B}}, \tilde{P})$, that $(X_\varepsilon(0), X_\varepsilon(1))$ converges to $(X_o, D\varphi(X_o))$ as $\varepsilon \rightarrow 0$, $\tilde{P}$ -a.s., by Skhorohod's theorem (see e.g. [12, p. 9]). Put

$$A := \{y \in \mathbf{R}^d | \varphi(y) < \infty, \partial\varphi(y) = \{D\varphi(y)\}\}.$$

Then $X_o \in A$ a.s. from (A.1) and $\cap_{r>0} \partial\varphi(U_r(x)) = \{D\varphi(x)\}$ for any $x \in A$ (see [25, p. 54]), from which the following holds:

$$\begin{aligned} & E[< X_\varepsilon(1), D\varphi(X_\varepsilon(0)) - X_\varepsilon(0) > f_R(X_\varepsilon(1))f_R(X_\varepsilon(0))] \\ &= \tilde{E}[< X_\varepsilon(1), D\varphi(X_\varepsilon(0)) - X_\varepsilon(0) > f_R(X_\varepsilon(1))f_R(X_\varepsilon(0)) : X_o \in A] \\ &\rightarrow \tilde{E}[< D\varphi(X_o), D\varphi(X_o) - X_o > f_R(D\varphi(X_o))f_R(X_o) : X_o \in A] \\ &\quad (\text{as } \varepsilon \rightarrow 0) \\ &\rightarrow E[< D\varphi(X_o), D\varphi(X_o) - X_o >] \quad (\text{as } R \rightarrow \infty). \end{aligned}$$

(3.27)-(3.29) implies (2.5)-(2.6).

Q. E. D.

We give technical lemmas and then prove Proposition 2.1.

Lemma 3.3 (see [17, Lemma 2.5]). *Suppose that (A.2) holds and replace $P_{1,\varepsilon}$ by P_1 in (1.6). Then for any $\varepsilon > 0$,*

$$V_\varepsilon(P_0, P_1) = 2\varepsilon E \left[\log \frac{h_\varepsilon(1, \tilde{X}_\varepsilon(1))}{h_\varepsilon(0, \tilde{X}_\varepsilon(0))} \right], \quad (3.30)$$

where $\tilde{X}_\varepsilon$ is the unique weak solution to (1.7),

$$h_\varepsilon(t, x) := \int_{\mathbf{R}^d} g_{\varepsilon(1-t)}(x - y) \tilde{\nu}_{1,\varepsilon}(dy),$$

and $(\tilde{\nu}_{0,\varepsilon}, \tilde{\nu}_{1,\varepsilon})$ is a solution to (1.6). $V_\varepsilon(P_0, P_1)$ is also the infimum of

$$\int_0^1 \int_{\mathbf{R}^d} |b(t, x)|^2 q(t, x) dt dx \quad (3.31)$$

over all (b, q) for which

$$q(t, x) \geq 0 \quad dx - a.e., \quad \int_{\mathbf{R}^d} q(t, x) dx = 1 \quad \text{for all } t \in [0, 1], \quad (3.32)$$

$$q(0, x) dx = P_0(dx), \quad q(1, x) dx = P_1(dx), \quad (3.33)$$

and for which the following holds: for any $f \in C_o^\infty(\mathbf{R}^d)$ and any $t \in [0, 1]$,

$$\begin{aligned} & \int_{\mathbf{R}^d} f(x)(q(t, x) - q(0, x))dx \\ &= \int_0^t ds \int_{\mathbf{R}^d} \left(\frac{\varepsilon}{2} \Delta f(x) + \langle b(t, x), Df(x) \rangle \right) q(s, x) dx, \end{aligned} \quad (3.34)$$

where $\Delta := \sum_{i=1}^d \partial^2 / \partial x_i^2$.

Remark 3.3 Suppose that (A.1) and (A.2) hold and that $\text{supp}(P_0) \cup \text{supp}(P_1)$ is bounded. Then it is known that $\tilde{V}(P_0, P_1)$ is the infimum of (3.31) over all (b, q) for which (3.32)-(3.34) hold for $\varepsilon = 0$ and for which $\cup_{0 \leq t \leq 1} \text{supp}(q(t, \cdot))$ is bounded (see [5] or [25, p. 239]).

Lemma 3.4 Suppose that (A.0), (A.2) and (A.3) hold. Then for any $\varepsilon > 0$, $V_\varepsilon(P_0, P_1)$ is finite. In particular, $V_1(P_{1,1}, P_1)$ is finite.

Proof. Put $\tilde{\mu}_\varepsilon(dx dy) := \tilde{\nu}_{0,\varepsilon}(dx) g_\varepsilon(x - y) \tilde{\nu}_{1,\varepsilon}(dy)$ (see (3.30) for notation). Replace μ_ε by $\tilde{\mu}_\varepsilon$ in (3.1) and denote by $\tilde{H}_{m,\varepsilon}$ a function obtained from (3.1). Then, from (3.4), (3.7) and (3.30),

$$\begin{aligned} V_\varepsilon(P_0, P_1) &= E[|\tilde{X}_\varepsilon(0)|^2 + |\tilde{X}_\varepsilon(1)|^2 - 2\tilde{H}_{\infty,\varepsilon}(\tilde{X}_\varepsilon(0), \tilde{X}_\varepsilon(1))] \quad (3.35) \\ &\quad + 2\varepsilon \int_{\mathbf{R}^d} \log\left(\frac{dP_1}{dx}\right) P_1(dx) + 2\varepsilon \log \sqrt{2\pi\varepsilon}^d. \end{aligned}$$

From (3.1), (3.6), (3.12) and (A.0), for sufficiently large $m \geq 1$,

$$E[\tilde{H}_{\infty,\varepsilon}(\tilde{X}_\varepsilon(0), \tilde{X}_\varepsilon(1))] \geq E[\tilde{H}_{m,\varepsilon}(\tilde{X}_\varepsilon(0), \tilde{X}_\varepsilon(1))] > -\infty. \quad (3.36)$$

Q. E. D.

(Proof of Proposition 2.1). Most part of the proof is almost the same as that of Theorem 2.2. The only thing we have to prove is the following:

$$\limsup_{\varepsilon \rightarrow 0} V_\varepsilon(P_0, P_1) \leq V(P_0, P_1). \quad (3.37)$$

Take ψ for which $P_0\psi^{-1} = P_1$, which is possible from (2.2). For $r \in (0, 1/2)$, solve Schrödinger's functional equation:

$$\begin{aligned} P_{1,\varepsilon(1-r)}(dx) &= \left(\int_{\mathbf{R}^d} g_{\varepsilon r}(x-y) \nu_{1,r,\varepsilon}(dy) \right) \nu_{0,r,\varepsilon}(dx), \\ P_1(dy) &= \left(\int_{\mathbf{R}^d} g_{\varepsilon r}(x-y) \nu_{0,r,\varepsilon}(dx) \right) \nu_{1,r,\varepsilon}(dy). \end{aligned} \quad (3.38)$$

For $t \in [0, 1-r]$, put

$$X_{r,\varepsilon}(t) := X_o + t \frac{\psi(X_o) - X_o}{1-r} + \sqrt{\varepsilon} W(t), \quad (3.39)$$

and solve the following: for $t \in [1-r, 1]$

$$X_{r,\varepsilon}(t) = X_{r,\varepsilon}(1-r) + \int_{1-r}^t b_{r,\varepsilon}(s, X_{r,\varepsilon}(s)) ds + \sqrt{\varepsilon}(W(t) - W(1-r)), \quad (3.40)$$

where

$$b_{r,\varepsilon}(s, x) := D_x \log \left(\int_{\mathbf{R}^d} g_{\varepsilon(1-s)}(x-y) \nu_{1,r,\varepsilon}(dy) \right).$$

Then, from Lemma 3.3,

$$V_\varepsilon(P_0, P_1) \leq \frac{E[|\psi(X_o) - X_o|^2]}{1-r} + E \left[\int_{1-r}^1 |b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))|^2 ds \right] \quad (3.41)$$

since $X_{r,\varepsilon}(0) = X_o$ and $PX_{r,\varepsilon}(1)^{-1} = P_1$.

We prove the following to complete the proof: for any $r \in (0, 1/2)$,

$$\lim_{\varepsilon \rightarrow 0} E \left[\int_{1-r}^1 |b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))|^2 ds \right] = 0. \quad (3.42)$$

$$p_{r,\varepsilon}(t, x) := \int_{\mathbf{R}^d} g_{\frac{\varepsilon(1-r)(1-t)}{r}}(x-y) P_1(dy) \quad (3.43)$$

is a weak solution to the following: for $t \in [1-r, 1)$,

$$\frac{\partial p_{r,\varepsilon}(t, x)}{\partial t} = \frac{\varepsilon}{2} \Delta p_{r,\varepsilon}(t, x) - \operatorname{div} \left\{ \left(\frac{\varepsilon}{2r} \right) \frac{D_x p_{r,\varepsilon}(t, x)}{p_{r,\varepsilon}(t, x)} p_{r,\varepsilon}(t, x) \right\}. \quad (3.44)$$

Hence, from Lemmas 3.3 and 3.4, for $\varepsilon < 1$,

$$\begin{aligned}
& E\left[\int_{1-r}^1 |b_{r,\varepsilon}(s, X_{r,\varepsilon}(s))|^2 ds\right] \\
& \leq \int_{1-r}^1 dt \int_{\mathbf{R}^d} \left| \left(\frac{\varepsilon}{2r}\right) \frac{D_x p_{r,\varepsilon}(t, x)}{p_{r,\varepsilon}(t, x)} \right|^2 p_{r,\varepsilon}(t, x) dx \\
& = \frac{\varepsilon}{4r(1-r)} \int_{1-\varepsilon(1-r)}^1 ds \int_{\mathbf{R}^d} \left| \frac{D_x p_{\frac{1}{2},1}(s, x)}{p_{\frac{1}{2},1}(s, x)} \right|^2 p_{\frac{1}{2},1}(s, x) dx \rightarrow 0 \quad (\text{as } \varepsilon \rightarrow 0),
\end{aligned} \tag{3.45}$$

where we used the following change of variable:

$$\frac{\varepsilon(1-r)(1-t)}{r} = 1-s,$$

and the following:

$$\begin{aligned}
& \int_{1-\varepsilon(1-r)}^1 ds \int_{\mathbf{R}^d} \left| \frac{D_x p_{\frac{1}{2},1}(s, x)}{p_{\frac{1}{2},1}(s, x)} \right|^2 p_{\frac{1}{2},1}(s, x) dx \\
& \leq \int_0^1 ds \int_{\mathbf{R}^d} \left| \frac{D_x p_{\frac{1}{2},1}(s, x)}{p_{\frac{1}{2},1}(s, x)} \right|^2 p_{\frac{1}{2},1}(s, x) dx = V_1(P_{1,1}, P_1) < \infty.
\end{aligned}$$

Q. E. D.

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