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Weighted Strichartz estimates and existence of self-similar solutions for semilinear wave equations

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Abstract

We study the existence of self-similar solutions to the Cauchy problem for semilinear wave equations with power type nonlinearity. Radially symmetric self-similar solutions are obtained in odd space dimensions when the power is greater than the critical one that are widely referred to in other existence problems of global solutions to nonlinear wave equations with small data. This result is a partial generalization of [11] to odd space dimensions. To construct self-similar solutions, we prove the weighted Strichartz estimates in terms of weak Lebesgue spaces over space-time.

1 Introduction and the main result

We consider the existence of self-similar solutions to the Cauchy problem for semilinear wave equations

$$\Box u = f(u), \quad (t,x) \in (0,\infty) \times \mathbf{R}^n \equiv \mathbf{R}^{1+n}_+, \tag{1.1}$$

$$u|_{t=0} = \varepsilon\phi, \ \partial_t u|_{t=0} = \varepsilon\psi, \quad x \in \mathbf{R}^n, \tag{1.2}$$

where $n \geq 2$, $\Box = \partial_t^2 - \Delta$ is the d'Alembertian with Laplacian Δ in $\mathbf{R}^n$, $\varepsilon > 0$ is a small parameter, and $f(u)$ is homogeneous of degree p with respect to u and

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satisfies the estimates

$$|f(u)| \le C|u|^p,$$
$$|f(u) - f(v)| \le C(|u|^{p-1} + |v|^{p-1})|u - v|,$$

where C is independent of u and v and $p > 1$. Typical examples of $f(u)$ are given by $\pm u^p$, $\pm|u|^p$, $\pm|u|^{p-1}\,u$, etc. We now illustrate how self-similarity comes up with these single power nonlinearities.

If u is a solution of the equation (1.1), then u_λ, defined by

$$u_\lambda(t,x) \equiv \lambda^{\frac{2}{p-1}} u(\lambda t, \lambda x),$$

is also a solution of (1.1) for any $\lambda > 0$. That is to say, the equation (1.1) is invariant with respect to the scale transform $u \mapsto u_\lambda$. In particular, a solution u is called a self-similar solution if $u_\lambda \equiv u$ for all $\lambda > 0$. From the definition, the Cauchy data of self-similar solutions must be homogeneous functions. In other words, we need to treat homogeneous functions as initial data to construct self-similar solutions to the Cauchy problem (1.1), (1.2). In this paper, we consider the data of the form

$$\phi(x) = C_1|x|^{-\frac{2}{p-1}}, \quad \psi(x) = C_2|x|^{-\frac{2}{p-1}-1} \tag{1.3}$$

with C_1, $C_2 \in \mathbf{R}$. The data (1.3) is the same as (3) in Pecher [10]. Moreover, the data (1.3) fall within the critical case concerning the decay rate at infinity in space. See Takamura [15], for example.

This type of homogeneous initial data is usefull to construct self-similar solutions. Such an idea for the construction of self-similar solutions of evolution equations goes back to [5], [1] for the Navier-Stokes equations and [2] for semilinear Schrödinger equations.

As for the existence of self-similar solutions to the Cauchy problem (1.1), (1.2), several results are known. First, Pecher [10] showed the existence of self-similar solutions for $p > (4+\sqrt{13}\,)/3$ when $n = 3$. This lower bound on p, which is denoted by $p_1(n)$ in general dimensions n, is the one that appeared in Mochizuki-Motai [9] in connection with the scattering theory for (1.1). To be more specific, $p_1(n)$ is given by the positive root of the following quadratic equation in p:

$$n(n-1)p^2 - (n^2 + 3n - 2)p + 2 = 0.$$

Pecher's result is extended for general dimensions by Ribaud-Youssfi [13].

Next, Pecher [11] also showed the existence of self-similar solutions for $1+\sqrt{2} < p \le 2$ when $n = 3$ and gave a counter-example indicating that the lower bound on p is sharp. This lower bound, which is denoted by $p_0(n)$ in general dimensions n, is known as the critical exponent concerning the existence of global solutions for compactly supported, smooth, small data. To be more specific, $p_0(n)$ is given by the positive root of the following quadratic equation in p:

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

Note that $p_0(n) < p_1(n)$ holds in all dimensions. Hidano [6] also showed the existence of self-similar solutions for $p_0(n) < p < \frac{n+3}{n-1}$ when $n = 2$, 3.

The purpose of this paper is to construct radially symmetric global solutions of the Cauchy problem (1.1), (1.2) with (1.3) for $p_0(n) < p < \frac{n+3}{n-1}$ in odd space dimensions.

Before stating our main result, we introduce weak Lebesgue spaces. Weak Lebesgue spaces L^p_w are defined by

$$L^p_w = \bigl\{ f \in L^1_{\mathrm{loc}}; \|f\|_{L^p_w} \equiv \sup_{\lambda>0} \lambda \bigl| \{x;\, |f(x)| > \lambda\} \bigr|^{1/p} < \infty \bigr\},$$

for $1 \le p < \infty$, where $|\cdot|$ denotes the Lebesgue measure. Although $\|\cdot\|_{L^p_w}$ does not satisfy the triangle inequality, there exists a norm equivalent to $\|\cdot\|_{L^p_w}$ for $p > 1$ and with this norm the space L^p_w becomes a Banach space.

Now we are in a position to state our main result.

Theorem 1. *Let $n \ge 3$ be odd and let $p_0(n) < p < \frac{n+3}{n-1}$. Then, there exists a unique solution u of the integral equation associated with the Cauchy problem (1.1), (1.2) with (1.3) such that*

$$|t^2 - |x|^2|^\gamma u \in L^{p+1}_w(\mathbf{R}^{1+n}_+),$$

if $\varepsilon > 0$ is sufficiently small, where $\gamma = \frac{1}{p-1} - \frac{n+1}{2(p+1)}$.

The norm of the weighted weak Lebesgue space to which the solution u belongs is invariant with respect to the scale transform $u \longmapsto u_\lambda$. This invariance is important to treat self-similar solutions and requires a direct use of the weight of homogeneous type. Since self-similar solutions u of (1.1) are to be homogeneous functions in time and space variables by definition, we observe that $|t^2 - |x|^2|^\gamma\, u$ does not belong to the usual Lebesgue spaces on $\mathbf{R}^{1+n}_+$, and therefore it is natural to use weak Lebesgue spaces instead.

Our method to prove Theorem 1 is based on the use of weighted Strichartz estimates in terms of weak Lebesgue spaces on $\mathbf{R}^{1+n}_+$. Since we obtain weighted Strichartz estimates only in odd dimensional and radially symmetric case, our main result is also restricted to these cases.

Part of the contents of this paper has been announced in [8].

2 Estimates of solutions for free wave equation

In this section, we show that solutions to the Cauchy problem for the free wave equation belong to some weighted weak Lebesgue spaces.

Let v be a solution of the following Cauchy problem of the free wave equation

$$\Box v = 0 \quad \text{in } \mathbf{R}^{1+n}_+, \tag{2.1}$$

$$v|_{t=0} = \phi, \quad \partial_t v|_{t=0} = \psi \quad \text{in } \mathbf{R}^n. \tag{2.2}$$

Throughout this section, we suppose that the Cauchy data ϕ and ψ are smooth functions away from the origin and are homogeneous of degrees $-\alpha$ and $-\alpha-1$, respectively, where $0 < \alpha < n-1$.

Theorem 2. *Let $n \geq 2$ and let α satisfy $\frac{n-1}{2} < \alpha < \min(\frac{n+1}{2}, n-1)$. Then, for $1 - \frac{\alpha+2}{n+1} < \frac{1}{q} < 1 - \frac{\alpha}{n-1}$, the solution v of (2.1), (2.2) satisfies*

$$\left|t^2 - |x|^2\right|^\gamma v \in L^q_w(\mathbf{R}^{1+n}_+),$$

where $\gamma = \frac{\alpha}{2} - \frac{n+1}{2q}$.

Remark 1. (1) *Let D^α_λ be the dilation operator defined by*

$$D^\alpha_\lambda v(t,x) = \lambda^\alpha v(\lambda t, \lambda x), \quad \lambda > 0.$$

Then $D^\alpha_\lambda v \equiv v$ holds for all $\lambda > 0$ by homogeneity. The condition $\gamma = \frac{\alpha}{2} - \frac{n+1}{2q}$ makes the norm of the function space to which v belongs invariant, i. e.

$$\left\| |t^2 - |x|^2|^\gamma D^\alpha_\lambda v \right\|_{L^q_w(\mathbf{R}^{1+n}_+)} = \left\| |t^2 - |x|^2|^\gamma v \right\|_{L^q_w(\mathbf{R}^{1+n}_+)}, \quad \lambda > 0.$$

(2) *When we apply Theorem 2 for nonlinear problem (1.1), (1.2) with $q = p+1$ and $\alpha = 2/(p-1)$, the condition $\frac{n-1}{2} < \alpha < \min(\frac{n+1}{2}, n-1)$ is equivalent to $\max(\frac{n+5}{n+1}, \frac{n+1}{n-1}) < p < \frac{n+3}{n-1}$. Note that the critical exponent $p_0(n)$ is greater than the lower bound of this interval, while the condition $1 - \frac{\alpha+2}{n+1} < \frac{1}{q} < 1 - \frac{\alpha}{n-1}$ is equivalent to $p_0(n) < p < \frac{n+3}{n-1}$.*

To prove Theorem 2 we use the following pointwise estimate of v.

Lemma 2.1. *Let $n \geq 2$ and let α satisfy $\frac{n-1}{2} < \alpha < \min(\frac{n+1}{2}, n-1)$. Then v satisfies the estimate*

$$|v(t,x)| \leq C(t+|x|)^{-\frac{n-1}{2}} |t-|x||^{-\alpha+\frac{n-1}{2}}, \quad (t,x) \in \mathbf{R}^{1+n}_+.$$

For the proof of Lemma 2.1, see [7, 8].

Proof of Theorem 2. From the definition of weak Lebesgue spaces, it suffices to show that

$$\sup_{\lambda>0} \lambda |\{(t,x) \in \mathbf{R}^{1+n}_+;\ |t^2-|x|^2|^\gamma\, |v(t,x)| > \lambda\}|^{1/q} < \infty. \tag{2.3}$$

Now we fix $\lambda > 0$ and we consider the distribution function dividing $\mathbf{R}^{1+n}_+$ into $(0, \lambda^{-\frac{q}{n+1}}] \times \mathbf{R}^n$ and $(\lambda^{-\frac{q}{n+1}}, \infty) \times \mathbf{R}^n$ and estimate contributions separately.

We first consider the case $t > \lambda^{-\frac{q}{n+1}}$. By Lemma 2.1 we have the estimate

$$|t^2-|x|^2|^\gamma\, |v(t,x)| \leq C\, t^{-\frac{n-1}{2}+\gamma}\, |t-|x||^{-\alpha+\frac{n-1}{2}+\gamma}. \tag{2.4}$$

Note that the inequalities

$$-\tfrac{n-1}{2} + \gamma < 0, \quad -\alpha + \tfrac{n-1}{2} + \gamma < 0$$

hold by assumption. Since $t^{-\frac{n-1}{2}+\gamma}\, |t-|x||^{-\alpha+\frac{n-1}{2}+\gamma} > \lambda$ is equivalent to

$$|t-|x|| < \lambda^{-1/(\alpha-\frac{n-1}{2}-\gamma)}\, t^{-(\frac{n-1}{2}-\gamma)/(\alpha-\frac{n-1}{2}-\gamma)} \equiv R_1(t,\lambda),$$

we estimate

$$\begin{aligned}
&|\{(t,x) \in (\lambda^{-\frac{q}{n+1}}, \infty) \times \mathbf{R}^n;\ |t^2-|x|^2|^\gamma\, |v(t,x)| > \lambda\}| \\
&\leq C \int_{\lambda^{-q/(n+1)}}^{\infty} \Big(\int_{t-R_1(t,\lambda)}^{t+R_1(t,\lambda)} r^{n-1}\, dr \Big) dt \\
&\leq C \int_{\lambda^{-q/(n+1)}}^{\infty} t^{n-1}\, R_1(t,\lambda) dt,
\end{aligned}$$

where we have used the fact that $R_1(t,\lambda) < t$, which is equivalent to $t > \lambda^{-\frac{q}{n+1}}$. The last integral converges and is evaluated by a constant multiple of λ^{-q}, since the assumption $\frac{1}{q} < 1 - \frac{\alpha}{n-1}$ implies

$$n - 1 - (\tfrac{n-1}{2} - \gamma)/(\alpha - \tfrac{n-1}{2} - \gamma) < -1.$$

In the case where $0 < t \le \lambda^{-\frac{q}{n+1}}$, we use the estimate

$$|t^2 - |x|^2|^\gamma \, |v(t,x)| \le C \, (t + |x|)^{-\frac{n-1}{2}+\gamma+\delta} \, |t - |x||^{-\alpha+\frac{n-1}{2}+\gamma-\delta}, \tag{2.5}$$

which follows from Lemma 2.1 for any $\delta > 0$, since $|t - |x|| \le t + |x|$.

Now we set $\delta = -\frac{\alpha}{2} + \frac{n-1}{2} + \frac{n}{2q}$. Then $\delta > 0$,

$$-\tfrac{n-1}{2} + \gamma + \delta = -\tfrac{1}{2q} < 0, \quad -\alpha + \tfrac{n-1}{2} + \gamma - \delta = -\tfrac{2n+1}{2q} < 0,$$

and the right hand side of (2.5) is bounded by a constant multiple of

$$t^{-\frac{1}{2q}} \, |t - |x||^{-\frac{2n+1}{2q}}.$$

Since $t^{-\frac{1}{2q}} \, |t - |x||^{-\frac{2n+1}{2q}} > \lambda$ is equivalent to

$$|t - |x|| < \lambda^{-\frac{2q}{2n+1}} \, t^{-\frac{1}{2n+1}} \equiv R_2(t,\lambda),$$

we estimate

$$\begin{aligned}
&|\{(t,x) \in (0, \lambda^{-\frac{q}{n+1}}] \times \mathbf{R}^n; \ |t^2 - |x|^2|^\gamma \, |v(t,x)| > \lambda\}| \\
&\le C \int_0^{\lambda^{-q/(n+1)}} \Big(\int_0^{t+R_2(t,\lambda)} r^{n-1} \, dr \Big) dt \\
&\le C \int_0^{\lambda^{-q/(n+1)}} R_2(t,\lambda)^n dt,
\end{aligned}$$

where we have used the fact that $R_2(t,\lambda) \ge t$, which is equivalent to $t \le \lambda^{-\frac{q}{n+1}}$. The last integral also converges and is evaluated by a constant multiple of λ^{-q}.

Therefore, combining the above estimates, we obtain (2.3). □

3 Weighted Strichartz estimates

In this section we show the weighted Strichartz estimates between weak Lebesgue spaces.

Let w be a solution to the following Cauchy problem of the inhomogeneous wave equations with zero data:

$$\Box w = F \quad \text{in } \mathbf{R}^{1+n}_+, \tag{3.1}$$

$$w|_{t=0} = \partial_t w|_{t=0} \equiv 0 \quad \text{in } \mathbf{R}^n. \tag{3.2}$$

For $1 \le q \le \infty$ we denote by q' the dual exponent defined by $1/q + 1/q' = 1$.

Theorem 3. *Let $n \geq 3$ be odd and let $2 < q < \frac{2(n+1)}{n-1}$. For $\frac{n-1}{q} < \alpha < \frac{n-1}{q'}$ we set*

$$a = \tfrac{\alpha}{2} - \tfrac{n+1}{2q}, \quad b = \tfrac{\alpha}{2} + \tfrac{n+1}{2q} - \tfrac{n-1}{2}.$$

Then, there exists a constant $C > 0$ such that

$$\big\| |t^2 - |x|^2|^a w \big\|_{L^q_w(\mathbf{R}^{1+n}_+)} \leq C \big\| |t^2 - |x|^2|^b F \big\|_{L^{q'}_w(\mathbf{R}^{1+n}_+)}, \tag{3.3}$$

for any function F satisfying the following conditions: (i) *$F(t, \cdot)$ is radial in space, i.e.*

$$F(t,x) = \widetilde{F}(t, |x|), \quad (t,x) \in \mathbf{R}^{1+n}_+,$$

where $\widetilde{F}$ is a function on $(0,\infty) \times (0,\infty)$.
(ii) *F is homogeneous of degree $-\alpha - 2$, i.e.*

$$F(\lambda t, \lambda x) = \lambda^{-\alpha-2} F(t,x), \quad (t,x) \in \mathbf{R}^{1+n}_+, \ \lambda > 0. \tag{3.4}$$

Remark 2. (1) *The exponents a and b are determined to make both norms in (3.3) invariant with respect to the following scale transforms which preserve the equation (3.1):*

$$w(t,x) \longmapsto \lambda^{\alpha} w(\lambda t, \lambda x), \quad F(t,x) \longmapsto \lambda^{\alpha+2} F(\lambda t, \lambda x).$$

This fact is consistent with the assumption (3.4) which implies the solution w is also invariant with respect to the scale transform above.
(2) *When we apply Theorem 3 to nonlinear problem (1.1), (1.2) with $q = p+1$ and $\alpha = 2/(p-1)$, where p is that of (1.1), then $a = \gamma$, $b = p\gamma$, and we have*

$$\big\| |t^2 - |x|^2|^{\gamma} w \big\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} \leq C \big\| |t^2 - |x|^2|^{p\gamma} F \big\|_{L^{\frac{p+1}{p}}_w(\mathbf{R}^{1+n}_+)}, \tag{3.5}$$

where $\gamma = \frac{1}{p-1} - \frac{n+1}{2(p+1)}$. Note that the condition $\alpha < \frac{n-1}{q'}$ is equivalent to $p > p_0(n)$.

In what follows, we explain the outline of the proof of Theorem 3. To prove Theorem 3 we first prepare the following lemma.

Lemma 3.1. *Let $n \geq 3$ be odd. For $2 < q \leq \frac{2(n+1)}{n-1}$ let a and b satisfy the following conditions:*

$$a - b + \tfrac{n+1}{q} = \tfrac{n-1}{2}, \quad \tfrac{n}{q} - \tfrac{n-1}{2} < b < \tfrac{1}{q}.$$

Then, there exists a constant $C > 0$ such that

$$\big\| |t^2 - |x|^2|^a w \big\|_{L^q(\mathbf{R}^{1+n}_+)} \leq C \big\| |t^2 - |x|^2|^b F \big\|_{L^{q'}(\mathbf{R}^{1+n}_+)}, \tag{3.6}$$

for any function F with radial symmetry in space.

A similar estimate to Lemma 3.1 has been shown by Georgiev-Lindblad-Sogge [4, Theorem 1.4]. See also Tataru [16]. In the above lemma their support condition $\operatorname{supp} F \subset \{(t,x);\ |x| < t\}$ is removed at the cost of an additional lower bound $b > \frac{n}{q} - \frac{n-1}{2}$. Although the proof of Lemma 3.1 is essentially the same as theirs, we give the proof for the completeness. As a new ingredient, we use the following lemma to overcome the difficulty caused by the lack of assumption concerning the support on F.

Lemma 3.2 ([14]). *Let $0 < \lambda < n$, $1 < r, s < \infty$. Let α and β satisfy $\alpha < n/s'$, $\beta < n/r'$, $\alpha + \beta \geq 0$, and $1/s + 1/r + (\lambda + \alpha + \beta)/n = 2$. Then,*

$$\Big| \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \frac{f(x)g(y)}{|x|^\alpha\, |x-y|^\lambda\, |y|^\beta} dx\, dy \Big| \leq C \|f\|_{L^s(\mathbf{R}^n)} \|g\|_{L^r(\mathbf{R}^n)}.$$

Proof of Lemma 3.1. For simplicity we use the notation F to denote $\widetilde{F}$. It is known that the solution w is explicitly represented as

$$w(t,r) = r^{-\frac{n-1}{2}} \int_0^t \int_{|t-s-r|}^{t-s+r} P_{\frac{n-3}{2}}(\mu)\, F(s,\lambda)\, \lambda^{\frac{n-1}{2}} d\lambda\, ds, \tag{3.7}$$

where $r = |x|$, $\mu = (r^2 + \lambda^2 - (t-s)^2)/2r\lambda$, and P_k is the Legendre polynomial of degree k defined by

$$P_k(z) = \frac{1}{2^k k!} \frac{d^k}{dz^k} (z^2 - 1)^k,$$

for $k \geq 0$. As for the representation (3.6) we refer the reader to Takamura [15, Lemma 2.2] with the Duhamel principle for instance. Here, we notice that

$$|P_{\frac{n-3}{2}}(\mu)| \leq 1 \quad \text{for } |t-s-r| \leq \lambda \leq t-s+r, \tag{3.8}$$

since $|P_k(z)| \leq 1$ for $|z| \leq 1$, and $|\mu| \leq 1$ for $|t-s-r| \leq \lambda \leq t-s+r$. To derive the estimate (3.6) it is sufficient to show that

$$\begin{aligned} &\Big| \int_0^\infty \int_0^\infty |t^2 - r^2|^a\, w(t,r)\, \Phi(t,r)\, r^{n-1} dr\, dt \Big| \\ &\qquad\qquad \leq C \| |t^2 - r^2|^b\, r^{\frac{n-1}{q'}} F \|_{L^{q'}} \| r^{\frac{n-1}{q'}}\, \Phi \|_{L^{q'}} \end{aligned} \tag{3.9}$$

for all $\Phi \in C_0^\infty((0,\infty) \times (0,\infty))$ by duality and radial symmetry. By the representation (3.7) and (3.8), the left hand side of (3.9) is bounded by

$$\begin{aligned} &C \int_0^\infty \int_0^\infty \int_0^t \int_{|t-s-r|}^{t-s+r} r^{\frac{n-1}{2}}\, \lambda^{\frac{n-1}{2}}\, |t^2 - r^2|^a\, |F(s,\lambda)| |\Phi(t,r)|\, d\lambda\, ds\, dr\, dt \\ &= C \int_0^\infty \int_0^\infty \int_0^t \int_{|t-s-r|}^{t-s+r} \frac{\big(|s^2 - \lambda^2|^b\, \lambda^{\frac{n-1}{q'}}\, |F(s,\lambda)|\big) \big(r^{\frac{n-1}{q'}}\, |\Phi(t,r)|\big)}{r^\delta\, \lambda^\delta\, |t^2 - r^2|^{-a}\, |s^2 - \lambda^2|^b}\, d\lambda\, ds\, dr\, dt, \end{aligned} \tag{3.10}$$

where $\delta = (n-1)(1/2 - 1/q)$. Then, applying the change of variables

$$u = t + r,\ v = t - r,\ \xi = s + \lambda,\ \eta = s - \lambda,$$

and the substitutions

$$|s^2 - \lambda^2|^b \, \lambda^{\frac{n-1}{q'}} \, |F(s,\lambda)| = G(\xi,\eta), \quad r^{\frac{n-1}{q'}} \, |\Phi(t,r)| = H(u,v),$$

we see that (3.10) equals to

$$C\Big(\int_0^\infty \int_0^u \int_v^u \int_{-\xi}^v + \int_0^\infty \int_{-u}^0 \int_{-v}^u \int_{-\xi}^v \Big) \times \frac{G(\xi,\eta)\, H(u,v)}{|u-v|^\delta \, |\xi-\eta|^\delta |u|^{-a} \, |v|^{-a} \, |\xi|^b \, |\eta|^b} \, d\eta \, d\xi \, dv \, du. \tag{3.11}$$

In both of the domains of the integration above, the condition $\eta \le v \le \xi \le u$ holds and therefore

$$|u-v|^{-\delta} \le |u-\xi|^{-\delta}, \quad |\xi-\eta|^{-\delta} \le |v-\eta|^{-\delta}, \tag{3.12}$$

since $\delta > 0$. By (3.12) and applying Lemma 3.2, we estimate (3.11) as

$$\begin{aligned} & C \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{|v|^{-a} \, |v-\eta|^\delta \, |\eta|^b} \Big(\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{G(\xi,\eta)\, H(u,v)}{|u|^{-a} \, |u-\xi|^\delta \, |\xi|^b} \, d\xi \, du \Big) d\eta \, dv \\ & \le C \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\|G(\cdot,\eta)\|_{L^{q'}} \|H(\cdot,v)\|_{L^{q'}}}{|v|^{-a} \, |v-\eta|^\delta \, |\eta|^b} \, d\eta \, dv \\ & \le C \|G\|_{L^{q'}} \|H\|_{L^{q'}} = C \| |t^2 - r^2|^b \, r^{\frac{n-1}{q'}} \, F \|_{L^{q'}} \| r^{\frac{n-1}{q'}} \, \Phi \|_{L^{q'}}, \end{aligned}$$

where we have used the facts that $2 < q < \frac{2(n+1)}{n-1}$ is equivalent to $0 < \delta < \frac{n-1}{n+1}$, that $-a < \frac{1}{q}$ is equivalent to $b > \frac{n}{q} - \frac{n-1}{2}$, and that $-a + b = \frac{n+1}{q} - \frac{n-1}{2} > 0$. This completes the proof of Lemma 3.1. □

Our purpose here is to derive Theorem 3 by interpolation between two estimates given by Lemma 3.1. To describe the interpolation spaces of weighted Lebesgue spaces we prepare some notations.

We call a measurable function ω a weight function if ω is nonnegative and satisfies $|\{\omega(x) = 0\} \cup \{\omega(x) = \infty\}| = 0$, where $|\cdot|$ denotes the Lebesgue measure. For a σ-finite measure μ and a weight function ω, we define the weighted Lebesgue space $L^p(\omega,\mu)$ and the weighted weak Lebesgue space $L^p_w(\omega,\mu)$ by

$$L^p(\omega,\mu) = \{f;\ \|f\|_{L^p(\omega,\mu)} \equiv \Big(\int \omega^p |f|^p d\mu \Big)^{1/p} < \infty\},$$

$$L^p_w(\omega,\mu)=\{f;\ \|f\|_{L^p_w(\omega,\mu)}\equiv\sup_{\lambda>0}\lambda\,\mu\big(\{x;\omega(x)|f(x)|>\lambda\}\big)^{1/p}<\infty\},$$

for $1\le p<\infty$. In the case $\omega\equiv 1$, we denote

$$L^p(\omega,\mu)=L^p(\mu),\quad L^p_w(\omega,\mu)=L^p_w(\mu).$$

Then, the real interpolation spaces of weighted Lebesgue spaces are characterized by weighted weak Lebesgue spaces as follows.

Lemma 3.3 ([3]). *Let ω_0, ω_1 be weight functions. Let $1\le p_0<p_1<\infty$, $1/p=(1-\theta)/p_0+\theta/p_1$ with $0<\theta<1$. Then the real interpolation space of weighted Lebesgue spaces is realized as*

$$\big(L^{p_0}(\omega_0,\mu),L^{p_1}(\omega_1,\mu)\big)_{\theta,\infty}=L^p_w\Big(\Big(\frac{\omega_1^{p_1}}{\omega_0^{p_0}}\Big)^{\frac{1}{p_1-p_0}},\Big(\frac{\omega_0}{\omega_1}\Big)^{\frac{p_0p_1}{p_1-p_0}}\mu\Big)$$

with equivalent norms.

A direct application of Lemma 3.3 to Lemma 3.1, however, is insufficient for our purpose, since part of weight function influences the measure of the weighted weak Lebesgue space above. To resolve this problem we use the following lemma.

Lemma 3.4. *Let n be a positive integer and let $1\le q<\infty$. For α, $\beta\in\mathbf{R}$ with $\alpha\ne 0$, $q\alpha+\beta=n$, we assume that f and the weight function ω are homogeneous of degrees $-\alpha$ and $-\beta$, respectively. Then there exist constants C, $C'>0$ which are independent of f and ω such that*

$$C\|f\|_{L^q_w(\omega dx)}\le\|f\|_{L^q_w(\omega^{1/q},dx)}\le C'\|f\|_{L^q_w(\omega dx)},\tag{3.13}$$

where dx denotes the Lebesgue measure on $\mathbf{R}^n$.

Proof of Lemma 3.4. We first prove the first inequality of (3.13). For $j\in\mathbf{Z}$ we set

$$E_j=\{x\in\mathbf{R}^n;\ 2^j\le\omega(x)<2^{j+1}\}.$$

Then,

$$\begin{aligned}\int_{\{|f|>\lambda\}}\omega(x)\,dx&=\sum_{j=-\infty}^{\infty}\int_{\{|f|>\lambda\}\cap E_j}\omega(x)\,dx\\&\le\sum_{j=-\infty}^{\infty}2^{j+1}\int_{\{|f|>\lambda\}\cap E_j}dx\\&\le\sum_{j=-\infty}^{\infty}2^{j+1}\int_{\{2^{-j/q}\,\omega^{1/q}\,|f|>\lambda\}\cap E_j}dx,\end{aligned}\tag{3.14}$$

since $2^j \le \omega < 2^{j+1}$ on E_j. Applying the change of variables $x = 2^{-j/n}y$ and using the homogeneity of f and ω, the right hand side of the last inequality of (3.14) is equal to

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} 2^{j+1} \int_{\{2^{-\frac{j}{q}}\, 2^{(\frac{\beta}{nq}+\frac{\alpha}{n})j}\omega(y)^{\frac{1}{q}}|f(y)|>\lambda\}\cap\widetilde{E}_j} \big(2^{-\frac{j}{n}}\big)^n\, dy \\ &= \sum_{j=-\infty}^{\infty} 2 \int_{\{\omega^{1/q}\,|f|>\lambda\}\cap\widetilde{E}_j} dy, \end{aligned} \tag{3.15}$$

since $q\alpha + \beta = n$, where

$$\widetilde{E}_j = \{x \in \mathbf{R}^n;\ r^j \le \omega(x) < 2\, r^j\}$$

with $r = 2^{1-\beta/n}$. To estimate (3.15) we divide the range of β into three cases.
(i) When $\beta < n$, which implies $r > 1$, there exists $N \in \mathbf{N}$ such that $\widetilde{E}_j \cap \widetilde{E}_{j+N} = \emptyset$. In fact, we can choose $N \in \mathbf{N}$ satisfying $2 < r^N$. Thus, we obtain (3.15) is bounded by

$$2N|\{\omega^{1/q}\,|f| > \lambda\}|. \tag{3.16}$$

(ii) When $n < \beta < 2n$, which implies $1/2 < r < 1$, we observe that $\widetilde{E}_j \cap \widetilde{E}_{j+1} \neq \emptyset$. If we choose $N \in \mathbf{N}$ satisfying $2r^N < 1$, then $\widetilde{E}_j \cap \widetilde{E}_{j+N} = \emptyset$. Thus, we obtain (3.15) is bounded by (3.16).
(iii) When $\beta \ge 2n$, which implies $0 < r \le 1/2$, we observe that $\widetilde{E}_j \cap \widetilde{E}_{j+1} = \emptyset$. Thus, we obtain (3.15) is bounded by (3.16) with $N = 1$.

Note that $\beta = n$ is excluded by our assumption $\alpha \neq 0$.

Therefore, we obtain

$$\begin{aligned}\|f\|_{L^q_w(\omega dx)} &= \sup_{\lambda>0} \lambda\Big(\int_{\{|f|>\lambda\}} \omega(x)\, dx\Big)^{1/q} \\ &\le C \sup_{\lambda>0} \lambda|\{\omega^{1/q}\,|f| > \lambda\}|^{1/q} = C\|f\|_{L^q_w(\omega^{1/q},\, dx)}.\end{aligned}$$

The second inequality of (3.13) is proved similarly. In fact, using the inequality $\omega \le 2^{j+1}$ on E_j, and homogeneity of f, we obtain

$$\begin{aligned}\lambda^q\,|\{\omega^{1/q}\,|f| > \lambda\}| &= \sum_{j=-\infty}^{\infty} \lambda^q \int_{\{\omega^{1/q}\,|f|>\lambda\}\cap E_j} dx \\ &\le \sum_{j=-\infty}^{\infty} \lambda^q \int_{\{2^{(j+1)/q}\,|f|>\lambda\}\cap E_j} dx \\ &= \sum_{j=-\infty}^{\infty} 2\widetilde{\lambda}^q \int_{\{|f(2^{-j/\alpha q}\,x)|>\widetilde{\lambda}\}\cap E_j} dx, \end{aligned} \tag{3.17}$$

where $\widetilde{\lambda} = 2^{-1/q}\lambda$. By the change of variables $2^{-j/\alpha q}x = y$, (3.17) is equal to

$$\sum_{j=-\infty}^{\infty} 2\widetilde{\lambda}^q \int_{\{|f(y)|>\widetilde{\lambda}\}\cap \widetilde{E}'_j} \big(2^{\frac{j}{\alpha q}}\big)^n dy \le \sum_{j=-\infty}^{\infty} 2\widetilde{\lambda}^q \int_{\{|f|>\widetilde{\lambda}\}\cap \widetilde{E}'_j} 2^{\frac{n}{\alpha q}j}\, 2^{-(1+\frac{\beta}{\alpha q})j}\omega(y)\, dy$$
$$= \sum_{j=-\infty}^{\infty} 2\widetilde{\lambda}^q \int_{\{|f|>\widetilde{\lambda}\}\cap \widetilde{E}'_j} \omega(y)\, dy, \tag{3.18}$$

since $q\alpha + \beta = n$, where

$$\widetilde{E}'_j = \{x \in \mathbf{R}^n; r'^j \le \omega(x) < 2\, r'^j\}$$

with $r' = 2^{1+\beta/\alpha q}$. Since our assumption $\alpha q + \beta = n$ assures $r' \ne 1$, by the preceding arguments (3.18) is bounded by

$$C\widetilde{\lambda}^q \int_{\{|f|>\widetilde{\lambda}\}} \omega(y)\, dy.$$

Therefore, we obtain

$$\|f\|_{L^q_w(\omega^{1/q}, dx)} = \sup_{\lambda>0} \lambda |\{\omega^{1/q}\, |f| > \lambda\}|^{1/q}$$
$$\le C \sup_{\widetilde{\lambda}>0} \widetilde{\lambda}\Big(\int_{\{|f|>\widetilde{\lambda}\}} \omega(y)\, dy\Big)^{1/q} = C\|f\|_{L^q_w(\omega dx)}.$$

This completes the proof of Lemma 3.4. □

Proof of Theorem 3. Let q, α, a, b satisfy the assumptions of Theorem 3. Then we take q_i, a_i, b_i, for $i = 0,\ 1$, satisfying

$$\tfrac{1}{q} = \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1}, \quad a = (1-\theta)\, a_0 + \theta\, a_1, \quad b = (1-\theta)\, b_0 + \theta\, b_1,$$
$$a_i - b_i + \tfrac{n+1}{q_i} = \tfrac{n-1}{2}, \quad \tfrac{n}{q_i} - \tfrac{n-1}{2} < b_i < \tfrac{1}{q_i},$$

for some $\theta \in (0, 1)$. By Lemma 3.1 we have

$$\big\| |t^2 - r^2|^{a_i} r^{\frac{n-1}{q_i}} w\big\|_{L^{q_i}(dtdr)} \le C \big\| |t^2 - r^2|^{b_i} r^{\frac{n-1}{q'_i}} F\big\|_{L^{q'_i}(dtdr)}, \quad i = 0,\ 1,$$

using polar coordinates. Then, by Lemma 3.3, interpolating the above inequalities, we have

$$\big\| |t^2 - r^2|^{\frac{a_1\, q_1 - a_0\, q_0}{q_1 - q_0}} w\big\|_{L^q_w(|t^2-r^2|^{q_0\, q_1\, (a_0-a_1)/(q_1-q_0)}\, r^{n-1} dtdr)}$$
$$\le C \big\| |t^2 - r^2|^{\frac{b_1\, q'_1 - b_0\, q'_0}{q'_1 - q'_0}} F\big\|_{L^{q'}_w(|t^2-r^2|^{q'_0\, q'_1\, (b_0-b_1)/(q'_1-q'_0)}\, r^{n-1} dtdr)}.$$

Here, we notice by (3.7) that the homogeneity of F of degree $-\alpha-2$ implies the homogeneity of w of degree $-\alpha$. From these homogeneities of w, F, and the weights, we are able to apply Lemma 3.4 to obtain

$$\big\||t^2-r^2|^a w\big\|_{L^q_w(r^{n-1}dtdr)} \le C\big\||t^2-r^2|^b F\big\|_{L^{q'}_w(r^{n-1}dtdr)}, \tag{3.19}$$

since

$$\frac{a_1\,q_1-a_0\,q_0}{q_1-q_0}+\frac{1}{q}\,\frac{q_0\,q_1\,(a_0-a_1)}{q_1-q_0}=a, \quad \frac{b_1\,q_1'-b_0\,q_0'}{q_1'-q_0'}+\frac{1}{q'}\,\frac{q_0'\,q_1'\,(b_0-b_1)}{q_1'-q_0'}=b.$$

The inequality (3.19) is equivalent to the inequality (3.3), which completes the proof of Theorem 3. □

4 Proof of Theorem 1

In this section, we prove Theorem 1. We define the sequence $\{u_j\}$ inductively by

$$u_j(t)=u_0(t)+\int_0^t(-\Delta)^{-\frac12}\sin[(t-s)(-\Delta)^{\frac12}]\,f(u_{j-1}(s))\,ds, \quad j\ge 1,$$
$$u_0(t)=\cos[t(-\Delta)^{\frac12}]\,\varepsilon\phi+(-\Delta)^{-\frac12}\sin[t(-\Delta)^{\frac12}]\,\varepsilon\psi.$$

We observe that $u_j(\lambda t,\lambda x)=\lambda^{-2/(p-1)}u_j(t,x)$ holds inductively for $j\ge 0$ by the homogeneity of ϕ, ψ, and f. Note that this fact enables us to apply Theorem 3.

By an equivalent triangle inequality we have

$$\begin{aligned}\big\||t^2-|x|^2|^\gamma u_j\big\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)} &\le C\,\big\||t^2-|x|^2|^\gamma u_0\big\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}\\ &+C\,\Big\||t^2-|x|^2|^\gamma\int_0^t(-\Delta)^{-\frac12}\sin[(t-s)(-\Delta)^{\frac12}]\,f(u_{j-1}(s))\,ds\Big\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)},\end{aligned} \tag{4.1}$$

where $\gamma=\frac{1}{p-1}-\frac{n+1}{2(p+1)}$. The first term on the right hand side of (4.1) is finite by Theorem 2 and we set

$$C\,\big\||t^2-|x|^2|^\gamma u_0\big\|_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}=C_0\,\varepsilon.$$

In fact, the assumptions of Theorem 2 is satisfied as long as $p_0(n)<p<\frac{n+3}{n-1}$, when we set $\alpha=2/(p-1)$, $q=p+1$ (see Remark 1 (2)). Applying (3.5), we see that the second term on the right hand side of (4.1) is bounded by a constant multiple of

$$\big\||t^2-|x|^2|^{p\gamma}\,|u_{j-1}|^p\big\|_{L^{(p+1)/p}_w(\mathbf{R}^{1+n}_+)}=\big\||t^2-|x|^2|^\gamma\,u_{j-1}\big\|^p_{L^{p+1}_w(\mathbf{R}^{1+n}_+)}.$$

In fact, the assumptions of Theorem 3 is also satisfied as long as $p_0(n) < p < \frac{n+3}{n-1}$, when we set $\alpha = 2/(p-1)$, $q = p+1$ (see Remark 2 (2)). Thus, we obtain

$$\left\| |t^2 - |x|^2|^\gamma u_j \right\|_{L_w^{p+1}(\mathbf{R}_+^{1+n})} \le 2\, C_0\, \varepsilon$$

for all $j \ge 1$, if ε is sufficiently small.

On the other hand, applying Theorem 3 and Hölder's inequality in weak Lebesgue spaces, we obtain

$$\begin{aligned}
&\left\| |t^2 - |x|^2|^\gamma (u_{j+1} - u_j) \right\|_{L_w^{p+1}(\mathbf{R}_+^{1+n})} \\
&\le C \left\| |t^2 - |x|^2|^{p\gamma} (f(u_j) - f(u_{j-1})) \right\|_{L_w^{(p+1)/p}(\mathbf{R}_+^{1+n})} \\
&\le C \left\| |t^2 - |x|^2|^{(p-1)\gamma} (|u_j|^{p-1} + |u_{j-1}|^{p-1}) \right\|_{L_w^{(p+1)/(p-1)}(\mathbf{R}_+^{1+n})} \\
&\qquad\qquad \times \left\| |t^2 - |x|^2|^\gamma (u_j - u_{j-1}) \right\|_{L_w^{p+1}(\mathbf{R}_+^{1+n})} \\
&\le C\, \varepsilon^{p-1} \left\| |t^2 - |x|^2|^\gamma (u_j - u_{j-1}) \right\|_{L_w^{p+1}(\mathbf{R}_+^{1+n})}.
\end{aligned}$$

Thus, we conclude that $\{u_j\}$ is a Cauchy sequence in the weighted weak Lebesgue space on $\mathbf{R}_+^{1+n}$ for sufficiently small ε and that the limit u is the desired solution.

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