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# Right accessibility of semicontinuous initial data for Hamilton-Jacobi equations

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## Abstract

We study Hamilton-Jacobi equations with upper semicontinuous initial data without convexity assumptions on the Hamiltonian. We analyse the behavior of generalized *u.s.c.* solutions at the initial time  $t = 0$ , and find necessary and sufficient conditions on the Hamiltonian such that the solution attains the initial data along a sequence (right accessibility).

## Introduction

In this paper we study the Cauchy problem for Hamilton-Jacobi (briefly, HJ) equations

$$(CP) \begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^N \times ]0, T[, \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^N, \end{cases}$$

with upper semicontinuous initial data  $g$ . The standard theory of viscosity solutions covers the cases where the solution is continuous, in particular for  $g$  continuous or Hamiltonian  $H(x, p)$  coercive and superlinear as  $|p| \rightarrow \infty$ , see [L1, L2, CIL, Ba, BCESS, BCD] and their references. To treat problems where the solution is expected to be discontinuous one has to weaken further the notion of viscosity solution. This was done successfully by Barron and Jensen [BJ] for  $H$  concave in  $p$  (or convex, if  $g$  is *lsc*), and their notion of bilateral supersolution of the HJ equation has been used and extended by many authors, see, e.g., [F, Sor, Ba, BCD] and the references therein. In this framework the initial data of the Cauchy problem are attained in the following sense: for all  $x_o$  there are sequences  $x_n \rightarrow x_o$  and  $t_n \rightarrow 0$ ,  $t_n > 0$ , such that  $u(x_n, t_n) \rightarrow g(x_o)$ .

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This is the property that we call right accessibility of the initial data, following the terminology of [GS].

If the Hamiltonian is neither concave nor convex in  $p$  there is not a satisfactory definition of discontinuous solution of the HJ equation in terms of local properties. However, a generalised solution of the Cauchy problem can be defined in many reasonable ways: as pointwise limit of sub- and supersolutions (the generalized minimax solution [RS, S]), as maximal subsolution and infimum of all supersolutions (the envelope solution [BCD, Be]), by means of the level set method (the  $L$ -solution [GS]). When all these approaches work they all single out the same generalized solution. Its local properties need further investigations, and in this paper we focus on the behavior at  $t = 0$ .

The right accessibility of the initial data for nonconcave Hamiltonians was studied by Giga and Sato [GS] in the case that  $H = H(p)$  is independent of  $x$  and has a recession function  $\lim_{\lambda \rightarrow 0+} \lambda H(p/\lambda)$ . They prove that all *u.s.c.* data are accessible if and only if the Wulff shape  $W_\alpha$  associated to  $\alpha(p) = -H(-p)$  is nonempty, where  $W_\alpha$  is

$$W_\alpha := \{z \in \mathbb{R}^N : \sup_{p \in \mathbb{R}^N} (z \cdot p - \alpha(p)) \leq 0\}.$$

Their proof is based on the special initial data  $g_o(x) = -\infty$  for all  $x \neq x_o$ ,  $g_o(x_o) = K$ . If  $W_\alpha = \emptyset$  the  $L$ -solution  $u$  of (CP) is the constant  $-\infty$ , otherwise  $u(x_o + sq, s) = K$  for all  $q \in W_\alpha$  and all  $s > 0$ .

We treat the case of  $x$ -dependent Hamiltonians  $H(x, p)$  Lipschitzian in  $p$ . We show that the generalized solution  $u$  corresponding to the initial data  $g_o$  is the constant  $K$  along any Lipschitz arc  $(x(s), s)$  satisfying the differential inclusion

$$x'(s) \in W_{\alpha(x(s), \cdot)} \quad \text{for a.e. } s \in [0, t], \quad x(0) = x_o,$$

for some  $t > 0$ , where  $\alpha(x, p) = -H(x, -p)$ . It turns out that the existence of such an arc for all  $x_o$  implies the right accessibility of all *u.s.c.* initial data  $g$  that do not take the value  $+\infty$ . We also give a slightly more technical condition that is necessary and sufficient for the propagation of the maximum of  $g_o$ .

Our methods are completely different from those of [GS]. We first prove in Section 1 a representation formula for the generalized solution of (CP) as the value function of a differential game, following Evans and Souganidis [ES]. This formula is the key tool for deriving in Section 2 several conditions for right accessibility. In Section 3 these conditions are illustrated on many examples.

## 1 Generalized solutions and a representation formula

Consider the Cauchy problem

$$(CP) \begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^N \times ]0, T[ \\ u(x, 0) = g(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

Throughout the paper we make the following assumptions on the Hamiltonian:

$$|H(x, p) - H(x, q)| \leq L|p - q| \quad (1)$$

for all  $x, p, q \in \mathbb{R}^N$  and

$$|H(x, p) - H(y, p)| \leq C_R(1 + |p|)|x - y| \quad (2)$$

for all  $|x|, |y| \leq R, p \in \mathbb{R}^N$ .

**Definition 1** For  $g : \mathbb{R}^N \rightarrow [-\infty, +\infty[$  u.s.c., let  $g_n \in C(\mathbb{R}^N)$  be a nonincreasing sequence converging pointwise to  $g$ , and let  $u_n$  be the viscosity solution of (CP) with initial data  $g_n$ . The infimum  $u$  of  $u_n$  is a generalized solution of (CP).

**Remark 2** Of course  $u : \mathbb{R}^N \times [0, T[ \rightarrow [-\infty, +\infty[$  is u.s.c.; if  $u(x, t) > -\infty$  for all  $x, t$  then it is a subsolution of (CP), i.e., a viscosity subsolution of the PDE (see, e.g., Proposition V.2.16 of [BCD]) such that  $u(x, 0) \leq g(x)$  for all  $x$ . Moreover, under the current assumptions the Comparison Principle holds for (CP), i.e., any subsolution of (CP) is below any supersolution (see [I] or Theorem III.3.15 and Chapter V of [BCD]). Then any subsolution is below  $u_n$  for all  $n$ , so  $u$  is the maximal subsolution of (CP). Finally, by the Comparison principle again,  $u$  is below any supersolution, so it is also the infimum of all supersolutions of (CP). A function with these properties is called an *envelope solution*, or Perron-Wiener-Brelot solution; its existence, uniqueness and other properties were studied in [BCD, BB] for Dirichlet problems and by Bettini [Be] for the current Cauchy problem with bounded  $g$  and more general Hamiltonian, see also [PQ] for applications to differential games.

When  $u$  is finite it is also the *generalized minimax solution* of Rozyev and Subbotin [RS], i.e., there exist a sequence of subsolutions of (CP) and a sequence of supersolutions converging pointwise to  $u$ ; here semisolutions are meant in the Subbotin's minimax sense, but this is equivalent to the viscosity sense, see [S]. For bounded  $g$  and more general Hamiltonian the existence and uniqueness of generalized minimax solutions were studied in [RS].

**Remark 3** For Hamiltonians with the further property that the recession function  $\lim_{\lambda \rightarrow 0+} \lambda H(x, p/\lambda)$  exists and is finite, and for any u.s.c.  $g : \mathbb{R}^N \rightarrow [-\infty, +\infty]$ , Giga and Sato [GS] proved the existence and uniqueness of the *L-solution* of (CP), a notion defined by means of the level set method. The generalized solution defined above coincides with the *L-solution* provided that there exists a unique continuous viscosity solution of (CP) when the initial data is continuous. Such situation arises when  $H(x, 0)$  is bounded (see Theorem 4 (ii)). Indeed, by the existence of solutions with continuous data, we can consider the infimum  $u$  of the viscosity solutions  $u_\varepsilon$  of (CP) with initial data  $g_\varepsilon \in C(\mathbb{R}^n)$ , a nonincreasing one parameter family converging pointwise to  $g$  as  $\varepsilon \rightarrow 0$ . Clearly  $u$  is a generalized solution. Since  $u_\varepsilon + \varepsilon$  is also a viscosity solution of the HJ equation, we may assume that  $g_\varepsilon - g_{\varepsilon'} \geq \varepsilon - \varepsilon'$  for  $\varepsilon > \varepsilon' > 0$ . Then we can use Lemma 4.4 of [GS] and conclude that  $u$  is the *L-solution*.

The goal of this section is a representation formula for the (generalized) solution of (CP). It will be our main tool in the study of the right accessibility in the next section, and it also gives a proof of the existence and uniqueness of the solution.

We follow Evans and Souganidis [ES] and first rewrite the Hamiltonian

$$H(x, p) := \max_{a \in \mathbb{R}^N} \min_{|b| \leq L} \{b \cdot p - b \cdot a + H(x, a)\}, \quad (3)$$

as in Lemma 5.1 of [ES]. Next we associate to (CP) a differential game governed by the system

$$(S) \begin{cases} y'(t) = -b(t) & \text{for } t > 0, |b(t)| \leq L, \\ y(0) = x \end{cases}$$

and with running cost

$$\ell(x, a, b) = -H(x, a) + b \cdot a, \quad a \in \mathbb{R}^N, b \in \overline{B}(0, L).$$

The value function of the game is defined as follows. Set

$$\mathcal{A} := L^1([0, +\infty[, \mathbb{R}^N),$$

$$\mathcal{B} := L^\infty([0, +\infty[, \overline{B}(0, L)),$$

and call  $\Delta$  the set of *non-anticipating strategies* for the 2nd player, that is,  $\beta \in \Delta$  is a function

$$\beta : \mathcal{A} \rightarrow \mathcal{B}$$

such that for all  $t > 0$  and  $a, \tilde{a} \in \mathcal{A}$ ,  $a(s) = \tilde{a}(s)$  for all  $s \leq t$  implies  $\beta[a](s) = \beta[\tilde{a}](s)$  for all  $s \leq t$ .

We denote the solution of (S) with  $y_x(\cdot; b)$ , so

$$y_x(s) := y_x(s; \beta[a]) = x - \int_0^s \beta[a](\tau) d\tau. \quad (4)$$

The (*upper*) *value* of our game is

$$U(x, t) := \sup_{\beta \in \Delta} \inf_{a \in \mathcal{A}} \left\{ \int_0^t [\beta[a](s) \cdot a(s) - H(y_x(s), a(s))] ds + g(y_x(t)) \right\},$$

where  $g : \mathbb{R}^N \rightarrow [-\infty, +\infty]$  is called the terminal cost of the game. Next theorem says that the generalized solution  $u$  of (CP) coincides with the value function  $U$ .

**Theorem 4** *Assume (1), (2), and  $H(x, 0)$  bounded. Then*

- (i) *if  $g$  is locally bounded then  $U$  is locally bounded;*
- (ii) *if  $g \in C(\mathbb{R}^N)$  then  $U$  is the unique continuous solution of (CP)*
- (iii) *if  $g : \mathbb{R}^N \rightarrow [-\infty, +\infty[$  is u.s.c. then  $U$  is the unique generalized solution of (CP).*

**Remark 5** The statement (i) and the continuity of  $U$  for continuous  $g$  are not completely standard because  $\ell$  is neither bounded nor continuous in  $x$  uniformly with respect to the control  $a$ . The proof of (iii) is essentially the same as the proof of Theorem 4.21 in [Be].

**Remark 6** The proof of the theorem shows also that the sup in the definition of  $U$  is attained, so we have the following representation formula for the generalized solution  $u$  of (CP)

$$u(x, t) = \max_{\beta \in \Delta} \inf_{a \in \mathcal{A}} \left\{ \int_0^t [\beta[a](s) \cdot a(s) - H(y_x(s), a(s))] ds + g(y_x(t)) \right\}. \quad (5)$$

Note that the integral has no a priori bounds but it is always finite by (1) and the integrability of  $a$ , so the payoff in  $\{\dots\}$  makes sense even at points where  $g = -\infty$ .

**Remark 7** In the proof we recall a Comparison Principle for continuous solutions of (CP). If we apply it to the continuous approximations corresponding to the initial data  $g \leq \tilde{g}$ , we get the comparison between the generalised solutions  $u \leq \tilde{u}$ .

**Proof of Theorem 4.** The key estimates that allow us to dispense with the boundedness of  $\ell$  are the following. Let  $M$  be a bound on  $|H(\cdot, 0)|$ , and for  $\varepsilon > 0$  let  $\beta^* \in \Delta$  be a strategy that makes the quantity  $\inf_{a \in \mathcal{A}} \{\dots\}$  in the definition of  $U$  larger than  $U - \varepsilon$ . Use the control  $a(s) \equiv 0$  to get

$$U(x, t) \leq - \int_0^t H(y_x(s; \beta^*[0]), 0) ds + g(y_x(t; \beta^*[0])) + \varepsilon.$$

Since  $|y_x(s; b)| \leq |x| + Lt$  for all  $b \in \mathcal{B}$ ,  $U$  is locally bounded from above if  $g$  is locally bounded from above. Now take the strategy  $\bar{\beta}[a](s) := La(s)/|a(s)|$  and note that (1) implies

$$\bar{\beta}[a](s) \cdot a(s) - H(y_x(s; \bar{\beta}[a], a), a(s)) \geq -H(y_x(s; \bar{\beta}[a], a), 0) \geq -M \quad \forall a \in \mathcal{A}.$$

Then

$$U(x, t) \geq -Mt + \inf_{a \in \mathcal{A}} g(y_x(t; \bar{\beta}[a]))$$

and  $U$  is locally bounded from below if  $g$  is locally bounded from below. This completes the proof of (i).

To prove (ii) we denote with  $\omega_g(\cdot, R)$  the modulus of continuity of  $g$  in the ball  $\bar{B}(0, R)$  and observe that the preceding estimates give

$$g(x) - Mt - \omega_g(Lt, |x| + Lt) \leq U(x, t) \leq Mt + g(x) + \omega_g(Lt, |x| + Lt).$$

Then

$$|U(z, t) - g(x)| \rightarrow 0 \quad \text{as } z \rightarrow x, t \rightarrow 0.$$

This implies  $U^*(x, 0) = U_*(x, 0)$  for all  $x$ , where  $U^*$  and  $U_*$  are the semicontinuous envelopes of the value function. We claim that  $U^*$  is a subsolution and  $U_*$  is a supersolution of (CP) by standard arguments: first one proves the Dynamic Programming Principle as in Theorem 3.1 of [ES] or Theorem VIII.1.9 of [BCD], then one derives from it the inequalities in viscosity sense as in Theorem 4.1 of [ES] or Theorems V.2.6 and VIII.1.10 in [BCD]. The only additional difficulty comes from the lack of a compact constraint on the controls of the 1st player, and it is overcome by means of Lindelöf's theorem as in Exercise VIII.1.2 of [BCD]. Then the Comparison Principle of Exercise V.1.7 in [BCD] (based on a result of Ishii [I]) gives  $U^* \leq U_*$ ; since the opposite inequality is trivial, the two envelopes coincide, so  $U$  is continuous and it is the unique viscosity solution of (CP) for continuous  $g$ .

(iii) For *u.s.c.*  $g$  taking values in  $[-\infty, +\infty[$  we take  $g_n \in C(\mathbb{R}^N)$ , such that  $g_n \searrow g$ ; this is possible by classical result in general topology [Bou]. Let  $U_n$  be the upper value corresponding to  $g_n$ . By (ii)  $U_n$  coincides with the continuous solutions  $u_n$  of (CP) with initial data  $g_n$ , so we must prove that  $U_n \searrow U$ . The monotonicity of the sequence  $U_n$  follows from the Comparison Principle.

We consider the game where the 2nd player uses relaxed controls, namely, measurable functions of time taking values in  $\overline{B}(0, L)^{(r)}$ , the set of Radon probability measures on  $\overline{B}(0, L)$ . The extension of the system and of the running cost to this set is done in a standard way [BCD, EK]. By the convexity of  $\overline{B}(0, L)$  and of  $\ell(x, a, \overline{B}(0, L))$  the new game with relaxed controls is equivalent to the original one, and the value functions are unchanged. We endow  $\overline{B}(0, L)^{(r)}$  with the weak star topology. By a classical result on relaxed controls, see e.g. [EK], the set of strategies  $\Delta$  is compact and the sup is attained in the definition of  $U$  and of  $U_n$ , as in the formula (5).

Since the sequence  $U_n$  is decreasing and bounded from below by  $U$ , it remains to prove that  $\inf_n U_n \leq U$ . We fix  $(x, t)$  and an optimal strategy  $\beta_n$  for  $U_n$ , and fix  $a \in \mathcal{A}$ . By the compactness theorem for relaxed strategies [EK] we can extract a subsequence  $\beta_n$  such that

$$\beta_n[a] \rightarrow \beta[a] \quad \text{weak star in } L^\infty([0, T], \overline{B}(0, L)^{(r)}) \quad (6)$$

and

$$y_x(\cdot; \beta_n[a]) \rightarrow y_x(\cdot; \beta[a]) \quad \text{uniformly on } [0, T]. \quad (7)$$

If we call  $J$  and  $J_n$  the relaxed payoffs of the games, since

$$U_n(x, t) \leq J_n(x, t; a, \beta_n[a]),$$

it's enough to prove that

$$\liminf_n J_n(x, t; a, \beta_n[a]) \leq J(x, t; a, \beta[a]) \quad \forall a \in \mathcal{A}. \quad (8)$$

The fact that

$$\liminf_n g_n(y_x(t; \beta_n[a])) \leq g(y_x(t; \beta[a]))$$

follows easily from the monotonicity of the sequence  $g_n$ , the continuity of each  $g_n$ , and the semicontinuity of  $g$ . To deal with the integral part of  $J_n - J$  we add and subtract to the integrand the function  $\ell(y_x(s; \beta[a]), a(s), \beta_n[a](s))$  and split in two pieces. Thanks to some cancelations the first piece is

$$c_n := \int_0^t [-H(y_x(s; \beta_n[a]), a(s)) + H(y_x(s; \beta[a]), a(s))] ds$$

and the second is

$$d_n := \int_0^t (\beta_n[a](s) \cdot a(s) - \beta[a](s) \cdot a(s)) ds.$$

The first piece tends to 0 because, by (2),

$$|c_n| \leq C_R(t + \int_0^t |a(s)| ds) \sup_{0 \leq s \leq t} |y_x(s; \beta_n[a]) - y_x(s; \beta[a])|,$$

where  $R = |x| + Lt$ , and the r.h.s. tends to 0 because of (7) and the integrability of  $a$ . Finally, the convergence  $|d_n| \rightarrow 0$  follows easily from (6), the integrability of  $a$  and the boundedness of  $\overline{B}(0, L)$  (see Section 4.2 of [Be] for more details).  $\square$

## 2 Necessary and sufficient conditions for right accessibility

We want to study the right accessibility of the initial data of (CP), following [GS]. We say that an initial function  $g$  is *right accessible at  $x_o$*  for the generalized solution  $u$  of (CP) if  $(u|_{t>0})^*(x_o, 0) = g(x_o)$ , i.e., there are sequences of  $t_n > 0$  and  $x_n \in \mathbb{R}^N$  such that  $t_n \rightarrow 0$ ,  $x_n \rightarrow x_o$ , and  $u(x_n, t_n) \rightarrow g(x_o)$ ; the initial function is right accessible, briefly RA, if it is RA at all  $x_o \in \mathbb{R}^N$ .

As shown by [GS] the crucial initial data to consider are of the form

$$g_o(x) := \begin{cases} K & \text{if } x = x_o, \\ -\infty & \text{if } x \neq x_o. \end{cases}$$

For these data we first consider the slightly stronger property of *propagation of the singular maximum*: the maximum  $K$  of  $g_o$  propagates if there are sequences of  $t_n > 0$  and  $x_n \in \mathbb{R}^N$  such that  $t_n \rightarrow 0$ ,  $x_n \rightarrow x_o$ , and  $u(x_n, t_n) = K$  for all  $n$ . Of course, if the maximum of  $g_o$  propagates then  $g_o$  is right accessible at  $x_o$ . The connection between the special data of the form  $g_o$  and general data is the following.

**Proposition 8** Assume (1), (2), and  $H(x, 0)$  bounded. If the initial data of the form  $g_o$  are RA for all  $K \in \mathbb{R}$  and  $x_o \in \mathbb{R}^N$ , then any u.s.c.  $g : \mathbb{R}^N \rightarrow [-\infty, +\infty[$  is RA.

**Proof.** Since  $u$  is u.s.c. in  $\mathbb{R}^N \times [0, T[$ ,

$$\limsup_n u(x_n, t_n) \leq g(x_o), \quad \text{for all } x_n \rightarrow x_o, t_n \searrow 0.$$

For fixed  $x_o$  we consider the initial data  $g_o$  with  $K = g(x_o)$  and call  $w$  the corresponding solution of (CP). By assumption there are sequences  $x_n \rightarrow x_o$ ,  $t_n \searrow 0$ , such that

$$\lim_n w(x_n, t_n) = K = g(x_o).$$

By Remark 7 and  $g_o \leq g$  we get  $w \leq u$ , so

$$\liminf_n u(x_n, t_n) \geq g(x_o),$$

which completes the proof.  $\square$

Under the stronger condition that  $H$  is positively 1-homogeneous in  $p$ , i.e.,

$$\lambda H(x, p) = H(x, \lambda p), \quad \forall x, p, \quad \forall \lambda \geq 0,$$

the property of propagation of the singular maximum for all data  $g_o$  is not only sufficient but also necessary for the right accessibility of all u.s.c. initial data. In fact, in this case the RA of  $g_o$  at  $x_o$  is equivalent to the propagation of the singular maximum by the following Lemma.

**Lemma 9** Assume (1), (2) with  $C_R$  independent of  $R$ , and  $H$  positively 1-homogeneous in  $p$ . Then the generalized solution  $u$  of (CP) with  $g = g_o$  takes values in  $\{K, -\infty\}$ , i.e., for all  $(x, t)$  either  $u(x, t) = K$  or  $u(x, t) = -\infty$ .

**Proof.** We may assume that  $K = 0$  by translation of the dependent variable. Let  $v$  be the solution of (CP) with initial data  $v(x, 0) = -|x - x_o|$ . Let  $\theta_n$  be the nondecreasing continuous function of the form

$$\theta_n(s) = \min\left(\frac{1}{n}, \frac{1}{n} + ns\right).$$

By the invariance lemma (see Lemma 4.1 in [GS] or [ES]) the function  $w_n(x, t) := \theta_n(v(x, t))$  is the solution of (CP) with the initial data

$$g_n(x) := \theta_n(-|x - x_o|)$$

because  $H$  is positively 1-homogeneous. The sequence  $g_n$  is decreasing and converges to  $g_o$ , because  $\theta_n \downarrow \theta_0$  with

$$\theta_0(s) = \begin{cases} 0, & s \geq 0 \\ -\infty, & s < 0. \end{cases}$$

Then the generalized solution  $u$  of (CP) with initial data  $g_0$  is the monotone decreasing limit of  $w_n$ . From the formula for  $w_n$  and the limit  $\theta_n \downarrow \theta_0$  we see that  $u$  takes values either zero or  $-\infty$ .  $\square$

We will study the propagation of the singular maximum under the assumption

$$0 \leq H(x, 0) \leq M \quad \forall x \in \mathbb{R}^N. \quad (9)$$

All the main results of the paper follow from the next theorem.

**Theorem 10** *Assume (1), (2), and (9). Then the maximum of  $g_0$  propagates if and only if there exist  $t > 0$ ,  $x \in \mathbb{R}^N$ , and  $\beta \in \Delta$  such that*

$$\begin{cases} \int_0^t H(x_o + \int_s^t \beta[a](\tau) d\tau, a(s)) ds \leq \int_0^t \beta[a](s) \cdot a(s) ds, \\ x_o + \int_0^t \beta[a](s) ds = x, \end{cases} \quad (10)$$

for all  $a \in \mathcal{A}$ . Moreover, if this condition holds, then

$$u(x_o + \int_\tau^t \beta[0](s) ds, t - \tau) = K \quad \forall \tau \in [0, t]. \quad (11)$$

**Proof.** We choose  $g_n \searrow g$  with  $g_n \leq K$ . By (9) the constant  $K$  is a supersolution of (CP), so  $u_n \leq K$  by a Comparison Principle (e.g., Exercise V.1.7 in [BCD]). Therefore

$$u(x, t) \leq K \quad \forall (x, t). \quad (12)$$

We claim that, if  $u(x, t) = K$  for some  $t > 0$ , then there is a strategy  $\beta^* \in \Delta$  such that

$$u(y_x(\tau; \beta^*[0]), t - \tau) = K \quad \forall \tau \in [0, t]. \quad (13)$$

To prove the claim we begin with using the representation formula (5) and writing the Dynamic Programming Principle

$$u(x, t) = \sup_{\beta \in \Delta} \inf_{a \in \mathcal{A}} \left\{ \int_0^\tau [\beta[a](s) \cdot a(s) - H(y_x(s), a(s))] ds + u(y_x(\tau), t - \tau) \right\},$$

for all  $\tau \in [0, t]$ , where  $y_x(\cdot) = y_x(\cdot; \beta[a])$  is defined by (4) (we observed in the proof of Theorem 4 that this holds by the same arguments as in [ES] or [BCD]). Next we observe that, if the max in (5) is attained at  $\beta^*$ , then also the max in the Dynamic Programming Principle is attained at  $\beta^*$ , that is,  $u(x, t) = U(x, t, \tau)$  for all  $\tau \in [0, t]$ , where

$$U(x, t, \tau) := \inf_{a \in \mathcal{A}} \left\{ \int_0^\tau [\beta^*[a](s) \cdot a(s) - H(y_x^*(s), a(s))] ds + u(y_x^*(\tau), t - \tau) \right\}$$

and  $y_x^*(\cdot) = y_x(\cdot; \beta^*[a])$ . To prove the missing inequality  $u \leq U$  we fix  $\tau$  and  $\varepsilon > 0$ . We define the strategy

$$\tilde{\beta}[a](s) := \beta^*[a(\cdot + \tau)](s + \tau)$$

and observe it is non-anticipating. Now we choose  $\bar{a}$  such that

$$U(x, t, \tau) \geq \int_0^\tau \ell(y_x(s; \beta^*[\bar{a}]), \bar{a}(s), \beta^*[\bar{a}](s)) ds + u(z, t - \tau) - \frac{\varepsilon}{2},$$

where  $z := y_x(\tau; \beta^*[\bar{a}])$ , and choose  $\tilde{a}$  such that

$$u(z, t - \tau) \geq \int_0^{t-\tau} \ell(y_z(s; \tilde{\beta}[\tilde{a}]), \tilde{a}(s), \tilde{\beta}[\tilde{a}](s)) ds + g(y_z(t - \tau; \tilde{\beta}[\tilde{a}])) - \frac{\varepsilon}{2}.$$

We define

$$a^*(s) = \bar{a}(s) \text{ if } s \leq \tau, \quad a^*(s) = \tilde{a}(s - \tau) \text{ if } s > \tau$$

and add up the inequalities to get

$$U(x, t, \tau) + \varepsilon \geq \int_0^t \ell(y_x(s; \beta^*[a^*]), a^*(s), \beta^*[a^*](s)) ds + g(y_x(t; \beta^*[a^*])) \geq u(x, t).$$

By the arbitrariness of  $\varepsilon$  we can conclude that  $u(x, t) = U(x, t, \tau)$ .

Now we observe that the integrand in the definition of  $U$  is non-positive for the control  $a(\cdot) \equiv 0$  by (9). Therefore, if

$$u(y_x(\tau; \beta^*[0]), t - \tau) < K$$

for some  $\tau$ , then also  $u(x, t) < K$ , which proves the claim. Therefore the max of  $g_o$  propagates if and only if there exist  $t > 0$  and  $x$  such that  $u(x, t) = K$ .

From the representation formula (5) and (12),  $u(x, t) = K$  if and only if there exists  $\beta^* \in \Delta$  such that  $y_x(t; \beta^*[a]) = x_o$  for all  $a \in \mathcal{A}$  and

$$\int_0^t [\beta^*[a](s) \cdot a(s) - H(y_x(s; \beta^*[a]), a(s))] ds \geq 0 \quad \forall a \in \mathcal{A}. \quad (14)$$

From (4) the first condition coincides with the second condition in (10), and this allows to write

$$y_x(s; \beta^*[a]) = x_o + \int_s^t \beta^*[a](\tau) d\tau.$$

Then (14) is equivalent to the first condition in (10), and from (13) we obtain (11).  $\square$

Now we derive several corollaries. The first is a necessary and sufficient condition of right accessibility.

**Corollary 11** *Assume (1), (2) with  $C_R$  independent of  $R$ , and  $H$  positively 1-homogeneous in  $p$ . Then any u.s.c. function  $g : \mathbb{R}^N \rightarrow [-\infty, +\infty[$  is right accessible if and only if for all  $x_o \in \mathbb{R}^N$  there exist  $t > 0$ ,  $x \in \mathbb{R}^N$ , and  $\beta \in \Delta$  such that (10) holds for all  $a \in \mathcal{A}$ .*

**Proof.** It is enough to combine Proposition 8, Lemma 9, and Theorem 10.  $\square$

In the next results we reformulate in various ways the conditions for the propagation of the maximum in Theorem 10.

**Corollary 12** *Assume (1), (2), and (9). Then a necessary and sufficient condition for the propagation of the maximum of  $g_o$  is the existence of  $x \in \mathbb{R}^N$ ,  $t > 0$ , and for all  $\phi \in L^1([0, t], \mathbb{R}^N)$  of a Lipschitz curve  $x(\cdot; \phi) : [0, t] \rightarrow \mathbb{R}^N$  with Lipschitz constant  $\leq L$  such that  $x(0; \phi) = x_o$ ,  $x(t; \phi) = x$ ,*

$$\int_0^t H(x(s; \phi), \phi(s)) ds \leq \int_0^t x'(s; \phi) \cdot \phi(s) ds \quad \forall \phi \in L^1([0, t], \mathbb{R}^N), \quad (15)$$

*and with the property that, for all  $\tau \in [0, t]$  and  $\phi, \tilde{\phi} \in L^1([0, t], \mathbb{R}^N)$ ,  $\phi(s) = \tilde{\phi}(s)$  for all  $s \in [\tau, t]$  implies  $x(s; \phi) = x(s; \tilde{\phi})$  for all  $s \in [\tau, t]$ .*

**Proof.** The necessity is easily obtained from Theorem 10 by defining

$$x(s; \phi) := y_x(t - s; \beta[a]) = x_o + \int_{t-s}^t \beta[a](\tau) d\tau, \quad a(\tau) := \phi(t - \tau), \quad s, \tau \in [0, t],$$

and  $a$  can be extended arbitrarily for  $\tau > t$ . For the sufficiency we define

$$\beta[a](s) := x'(t - s; \phi), \quad \phi(\tau) := a(t - \tau), \quad s, \tau \in [0, t],$$

and  $\beta$  constant for  $s > t$ , and we reduce again to Theorem 10 by simple changes of variables.  $\square$

The next result gives a sufficient condition of propagation in terms of the Wulff shape  $W_{\alpha(x, \cdot)}$

$$W_{\alpha(x, \cdot)} := \{z \in \mathbb{R}^N : \sup_{p \in \mathbb{R}^N} (z \cdot p - \alpha(x, p)) \leq 0\}$$

associated with the function  $p \rightarrow \alpha(x, p) := -H(x, -p)$ .

**Corollary 13** *Assume (1), (2), and (9). Then a sufficient condition for the propagation of the maximum of  $g_o$  is the existence for some  $t > 0$  of a Lipschitz curve  $x(\cdot) : [0, t] \rightarrow \mathbb{R}^N$  with Lipschitz constant  $\leq L$  such that  $x(0) = x_o$  and*

$$\int_0^t H(x(s), \phi(s)) ds \leq \int_0^t x'(s) \cdot \phi(s) ds \quad \forall \phi \in L^1([0, t], \mathbb{R}^N). \quad (16)$$

*This inequality is satisfied, in particular, if*

$$H(x(s), p) \leq x'(s) \cdot p \quad \text{for a.e. } s \in [0, t] \quad \forall p \in \mathbb{R}^N, \quad (17)$$

*that is,  $x(\cdot)$  solves the differential inclusion*

$$x'(s) \in W_{\alpha(x(s), \cdot)} \quad \text{for a.e. } s \in [0, t]. \quad (18)$$

**Proof.** The first statement follows from Corollary 12 in the special case of  $x(\cdot; \phi)$  independent of  $\phi$ . The other statements are trivial.  $\square$

The last statement of Theorem 10 says that set of points where the maximum  $K$  of the initial data propagates contains some Lipschitz arc starting from  $(x_o, 0)$ : it is enough to take  $x(s) := x_o + \int_{t-s}^t \beta[0](\rho) d\rho$  and use (11) to get

$$u(x(s), s) = K \quad \forall s \in [0, t].$$

The next result states that, if the propagation set is reduced to a single such arc, and we are in the homogeneous case of Lemma 9, then the sufficient condition of the previous corollary is also necessary.

**Proposition 14** *Under the assumptions of Lemma 9, if the propagation set  $\{(y, s) : u(y, s) = K\}$  in a neighborhood of  $(x_o, 0)$  coincides with  $\{(x(s), s) : 0 \leq s \leq t\}$  for some  $t > 0$  and a Lipschitz arc  $x(\cdot)$  such that  $x(0) = x_o$ , then  $x(\cdot)$  satisfies (17) and (18).*

**Proof.** Fix any  $s$  where  $x(\cdot)$  is differentiable and consider any  $(p, \mu) \in \mathbb{R}^{N+1}$  such that

$$p \cdot x'(s) + \mu = 0.$$

By Taylor's expansion of  $x(\cdot)$  at  $s$

$$p \cdot (x(\tau) - x(s)) + \mu(\tau - s) = o(|\tau - s|).$$

By Lemma 9  $u(x(\tau), \tau) = K$  and  $u = -\infty$  elsewhere in a neighborhood of  $(x(s), s)$ . Then it is easy to see that  $(p, \mu) \in D^+u(x(s), s)$ , where  $D^+u$  denotes the superdifferential of  $u$  (see, e.g., [BCD]). Therefore

$$(p, -p \cdot x'(s)) \in D^+u(x(s), s) \quad \text{for all } p \in \mathbb{R}^N,$$

and since  $u$  is a subsolution of the H-J equation we get

$$-p \cdot x'(s) + H(x(s), p) \leq 0,$$

which proves the desired inequality (17).  $\square$

A more general necessary condition is given by the next result.

**Corollary 15** *Assume (1), (2), and (9). Then a necessary condition for the propagation of the maximum of  $g_o$  is the existence of  $t > 0$ ,  $q \in \mathbb{R}^N$ , and a map  $\gamma : \mathbb{R}^N \rightarrow \mathcal{B}$  such that*

$$\begin{cases} \frac{1}{t} \int_0^t H(x_o + \int_s^t \gamma[p](\tau) d\tau, p) ds \leq q \cdot p \\ \frac{1}{t} \int_0^t \gamma[p](s) ds = q \end{cases} \quad (19)$$

for all  $p \in \mathbb{R}^N$ .

**Proof.** The necessary condition is obtained from Theorem 10 in the special case  $a(s) \equiv p$  by setting  $q := \frac{x - x_o}{t}$ .  $\square$

**Remark 16** From the proof of Theorem 10 it is easy to see the following facts: (i) the maximum of  $g_o$  propagates if and only if there exist  $x \in \mathbb{R}^N$  and  $t > 0$  such that  $u(x, t) = K$ ; (ii) the propagation set is arc-wise connected; (iii) in the case of propagation, besides (10) we have

$$\forall \varepsilon > 0 \exists a_\varepsilon : \int_0^t H(x_o + \int_s^t \beta[a_\varepsilon](\tau) d\tau, a_\varepsilon(s)) ds + \varepsilon \geq \int_0^t \beta[a_\varepsilon](s) \cdot a_\varepsilon(s) ds.$$

### 3 Examples

This section collects several special cases where the necessary and the sufficient conditions for propagation and right accessibility can be explicitly checked.

**Remark 17** In the case  $H = H(p)$  in a neighborhood of  $x_o$  we obtain the following local versions of results by Giga and Sato [GS]. From Corollary 13 we see that  $g_o$  is right accessible at  $x_o$  if there exists  $q \in \mathbb{R}^N$  such that

$$H(p) \leq q \cdot p \quad \forall p \in \mathbb{R}^N,$$

i.e.,  $W_\alpha \neq \emptyset$ . (We just take  $x(s) := x_o + sq$ ). This is stated in Theorem 6.7 of [GS] for  $H = H(p)$  for all  $x$  and admitting a recession function as in Remark 3, and it is also proved this condition is necessary. Here the necessity follows from Corollary 15 and Lemma 9 if  $H$  is independent of  $x$  and also positively 1-homogeneous for all  $x$  in a neighborhood of  $x_o$ . This recovers completely and extends Theorem 6.2 of [GS].

**Example 18** By Corollary 15, a simple necessary condition for propagation is the existence of  $t > 0$  and  $\gamma : \mathbb{R}^N \rightarrow \mathcal{B}$  such that

$$H(x_o + \int_s^t \gamma[0](\tau) d\tau, 0) = 0 \quad \text{for a.e. } s.$$

Thus the max of  $g_o$  does not propagate if  $H(x, 0) > 0$  in a neighborhood of  $x_o$ .

**Example 19** A simple sufficient condition for propagation is  $H(x_o, p) = 0$  for all  $p$ . In fact we take  $\beta \equiv 0$  in Theorem 10 and get that the max of  $g_o$  propagates and  $u(x_o, s) \equiv K$  for  $s \geq 0$ .

**Remark 20** If  $H(x, \cdot)$  is *concave* and  $H(x, 0) = 0$  for all  $x$  near  $x_o$ , then a solution of the differential inclusion

$$x'(s) \in D_p^+ H(x(s), 0) \quad \text{for a.e. small } s > 0, \quad x(0) = x_o$$

satisfies Corollary 13. In fact, by the definition of  $D_p^+$  (the superdifferential with respect to the variables  $p$ ) and the concavity of  $H(x, \cdot)$

$$H(x(s), p) \leq H(x(s), 0) + x'(s) \cdot p \quad \forall p$$

so (17) is satisfied.

In particular, if  $H(x, \cdot)$  is also differentiable at 0 for all  $x$  near  $x_o$  and  $x \rightarrow D_p H(x, 0)$  is continuous, the differential inclusion becomes an ODE that has solutions, so the maximum of  $g_o$  propagates.

Another special case is

$$H(x, p) = f(x)H_1(p), \quad f \geq 0 \text{ near } x_o, \quad H_1 \text{ concave.}$$

Here we can take any  $\zeta \in D^+ H_1(0)$  and solve the ODE

$$x'(s) = f(x(s))\zeta, \quad x(0) = x_o.$$

Thus in these cases the max of  $g_o$  propagates and  $g_o$  is right accessible at  $x_o$ , as one expects from the theory of bilateral supersolutions for Hamiltonians concave in  $p$  for all  $x$  [BJ, BCD]. Note that our results are local, in the sense that we assume the concavity in  $p$  only for  $x$  near  $x_o$ .

**Remark 21** Here we consider two cases of *convex*  $H(x, \cdot)$  for all  $x$  near  $x_o$  and show that the max of  $g_o$  propagates only if  $H(x, \cdot)$  is affine. We first take

$$H(x, p) = V(x) + H_2(p), \quad H_2 \text{ convex}, \quad H_2(0) = 0, \quad V \geq 0,$$

and fix  $\zeta \in D^- H_2(0)$ , where  $D^-$  denotes the subdifferential (see, e.g., [BCD]). We consider the map  $\gamma$ , the vector  $q$ , and  $t > 0$  in the necessary condition of propagation of Corollary 15, and get

$$\zeta \cdot p \leq \frac{1}{t} \int_0^t V(x_o + \int_s^t \gamma[p](\tau) d\tau) ds + \zeta \cdot p$$

$$\leq \frac{1}{t} \int_0^t V(x_o + \int_s^t \gamma[p](\tau) d\tau) ds + H_2(p) \leq q \cdot p, \quad \forall p \in \mathbb{R}^N,$$

where the 2nd inequality comes from the definition of  $D^-$  and the convexity of  $H_2$ . Then  $\zeta = q$ , so we use again  $V \geq 0$  to get

$$H_2(p) = \zeta \cdot p, \quad \forall p \in \mathbb{R}^N, \quad (20)$$

$$V(x_o + \int_s^t \gamma[p](\tau) d\tau) = 0, \quad \forall p \in \mathbb{R}^N, s \in [0, t].$$

The second example is

$$H(x, p) = f(x)H_2(p), \quad H_2 \text{ convex}, \quad H_2 \geq 0, \quad f(x) \geq c > 0$$

and we fix again  $\zeta \in D^- H_2(0)$ . From the necessary condition of RA (19) we get

$$c\zeta \cdot p \leq cH_2(p) \leq \frac{1}{t} \int_0^t f(x_o + \int_s^t \gamma[p](\tau) d\tau) ds H_2(p) \leq q \cdot p, \quad \forall p \in \mathbb{R}^N,$$

which implies  $c\zeta = q$  and therefore (20) holds again and

$$f(x_o + \int_s^t \gamma[p](\tau) d\tau) = 1, \quad \forall p \in \mathbb{R}^N, s \in [0, t].$$

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