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Weighted Sobolev-Lieb-Thirring inequalities

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Abstract. We give a weighted version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions. In the proof of our result we use φ -transform of Frazier-Jawerth.

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1 Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring inequality.

Theorem 1.1 ([2]). *Let $n \in \mathbb{N}$, $s > 0$ and p with*

$$\max\left(1, \frac{n}{2s}\right) < p \leq 1 + \frac{n}{2s}.$$

Then there exists a positive constant $c = c(p, n, s)$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^s(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$(1) \quad \left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \|(-\Delta)^{s/2} \phi_i\|^2$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

In this theorem $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order s and $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^n)$.

In [8] Lieb and Thirring proved this theorem for $s = 1$ and applied it to the problem of the stability of matter. Ghidaglia, Marion, and Temam proved (1) for $s \in \mathbb{N}$ under

the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$\sum_{i,j=1}^N \xi_i \overline{\xi_j}(\phi_i, \phi_j) \leq \sum_{i=1}^N |\xi_i|^2$$

holds for all $\xi_i \in \mathbb{C}, i = 1, \dots, N$ ([4]). They applied the inequality (1) to the estimate of the dimension of attractors associated with partial differential equations(c.f.[13]). In this paper we shall give a weighted version of (1) under suborthonormal condition on $\{\phi_i\}$. In the proof of our theorem we shall use Frazier-Jawerth's φ -transform([3]).

For the statement of our result we need to recall the definition of A_p -weights(c.f. [5], [10]). By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let w be a non-negative, locally integrable function on $\mathbb{R}^n$. We say that w is an A_p -weight for $1 < p < \infty$ if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(x) dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_p -constant of w . For example, $w(x) = |x|^\alpha$ is an A_p -weight when $-n < \alpha < n(p-1)$.

We say that w is an A_1 -weight if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(y) dy \leq Cw(x) \quad a.e. x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_1 -constant of w . Let A_p be the class of A_p -weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

For a nonnegative, locally integrable function w on $\mathbb{R}^n$ we define

$$L^p(w) = \left\{ f : \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x) dx < \infty \right\}.$$

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ the cube Q defined by

$$Q = Q_{\nu k} = \{(x_1, \dots, x_n) : k_i \leq 2^\nu x_i < k_i + 1, i = 1, \dots, n\}$$

is called a dyadic cube in $\mathbb{R}^n$. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$. For any $Q \in \mathcal{Q}$ there exists a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is double of that of Q . We call Q' the parent of Q .

For $s > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ we define via inverse Fourier transform

$$(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).$$

Let $w \in A_2$ and $\mathcal{H}^s(w)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) dx + \|f\|^2 \right\}^{1/2}.$$

We remark that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x) dx < \infty$$

because

$$|(-\Delta)^{s/2} f(x)| \leq \frac{c}{(1+|x|)^n} \quad (x \in \mathbb{R}^n)$$

and

$$\int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{2n}} dx < \infty$$

(c.f. [10, p.209]).

Let $f \in \mathcal{H}^s(w)$ and $\{f_i\}_{i=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\|f - f_i\|_{\mathcal{H}^s(w)} \rightarrow 0$ ($i \rightarrow \infty$). This means that there exist $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(w)$ such that $\|g_1 - f_i\| \rightarrow 0$ and

$$\int_{\mathbb{R}^n} |g_2(x) - (-\Delta)^{s/2} f_i(x)|^2 w(x) dx \rightarrow 0$$

as $i \rightarrow \infty$. We denote $(-\Delta)^{s/2} f = g_2$. We remark that $g_1 \equiv 0$ means $g_2 \equiv 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} g_2 \bar{\varphi} dx = \lim_{i \rightarrow \infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \bar{\varphi} dx = \lim_{i \rightarrow \infty} \int_{\mathbb{R}^n} \overline{f_i (-\Delta)^{s/2} \varphi} dx = 0.$$

Hence we have $g_2 \equiv 0$. This means that we can identify $\mathcal{H}^s(w)$ as a subspace of $L^2(\mathbb{R}^n)$.

The following is the main result of this paper.

Theorem 1.2. *Let $n \in \mathbb{N}, s > 0$, and*

$$\max(1, \frac{n}{2s}) < p \leq 1 + \frac{n}{2s}.$$

Let $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \geq n$, then we assume that $w^{-n/(2s)} \in A_p$ and

$$(2) \quad \int_{Q'} w dx \leq 2^{2s} \int_Q w dx$$

for all dyadic cubes $Q \in \mathcal{Q}$ and its parent Q' .

Then there exists a positive constant c such that for every family $\{\phi_i\}_{i=1}^N$ in $\mathcal{H}^s(w)$ which is suborthonormal in $L^2(\mathbb{R}^n)$, we have

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$$

and c depends only on n, s, p, A_2 -constant of w , and $A_{n/(2s)}$ or A_p -constant of $w^{-n/(2s)}$.

When $2s < n$, an example of weight function w is given by $w(x) = |x|^\alpha$ for $-n+2s < \alpha < 2s$. When $2s > n$, an example of weight function w is given by $w(x) = |x|^\alpha$ for $0 \leq \alpha < \min\{2s - n, n\}$ (c.f. [12, Section 4]). When $2s = n$, the condition (2) means w is equivalent to a constant almost everywhere (c.f. [12, Proposition 4.1]).

2 Preliminaries

Let ψ be a function which satisfies the following conditions.

(A1) $\psi \in \mathcal{S}(\mathbb{R}^n)$.

(A2) $\text{supp } \hat{\psi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$

(A3) $|\hat{\psi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.

(A4) $\sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 = 1$ for all $\xi \neq 0$.

For $\nu \in \mathbb{Z}, k \in \mathbb{Z}^n$ and $Q = Q_{\nu k}$, we set

$$\psi_Q(x) = 2^{\nu n/2} \psi(2^\nu x - k) \quad (x \in \mathbb{R}^n).$$

Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where f is a locally integrable function on $\mathbb{R}^n$ and the supremum is taken over all cubes Q which contain x .

Proposition 2.1.

(i) Let $1 < p < \infty$ and w be a non-negative locally integrable function on $\mathbb{R}^n$. Then there exists a positive constant c such that

$$\int_{\mathbb{R}^n} M(f)^p w dx \leq c \int_{\mathbb{R}^n} |f|^p w dx$$

for all $f \in L^p(w)$ if and only if $w \in A_p$. The constant c depends only on n, p and A_p -constant of w .

- (ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1, p)$ such that $w \in A_q$.
- (iii) Let $0 < \tau < 1$ and f be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $(M(f))^\tau \in A_1$ and the A_1 -constant of $(M(f))^\tau$ depends only on n and τ .
- (iv) Let $1 \leq p < \infty$ and $w \in A_p$. Then there exists a positive constant c such that

$$\int_{2Q} w \, dx \leq c \int_Q w \, dx$$

for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of Q and c depend only on n and A_p -constant of w .

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3 Proof of Theorem 1.2

The suborthonormal condition on $\{\phi_i\}$ is equivalent to the inequality

$$\sum_{i=1}^N |(\phi_i, f)|^2 \leq \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$ (c.f.[1, p57]).

We shall prove the inequality

$$\begin{aligned} & \left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\ (3) \quad & \leq c K^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i(x)|^2 w(x) \, dx \end{aligned}$$

under the assumption

$$(4) \quad \sum_{i=1}^N |(\phi_i, f)|^2 \leq K \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$ where K is a positive constant. This is equivalent to the statement of Theorem 1.2. We remark that K may depend on $\{\phi_i\}$. For example, the inequality (3) says that

$$(5) \quad \left\{ \int_{\mathbb{R}^n} |\phi|^{2p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \leq c \|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi|^2 w \, dx$$

holds for all $\phi \in \mathcal{H}^s(w)$ under suitable condition on s, p, n and w because

$$|(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C_0^\infty(\mathbb{R}^n), i = 1, \dots, N$. Let

$$V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2s(p-1))}$$

where the value of the constant $\delta_1 > 0$ will be given later. Since $\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$ is a bounded function with compact support and $w^{n/(2s(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have

$$\int_{\mathbb{R}^n} V^p w^{-n/(2s)} dx < \infty.$$

We may also assume that

$$0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)} dx.$$

By (ii) of Proposition 2.1 there exists a constant κ such that $1 < \kappa < p$ and $w^{-n/(2s)} \in A_{p/\kappa}$. We set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 2.1 leads to

$$(6) \quad \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)} dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)} dx < \infty.$$

Furthermore v is an A_1 -weight by (iii) of Proposition 2.1.

We have the following lemmas.

Lemma 3.1. *For $s > 0$ and $w \in A_2$ there exists a positive constant α such that*

$$\alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where α is given by

$$\alpha^{-1} = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and c is a constant depending only on n, s and A_2 -constant of w .

Lemma 3.2. *For $v \in A_2$ there exist positive constants β and β' such that*

$$\beta' \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v dx \leq \int_{\mathbb{R}^n} |f|^2 v dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where β is given by

$$\beta = c \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and c is a constant depending only on n and A_2 -constant of v .

The proof of Lemmas 3.1 and 3.2 are in [11, Proposition 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader's convenience because the dependence of ψ in α and β is not explained in [11].

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f|^2 V dx \leq \int_{\mathbb{R}^n} |f|^2 v dx \leq \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v dx,$$

where we used Lemma 3.2. Hence by Lemma 3.1

$$(7) \quad \begin{aligned} & \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w dx - \int_{\mathbb{R}^n} V |f|^2 dx \\ & \geq \alpha \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w dx - \beta \sum_{Q \in \mathcal{Q}} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q v dx. \end{aligned}$$

Now we set

$$(8) \quad \mathcal{I} = \left\{ Q \in \mathcal{Q} : \beta \int_Q v dx > \alpha |Q|^{-2s/n} \int_Q w dx \right\}.$$

Let $\{\mu_k\}_{1 \leq k}$ be the non-decreasing rearrangement of

$$\left\{ \alpha |Q|^{-2s/n-1} \int_Q w dx - \beta |Q|^{-1} \int_Q v dx \right\}_{Q \in \mathcal{I}}.$$

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When

$$\mu_k = \alpha |Q|^{-2s/n-1} \int_Q w dx - \beta |Q|^{-1} \int_Q v dx,$$

we define $\Psi_k = \psi_Q$. Then we have by (7)

$$(9) \quad \begin{aligned} & \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V |\phi_i|^2 dx \\ & \geq \sum_{i=1}^N \sum_{Q \in \mathcal{Q}} |(\phi_i, \psi_Q)|^2 \left\{ \alpha |Q|^{-2s/n-1} \int_Q w dx - \beta |Q|^{-1} \int_Q v dx \right\} \\ & \geq \sum_{i=1}^N \sum_k \mu_k |(\phi_i, \Psi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\phi_i, \Psi_k)|^2 \\ (10) \quad & \geq -K \|\psi\|^2 \sum_k |\mu_k| \geq -K \|\psi\|^2 \left(\sum_k |\mu_k|^\gamma \right)^{1/\gamma}, \end{aligned}$$

where $\gamma = p - n/(2s) \in (0, 1]$ and we used (4).

Now the following lemma holds.

Lemma 3.3.

$$\sum_k |\mu_k|^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx,$$

where c is given by

$$c = c' \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^{n/s+2p}$$

and c' depends only on n, s, p and w .

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (6) the last quantity in (10) is estimated from below by

$$\begin{aligned} & -cK \left(\int_{\mathbb{R}^n} V^p w^{-n/(2s)} dx \right)^{1/\gamma} \\ &= -cK \delta_1^{p/(p-n/(2s))} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} dx \right)^{1/(p-n/(2s))}, \end{aligned}$$

where

$$c = c' \|\psi\|^2 \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^{(4ps+2n)/(2ps-n)}$$

and c' depends only on n, s, p and w . We may take the infimum of the above constant with respect to possible ψ and replace c by this infimum.

Let

$$\delta_1 = \delta_2 K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} dx \right)^{2s(p-n/(2s)-1)/n},$$

where δ_2 is a positive constant. Then we have by (9)

$$\begin{aligned} & \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w dx \\ & \geq \delta_2 K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} dx \right)^{2s(p-1)/n} \\ & \quad - cK \delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} dx \right)^{2s(p-1)/n} \\ & = \{\delta_2 - c\delta_2^{p/(p-n/(2s))}\} K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))} dx \right)^{2s(p-1)/n}. \end{aligned}$$

If we take δ_2 small enough, then we get the inequality (3) because $1 < p/(p-n/(2s))$.

Next we shall show (3) for $\phi_i \in \mathcal{H}^s(w)$, $i = 1, \dots, N$. First we show

$$(11) \quad \mathcal{H}^s(w) \subset L^{2p/(p-1)}(w^{n/(2s(p-1))}).$$

Let $h \in \mathcal{H}^s(w)$. Then there exists a sequence $\{h_m\}_{m=1}^\infty \subset C_0^\infty(\mathbb{R}^n)$ such that $\|h - h_m\|_{\mathcal{H}^s(w)} \rightarrow 0$ ($m \rightarrow \infty$). Since we proved that (5) holds for $h_m \in C_0^\infty(\mathbb{R}^n)$, we get

$$\left\{ \int_{\mathbb{R}^n} |h_m|^{2p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \leq c \|h_m\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h_m|^2 w dx,$$

where c does not depend on h_m . Since $4sp/n - 2 > 0$ and $\{h_m\}$ is a Cauchy sequence in $\mathcal{H}^s(w)$, the above inequality says that $\{h_m\}$ is a Cauchy sequence in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. Let g be the limit of $\{h_m\}$ in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. For any compact set E in $\mathbb{R}^n$ we have

$$\begin{aligned} \int_E |g - h_m| dx &\leq \left(\int_E |g - h_m|^{2p/(p-1)} w^{n/(2s(p-1))} dx \right)^{(p-1)/(2p)} \\ &\quad \times \left(\int_E w^{-n/(2s(p+1))} dx \right)^{(p+1)/(2p)}. \end{aligned}$$

Since $w^{-n/(2s)}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ or $w^{-n/(2s)} \in A_p$, we get $h_m \rightarrow g$ in $L_{loc}^1(\mathbb{R}^n)$ as $m \rightarrow \infty$. Hence we have $g = h$ and (11). Furthermore we have

$$(12) \quad \left\{ \int_{\mathbb{R}^n} |h|^{2p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \leq c \|h\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} h|^2 w dx.$$

We fix a positive number ε . Let $\chi_1, \dots, \chi_N$ be functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{i=1}^N \|\phi_i - \chi_i\|_{\mathcal{H}^s(w)}^2 < \varepsilon.$$

Now the inequalities

$$\begin{aligned} \sum_{i=1}^N |(\chi_i, f)|^2 &\leq 2 \sum_{i=1}^N |(\chi_i - \phi_i, f)|^2 + 2 \sum_{i=1}^N |(\phi_i, f)|^2 \\ (13) \quad &\leq 2 \sum_{i=1}^N \|\chi_i - \phi_i\|^2 \|f\|^2 + 2K \|f\|^2 \leq 2(K + \varepsilon) \|f\|^2 \end{aligned}$$

hold for all $f \in L^2(\mathbb{R}^n)$. On the other hand

$$\begin{aligned}
& \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \\
& \leq \left\{ \sum_{i=1}^N \left(\int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} dx \right)^{(p-1)/p} \right\}^{2sp/n} \\
& \leq N^{2sp/n-1} \sum_{i=1}^N \left(\int_{\mathbb{R}^n} |\phi_i - \chi_i|^{2p/(p-1)} w^{n/(2s(p-1))} dx \right)^{2s(p-1)/n} \\
& \leq cN^{2sp/n-1} \sum_{i=1}^N \|\phi_i - \chi_i\|^{4sp/n-2} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w dx \\
& \leq cN^{2sp/n-1} \varepsilon^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i - (-\Delta)^{s/2} \chi_i|^2 w dx \\
& \leq cN^{2sp/n-1} \varepsilon^{2sp/n},
\end{aligned}$$

where we used (12).

Therefore

$$\begin{aligned}
& \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \\
& \leq \left\{ \int_{\mathbb{R}^n} \left(2 \sum_{i=1}^N |\phi_i - \chi_i|^2 + 2 \sum_{i=1}^N |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \\
& \leq 2^{2sp/n} \left[\left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{(p-1)/p} \right. \\
& \quad \left. + \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{(p-1)/p} \right]^{2sp/n} \\
& \leq 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i - \chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \\
& \quad + 2^{4sp/n-1} \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\chi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} dx \right\}^{2s(p-1)/n} \\
(14) \quad & \leq c2^{4sp/n-1} N^{2sp/n-1} \varepsilon^{2sp/n} + c2^{6sp/n-2} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 w dx,
\end{aligned}$$

where we used (13) and (3) for χ_i . Since

$$\begin{aligned}
& \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i|^2 w \, dx \\
& \leq 2 \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \chi_i - (-\Delta)^{s/2} \phi_i|^2 w \, dx + 2 \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx \\
& \leq 2\varepsilon + 2 \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx,
\end{aligned}$$

we have by (14)

$$\begin{aligned}
& \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
& \leq c 2^{4sp/n-1} N^{2sp/n-1} \varepsilon^{2sp/n} + c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \varepsilon \\
& \quad + c 2^{6sp/n-1} (K + \varepsilon)^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx.
\end{aligned}$$

Since we can take ε arbitrary small, we conclude

$$\begin{aligned}
& \left\{ \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\phi_i|^2 \right)^{p/(p-1)} w^{n/(2s(p-1))} \, dx \right\}^{2s(p-1)/n} \\
& \leq c 2^{6sp/n-1} K^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2} \phi_i|^2 w \, dx.
\end{aligned}$$

Hence we get (3).

4 Proof of Lemma 3.3

The arguments of the proof is similar to those in [11] and [12]. First we consider the case $n > 2s$. For $\lambda > 0$ we set

$$(15) \quad \mathcal{I}_\lambda = \{Q \in \mathcal{Q} : \alpha |Q|^{-2s/n-1} \int_Q w \, dx - \beta |Q|^{-1} \int_Q v \, dx < -\lambda\}.$$

Then we have for $Q \in \mathcal{I}_\lambda$

$$\alpha |Q|^{-2s/n-1} \int_Q w \, dx < |Q|^{-1} \int_Q (\beta v - \lambda)_+ \, dx,$$

where

$$(\beta v - \lambda)_+(x) = \max\{0, \beta v(x) - \lambda\}.$$

Since $p = n/(2s) + \gamma$, $\gamma \in (0, 1]$, and

$$\beta^{-p} \gamma \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)} dx \lambda^{\gamma-1} d\lambda \leq \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx < \infty,$$

we have

$$\int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx < \infty$$

for all $\lambda > 0$. By the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ and (ii) of Proposition 2.1, there exists a $\kappa' \in (1, n/(2s))$ such that $w^{-n/(2s)} \in A_{n/(2s\kappa')}$. We set

$$v_\lambda^*(x) = M((\beta v - \lambda)_+^{\kappa'}(x))^{1/\kappa'}.$$

Then

$$(16) \quad \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \leq c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx < \infty$$

and $v_\lambda^* \in A_1$ by (iii) of Proposition 2.1, where c_1 depends only on n, s and $A_{n/(2s)}$ -constant of $w^{-n/(2s)}$.

We can show that $\mathcal{I}_\lambda$ is a finite set as follows. Let $Q \in \mathcal{I}_\lambda$. Then we have

$$\begin{aligned} \alpha |Q|^{-2s/n} \int_Q w dx &\leq \int_Q v_\lambda^* dx \\ &\leq \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \right\}^{2s/n} \left\{ \int_Q w^{n/(n-2s)} dx \right\}^{(n-2s)/n}. \end{aligned}$$

Since $w^{-n/(2s)} \in A_{n/(2s)}$, the last quantity is bounded by

$$\begin{aligned} c_2 \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \right\}^{2s/n} |Q| \left(\int_Q w^{-n/(2s)} dx \right)^{-2s/n} \\ \leq c_2 \left\{ \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \right\}^{2s/n} |Q|^{-2s/n} \int_Q w dx, \end{aligned}$$

where we used the inequality

$$1 \leq \frac{1}{|Q|} \int_Q w dx \left(\frac{1}{|Q|} \int_Q w^{-n/(2s)} dx \right)^{2s/n}.$$

The above calculation says

$$1 \leq c_3 \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx,$$

where $c_3 = c' \alpha^{-n/(2s)}$ and c' is the $A_{n/(2s)}$ -constant of $w^{-n/(2s)}$.

First we assume that $\mathcal{I}_\lambda$ includes infinite disjoint cubes $\{Q_i\}_{i=1}^\infty$. Then we have

$$\infty = \sum_{i=1}^\infty 1 \leq \sum_{i=1}^\infty c_3 \int_{Q_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \leq c_3 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx < \infty.$$

This is a contradiction. Hence $\mathcal{I}_\lambda$ does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes $\{Q_i\}_{i=1}^\infty \subset \mathcal{I}_\lambda$ such that $Q_i \neq Q_j$ ($i \neq j$) and $Q_1 \subset Q_2 \subset Q_3 \subset \dots$. Let $\tilde{Q}_i$ be a half size dyadic sub-cube of Q_{i+1} such that $Q_i \cap \tilde{Q}_i = \emptyset$. Since $Q_{i+1} \in \mathcal{I}_\lambda$, we have

$$\alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \leq \int_{Q_{i+1}} v_\lambda^* dx.$$

Now we get

$$\int_{Q_{i+1}} v_\lambda^* dx \leq \int_{3\tilde{Q}_i} v_\lambda^* dx \leq c_4 \int_{\tilde{Q}_i} v_\lambda^* dx,$$

where we used the doubling property of v_λ^* . Since

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \geq 2^{-2s} |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w dx,$$

we get

$$c_5 |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w dx \leq \int_{\tilde{Q}_i} v_\lambda^* dx.$$

The similar calculation as before leads to

$$1 \leq c_6 \int_{\tilde{Q}_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx,$$

where $c_6 = c'' \alpha^{-n/(2s)}$ and c'' depends only on n, s , and w . Since $\{\tilde{Q}_i\}_{i=1}^\infty$ is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \subset Q_2 \subset Q_3 \subset \dots$ has a maximal element. Similarly we can show that any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \supset Q_2 \supset Q_3 \supset \dots$ has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in $\mathcal{I}_\lambda$ with respect to the inclusion relation is finite. Hence $\mathcal{I}_\lambda$ is a finite set. We remark that the non-decreasing rearrangement of $\mathcal{I}$ in (8) is possible because $\mathcal{I}_\lambda$ is a finite set for every $\lambda > 0$.

Let $N(\lambda) = \#\mathcal{I}_\lambda$, that is, the number of elements of $\mathcal{I}_\lambda$. Let $\tilde{\mathcal{I}}_\lambda$ be the set of all $Q \in \mathcal{I}_\lambda$ which satisfy the following condition: there exists a half size dyadic sub-cube $\tilde{Q} \subset Q$ such that $\tilde{Q} \notin \mathcal{I}_\lambda$ and $\tilde{Q}$ does not contain any dyadic cube in $\mathcal{I}_\lambda$. Then we have the following lemma.

Lemma 4.1. $\#\mathcal{I}_\lambda \leq 2\#\tilde{\mathcal{I}}_\lambda$.

Lemma 4.1 is proved in Rochberg and Taibleson's paper([9, Lemma 1]).

Let $Q \in \tilde{\mathcal{I}}_\lambda$ and $\tilde{Q}$ be a dyadic cube which satisfies the condition in the definition of $\tilde{\mathcal{I}}_\lambda$. Then by similar calculations as before we get

$$1 \leq c_6 \int_{\tilde{Q}} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx.$$

For every $Q \in \tilde{\mathcal{I}}_\lambda$ we choose a $\tilde{Q}$ as above. Let $\{\tilde{Q}_j\}_{j \in J}$ be the set of all such cubes $\tilde{Q}$. Then the cubes in $\{\tilde{Q}_j\}_{j \in J}$ are mutually disjoint. Therefore we get

$$\begin{aligned} \#\tilde{\mathcal{I}}_\lambda &= \#J \leq \sum_{j \in J} c_6 \int_{\tilde{Q}_j} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \\ &\leq c_6 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)} dx \leq c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx, \end{aligned}$$

where we used (16). Hence we have

$$N(\lambda) \leq 2c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx.$$

Therefore we conclude

$$\begin{aligned} \sum_k |\mu_k|^\gamma &= \int_0^\infty \gamma \lambda^{\gamma-1} N(\lambda) d\lambda \\ &\leq 2c_7 \int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)} dx \gamma \lambda^{\gamma-1} d\lambda \\ &\leq c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx, \end{aligned}$$

where $c_8 = c''' \alpha^{-n/(2s)} \beta^p$ and c''' depends only on n, s, p and w .

Next we consider the case $n \leq 2s$. We remark that $v(x) > 0$ for all $x \in \mathbb{R}^n$. In fact if $v(x_0) = 0$ at some point x_0 , then by the definition of the maximal operator we have $V \equiv 0$, that is, $\phi_i \equiv 0, i = 1, \dots, N$.

We also remark that $\mathcal{I}$ in (8) is not empty. In fact if $\mathcal{I}$ is empty, then we have

$$\beta \int_Q v dx \leq \alpha |Q|^{-2s/n} \int_Q w dx$$

for all $Q \in \mathcal{Q}$. Let $Q_0 \in \mathcal{Q}$ and $Q_0 \subset Q_1 \subset Q_2 \subset \dots$ be the infinite sequence of dyadic cubes such that Q_{i+1} is the parent of Q_i for all $i = 0, 1, 2, \dots$. By (2) we have

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w dx \leq |Q_i|^{-2s/n} \int_{Q_i} w dx$$

for all i . Hence we have

$$(17) \quad \beta \int_{Q_i} v \, dx \leq \alpha |Q_0|^{-2s/n} \int_{Q_0} w \, dx$$

for all i . On the other hand, since $v \in A_1$, there exists a constant $d > 1$ such that

$$d \int_{Q_i} v \, dx \leq \int_{Q_{i+1}} v \, dx$$

for all i (c.f.[5, p141]). Hence we have

$$d^i \int_{Q_0} v \, dx \leq \int_{Q_i} v \, dx$$

and

$$\lim_{i \rightarrow \infty} \int_{Q_i} v \, dx = \infty,$$

which contradicts to (17). Therefore $\mathcal{I}$ is not empty.

Let $Q \in \mathcal{I}$ and Q' be the parent of Q . Then we have

$$\alpha |Q'|^{-2s/n} \int_{Q'} w \, dx \leq \alpha |Q|^{-2s/n} \int_Q w \, dx < \beta \int_Q v \, dx \leq \beta \int_{Q'} v \, dx,$$

where we used the assumption (2). Hence we have $Q' \in \mathcal{I}$, which means that $\mathcal{I}$ is an infinite set.

Lemma 4.2. *There exists a $c > 0$ such that*

$$\sum_{Q \in \mathcal{I}} \left(\frac{1}{|Q|} \int_Q v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx,$$

where $c = c' \alpha^{-n/(2s)} \beta^{n/(2s)}$ and c' depends only on n, p, s and w .

This lemma is proved in [12, Lemma 3.3]. Let $\mathcal{I}_\lambda$ be the set defined by (15).

Lemma 4.3. *For each $\lambda > 0$, $\mathcal{I}_\lambda$ is a finite set.*

Lemma 4.3 is easily proved by Lemma 4.2 (c.f.[12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of $\mathcal{I}$ is possible.

By Lemma 4.2 we conclude

$$\begin{aligned} \sum_{k=1}^{\infty} |\mu_k|^\gamma &= c \sum_{Q \in \mathcal{I}} \left(\beta |Q|^{-1} \int_Q v \, dx - \alpha |Q|^{-2s/n-1} \int_Q w \, dx \right)^\gamma \\ &\leq c \sum_{Q \in \mathcal{I}} \left(\beta |Q|^{-1} \int_Q v \, dx \right)^\gamma \leq c \int_{\mathbb{R}^n} v^p w^{-n/(2s)} \, dx, \end{aligned}$$

where $c = c'' \alpha^{-n/(2s)} \beta^p$ and c'' depends only on n, p, s and w . This ends the proof of Lemma 3.3.

5 Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

Lemma 5.1. *Let $w \in A_2$ and $m \in C^n(\mathbb{R}^n \setminus \{0\})$. Suppose that*

$$B = \max_{|\sigma| \leq n} \sup_{0 < r} r^{2|\sigma| - n} \int_{r \leq |\xi| \leq 2r} \left| \left(\frac{\partial}{\partial \xi} \right)^\sigma m(\xi) \right|^2 d\xi < \infty.$$

Then the operator T defined by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ is bounded from $L^2(w)$ to $L^2(w)$ and the operator norm $\|T\|$ is bounded by $CB^{1/2}$ where C is a constant which depends only on n and w .

The proof of Lemma 5.1 is in [6] or [7].

For $\nu \in \mathbb{Z}$ we define $\psi_\nu(x) = 2^{n\nu}\psi(2^\nu x)$. Let $w \in A_2$ and $s \geq 0$. Frazier and Jawerth proved that there exist positive constants c_1 and c_2 such that

$$\begin{aligned} c_1 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w dx &\leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} 2^{2s\nu} |f * \psi_\nu(x)|^2 \right\} w(x) dx \\ &\leq c_2 \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where c_1 and c_2 depend only on n, s and w ([3, Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let $\{r_\nu(t)\}$ be the Rademacher functions on $[0, 1]$ indexed by $\nu \in \mathbb{Z}$ and

$$T_t f(x) = \sum_{\nu \in \mathbb{Z}} r_\nu(t) f * \psi_\nu(x).$$

Then T_t satisfies the condition of Lemma 5.1. Hence

$$\int_{\mathbb{R}^n} |T_t f(x)|^2 w(x) dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx,$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where $M = \max_{|\sigma| \leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$ and C is a positive constant depending only on n and w . By integrating from 0 to 1 with respect to t , we get

$$\int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) dx \leq CM \int_{\mathbb{R}^n} |f(x)|^2 w(x) dx.$$

By the duality argument and the fact $w^{-1} \in A_2$ we obtain

$$\int_{\mathbb{R}^n} |f(x)|^2 w(x) dx \leq CM \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu(x)|^2 \right\} w(x) dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Hence we have

$$c_3 M^{-1} \int_{\mathbb{R}^n} |f|^2 w \, dx \leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |f * \psi_\nu|^2 \right\} w \, dx \leq c_4 M \int_{\mathbb{R}^n} |f|^2 w \, dx,$$

where c_3 and c_4 are constants depending only on n and w .

Therefore we get

$$\begin{aligned} c_3 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx &\leq \int_{\mathbb{R}^n} \left\{ \sum_{\nu \in \mathbb{Z}} |(-\Delta)^{s/2} f * \psi_\nu|^2 \right\} w \, dx \\ &\leq c_4 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ (c.f. [11]).

Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\text{supp } \Phi \subset \{\xi : 1/4 \leq |\xi| \leq 4\}$ and $\Phi(\xi) = 1$ for $1/2 \leq |\xi| \leq 2$. For $\nu \in \mathbb{Z}$ the multiplier $m_\nu(\xi) = 2^{-s\nu} |\xi|^s \Phi(\xi/2^\nu)$ satisfies the condition of Lemma 5.1. Hence we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2 w(x) \, dx \leq c_5 \int_{\mathbb{R}^n} 2^{2s\nu} |f * \psi_\nu(x)|^2 w(x) \, dx,$$

where $c_5 = c_6 \inf_{\Phi} \max_{|\sigma| \leq n} \|\partial^\sigma \Phi\|_\infty^2$ and c_6 is a positive constant depending only on n, s and w and the infimum is taken over all possible Φ .

Similarly there exists a positive constant c_7 depending only on n, s and w such that

$$\int_{\mathbb{R}^n} 2^{2s\nu} |f * \psi_\nu(x)|^2 w(x) \, dx \leq c_7 \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f * \psi_\nu(x)|^2 w(x) \, dx.$$

Hence we get

$$\begin{aligned} c_8 M^{-1} \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx &\leq \sum_{Q \in \mathcal{Q}} |Q|^{-2s/n} |(f, \psi_Q)|^2 \frac{1}{|Q|} \int_Q w \, dx \\ &\leq c_9 M \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f|^2 w \, dx \end{aligned}$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where c_8 and c_9 are positive constant depending only on n, s and w . This ends the proof of Lemmas 3.1 and 3.2.

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