HOKKAIDO UNIVERSITY

Title	Coordinate Change of Gauss-Manin System and Generalized Mirror Transformation
Author(s)	Jinzenji, Masao
Citation	Hokkaido University Preprint Series in Mathematics, 619, 1-19
Issue Date	2003-12-01
DOI	https://doi.org/10.14943/83764
Doc URL	https://hdl.handle.net/2115/69368
Type	departmental bulletin paper
File Information	re619.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

Coordinate Change of Gauss-Manin System and Generalized Mirror Transformation

Masao Jinzenji

Division of Mathematics, Graduate School of Science
Hokkaido University
Kita-ku, Sapporo, 060-0810, Japan
e-mail address: jin@math.sci.hokudai.ac.jp

November 12, 2003

Abstract

In this paper, we explicitly derive the generalized mirror transformation of quantum cohomology of general type projective hypersurfaces, proposed in our previous article, as an effect of coordinate change of the virtual Gauss-Manin system.

1 Introduction

This paper is the completion of our previous work [9], [8] on the quantum cohomology of general type hypersurfaces. To be more precise, we construct an algorithm to compute Kähler Gromov-Witten invariants (structure constants of Kähler sub-ring of small quantum cohomology ring) of degree k hypersurface in CP^{N-1} with $k > N$ (we denote it by M_N^k) for rational curves of arbitrary degree only by using Givental's ODE [5]

$$\Big((\partial_x)^{N-1} - k \cdot e^x \cdot (k\partial_x + k - 1)(k\partial_x + k - 2) \cdots (k\partial_x + 1)\Big) w(x) = 0, \tag{1.1}$$

as the starting point.[1] Of course, there are already several literatures [4], [13] and [2] on the quantum cohomology ring of general type projective hypersurfaces. But in [4] and [13], connection with mirror computation or (1.1) seems to be implicit. As for [2], direct connection with hypergeometric class or with (1.1) is suggested, but their construction is a little abstract and not practical for explicit prediction. From this point of view, our construction presented in this paper is explicit and can be used to compute any genus 0 Gromov-Witten invariants of M_N^k directly from (1.1).

Our construction is fundamentally based on the line of thoughts exposed in the course of our previous papers [3], [7],[9],[8] and [10]. Main ingredients of our construction consist of virtual structure constants and generalized mirror transformation, which were introduced and partially constructed in [9]. The virtual structure constants have their origin in our work [3] with A.Collino. In [3], A.Collino and myself constructed recursive formulas that express the structure constants of small quantum cohomology ring of M_N^k $(N - k) \geq 2$ in terms of the ones of M_{N+1}^k up to degree 5 rational curves (explicit form of recursive formula for rational curves of arbitrary degree was conjectured in [7] and was proved in [10]). The virtual structure constants are defined as rational number obtained from iterative use of these recursive formulas to the region $(N - k \leq 1)$. In [9], we conjectured that

[1] From now on, we omit the word "Kähler" and limit ourselves to consider Gromov-Witten invariants of M_N^k with insertion of $\mathcal{O}_{e^j}$, where e is the Kähler class of M_N^k.

the virtual structure constants are deeply related to (1.1) and that they correspond to three point functions of the B-model in the mirror conjecture. These conjecture were proved to be true in [10]. In other words, we have constructed an algorithm to compute the virtual structure constants for arbitrary N and k only by using (1.1). These results correspond to the first part of the mirror computation of the small quantum cohomology of general type projective hypersurface M_N^k $(k > N)$.

The second part of the mirror computation is the generalized mirror transformation, that correspond to the formula to convert the virtual structure constants into the actual structure constants of the small quantum cohomology ring of M_N^k in our context. Note that in the Calabi-Yau case, this process of mirror computation is realized by the coordinate change of the B-model deformation parameter into flat coordinate of the A-model, or Kähler deformation parameter. The notion of the generalized mirror transformation for M_N^k $(k > N)$ was first introduced in [9] and explicit form of the generalized mirror transformation was determined up to degree 3 rational curves by comparing the value of the virtual structure constants with the numerical results of some three point Gromov-Witten invariants of M_N^k. In [8], we determined the form of the generalized mirror transformation up to degree 5 rational curves by using curious coincidence between the terms appearing in the results of [9] and many-point Gromov-Witten invariants obtained from application of the associativity equation [12] and the Kähler equation to some simple combination of the virtual structure constants. This idea turned out to be very effective, but unfortunately, we found that such coincidence was not complete and that some non-trivial modification was needed as degree of rational curves grows. Therefore, our search for the generalized mirror transformation temporarily stopped.

Next break-through comes from application of the Gauss-Manin system to the quantum cohomology ring of M_N^k [10]. In [10], we proposed an idea of the virtual Gauss-Manin system, that has the virtual structure constants as the matrix elements of the Gauss-Manin connection. The virtual Gauss-Manin system, when applied to the Calabi-Yau hypersurfaces M_k^k, is effective not only in the B-model computation, but also in finding out the Jacobian between the B-model deformation parameter and the flat coordinate of the A-model, as was suggested in the last section of [10]. Therefore, we searched for a way to extend the construction of the mirror transformation of M_k^k via coordinate change of the virtual Gauss-Manin system to the case of M_N^k $(k > N)$. In the sequel, a hint of the answer was given by the theory of Iritani [6], which is a concrete exposition of the idea of Givental and Coates [2]. According to Iritani, we have to introduce not only the Kähler deformation parameter x^1, but also the parameter x^j $(j = 2, \cdots, N-2)$ that corresponds to e^j, i.e., j times wedge product of the Kähler class e. With this idea, we extend the virtual Gauss-Manin system by constructing the virtual Gauss-Manin connection corresponding to the deformation by x^j. With these preparation, we can read off the matrix elements of the Jacobian $\frac{\partial t^j}{\partial x^i}$ between x^j and the A-model deformation parameter t^j from the matrix elements of the virtual Gauss-Manin connection. Moreover, we can integrate out the Jacobian and obtain the rule of coordinate change $x^i = x^i(t^1, \cdots, t^{N-2})$. The results so obtained force us to introduce many point Gromov-Witten invariants obtained from application of the associativity equation and the modified Kähler equation, that was derived from Iritani's framework, to the matrix elements of Gauss-Manin connection associated with deformation of t^1. After some straightforward computations, we can naturally reproduce the results up to degree 5 rational curves given in [8]. Of course, our construction given in this paper can be applied to predict arbitrary genus 0 Gromov-Witten invariants of M_N^k $(k > N)$. Indeed, we computed some three point Gromov-Witten invariants of M_{13}^{14} for degree 6 rational curves by using our construction, and checked coincidence with the numerical results obtained from fixed point computation by Köntsevich [11]. We also expect that our construction will be explained from compactification of moduli space of rational curves in CP^{N-1} [9],[1].

This paper is organized as follows. In Section 2, we introduce the notation, the definition and the results obtained in our previous works. Especially, the definition of the virtual structure constants, the virtual Gauss-Manin system and the virtual Gromov-Witten invariants are given, and their relation to Givental's ODE (1.1) is discussed. We also mention some conjectures on the form of the generalized mirror transformation given in [9]. In Section 3, we expose our algorithm to construct the generalized mirror transformation for rational curves of arbitrary degree from the virtual Gauss-Manin system. Next, we derive our previous results in [8]. Note that our presentation in this section is slightly different from the outline given here, because we want to respect priority of the work of Iritani. Of course, the result of computation does not change. In Section 4, we briefly review the results of Iritani and explain how our algorithm in Section 3 naturally follows from his framework.

Acknowledgement We would like to thank Prof. T.Eguchi and Prof. M.Noumi for discussions at an early stage of this work. We also thank Prof. Y.Tonegawa and Hokkaido University Computer Center for assistance on numerical computation using fixed point theorem. Finally, we especially thank Dr. H.Iritani for suggestions that led me to a key idea to finish this work.

2 Overview of Our Previous Results

2.1 Definitions

In this subsection, we introduce the quantum Kähler sub-ring of the quantum cohomology ring of a degree k hypersurface in CP^{N-1}. Let M_N^k be a hypersurface of degree k in CP^{N-1}. We denote by $QH_e^*(M_N^k)$ the subring of the quantum cohomology ring $QH^*(M_N^k)$ generated by $\mathcal{O}_e$ induced from the Kähler form e (or, equivalently the intersection $H \cap M_N^k$ between a hyperplane class H of CP^{N-1} and M_N^k). Additive basis of $QH_e^*(M_N^k)$ is given by $\mathcal{O}_{e^j}$ $(j = 0, 1, \cdots, N-2)$, which is induced from $e^j \in H^{j,j}(M_N^k)$. The multiplication rule of $QH_e^*(M_N^k)$ is determined by the Gromov-Witten invariant of genus 0 $\langle \mathcal{O}_e \mathcal{O}_{e^{N-2-m}} \mathcal{O}_{e^{m-1-(k-N)d}} \rangle_{d, M_N^k}$ and it is given as follows:

$$\begin{aligned} L_m^{N,k,d} &:= \frac{1}{k} \langle \mathcal{O}_e \mathcal{O}_{e^{N-2-m}} \mathcal{O}_{e^{m-1-(k-N)d}} \rangle_d, \\ \mathcal{O}_e \cdot 1 &= \mathcal{O}_e, \\ \mathcal{O}_e \cdot \mathcal{O}_{e^{N-2-m}} &= \mathcal{O}_{e^{N-1-m}} + \sum_{d=1}^{\infty} L_m^{N,k,d} q^d \mathcal{O}_{e^{N-1-m+(k-N)d}}, \\ q &:= \exp(t), \end{aligned} \tag{2.2}$$

where the subscript d counts the degree of the rational curves measured by e. Therefore, $q = \exp(t)$ is the degree counting parameter.

Definition 1 *We call $L_n^{N,k,d}$ the structure constant of weighted degree d.*

Since M_N^k is a complex $(N-2)$ dimensional manifold, we see that a structure constant $L_m^{N,k,d}$ is non-zero only if the following condition is satisfied:

$$\begin{aligned} & 1 \leq N-2-m \leq N-2, 1 \leq m-1+(N-k)d \leq N-2, \\ \Longleftrightarrow \quad & max\{0, 2-(N-k)d\} \leq m \leq min\{N-3, N-1-(N-k)d\}. \end{aligned} \tag{2.3}$$

We rewrite (2.3) into

$$\begin{aligned} (N-k \geq 2) &\Longrightarrow 0 \leq m \leq (N-1)-(N-k)d \\ (N-k=1, d=1) &\Longrightarrow 1 \leq m \leq N-3 \\ (N-k=1, d \geq 2) &\Longrightarrow 0 \leq m \leq N-1-(N-k)d \\ (N-k \leq 0) &\Longrightarrow 2+(k-N)d \leq m \leq N-3. \end{aligned} \tag{2.4}$$

From (2.4), we easily see that the number of the non-zero structure constants $L_m^{N,k,d}$ is finite except for the case of $N = k$. Moreover, if $N \geq 2k$, the non-zero structure constants come only from the $d = 1$ part and the non-vanishing $L_m^{N,k,1}$ is determined by k and independent of N.

2.2 Virtual Structure Constants and Givental's ODE

In this subsection, we introduce the virtual structure constants $\tilde{L}_m^{N,k,d}$ which is non-zero only if $0 \leq m \leq N-1+(k-N)d$. The original definition of $\tilde{L}_m^{N,k,d}$ in [10] is given by the initial condition:

$$\sum_{m=0}^{k-1} \tilde{L}_m^{N,k,1} w^m = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)), \quad (N-k \geq 2), \tag{2.5}$$

and the recursive formulas that describe $\tilde{L}_m^{N,k,d}$ as a weighted homogeneous polynomial in $\tilde{L}_n^{N+1,k,d'}$ $(d' \leq d)$ of degree d. See [7] for the explicit form of the recursive formulas. In [10], we showed that the virtual structure constants are directly connected with the Givental's ODE:

$$\Big((\partial_x)^{N-1} - k \cdot e^x \cdot (k\partial_x + k - 1)(k\partial_x + k - 2) \cdots (k\partial_x + 1)\Big) w(x) = 0, \tag{2.6}$$

for arbitrary N and k via the virtual Gauss-Manin system defined as follows:

Definition 2 *We call the following rank 1 ODE for vector valued function $\tilde{\psi}_m(x)$*

$$\partial_x \tilde{\psi}_{N-2-m}(x) \quad = \quad \tilde{\psi}_{N-1-m}(x) + \sum_{d=1}^{\infty} \exp(dx) \cdot \tilde{L}_m^{N,k,d} \cdot \tilde{\psi}_{N-1-m-(N-k)d}(x). \tag{2.7}$$

the virtual Gauss-Manin system associated with the quantum Kähler sub-ring of M_N^k, where m runs through $0 \leq m \leq N-2$ if $N-k \geq 1$, $0 \leq m \leq N-1$ if $N-k=0$, and $m \in \mathbf{Z}$ if $N-k<0$.

Here, we restate the main result in [10].

Theorem 1 *We can derive the following identity from the virtual Gauss-Manin system (2.7).*

$$\tilde{\psi}_{N-1}(x) = \Big((\partial_x)^{N-1} - k \cdot e^x \cdot (k\partial_x + k - 1) \cdots (k\partial_x + 2) \cdot (k\partial_x + 1)\Big) \partial_x^{\beta} \tilde{\psi}_{-\beta}(x) \tag{2.8}$$

where $\beta = 0$ if $N-k \geq 1$, $\beta = 1$ if $N-k=0$, and $\beta = \infty$ if $N-k<0$.

We can also compute $\tilde{L}_m^{N,k,d}$ only by using the above theorem, and this process is the analogue of the B-model computation of the Calabi-Yau case.

Corollary 1 *The virtual structure constants $\tilde{L}_n^{N,k,d}$ are fully reconstructed from the identity (2.8). As a result, we can compute all the virtual structure constants by using the relation:*

$$\begin{aligned}
&\sum_{n=0}^{k-1} \tilde{L}_n^{N,k,1} w^n = k \cdot \prod_{j=1}^{k-1} (jw + (k-j)), \\
&\sum_{m=0}^{N-1+(k-N)d} \tilde{L}_m^{N,k,d} z^m = \\
&\sum_{l=2}^{d} (-1)^l \sum_{0=i_0<\cdots<i_l=d} \sum_{j_l=0}^{N-1+(k-N)d} \cdots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \prod_{n=1}^{l} \Big((1 + (d - i_{n-1})(\frac{z-1}{d}))^{j_n - j_{n-1}} \cdot \tilde{L}_{j_n+(N-k)i_{n-1}}^{N,k,i_n - i_{n-1}} \Big).
\end{aligned} \tag{2.9}$$

Note that the condition for non-vanishing $\tilde{L}_m^{N,k,d}$: $0 \leq m \leq N-1+(k-N)d$, implies that we have infinite number of $\tilde{L}_m^{N,k,d}$'s in the $k > N$ case. In [9], we proposed a conjecture of the formula to convert the virtual structure constants $\tilde{L}_m^{N,k,d}$ into the real structure constants $L_m^{N,k,d}$. To restate the conjecture in [9], we introduce some combinatorial definitions.

Definition 3 *Let P_d be the set of partitions of positive integer d:*

$$P_d = \{\sigma_d = (d_1, d_2, \cdots, d_{l(\sigma_d)}) \ \mid \ 1 \leq d_1 \leq d_2 \leq \cdots \leq d_{l(\sigma_d)} \ , \ \sum_{j=1}^{l(\sigma_d)} d_j = d \ , \ d_j \in \mathbf{Z}\}. \tag{2.10}$$

From now on, we denote a partition σ_d by $d_1 + d_2 + \cdots + d_{l(\sigma_d)}$. In (2.10), we denote the length of the partition σ_d by $l(\sigma_d)$. If $d=0$, we define P_0 as the set that consists of one trivial partition 0 with length 0.

Definition 4 *Let $S(\sigma_d)$ be a rational number associated with the partition $\sigma_d \in P_d$, which is defined by the following generating function:*

$$\sum_{d=0}^{\infty}(\sum_{\sigma_d \in P_d} S(\sigma_d) \prod_{j=1}^{l(\sigma_d)} a_{d_j})z^d := \exp(\sum_{j=1}^{\infty} a_j z^j). \tag{2.11}$$

With these preparation, we restate the main conjecture in [9] on the structure of the generalized mirror transformation:

Conjecture 1 *The generalized mirror transformation takes the form*

$$L_n^{N,k,d} = \sum_{m=0}^{d-1} \sum_{\sigma_m \in P_m} (-1)^{l(\sigma_m)} \cdot d^{l(\sigma_m)} \cdot S(\sigma_m) \cdot \prod_{i=1}^{l(\sigma_m)} \left(\frac{\tilde{L}^{N,k,d_i}_{1+(k-N)d_i}}{d_i} \right) \cdot G^{N,k,d}_{d-m}(n;\sigma_m), \tag{2.12}$$

where $G^{N,k,d}_{d-m}(n;\sigma_m)$ is a polynomial of $\tilde{L}^{N,k,d}_n$ with weighted degree d.

Of course, we have to determine the polynomial $G^{N,k,d}_{d-m}(n;\sigma_m)$ to predict the real structure constants $L_n^{N,k,d}$. In [8], we have determined $G^{N,k,d}_{d-m}(n;\sigma_m)$ for arbitrary N and k up to $d=3$ case, and $G^{k-1,k,d}_{d-m}(n;\sigma_m)$ up to $d=5$ case. In these cases, we only need $\tilde{L}^{N,k,d}_n$'s that satisfy $1+(k-N)d \leq n \leq N-2$. Therefore, we only have to use finite number of $\tilde{L}^{N,k,d}_n$'s in the $k > N$ case in spite of the fact that we have infinite number of the virtual structure constants. This observation plays an important role in our construction given in the next section. Moreover, the above conjecture naturally follows from the construction. We will show the explicit algorithm to determine the unknown polynomial $G^{N,k,d}_{d-m}(n;\sigma_m)$ for arbitrary degree d in the next section.

2.3 Virtual Gromov-Witten Invariants

We introduce here the virtual Gromov-Witten invariants to make correspondence of the results of this article with the ones in [8]. Precisely speaking, these quantities do not play an essential role in the main result of this article, but they are convenient for describing our results of computation.

Definition 5 *The virtual Gromov-Witten invariant $v(\prod_{j=1}^n \mathcal{O}_{e^{a_j}})_d$ on M_N^k is the rational number that satisfy the condition:*
(i) initial condition

$$\begin{aligned}
&v(\mathcal{O}_{e^a}\mathcal{O}_{e^b}\mathcal{O}_{e^c})_0 = k \cdot \delta_{a+b+c,N-2}, \\
&v(\prod_{j=1}^{n} \mathcal{O}_{e^{a_j}})_0 = 0, \quad (n \neq 3), \\
&\frac{1}{k} v(\mathcal{O}_{e^{N-2-n}}\mathcal{O}_{e^{n-1-(k-N)d}}\mathcal{O}_e)_d = \tilde{L}^{N,k,d}_n - \tilde{L}^{N,k,d}_{1+(k-N)d}, \quad (d \geq 1),
\end{aligned} \tag{2.13}$$

(ii) flat metric condition

$$\begin{aligned}
&v(\mathcal{O}_{e^0}\mathcal{O}_{e^a}\mathcal{O}_{e^b})_0 = k \cdot \delta_{a+b,N-2}, \\
&v(\mathcal{O}_{e^0}\prod_{j=1}^{n} \mathcal{O}_{e^{a_j}})_d = 0, \quad (d \geq 1, \quad or \quad d=0, \quad n \neq 2),
\end{aligned} \tag{2.14}$$

(iii) topological selection rule

$$v(\prod_{j=1}^{n} \mathcal{O}_{e^{a_j}})_d \neq 0 \Longrightarrow (N-5)+(N-k)d = \sum_{j=1}^{n}(a_j - 1), \tag{2.15}$$

(iv) Kähler equation

$$v(\mathcal{O}_e \prod_{j=1}^{n} \mathcal{O}_{e^{a_j}})_d = d \cdot v(\prod_{j=1}^{n} \mathcal{O}_{e^{a_j}})_d, \tag{2.16}$$

(v) associativity equation

$$\begin{aligned}
&\sum_{d_1=0}^{d} \sum_{\{\alpha_*\} \coprod \{\beta_*\}=\{n_*\}} \sum_{i=0}^{N-2} v(\mathcal{O}_{e^a}\mathcal{O}_{e^b}(\prod_{\alpha_j \in \{\alpha_*\}} \mathcal{O}_{e^{\alpha_j}})\mathcal{O}_{e^i})_{d_1} v(\mathcal{O}_{e^{N-2-i}}(\prod_{\beta_j \in \{\beta_*\}} \mathcal{O}_{e^{\beta_j}})\mathcal{O}_{e^c}\mathcal{O}_{e^d})_{d-d_1} \\
&= \sum_{d_1=0}^{d} \sum_{\{\alpha_*\} \coprod \{\beta_*\}=\{n_*\}} \sum_{i=0}^{N-2} v(\mathcal{O}_{e^a}\mathcal{O}_{e^c}(\prod_{\alpha_j \in \{\alpha_*\}} \mathcal{O}_{e^{\alpha_j}})\mathcal{O}_{e^i})_{d_1} v(\mathcal{O}_{e^{N-2-i}}(\prod_{\beta_j \in \{\beta_*\}} \mathcal{O}_{e^{\beta_j}})\mathcal{O}_{e^b}\mathcal{O}_{e^d})_{d-d_1}, \\
&(a+b+c+d+\sum_{j=1}^{m}(n_j-1) = N-2+(N-k)d).
\end{aligned} \tag{2.17}$$

Next, we introduce the notation that was heavily used in [8],

Definition 6

$$V_{d-m}^{N,k,d}(n; d_1+d_2+\cdots+d_{l(\sigma_m)}) := \frac{1}{k\cdot(d-m)^{l(\sigma_m)-1}} v(\mathcal{O}_{N-2-n}\mathcal{O}_{n-1-(k-N)d} \prod_{j=1}^{l(\sigma_m)} \mathcal{O}_{1+(k-N)d_j})_{d-m}. \tag{2.18}$$

In [8], we proposed some conjectures on the relation between $V_{d-m}^{N,k,d}(n;\sigma_m)$ and $G_{d-m}^{N,k,d}(n;\sigma_m)$ in (2.12).

Conjecture 2 *If $l(\sigma_m) \le 1$ or $d-m=1$, $G_{d-m}^{N,k,d}(n;\sigma_m)$ is given by $V_{d-m}^{N,k,d}(n;\sigma_m)$.*

In the same way as Conjecture 1, we can see that this conjecture is natural and consistent with the construction given in the next section. As the last part of this subsection, we write down a recursive formula for $V_{d-m}^{k-1,k,d}(n;\sigma_m)$, that follows directly from the definition of the virtual structure constants, for later use:

Proposition 1

$$\begin{aligned}
&V_{d-m}^{k-1,k,d}(n; d_1+d_2+\cdots+d_{l(\sigma_m)}) \\
&= V_{d-m}^{k-1,k,d-1}(n; d_1+d_2+\cdots+d_{l(\sigma_m)-1}+(d_{l(\sigma_m)}-1)) \\
&+V_{d-m}^{k-1,k,d-d_{l(\sigma_m)}}(n-d_{l(\sigma_m)}; d_1+d_2+\cdots+d_{l(\sigma_m)-1}) \\
&-V_{d-m}^{k-1,k,d-d_{l(\sigma_m)}}(d+1; d_1+d_2+\cdots+d_{l(\sigma_m)-1}) \\
&+\sum_{j=1}^{d-m-1} \sum_{A \coprod B} (\frac{d-m-j}{d-m})^p (\frac{j}{d-m})^{l(\sigma_m)-p-1} V_{d-m-j}^{k-1,k,d-m-j+d_A+d_{l(\sigma_m)}-1}(n; d_{a_1}+\cdots+d_{a_p}+(d_{l(\sigma_m)}-1)) \\
&\times V_j^{k-1,k,j+d_B}(n+j-d+m-d_A-d_{l(\sigma_m)}; d_{b_1}+\cdots+d_{b_{l(\sigma_m)-1-p}}) \\
&-\sum_{j=1}^{d-m-1} \sum_{A \coprod B} (\frac{d-m-j}{d-m})^p (\frac{j}{d-m})^{l(\sigma_m)-p-1} V_{d-m-j}^{k-1,k,d}(n; d_{a_1}+\cdots+d_{a_p}+(m+j-d_A)) \\
&\times V_j^{k-1,k,j+d_B}(m+1+j-d_A; d_{b_1}+\cdots+d_{b_{l(\sigma_m)-1-p}}),
\end{aligned} \tag{2.19}$$

where $\sum_{A \coprod B}$ means the summation on all the way of separating the set $\{1,2,\cdots,l(\sigma_m)-1\}$ into two disjoint sets $A=\{a_1,\cdots,a_p\}$ and $B=\{b_1,\cdots,b_{l(\sigma_m)-1-p}\}$, and d_A (resp. d_B) is an integer $\sum_{j=1}^{p} d_{a_j}$ (resp. $\sum_{j=1}^{l(\sigma_m)-1-p} d_{b_j}$).

Remark 1 *In applying (2.19), we don't need to arrange integers d_j in ascending order, because the virtual Gromov-Witten invariants are invariant under permutation of insertion of operators.*

3 Algorithmic Derivation of the Generalized Mirror Transformation

In this section, we write down the algorithm to compute the generalized mirror transformation and reproduce the results obtained in [8] for M_{k-1}^k up to $d = 5$ case. We explain theoretical background of our algorithm, following the result of Iritani [6], in the next section.

As the first step, we truncate the virtual Gauss-Manin system (2.7) into the form:

$$\begin{aligned}\partial_x \psi_{N-2-m}(x) &= \psi_{N-1-m}(x) + \sum_{d=1}^{[\frac{m-1}{k-N}]} \exp(dx) \cdot \tilde{L}_m^{N,k,d} \cdot \psi_{N-1-m+(k-N)d}(x). \\ & (0 \leq m \leq N-2) \end{aligned} \tag{3.20}$$

This truncation means that we throw away all the $\tilde{L}_m^{N,k,d}$'s except for the ones that satisfy $(1 + (k-N)d \leq m \leq N-2)$. Moreover, $\tilde{\psi}_j(x)$ $(j \in \mathbf{Z})$ is replaced by $\psi_j(x)$ $(j = 0, \cdots N-2)$. In other words, we throw away $\tilde{\psi}_j(x)$'s, that are exotic as the usual Gauss-Manin system associated with the quantum cohomology ring of M_N^k. For example, we write down the form of truncated virtual Gauss-Manin system of M_8^9:

$$\begin{aligned}
\partial_x \psi_0(x) &= \psi_1(x) + \alpha e^x \psi_2(x) + \eta e^{2x} \psi_3(x) + \varphi e^{3x} \psi_4(x) + \pi e^{3x} \psi_5(x) + \epsilon e^{5x} \psi_6(x) \\
\partial_x \psi_1(x) &= \psi_2(x) + \beta e^x \psi_3(x) + \xi e^{2x} \psi_4(x) + \kappa e^{3x} \psi_5(x) + \pi e^{3x} \psi_6(x) \\
\partial_x \psi_2(x) &= \psi_3(x) + \gamma e^x \psi_4(x) + \xi e^{2x} \psi_5(x) + \varphi e^{3x} \psi_6(x) \\
\partial_x \psi_3(x) &= \psi_4(x) + \beta e^x \psi_5(x) + \eta e^{2x} \psi_6(x) \\
\partial_x \psi_4(x) &= \psi_5(x) + \alpha e^x \psi_6(x) \\
\partial_x \psi_5(x) &= \psi_6(x) \\
\partial_x \psi_6(x) &= 0.
\end{aligned} \tag{3.21}$$

Since the next step is rather complicated, we explain it by using the above example. First, we eliminate $\psi_6(x)$ from the 5th line of (3.21) by using the 6th line,

$$\begin{aligned}\partial_x \psi_4(x) &= \psi_5(x) + \alpha e^x \partial_x \psi_5(x) \\ &= (1 + \alpha e^x \partial_x) \psi_5(x),\end{aligned} \tag{3.22}$$

and obtain,

$$\begin{aligned}\psi_5(x) &= (1 + \alpha e^x \partial_x)^{-1} \partial_x \psi_4(x), \\ \psi_6(x) &= \partial_x (1 + \alpha e^x \partial_x)^{-1} \partial_x \psi_4(x).\end{aligned} \tag{3.23}$$

In this way, we can eliminate $\psi_5(x)$ and $\psi_6(x)$ in (3.21). We can also eliminate $\psi_4(x)$ by plugging (3.23) into 4th line of (3.21),

$$\partial_x \psi_3(x) = \left(1 + \beta e^x (1 + \alpha e^x \partial_x)^{-1} \partial_x + \eta e^{2x} \partial_x (1 + \alpha e^x \partial_x)^{-1} \partial_x\right) \psi_4(x), \tag{3.24}$$

and obtain,

$$\begin{aligned}
\psi_4(x) &= \left(1 + \beta e^x (1 + \alpha e^x \partial_x)^{-1} \partial_x + \eta e^{2x} \partial_x (1 + \alpha e^x \partial_x)^{-1} \partial_x\right)^{-1} \partial_x \psi_3(x), \\
\psi_5(x) &= (1 + \alpha e^x \partial_x)^{-1} \partial_x \left(1 + \beta e^x (1 + \alpha e^x \partial_x)^{-1} \partial_x + \eta e^{2x} \partial_x (1 + \alpha e^x \partial_x)^{-1} \partial_x\right)^{-1} \partial_x \psi_3(x), \\
\psi_6(x) &= \partial_x (1 + \alpha e^x \partial_x)^{-1} \partial_x \left(1 + \beta e^x (1 + \alpha e^x \partial_x)^{-1} \partial_x + \eta e^{2x} \partial_x (1 + \alpha e^x \partial_x)^{-1} \partial_x\right)^{-1} \partial_x \psi_3(x).
\end{aligned} \tag{3.25}$$

By continuing the elimination process, we can obtain the following relation between $\psi_0(x)$ and $\psi_1(x)$:

$$\psi_1(x) = F(e^x, \partial_x)\psi_0(x). \tag{3.26}$$

Then we dictate the following relation that characterize the flat coordinate t,

$$\partial_t = F(e^x, \partial_x), \quad \text{i.e.,} \quad \partial_t\psi_0(x) = \psi_1(x). \tag{3.27}$$

Explicitly, ∂_t in our example is given as follows:

$$\begin{aligned}\partial_t &= \partial_x - \alpha e^x\partial_x^2 + e^{2x}((\alpha\beta-\eta)\partial_x^3 + \alpha^2(\partial_x+1)\partial_x^2) - e^{3x}(\varphi\partial_x^4 - \eta\beta(\partial_x+1)\partial_x^3 - \eta\gamma\partial_x^4 + (\alpha\beta\gamma-\alpha\xi)\partial_x^4 \\ &\quad +\alpha\beta^2(\partial_x+1)\partial_x^3 + \alpha(\alpha\beta-\eta)((\partial_x+2)\partial_x^3 + (\partial_x+1)^2\partial_x^2) + \alpha^3(\partial_x+2)(\partial_x+1)\partial_x^2) + \cdots.\end{aligned} \tag{3.28}$$

Using (3.28), we can rewrite (3.21) into the form,

$$\begin{aligned}
\partial_t\psi_0(x) &= \psi_1(x) \\
\partial_t\psi_1(x) &= \psi_2(x) + (\beta-\alpha)(e^x - \alpha e^{2x} + (\alpha\beta-\eta+2\alpha^2)e^{3x})\psi_3(x) \\
&\quad +((\xi-\eta-\alpha(\gamma-\alpha))(e^{2x} - 2\alpha e^{3x}) + ((\alpha\beta-\eta+\alpha^2)(\beta-\alpha) + (\alpha\beta-\eta)(\gamma-\alpha))e^{3x})\psi_4(x) \\
&\quad +(\kappa-\varphi-\alpha(\xi-\eta) - \eta(\beta-\alpha) + \alpha^2(\gamma-\alpha))e^{3x}\psi_5(x) \\
\partial_t\psi_2(x) &= \psi_3(x) + (\gamma-\alpha)(e^x - \alpha e^{2x} + (\alpha\beta-\eta+2\alpha^2)e^{3x})\psi_4(x) \\
&\quad +((\xi-\eta-\alpha(\gamma-\alpha))(e^{2x} - 2\alpha e^{3x}) + (2(\alpha\beta-\eta) + \alpha^2)(\gamma-\alpha))e^{3x})\psi_5(x) \\
\partial_t\psi_3(x) &= \psi_4(x) + (\beta-\alpha)(e^x - \alpha e^{2x} + (\alpha\beta-\eta+2\alpha^2)e^{3x})\psi_5(x) + (\beta-\alpha)(\alpha\beta-\eta+\alpha^2)\psi_6(x) \\
\partial_t\psi_4(x) &= \psi_5(x) \\
\partial_t\psi_5(x) &= \psi_6(x) \\
\partial_t\psi_6(x) &= 0
\end{aligned} \tag{3.29}$$

Though these computations are rather hard because of non-commutativity between e^x and ∂_x, we can manage them with the aid of Maple package: Ore-algebra. In general, we obtain the following equations as the result of this step,

$$\begin{aligned}
\partial_t\psi_0(x) &= \psi_1(x), \\
\partial_t\psi_{N-2-m}(x) &= \psi_{N-1-m}(x) + \sum_{d=1}^{[\frac{m-2}{k-N}]} f_m^{N,k,d}(e^x)\cdot\psi_{N-1-m+d(k-N)}(x), \quad (1 \le m \le N-3), \\
\partial_t\psi_{N-2}(x) &= 0,
\end{aligned} \tag{3.30}$$

where $f_m^{N,k,d}(e^x)$ is the power series in e^x whose lowest power is more than d.

Definition 7 *We denote by* $\frac{1}{k}w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e)_d$ *the coefficient of* e^{dx} *in* $f_m^{N,k,d}(e^x)$.

In our example, they are given as follows:

$$\begin{aligned}
&\frac{1}{k}w(\mathcal{O}_e\mathcal{O}_e\mathcal{O}_{e^3})_1 = \beta-\alpha, \quad \frac{1}{k}w(\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2})_1 = \gamma-\alpha, \\
&\frac{1}{k}w(\mathcal{O}_e\mathcal{O}_e\mathcal{O}_{e^2})_2 = \xi-\eta-\alpha(\gamma-\alpha), \\
&\frac{1}{k}w(\mathcal{O}_e\mathcal{O}_e\mathcal{O}_e)_3 = \kappa-\varphi-\alpha(\xi-\eta) - \eta(\beta-\alpha) + \alpha^2(\gamma-\alpha).
\end{aligned} \tag{3.31}$$

Explicitly, we can write down $\frac{1}{k}w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e)_d$ in terms of virtual Gromov-Witten invariants up to $d=3$ cases for arbitrary N and k.

$$\frac{1}{k}w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)}}\mathcal{O}_e)_1 = V_1^{N,k,1}(n;0),$$

$$
\begin{aligned}
\frac{1}{k}w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-2(k-N)}}\mathcal{O}_e)_2 &= V_2^{N,k,2}(n;0) - \tilde{L}_{1+(k-N)}^{N,k,1}V_1^{N,k,2}(n;1),\\
\frac{1}{k}w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-3(k-N)}}\mathcal{O}_e)_3 &= V_3^{N,k,3}(n;0) - \tilde{L}_{1+(k-N)}^{N,k,1}V_2^{N,k,3}(n;1) - \tilde{L}_{1+2(k-N)}^{N,k,2}V_1^{N,k,3}(n;2)\\
&\quad +(\tilde{L}_{1+(k-N)}^{N,k,1})^2V_1^{N,k,3}(n;1+1). \qquad (3.32)
\end{aligned}
$$

We also show here the explicit results for $d=4,5$ rational curves in the case of $N=k-1$:

$$
\begin{aligned}
\frac{1}{k}w(\mathcal{O}_{e^{k-3-m}}\mathcal{O}_{e^{m-5}}\mathcal{O}_e)_4 &= V_4^{k-1,k,4}(n;0) - \tilde{L}_2^{k-1,k,1}V_3^{k-1,k,4}(n;1) - \tilde{L}_3^{k-1,k,2}V_2^{k-1,k,4}(n;2)\\
&\quad -\tilde{L}_4^{k-1,k,3}V_1^{k-1,k,4}(n;3)\\
&\quad +(\tilde{L}_2^{k-1,k,1})^2\Big(V_2^{k-1,k,3}(n;1) + V_2^{k-1,k,3}(n-1;1) - V_2^{k-1,k,2}(5;0) + A(n)\Big)\\
&\quad +2\tilde{L}_2^{k-1,k,1}\tilde{L}_3^{k-1,k,2}V_1^{k-1,k,4}(n;1+2)\\
&\quad -(\tilde{L}_2^{k-1,k,1})^3V_1^{k-1,k,4}(n;1+1+1),\\
\frac{1}{k}w(\mathcal{O}_{e^{k-3-m}}\mathcal{O}_{e^{m-6}}\mathcal{O}_e)_5 &= V_5^{k-1,k,5}(n;0) - \tilde{L}_2^{k-1,k,1}V_4^{k-1,k,5}(n;1) - \tilde{L}_3^{k-1,k,2}V_3^{k-1,k,5}(n;2)\\
&\quad -\tilde{L}_4^{k-1,k,3}V_4^{k-1,k,5}(n;3) - \tilde{L}_5^{k-1,k,4}V_1^{k-1,k,5}(n;4)\\
&\quad +(\tilde{L}_2^{k-1,k,1})^2\Big(V_3^{k-1,k,4}(n;1) + V_3^{k-1,k,4}(n-1;1) - V_3^{k-1,k,3}(6;0)\\
&\quad +B(n) + C(n) - V_1^{k-1,k,1}(3;0)\cdot(2hi_1(n) + 2hi_2(n) + hi_3(n))\Big)\\
&\quad +2\tilde{L}_2^{k-1,k,1}\tilde{L}_3^{k-1,k,2}\Big(V_2^{k-1,k,4}(n;2) + V_2^{k-1,k,4}(n-1;2) - V_2^{k-1,k,2}(6;0)\\
&\quad +2hi_1(n) + \frac{3}{2}hi_2(n) + hi_3(n) - \frac{1}{2}hi_4(n)\Big)\\
&\quad -(\tilde{L}_2^{k-1,k,1})^3\Big(V_2^{k-1,k,3}(n;1) + 2V_2^{k-1,k,3}(n-1;1) + V_2^{k-1,k,3}(n-2;1)\\
&\quad -2V_2^{k-1,k,2}(5;0) - V_2^{k-1,k,3}(6;1) + D(n) + 3hi_1(n) + 3hi_2(n) + hi_3(n)\Big)\\
&\quad -3(\tilde{L}_2^{k-1,k,1})^2\tilde{L}_3^{k-1,k,2}V_1^{k-1,k,5}(n;1+1+2)\\
&\quad +(\tilde{L}_2^{k-1,k,1})^4V_1^{k-1,k,5}(n;1+1+1+1), \qquad (3.33)
\end{aligned}
$$

where

$$
\begin{aligned}
A(n): &= V_1^{k-1,k,2}(n;1)\cdot V_1^{k-1,k,1}(n-3;0) + V_1^{k-1,k,1}(n;0)\cdot V_1^{k-1,k,2}(n-2;1)\\
&\quad -V_1^{k-1,k,4}(n;1+2)\cdot V_1^{k-1,k,1}(3;0) - V_1^{k-1,k,4}(n;3)\cdot V_1^{k-1,k,1}(4;0),\\
B(n): &= V_2^{k-1,k,3}(n;1)\cdot V_1^{k-1,1,1}(n-4;0) + V_1^{k-1,k,1}(n;0)\cdot V_2^{k-1,k,3}(n-2;1)\\
&\quad -\big(V_2^{k-1,k,4}(n;2) + V_2^{k-1,k,4}(n-1;2) - V_2^{k-1,k,2}(6;0)\big)\cdot V_1^{k-1,k,1}(3;0)\\
&\quad -V_1^{k-1,k,5}(n;4)\cdot V_2^{k-1,k,2}(5;0),\\
C(n) &:= V_1^{k-1,k,2}(n;1)\cdot V_2^{k-1,k,2}(n-3;0) + V_2^{k-1,k,2}(n;0)\cdot V_1^{k-1,k,2}(n-3;1)\\
&\quad -V_1^{k-1,k,5}(n;1+3)\cdot V_2^{k-1,k,2}(4;0) - V_2^{k-1,k,5}(n;3)\cdot V_1^{k-1,k,1}(4;0),\\
D(n) &:= V_1^{k-1,k,2}(n;1)\cdot V_1^{k-1,k,1}(n-3;0) + V_1^{k-1,k,1}(n;0)\cdot V_1^{k-1,k,2}(n-2;1)\\
&\quad -V_1^{k-1,k,4}(n;1+2)\cdot V_1^{k-1,k,1}(3;0) - V_1^{k-1,k,4}(n;3)\cdot V_1^{k-1,k,1}(4;0)\\
&\quad +V_1^{k-1,k,2}(n-1;1)\cdot V_1^{k-1,k,1}(n-4;0) + V_1^{k-1,k,1}(n-1;0)\cdot V_1^{k-1,k,2}(n-3;1)\\
&\quad -V_1^{k-1,k,4}(n-1;1+2)\cdot V_1^{k-1,k,1}(3;0) - V_1^{k-1,k,4}(n-1;3)\cdot V_1^{k-1,k,1}(4;0). \qquad (3.34)
\end{aligned}
$$

In (3.33), $hi_j(n)$ is a degree 2 homogeneous polynomial of $\tilde{L}_m^{k-1,k,1}$ satisfying $hi_j(6) = hi_j(7) = 0$ and is given by,

$$\begin{aligned}
hi_1(n) &= \tilde{L}_n^{k-1,k,1}\tilde{L}_{n-4}^{k-1,k,1} - \tilde{L}_3^{k-1,k,1}(\tilde{L}_n^{k-1,k,1} + \tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1} + \tilde{L}_{n-4}^{k-1,k,1}) \\
&\quad + \tilde{L}_2^{k-1,k,1}(\tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1}) \\
&\quad - \big(\tilde{L}_6^{k-1,k,1}\tilde{L}_2^{k-1,k,1} - \tilde{L}_3^{k-1,k,1}(\tilde{L}_6^{k-1,k,1} + \tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} + \tilde{L}_3^{k-1,k,1} + \tilde{L}_2^{k-1,k,1}) \\
&\quad + \tilde{L}_2^{k-1,k,1}(\tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} + \tilde{L}_3^{k-1,k,1})\big), \\
hi_2(n) &= \tilde{L}_n^{k-1,k,1}\tilde{L}_{n-3}^{k-1,k,1} + \tilde{L}_{n-1}^{k-1,k,1}\tilde{L}_{n-4}^{k-1,k,1} \\
&\quad - \tilde{L}_4^{k-1,k,1}(\tilde{L}_n^{k-1,k,1} + \tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1} + \tilde{L}_{n-4}^{k-1,k,1}) + \tilde{L}_2^{k-1,k,1}\tilde{L}_{n-2}^{k-1,k,1} \\
&\quad - \big(\tilde{L}_6^{k-1,k,1}\tilde{L}_3^{k-1,k,1} + \tilde{L}_5^{k-1,k,1}\tilde{L}_2^{k-1,k,1} \\
&\quad - \tilde{L}_4^{k-1,k,1}(\tilde{L}_6^{k-1,k,1} + \tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} + \tilde{L}_3^{k-1,k,1} + \tilde{L}_2^{k-1,k,1}) + \tilde{L}_2^{k-1,k,1}\tilde{L}_4^{k-1,k,1}\big), \\
hi_3(n) &= \tilde{L}_n^{k-1,k,1}\tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-1}^{k-1,k,1}\tilde{L}_{n-3}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1}\tilde{L}_{n-4}^{k-1,k,1} \\
&\quad - \tilde{L}_5^{k-1,k,1}(\tilde{L}_n^{k-1,k,1} + \tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1} + \tilde{L}_{n-4}^{k-1,k,1}) - \tilde{L}_2^{k-1,k,1}\tilde{L}_{n-2}^{k-1,k,1} \\
&\quad - \big(\tilde{L}_6^{k-1,k,1}\tilde{L}_4^{k-1,k,1} + \tilde{L}_5^{k-1,k,1}\tilde{L}_3^{k-1,k,1} + \tilde{L}_4^{k-1,k,1}\tilde{L}_2^{k-1,k,1} \\
&\quad - \tilde{L}_5^{k-1,k,1}(\tilde{L}_6^{k-1,k,1} + \tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} + \tilde{L}_3^{k-1,k,1} + \tilde{L}_2^{k-1,k,1}) - \tilde{L}_2^{k-1,k,1}\tilde{L}_4^{k-1,k,1}\big), \\
hi_4(n) &= \tilde{L}_{n-1}^{k-1,k,1}\tilde{L}_{n-3}^{k-1,k,1} - \tilde{L}_4^{k-1,k,1}(\tilde{L}_{n-1}^{k-1,k,1} + \tilde{L}_{n-2}^{k-1,k,1} + \tilde{L}_{n-3}^{k-1,k,1}) + \tilde{L}_3^{k-1,k,1}\tilde{L}_{n-2}^{k-1,k,1} \\
&\quad - \big(\tilde{L}_5^{k-1,k,1}\tilde{L}_3^{k-1,k,1} - \tilde{L}_4^{k-1,k,1}(\tilde{L}_5^{k-1,k,1} + \tilde{L}_4^{k-1,k,1} + \tilde{L}_3^{k-1,k,1}) + \tilde{L}_3^{k-1,k,1}\tilde{L}_4^{k-1,k,1}\big).
\end{aligned} \tag{3.35}$$

For later use, we also write down the explicit formula of some $V_{d-m}^{k-1,k,d}(n;\sigma_m)$'s [2]:

$$\begin{aligned}
V_2^{k-1,k,4}(n;1+1) &= V_2^{k-1,k,3}(n;1) + V_2^{k-1,k,3}(n-1;1) - V_2^{k-1,k,2}(5;0) + \frac{1}{2}A(n) \\
V_3^{k-1,k,5}(n;1+1) &= V_3^{k-1,k,4}(n;1) + V_3^{k-1,k,4}(n-1;1) - V_3^{k-1,k,3}(6;0), \\
&\quad + \frac{2}{3}B(n) + \frac{1}{3}C(n) - \frac{1}{3}V_1^{k-1,k,1}(3;0)\cdot(2hi_1(n) + hi_2(n) + hi_3(n) - hi_4(n)), \\
V_2^{k-1,k,5}(n;1+2) &= V_2^{k-1,k,4}(n;2) + V_2^{k-1,k,4}(n-1;2) - V_2^{k-1,k,2}(6;0) \\
&\quad + hi_1(n) + \frac{1}{2}hi_2(n) + \frac{1}{2}hi_3(n) - \frac{1}{2}hi_4(n), \\
V_2^{k-1,k,5}(n;1+1+1) &= V_2^{k-1,k,3}(n;1) + 2V_2^{k-1,k,3}(n-1;1) + V_2^{k-1,k,3}(n-2;1) \\
&\quad - 2V_2^{k-1,k,2}(5;0) - V_2^{k-1,k,3}(6;1) \\
&\quad + \frac{1}{2}D(n) + \frac{1}{4}(4hi_1(n) + 4hi_2(n) + hi_3(n) + hi_4(n)).
\end{aligned} \tag{3.36}$$

We can obtain (3.36) by direct application of (2.19). It is quite interesting that the structure of the terms proportional to $\prod_{j=1}^{l(\sigma_m)} \tilde{L}_{1+(k-n)d_j}^{N,k,d}$ in $\frac{1}{k}w(\mathcal{O}_{e^{N-2-n}}\mathcal{O}_{e^{n-1-(k-N)d}}\mathcal{O}_e)_d$ resembles the one of $V_{d-m}^{N,k,d}(n;\sigma_m)$. We would like the reader to compare the corresponding $G_{d-m}^{k-1,k,d}(n;\sigma_m)$'s in the conjecture (2.12), that were determined in [8]:

$$\begin{aligned}
G_2^{k-1,k,4}(n;1+1) &= V_2^{k-1,k,3}(n;1) + V_2^{k-1,k,3}(n-1;1) - V_2^{k-1,k,2}(5;0) + \frac{3}{4}A(n) \\
G_3^{k-1,k,5}(n;1+1) &= V_3^{k-1,k,4}(n;1) + V_3^{k-1,k,4}(n-1;1) - V_3^{k-1,k,3}(6;0), \\
&\quad + \frac{4}{5}B(n) + \frac{3}{5}C(n) - \frac{1}{5}V_1^{k-1,k,1}(3;0)\cdot(6hi_1(n) + 5hi_2(n) + 3hi_3(n) - hi_4(n)),
\end{aligned}$$

[2] In [8], we made a combinatorial mistake in evaluating $V_2^{k-1,k,5}(n;1+1+1)$, and the terms including $hi_j(n)$ are different from the fourth equality in (3.36). Therefore, (3.36) is the correct answer.

$$
\begin{aligned}
G_2^{k-1,k,5}(n;1+2) &= V_2^{k-1,k,4}(n;2)+V_2^{k-1,k,4}(n-1;2)-V_2^{k-1,k,2}(6;0)\\
&\quad +\frac{1}{5}(8hi_1(n)+5hi_2(n)+4hi_3(n)-3hi_4(n)),\\
G_2^{k-1,k,5}(n;1+1+1) &= V_2^{k-1,k,3}(n;1)+2V_2^{k-1,k,3}(n-1;1)+V_2^{k-1,k,3}(n-2;1)\\
&\quad -2V_2^{k-1,k,2}(5;0)-V_2^{k-1,k,3}(6;1)\\
&\quad +\frac{4}{5}D(n)+\frac{1}{25}(46hi_1(n)+46hi_2(n)+16hi_3(n)-2hi_4(n)).
\end{aligned}
\tag{3.37}
$$

As was suggested in [8], we can see that the coefficients of $A(n)$,$B(n)$, $C(n)$ and $D(n)$ in (3.36) change followingly in (3.37):

$$
\frac{1}{2}\to\frac{3}{4},\quad \frac{2}{3}\to\frac{4}{5},\quad \frac{1}{3}\to\frac{3}{5},\quad \frac{1}{2}\to\frac{4}{5}. \tag{3.38}
$$

The terms that consist of $hi_j(n)$ change in a more complicated manner. These changes seem to imply that $G_{d-m}^{k-1,k,d}(n;\sigma_m)$ does not satisfy the simple Kähler equation given in (2.16). At this stage, we introduce a key definition to resolve these puzzles observed in [8].

Definition 8 *Let $w(\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_m}})$ be the rational number obtained from applying the associativity equation and the modified Kähler equation:*

$$
\begin{aligned}
&w(\mathcal{O}_e\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_m}})_d\\
&=d\cdot w(\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_m}})_d-\sum_{f=1}^{d-1}\tilde{L}^{N,k,f}_{1+(k-N)f}\cdot w(\mathcal{O}_{e^{1+(k-N)f}}\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_m}})_{d-f},
\end{aligned}
\tag{3.39}
$$

to the initial condition $w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e)_d$.

Proposition 2

$$
\frac{1}{k}w(\mathcal{O}_{e^{N-2-n}}\mathcal{O}_{e^{n-1-(k-N)d}}\mathcal{O}_e\prod_{i=1}^{l(\sigma_{d-1})}\mathcal{O}_{e^{1+(k-N)d_i}})_1=V_1^{N,k,d}(n;d_1+d_2+\cdots+d_{l(\sigma_{d-1})}). \tag{3.40}
$$

proof) In $d=1$ case, the modified Kähler equation (3.39) reduces to (2.16). Moreover, the initial condition and the associativity equation are also the same as the ones for the virtual Gromov-Witten invariants. Therefore, the assertion of the proposition follows. □

With these set up, our main result of this article is given as follows:

Conjecture 3

$$
\begin{aligned}
&\langle\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e\rangle_d\\
&=\sum_{g=0}^{d-1}(-1)^{l(\sigma_g)}\sum_{\sigma_g\in P_g}S(\sigma_g)\Big(\prod_{i=1}^{l(\sigma_g)}\frac{\tilde{L}^{N,k,d_i}_{1+(k-N)d_i}}{d_i}\Big)w(\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e\prod_{i=1}^{l(\sigma_g)}\mathcal{O}_{e^{1+(k-N)d_i}})_{d-g}.
\end{aligned}
\tag{3.41}
$$

Remark 2 *Under the assumption of (3.41), (3.39) leads us to the standard Kähler equation for real Gromov-Witten invariants:*

$$
\begin{aligned}
&\langle\mathcal{O}_e\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e\rangle_d\\
&=\sum_{g=0}^{d-1}(-1)^{l(\sigma_g)}\sum_{\sigma_g\in P_g}S(\sigma_g)\Big(\prod_{i=1}^{l(\sigma_g)}\frac{\tilde{L}^{N,k,d_i}_{1+(k-N)d_i}}{d_i}\Big)w(\mathcal{O}_e\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e\prod_{i=1}^{l(\sigma_g)}\mathcal{O}_{e^{1+(k-N)d_i}})_{d-g}\\
&=d\cdot\langle\mathcal{O}_{e^{N-2-m}}\mathcal{O}_{e^{m-1-(k-N)d}}\mathcal{O}_e\rangle_d.
\end{aligned}
\tag{3.42}
$$

In order to derive (3.42), we introduce another representation of $\sigma_{d-g} \in P_{d-g}$:

$$\sigma_g = d_1 + \cdots + d_{l(\sigma_g)} = \sum_{j=1}^{\infty} m_j \cdot j. \tag{3.43}$$

Then the combinatorial factor $S(\sigma_g) \prod_{i=1}^{l(\sigma_g)} \frac{1}{d_i}$ is rewritten as follows:

$$S(\sigma_g) \prod_{i=1}^{l(\sigma_g)} \frac{1}{d_i} = \prod_{j=1}^{\infty} \frac{1}{(m_j)! \cdot j^{m_j}}. \tag{3.44}$$

In these set up, what we have to show is,

$$(d-g) \prod_{j=1}^{\infty} \frac{1}{(m_j)! \cdot j^{m_j}} + \sum_{i\ (m_i>0)} m_i \cdot i \prod_{j=1}^{\infty} \frac{1}{(m_j)! \cdot j^{m_j}} = d \prod_{j=1}^{\infty} \frac{1}{(m_j)! \cdot j^{m_j}}, \tag{3.45}$$

but this identity directly follows from $\sum_{i\ (m_i>0)} m_i \cdot i = g$.

In the remaining part of this section, we explicitly reproduce the results in [8] by using Conjecture 3. First, we derive the results of arbitrary N and k ($N-k<0$) up to $d=3$ case. In the $d=1$ case, (3.41) merely says,

$$\frac{1}{k}\langle \mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-(k-N)}} \mathcal{O}_e \rangle_1 = \frac{1}{k} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-(k-N)}} \mathcal{O}_e)_1 = V_1^{N,k,1}(n;0) = \tilde{L}_n^{N.k.1} - \tilde{L}_{1+(k-N)}^{N,k,1}. \tag{3.46}$$

In the $d=2$ case, we can derive

$$\begin{aligned}
&\frac{1}{k}\langle \mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-2(k-N)}} \mathcal{O}_e \rangle_2 \\
&= \frac{1}{k} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-2(k-N)}} \mathcal{O}_e)_2 - \frac{1}{k} \tilde{L}_{1+(k-N)}^{N,k,1} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-2(k-N)}} \mathcal{O}_e \mathcal{O}_{e^{1+(k-N)}})_1 \\
&= V_2^{N,k,2}(n;0) - \tilde{L}_{1+(k-N)}^{N,k,1} V_1^{N,k,2}(n;1) - \tilde{L}_{1+(k-N)}^{N,k,1} V_1^{N,k,2}(n;1) \\
&= V_2^{N,k,2}(n;0) - 2\tilde{L}_{1+(k-N)}^{N,k,1} V_1^{N,k,2}(n;1).
\end{aligned} \tag{3.47}$$

Then we turn into the $d=3$ case. In this case, we have to use the modified Kähler equation (3.39), associativity equation and (3.32).

$$\begin{aligned}
&\frac{1}{k}\langle \mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_e \rangle_3 \\
&= \frac{1}{k}\Bigg(w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_e)_3 - \tilde{L}_{1+(k-N)}^{N,k,1} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_e \mathcal{O}_{e^{1+(k-N)}})_2 \\
&-\frac{1}{2} \tilde{L}_{1+2(k-N)}^{N,k,2} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_e \mathcal{O}_{e^{1+2(k-N)}})_1 \\
&+\frac{1}{2} (\tilde{L}_{1+(k-N)}^{N,k,1})^2 w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_e \mathcal{O}_{e^{1+(k-N)}} \mathcal{O}_{e^{1+(k-N)}})_1 \Bigg) \\
&= \frac{1}{k}\Bigg(w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_e)_3 - 2\tilde{L}_{1+(k-N)}^{N,k,1} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_{e^{1+(k-N)}})_2 \\
&-\frac{1}{2} \tilde{L}_{1+2(k-N)}^{N,k,2} w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_{e^{1+2(k-N)}})_1 \\
&+\frac{3}{2} (\tilde{L}_{1+(k-N)}^{N,k,1})^2 w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-3(k-N)}} \mathcal{O}_{e^{1+(k-N)}} \mathcal{O}_{e^{1+(k-N)}})_1 \Bigg) \\
&= V_3^{N,k,3}(n;0) - \tilde{L}_{1+(k-N)}^{N,k,1} V_2^{N,k,3}(n;1) - \tilde{L}_{1+2(k-N)}^{N,k,2} V_1^{N,k,3}(n;2) + (\tilde{L}_{1+(k-N)}^{N,k,1})^2 V_1^{N,k,3}(n;1+1)
\end{aligned}$$

$$
\begin{aligned}
&-2\tilde{L}^{N,k,1}_{1+(k-N)}\big(V^{N,k,3}_2(n;1)-\tilde{L}^{N,k,1}_{1+(k-N)}V^{N,k,3}_1(n;1+1)\big)-\frac{1}{2}\tilde{L}^{N,k,2}_{1+2(k-N)}V^{N,k,3}_1(n;2)\\
&+\frac{3}{2}(\tilde{L}^{N,k,1}_{1+(k-N)})^2V^{N,k,3}_1(n;1+1)\\
&=V^{N,k,3}_3(n;0)-3\tilde{L}^{N,k,1}_{1+(k-N)}V^{N,k,3}_2(n;1)-\frac{3}{2}\tilde{L}^{N,k,2}_{1+2(k-N)}V^{N,k,3}_1(n;2)+\frac{9}{2}(\tilde{L}^{N,k,1}_{1+(k-N)})^2V^{N,k,3}_1(n;1+1).
\end{aligned}
\tag{3.48}
$$

Indeed, these results agree with the results in [8]. Next, we derive the generalized mirror transformation of M^k_{k-1} model for the $d=4,5$ cases.

$$
\begin{aligned}
&\frac{1}{k}\langle\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e\rangle_4\\
&=\frac{1}{k}\bigg(w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e)_4\\
&-\tilde{L}^{k-1,k,1}_2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e\mathcal{O}_{e^2})_3-\frac{1}{2}\tilde{L}^{k-1,k,2}_3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e\mathcal{O}_{e^3})_2\\
&-\frac{1}{3}\tilde{L}^{k-1,k,3}_4w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e\mathcal{O}_{e^4})_1\\
&+\frac{1}{2}(\tilde{L}^{k-1,k,1}_2)^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2})_2+\frac{1}{2}\tilde{L}^{k-1,k,1}_2\tilde{L}^{k-1,k,2}_3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^3})_1\\
&-\frac{1}{6}(\tilde{L}^{k-1,k,1}_2)^3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_1\bigg)\\
&=\frac{1}{k}\bigg(w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_e)_4\\
&-3\tilde{L}^{k-1,k,1}_2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_{e^2})_3-\tilde{L}^{k-1,k,2}_3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_{e^3})_2\\
&-\frac{1}{3}\tilde{L}^{k-1,k,3}_4w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_{e^4})_1\\
&+2(\tilde{L}^{k-1,k,1}_2)^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_2+2\tilde{L}^{k-1,k,1}_2\tilde{L}^{k-1,k,2}_3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-5}}\mathcal{O}_{e^2}\mathcal{O}_{e^3})_1\\
&-\frac{2}{3}(\tilde{L}^{k-1,k,1}_2)^3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_1\bigg)\\
&=V^{k-1,k,4}_4(n;0)-\tilde{L}^{k-1,k,1}_2V^{k-1,k,4}_3(n;1)-\tilde{L}^{k-1,k,2}_3V^{k-1,k,4}_2(n;2)-\tilde{L}^{k-1,k,3}_4V^{k-1,k,4}_1(n;3)\\
&+(\tilde{L}^{k-1,k,1}_2)^2\bigg(V^{k-1,k,3}_2(n;1)+V^{k-1,k,3}_2(n-1;1)-V^{k-1,k,2}_2(5;0)+A(n)\bigg)\\
&+2\tilde{L}^{k-1,k,1}_2\tilde{L}^{k-1,k,2}_3V^{k-1,k,4}_1(n;1+2)-(\tilde{L}^{k-1,k,1}_2)^3V^{k-1,k,4}_1(n;1+1+1)\\
&-3\tilde{L}^{k-1,k,1}_2\bigg(V^{k-1,k,4}_3(n;1)-\tilde{L}^{k-1,k,1}_2\big(V^{k-1,k,3}_2(n;1)+V^{k-1,k,3}_2(n-1;1)-V^{k-1,k,2}_2(5;0)+A(n)\big)\\
&-\tilde{L}^{k-1,k,2}_3V^{k-1,k,4}_1(n;1+2)+(\tilde{L}^{k-1,k,1}_2)^2V^{k-1,k,4}_1(n;1+1+1)\bigg)\\
&-\tilde{L}^{k-1,k,2}_3\big(V^{k-1,k,4}_2(n;2)-\tilde{L}^{k-1,k,1}_2V^{k-1,k,4}_1(n;1+2)\big)\\
&-\frac{1}{3}\tilde{L}^{k-1,k,3}_4V^{k-1,k,4}_1(n;3)\\
&+2(\tilde{L}^{k-1,k,1}_2)^2\big(2V^{k-1,k,3}_2(n;1)+2V^{k-1,k,3}_2(n-1;1)-2V^{k-1,k,2}_2(5;0)+A(n)\\
&-3\tilde{L}^{k-1,k,1}_2V^{k-1,k,4}_1(n;1+1+1)\big)+2\tilde{L}^{k-1,k,1}_2\tilde{L}^{k-1,k,2}_3V^{k-1,k,4}_1(n;1+2)\\
&-\frac{2}{3}(\tilde{L}^{k-1,k,1}_2)^3V^{k-1,k,4}_1(n;1+1+1)\\
&=V^{k-1,k,4}_4(n;0)-4\tilde{L}^{k-1,k,1}_2V^{k-1,k,4}_3(n;1)-2\tilde{L}^{k-1,k,2}_3V^{k-1,k,4}_2(n;2)-\frac{4}{3}\tilde{L}^{k-1,k,3}_4V^{k-1,k,4}_1(n;3)
\end{aligned}
$$

$$+8(\tilde{L}_2^{k-1,k,1})^2\left(V_2^{k-1,k,3}(n;1)+V_2^{k-1,k,3}(n-1;1)-V_2^{k-1,k,2}(5;0)+\frac{3}{4}A(n)\right)$$
$$+8\tilde{L}_2^{k-1,k,1}\tilde{L}_3^{k-1,k,2}V_1^{k-1,k,4}(n;1+2)-\frac{32}{3}(\tilde{L}_2^{k-1,k,1})^3V_1^{k-1,k,4}(n;1+1+1) \tag{3.49}$$

Note that we have used (3.39) to derive the second equality in (3.49). In this derivation, the mysterious change of coefficients $\frac{1}{2}$ of $A(n)$ in (3.36) into $\frac{3}{4}$ in (3.37) naturally arises as the result of the identity:

$$1+3+4\cdot\frac{1}{2}=\frac{4^2}{2!}\cdot\frac{3}{4}. \tag{3.50}$$

Derivation of the $d=5$ result can be done in almost the same way as in the $d=4$ case. As the first step, we use (3.39) once again,

$$\begin{aligned}
&\frac{1}{k}\langle\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\rangle_5\\
&=\frac{1}{k}\Bigg(w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e)_5\\
&-\tilde{L}_2^{k-1,k,1}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2})_4-\frac{1}{2}\tilde{L}_3^{k-1,k,2}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^3})_3\\
&-\frac{1}{3}\tilde{L}_4^{k-1,k,3}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^4})_2-\frac{1}{4}\tilde{L}_5^{k-1,k,4}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^5})_1\\
&+\frac{1}{2}(\tilde{L}_2^{k-1,k,1})^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2})_3+\frac{1}{2}\tilde{L}_2^{k-1,k,1}\tilde{L}_3^{k-1,k,2}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^3})_2\\
&+\frac{1}{3}\tilde{L}_2^{k-1,k,1}\tilde{L}_4^{k-1,k,3}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^4})_1+\frac{1}{8}(\tilde{L}_3^{k-1,k,2})^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^3}\mathcal{O}_{e^3})_1\\
&-\frac{1}{6}(\tilde{L}_2^{k-1,k,1})^3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_2-\frac{1}{4}(\tilde{L}_2^{k-1,k,1})^2\tilde{L}_3^{k-1,k,2}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^3})_1\\
&+\frac{1}{24}(\tilde{L}_2^{k-1,k,1})^4w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_1\Bigg)\\
&=\frac{1}{k}\Bigg(w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e)_5\\
&-4\tilde{L}_2^{k-1,k,1}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2})_4-\frac{3}{2}\tilde{L}_3^{k-1,k,2}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^3})_3\\
&-\frac{2}{3}\tilde{L}_4^{k-1,k,3}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^4})_2-\frac{1}{4}\tilde{L}_5^{k-1,k,4}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^5})_1\\
&+\frac{5}{2}(\tilde{L}_2^{k-1,k,1})^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_3+\frac{5}{2}\tilde{L}_2^{k-1,k,1}\tilde{L}_3^{k-1,k,2}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^3})_2\\
&+\frac{5}{3}\tilde{L}_2^{k-1,k,1}\tilde{L}_4^{k-1,k,3}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^4})_1+\frac{5}{8}(\tilde{L}_3^{k-1,k,2})^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^3}\mathcal{O}_{e^3})_3\\
&-\frac{5}{6}(\tilde{L}_2^{k-1,k,1})^3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_2-\frac{5}{4}(\tilde{L}_2^{k-1,k,1})^2\tilde{L}_3^{k-1,k,2}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^3})_1\\
&+\frac{5}{24}(\tilde{L}_2^{k-1,k,1})^4w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_1\Bigg).
\end{aligned} \tag{3.51}$$

Then we decompose each $w(\mathcal{O}_{e^{n_1}}\mathcal{O}_{e^{n_2}}\cdots\mathcal{O}_{e^{n_l}})$ in terms of $V_{d-m}^{k-1,k,d}(n;\sigma_m)$ by using the associativity equation and the modified Kähler equation (3.39). We briefly explain the remaining computation. At first, the coefficients of the terms in (3.51) add up to the ones in (2.12) as follows,

$$1+4=5,\ \ 1+\frac{3}{2}=\frac{5}{2},\ \ 1+\frac{2}{3}=\frac{5}{3},\ \ 1+\frac{1}{4}=\frac{5}{4},$$

$$1+4+3\cdot\frac{5}{2}=\frac{25}{2},\ \ 2+4+\frac{3}{2}+\frac{5}{2}\cdot 2=\frac{25}{2},\ \ 2+4+\frac{2}{3}+\frac{5}{3}=\frac{25}{3},\ \ 1+\frac{3}{2}+\frac{5}{8}=\frac{25}{8},$$
$$1+4+3\cdot\frac{5}{2}+2\cdot\frac{5}{2}+2^2\cdot\frac{5}{6}=\frac{125}{6},\ \ 3+2\cdot 4+\frac{3}{2}+3\cdot\frac{5}{2}+2\cdot\frac{5}{2}+\frac{5}{2}+\frac{5}{2}+\frac{5}{4}=\frac{125}{4},$$
$$1+4+3\cdot\frac{5}{2}+2\cdot\frac{5}{2}+2^2\cdot\frac{5}{6}+\frac{5}{2}+2\cdot\frac{5}{6}+\frac{5}{6}+\frac{5}{24}=\frac{625}{24}. \tag{3.52}$$

The change of coefficients $\frac{2}{3},\frac{1}{3},\frac{1}{2}$ of $B(n)$, $C(n)$, $D(n)$ in (3.36) into $\frac{4}{5},\frac{3}{5},\frac{4}{5}$ in (3.37) is naturally derived in the same way as in the $d=4$ case:

$$1+4+3\cdot\frac{5}{2}\cdot\frac{2}{3}=\frac{25}{2}\cdot\frac{4}{5},\ \ 1+4+3\cdot\frac{5}{2}\cdot\frac{1}{3}=\frac{25}{2}\cdot\frac{3}{5},$$
$$1+4+3\cdot\frac{5}{2}+(2\cdot\frac{5}{2}+2^2\cdot\frac{5}{6})\cdot\frac{1}{2}=\frac{125}{6}\cdot\frac{4}{5}. \tag{3.53}$$

Lastly, we verify the non-trivial change of the terms including $hi_1(n), hi_2(n), hi_3(n), hi_4(n)$ in (3.37). We discuss the case of $1+1+1$ sector in detail. In this case, we have four terms in (3.51) that contribute to this sector:

$$\frac{1}{k}\cdot w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e)_5,\ \ -\frac{1}{k}\cdot 4\tilde{L}_2^{k-1,k,1}w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2})_4$$
$$\frac{1}{k}\cdot\frac{5}{2}(\tilde{L}_2^{k-1,k,1})^2w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_3,\ \ -\frac{1}{k}\cdot\frac{5}{6}(\tilde{L}_2^{k-1,k,1})^3w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_2 \tag{3.54}$$

If we consider the contributions coming from the first two terms, we can see that we don't have to use the modified Kähler equation (3.39). Therefore, we can compute the contributions that includes $hi_j(n)$ only by using associativity equation and (3.33). The contributions from these terms turn out to be the same and given by,

$$\begin{aligned}
&V_1^{k-1,k,3}(n;1+1)\cdot V_1^{k-1,k,1}(n-4;0)+V_1^{k-1,k,2}(n;1)\cdot V_1^{k-1,k,2}(n-3;1)\\
&+V_1^{k-1,k,1}(n;0)\cdot V_1^{k-1,k,3}(n-2;1+1)\\
&-V_1^{k-1,k,5}(n;1+1+2)\cdot V_1^{k-1,k,1}(3;0)-V_1^{k-1,k,5}(n;1+3)\cdot V_1^{k-1,k,2}(4;1)\\
&-V_1^{k-1,k,5}(n;4)\cdot V_1^{k-1,k,3}(5;1+1)\\
&=3hi_1(n)+3hi_2(n)+hi_3(n).
\end{aligned} \tag{3.55}$$

Contrary to the above cases, we have to treat carefully the term $w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_3$ because we have to use (3.39) once in the computation. In this case, we take care of the following terms that appear in decomposing $w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_3$ by using associativity equation:

$$\begin{aligned}
&\frac{1}{k^2}\Big(w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^2}\mathcal{O}_e\mathcal{O}_{e^{n-4}})_2\cdot w(\mathcal{O}_{e^{k-n+1}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e)_1+w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^2}\mathcal{O}_e\mathcal{O}_{e^{n-3}})_1\cdot w(\mathcal{O}_{e^{k-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_e)_2\\
&+w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_e\mathcal{O}_{e^{n-3}})_2\cdot w(\mathcal{O}_{e^{k-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_e)_1+w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_e\mathcal{O}_{e^{n-2}})_1\cdot w(\mathcal{O}_{e^{k-n-1}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_e)_2\\
&-w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^3})_2\cdot w(\mathcal{O}_{e^{k-6}}\mathcal{O}_e\mathcal{O}_e)_1-w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^4})_1\cdot w(\mathcal{O}_{e^{k-7}}\mathcal{O}_e\mathcal{O}_e)_2\\
&-w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^4})_2\cdot w(\mathcal{O}_{e^{k-7}}\mathcal{O}_{e^2}\mathcal{O}_e\mathcal{O}_e)_1-w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^5})_1\cdot w(\mathcal{O}_{e^{k-8}}\mathcal{O}_{e^2}\mathcal{O}_e\mathcal{O}_e)_1\Big).
\end{aligned} \tag{3.56}$$

If we take the leading terms of (3.39) in reducing the degree 2 four point functions in (3.56) into three point functions and if we pick up the terms proportional to $\tilde{L}_2^{k-1,k,1}$ in these degree 2 three point function, we obtain the following contribution:

$$2(3hi_1(n)+3hi_2(n)+hi_3(n)). \tag{3.57}$$

On the other hand, we also have the terms proportional to $\tilde{L}_2^{k-1,k,1}$ by picking up the sub-leading term of (3.39) that appear in reducing the degree 2 four point function in (3.56). For example, we take $\tilde{L}_2^{k-1,k,1}w(\mathcal{O}_{e^{k-1-n}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^{n-4}})_1$ in the r.h.s. of the equation:

$$w(\mathcal{O}_{e^{k-1-n}}\mathcal{O}_{e^2}\mathcal{O}_e\mathcal{O}_{e^{n-4}})_2=2w(\mathcal{O}_{e^{k-1-n}}\mathcal{O}_{e^2}\mathcal{O}_{e^{n-4}})_2-\tilde{L}_2^{k-1,k,1}w(\mathcal{O}_{e^{k-1-n}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^{n-4}})_1. \tag{3.58}$$

After some computation, the following contribution appears,

$$
\begin{aligned}
&V_1^{k-1,k,3}(n;1+1)\cdot V_1^{k-1,k,1}(n-4;0)+V_1^{k-1,k,1}(n;0)\cdot V_1^{k-1,k,3}(n-2;1+1)\\
&-V_1^{k-1,k,5}(n;1+1+2)\cdot V_1^{k-1,k,1}(3;0)-V_1^{k-1,k,5}(n;4)\cdot V_1^{k-1,k,3}(5;1+1)\\
&=2hi_1(n)+2hi_2(n)+hi_3(n)-hi_4(n). \qquad (3.59)
\end{aligned}
$$

The contribution from $w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_2$ is obtained only by taking leading terms of (3.39) and of three point functions. Therefore, the computation is the same as the one of virtual Gromov-Witten invariants, and we have from (3.36),

$$
\begin{aligned}
&V_1^{k-1,k,3}(n;1+1)\cdot V_1^{k-1,k,1}(n-4;0)+2V_1^{k-1,k,2}(n;1)\cdot V_1^{k-1,k,2}(n-3;1)\\
&+V_1^{k-1,k,1}(n;0)\cdot V_1^{k-1,k,3}(n-2;1+1)\\
&-V_1^{k-1,k,5}(n;1+1+2)\cdot V_1^{k-1,k,1}(3;0)-2V_1^{k-1,k,5}(n;1+3)\cdot V_1^{k-1,k,2}(4;1)\\
&-V_1^{k-1,k,5}(n;4)\cdot V_1^{k-1,k,3}(5;1+1)\\
&=4hi_1(n)+4hi_2(n)+hi_3(n)+hi_4(n). \qquad (3.60)
\end{aligned}
$$

Adding up these five contributions, we can finally reproduce the part of the formula of $G_2^{k-1,k,5}(n;1+1+1)$ including $hi_j(n)$:

$$
\begin{aligned}
&(3hi_1(n)+3hi_2(n)+hi_3(n))+4(3hi_1(n)+3hi_2(n)+hi_3(n))\\
&+\frac{5}{2}\cdot 2(3hi_1(n)+3hi_2(n)+hi_3(n))+\frac{5}{2}(2hi_1(n)+2hi_2(n)+hi_3(n)-hi_4(n))\\
&+\frac{5}{6}(4hi_1(n)+4hi_2(n)+hi_3(n)+hi_4(n))\\
&=\frac{125}{6}(\frac{46}{25}hi_1(n)+\frac{46}{25}hi_2(n)+\frac{16}{25}hi_3(n)-\frac{2}{25}hi_4(n)). \qquad (3.61)
\end{aligned}
$$

Computation of the remaining $1+1$, $1+2$ sectors goes in the same way. In these cases, we don't have to consider the sub-leading terms of (3.39) because we don't have to use (3.39) except for $w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^2})_3$ and $w(\mathcal{O}_{e^{k-3-n}}\mathcal{O}_{e^{n-6}}\mathcal{O}_{e^2}\mathcal{O}_{e^3})_2$. Therefore, we write down the final process of the computation in the following:

$$
\begin{aligned}
&1+2 \text{ sector}\\
&2(2hi_1(n)+\frac{3}{2}hi_2(n)+hi_3(n)-\frac{1}{2}hi_4(n))+4(2hi_1(n)+hi_2(n)+hi_3(n)-hi_4(n))\\
&+\frac{3}{2}(2hi_1(n)+2hi_2(n)+hi_3(n))+\frac{5}{2}(2hi_1(n)+hi_2(n)+hi_3(n)-hi_4(n))\\
&=\frac{25}{2}(\frac{8}{5}hi_1(n)+hi_2(n)+\frac{4}{5}hi_3(n)-\frac{3}{5}hi_4(n)),\\
&1+1 \text{ sector}\\
&(2hi_1(n)+2hi_2(n)+hi_3(n))+4(2hi_1(n)+2hi_2(n)+hi_3(n))\\
&+\frac{5}{2}\cdot(2hi_1(n)+hi_2(n)+hi_3(n)-hi_4(n))\\
&=\frac{25}{2}(\frac{6}{5}hi_1(n)+hi_2(n)+\frac{3}{5}hi_3(n)-\frac{1}{5}hi_4(n)). \qquad (3.62)
\end{aligned}
$$

In this way, we have reproduced the results in [8]. We also did some numerical test for curves of higher degree. In the $d=6$ case, Conjecture 3 predicts

$$
\begin{aligned}
&L_8^{13,14,6}=389591981138964503377056394266126437146547452695642109769122604114\setminus\\
&006726662085863788773654538896243 2/9375, \qquad (3.63)
\end{aligned}
$$

which agrees with the numerical computation using the fixed-point theorem [11].

4 Iritani's Theory

In this section, we briefly explain how our conjecture (3.41) naturally follows from applying Iritani's result [6] to our specific case. Crucial point of his framework is introduction of deformation variable of x^j $(j = 1, \cdots, N-2)$, $(x^1 = x)$ that corresponds to the insertion of $\mathcal{O}_{e^j}$. First, we prepare a $(N-1) \times (N-1)$ matrix $\tilde{C}_1(e^{x^1})$ whose matrix element is given by the virtual structure constants:

$$(\tilde{C}_1(e^{x^1}))_m{}^n = \begin{cases} \tilde{L}_m^{N,k,d} e^{dx^1} & (n = m + 1 + (k-N)d) \\ 0 & \text{otherwise.} \end{cases} \tag{4.64}$$

Then we can express the truncated Gauss-Manin system (3.20) into a compact form:

$$\partial_{x^1} \vec{\psi} = \tilde{C}_1(e^{x^1}) \vec{\psi}. \tag{4.65}$$

Next, we construct matrices $C_j(e^{x^1})$ $(j = 1, \cdots, N-2)$ that correspond to the deformation of $\vec{\psi}$ by x^j:

$$\partial_{x^j} \vec{\psi} = \tilde{C}_j(e^{x^1}) \vec{\psi}, \tag{4.66}$$

whose mn element vanishes unless $n = m + j + (k-N)d$. Non-zero matrix elements of $C_j(e^{x^1})$ $(j = 1, \cdots, N-2)$ are uniquely determined by the conditions:

$$[\tilde{C}_i(e^{x^1}), \tilde{C}_j(e^{x^1})] = 0, \ \ (\tilde{C}_j(e^{x^1}))_m{}^{m+j} = 1. \tag{4.67}$$

With these set up, we can obtain Jacobi matrix between B-model deformation parameters x^j and flat coordinates t^j from the flat coordinate condition $(\frac{\partial \psi_0}{\partial t^j} = \psi_j)$:

$$\begin{aligned} &\frac{\partial \psi_0}{\partial x^i} = \frac{\partial t^j}{\partial x^i} \frac{\partial \psi_0}{\partial t^j} = \frac{\partial t^j}{\partial x^i} \psi_j = (\tilde{C}_i(e^{x^1}))_0{}^j \psi_j \\ &\Longrightarrow \frac{\partial t^j}{\partial x^i} = (\tilde{C}_i(e^{x^1}))_0{}^j. \end{aligned} \tag{4.68}$$

Using (4.68), we can compute matrix $\bar{C}_i(e^{x^1})$ that corresponds to deformation of t^i as follows:

$$\partial_{t^i} \vec{\psi} = \frac{\partial x^j}{\partial t^i} \partial_{x^j} \vec{\psi} = \frac{\partial x^j}{\partial t^i} \tilde{C}_j(e^{x^1}) \vec{\psi} = \bar{C}_i(e^{x^1}) \vec{\psi} \tag{4.69}$$

Of course, we can construct rank 3 symmetric tensor

$$\bar{C}_{ijm}(e^{x^1}) = (\bar{C}_i(e^{x^1}))_j{}^l \eta_{lm}, \ \ (\eta_{lm} = k \cdot \delta_{l+m,N-2}). \tag{4.70}$$

Surprisingly, the non-zero elements of $\bar{C}_{1jm}(e^{x^1})$ is given by

$$\bar{C}_{1,N-2-n,n-1-(k-N)d}(e^{x^1}) = w(\mathcal{O}_{e^{N-2-n}} \mathcal{O}_{e^{n-1-(k-N)d}} \mathcal{O}_e)_d, \tag{4.71}$$

that was obtained by the first step of our construction. On the other hand, we can integrate Jacobi matrix (4.68) and obtain the mirror map that translates x^j into t^j as follows:

$$x^i = t^i, \ (i \notin \{1 + (k-N)j \ \mid \ j \in \mathbf{Z} \ \}), \ \ x^{1+(k-N)j} = t^{1+(k-N)j} - \frac{\tilde{L}_{1+(k-N)j}^{N,k,j}}{j} \exp(jt^1). \tag{4.72}$$

At this stage, the second step of our construction is nothing but the process of perturbing $\bar{C}_{ijm}(e^{x^1})$ by $x_2, x_3, \cdots, x_{N-2}$ by using the associativity equation:

$$\begin{aligned} &\bar{C}_{ijm}(e^{x^1}, x_2, x_3, \cdots, x_{N-2}) \eta^{ml} \bar{C}_{lst}(e^{x^1}, x_2, x_3, \cdots, x_{N-2}) \\ &= \bar{C}_{ism}(e^{x^1}, x_2, x_3, \cdots, x_{N-2}) \eta^{ml} \bar{C}_{ljt}(e^{x^1}, x_2, x_3, \cdots, x_{N-2}), \\ &\eta^{ml} = \frac{1}{k} \cdot \delta_{l+m,N-2}, \end{aligned} \tag{4.73}$$

and the Kähler equation:

$$\frac{\partial}{\partial t^1}\bar{C}_{ijm}(t^1,t^2,\cdots,t^{N-2}) = \sum_{j=1}^{N-2}\frac{\partial x^j}{\partial t^1}\frac{\partial}{\partial x^j}\bar{C}_{ijm}(e^{x^1},x^2,\cdots,x^{N-2})$$

$$= \frac{\partial}{\partial x^1}\bar{C}_{ijm}(e^{x^1},x^2,\cdots,x^{N-2}) - \sum_{d=1}^{\infty}\tilde{L}^{N,k,d}_{1+(k-N)d}\exp(dx^1)\frac{\partial}{\partial x^{1+(k-N)d}}\bar{C}_{ijm}(e^{x^1},x^2,\cdots,x^{N-2}). \quad (4.74)$$

Note that (4.74) is nothing but the modified Kähler equation (3.39). The result of Iritani asserts that the real perturbed three point function $C_{ijm}(e^{t^1},t^2,\cdots,t^{N-2})$ is given by,

$$C_{ijm}(e^{t^1},t^2,\cdots,t^{N-2}) = \bar{C}_{ijm}(e^{x^1},x^2,\cdots,x^{N-2}),$$

$$(\ x^i = t^i,\ (i \notin \{1+(k-N)j \mid j \in \mathbf{Z}\ \}),\ \ x^{1+(k-N)j} = t^{1+(k-N)j} - \frac{\tilde{L}^{N,k,j}_{1+(k-N)j}}{j}\exp(jt^1)\ ). \quad (4.75)$$

If we expand the r.h.s. of (4.75) in terms of $x^2,\cdots,x^{N-2}$ and set $t^2 = t^3 = \cdots = t^{N-2} = 0$, we obtain (3.41) as the special case of $i = 1,\ j = N-2-n,\ m = n-1-(k-N)d$.

References

[1] A.Bertram. *Another way to enumerate rational curves with torus actions* Invent.Math. **142** (2000), no.3, 487-512.

[2] Tom Coates, Alexander B. Givental. *Quantum Riemann-Roch, Lefschetz and Serre* math.AG/0110142

[3] A. Collino, M.Jinzenji. *On the Structure of Small Quantum Cohomology Rings for Projective Hypersurfaces* Commun.Math.Phys.206:157-183,1999

[4] A. Gathmann. *Absolute and Relative Gromov-Witten Invariants of Very Ample Hypersurfaces* math.AG/9908054.

[5] Alexander B. Givental. *Equivariant Gromov - Witten Invariants* Internat. Math. Res.Notices 13 (1996),613–663.

[6] H.Iritani. *Quantum D-module and Generalized Mirror Transformation* in preparation.

[7] M.Jinzenji. *Completion of the Conjecture: Quantum Cohomology of Fano Hypersurfaces* Mod.Phys.Lett. A15 (2000) 101-120.

[8] M.Jinzenji. *Virtual Gromov-Witten Invariants and the Quantum Cohomology Rings of General Type Projective Hypersurfaces* Mod.Phys.Lett. A15 (2000) 629-650.

[9] M.Jinzenji. *On the Quantum Cohomology Rings of General Type Projective Hypersurfaces and Generalized Mirror Transformation* Int.J.Mod.Phys.A15:1557-1596,2000.

[10] M.Jinzenji. *Gauss-Manin System and the Virtual Structure Constants* Int.J.Math. 13 (2002) 445-478.

[11] M.Kontsevich. *Enumeration of Rational Curves via Torus Actions* The moduli space of curves, R.Dijkgraaf, C.Faber, G.van der Geer (Eds.), Progress in Math., v.129, Birkhäuser, 1995, 335-368.

[12] M. Kontsevich, Yu. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry* Commun.Math.Phys. 164 (1994) 525-562.

[13] B.Lian, K.Liu and S.T.Yau. *Mirror Principle III* Asian J. Math. **3** (1999), no.4, 771-800.