An explicit formula for the Koecher-Maaß Dirichlet series for the Ikeda lifting

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1 Introduction

Let $F(Z)$ be a modular form of weight k with respect to the symplectic group $\Gamma_n = Sp_n(\mathbf{Z})$. Then $F(Z)$ has the following Fourier expansion:

$$F(Z) = \sum_A a_F(A)\exp(2\pi i \ \mathrm{tr}(AZ)),$$

where A runs over all semi-positive definite half-integral matrices over $\mathbf{Z}$ of degree n and $\mathrm{tr}(X)$ denotes the trace of a matrix X. We then define the Koecher-Maaß Dirichlet series $L(F,s)$ by

$$L(F,s) = \sum_A \frac{a_F(A)}{e(A)(\det A)^s},$$

where A runs over a complete set of representatives of $SL_n(\mathbf{Z})$-equivalence classes of positive definite half-integral matrices of degree n, and $e(A)$ denotes the order of the special orthogonal group of A. We remark that in case $n = 1$, $L(F,s)$ is nothing but the Hecke L-function associated to F.

Now let $F(Z)$ be a certain lifting of a Hecke eigenform $f(z)$ with respect to Γ_m with some $m \leq n$. Namely, let $F(Z)$ be a Hecke eigenform whose standard L -function or spinor L-function is expressed by the standard L-function or the spinor L-function of $f(z)$. Then we present the following problem:

Problem. Express $L(F,s)$ in terms of some Dirichlet series attached to f.

In [5], we considered the case where F is the Klingen Eisenstein lifting of a cuspidal Hecke eigenform f with respect to Γ_1. In this paper, we consider the

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case where F is the Ikeda lifting of a cuspidal Hecke eigenform with respect to Γ_1; let k and n be even positive integers. Let

$$f(z) = \sum_{m=1}^{\infty} a_f(m)\exp(2\pi imz)$$

be a normalized cuspidal Hecke eigenform of weight $2k-n$ with respect to Γ_1. Then Duke and Imamoglu conjectured that there exits a cuspidal Hecke eigenform of weight k with respect to Γ_n whose standard L-function is

$$\zeta(s)\prod_{i=1}^{n} L(s+k-i, f),$$

where $\zeta(s)$ is Riemann's zeta function. Ikeda [4] did construct such a modular form $I(f)^n$. For the precise definition of $I(f)^n$, see Section 3. Thus we call $I(f)^n$ the Ikeda lifting. This is a generalization of the Saito-Kurokawa lifting. As noted in [4], this type of conjecture was also proposed by the first named author in terms of the Koecher-Maaß Dirichlet series. In this paper, we give an explicit formula for the Koecher-Maaß Dirichlet series for the Ikeda lifting. This also gives a certain generalization of Böcherer's result in [1].

To state our main result, let $\tilde{f}(z) = \sum_{m=1}^{\infty} c(m)\exp(2\pi imz)$ be the modular form of weight $k-n/2+1/2$ belonging to the Kohnen plus-space corresponding to f, and $E_{n/2+1/2}(z) = \sum_{m=0}^{\infty} H(n/2, m)\exp(2\pi imz)$ be the Cohen Eisenstein series of weight $n/2+1/2$. For the precise definition of the Cohen Eisenstein series see Section 3. Let $L(\tilde{f}, s)$ and $L(E_{n/2+1/2}, s)$ be the Hecke L series of $\tilde{f}$ and $E_{n/2+1/2}$, respectively; namely let

$$L(\tilde{f}, s) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

and

$$L(E_{n/2+1/2}, s) = \sum_{m=1}^{\infty} \frac{H(n/2, m)}{m^s}.$$

We then define the convolution product $L(\tilde{f}, E_{n/2+1/2}; s)$ as

$$L(\tilde{f}, E_{n/2+1/2}; s) = \zeta(2s-k+1)\sum_{m=1}^{\infty} \frac{c(m)H(n/2, m)}{m^s}.$$

This type of Dirichlet series was introduced by Shimura [15]. Now our main result in this paper is:

Theorem 1 *Let n and k be even positive integers. Put*

$$\lambda_n = 2^{(n-1)(n-2)/2}\pi^{-n(n+1)/4}\prod_{i=1}^{n}\Gamma(i/2)\prod_{i=1}^{n/2-1}\zeta(2i),$$

where $\Gamma(s)$ is the Gamma function. Then, under the above assumption, we have

$$L(I(f)^n, s)$$

$$= 2^{ns}\lambda_n[\frac{2^{n/2-1}\pi^{n/2}}{\Gamma(n/2)}L(\tilde{f}, E_{n/2+1/2}; s)\prod_{i=1}^{n/2-1} L(f, 2s-2i)$$

$$+((-1)^{n/2}+1)(-1)^{n(n-2)/8}\prod_{i=1}^{n/2} L(f, 2s-2i+1)].$$

It seems interesting to give an explicit linear version of our result in relation to the result in [11]. Now, by using [12], we obtain

Corollary 1. *Put*

$$\xi(s, f) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-n/2+1/2)L(\tilde{f}, E_{n/2+1/2}; s).$$

Then $\xi(f, s)$ has a meromorphic continuation to the whole s-plane and has the following functional equation:

$$\xi(f, k-s) = \xi(f, s).$$

The meromorphy of this type of series was derived in [15] by using so called the Rankin-Selberg integral expression in more general setting, and the functional equation may also be derived by using it. However, we remark here that the corollary remains valid even for the case $k = n$, which was excluded in [15]. We also remark that by using the algebraicity of the special values of $L(\tilde{f}, E_{n/2+1/2}; s)$ shown in [15], we obtain the algebraicity of the special values of $L(I(f)^n, s)$. To explain this more precisely, for the normalized Hecke eigenform f as above, let K_f be the field generated over $\mathbf{Q}$ by all the Fourier coefficients of f. As is well known, K_f is a totally real field. Furthermore, let $u_+(f)$ and $u_-(f)$ be the complex numbers defined in [15]. Then we have

Corollary 2. *In addition to the above notation and the assumption, assume that $k > n$. Then for any odd integer l such that $n-1 \leq l \leq 2k-n-1$, the value $L(I(f)^n, l/2)$ belongs to the vector space $K_f(u_-(f)\pi^{l-n/2}i)^{n/2} + K_f(u_+(f)\pi^{l-n/2-1}i)^{n/2}$ generated over K_f by $u_-(f)\pi^{l-n/2}i)^{n/2}$ and $(u_+(f)\pi^{l-n/2-1}i)^{n/2}$. In particular, if $n \equiv 2 \bmod 4$, the value $L(I(f)^n, l/2)/(u_-(f)\pi^{l-n/2})^{n/2}$ is a pure imaginary algebraic number.*

We note that the algebraicity of special values of Koecher-Maaß series is treated by Choie and Kohnen [3] in more general setting.

The method of the proof of Theorem 1 is similar to that in the proof of [6], Theorem 1, which expressed the Koecher-Maaß Dirichlet series for a Siegel modular form in terms of its standard L-function, its primitive Fourier coefficients, and the squared Möbius function in [5]. However the primitive Fourier coefficients of the Ikeda lifting is not so simple. Thus we need some modification. The present paper is organized as follows; in Section 2, modifying the result in [5], we express the Koecher-Maaß Dirichlet series $L(F, s)$ for a general Siegel modular form F in terms of the standard L-function and the *modified primitive Fourier coefficients* of F. (cf. Theorem 2). In Section 3, applying Theorem 2 to $F = I(f)^n$ with f a normalized cuspidal Hecke eigenform with respect to Γ_1, we express $L(I(f)^n, s)$ as a sum of certain Euler products (cf. Theorem 3.2). After giving several preliminary results in Section 4, we complete the proof of Theorem 1 in Section 4.

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Notation. For a real number r we denote by $[r]$ the greatest integer not exceeding r. For a subset S of a commutative ring R, put $S^{\square} = \{a^2; a \in S\}$. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m = 0$ or $n = 0$. In particular put $M_n(R) = M_{nn}(R)$. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset X mod $aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R); \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. We call an element of $2\mathcal{H}_n(R)$ an even-integral matrix over R. For a subset S of $M_n(R)$ we denote by $S^{\times}$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A \underset{R'}{\sim} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2 Koecher-Maaß Dirichlet series and the standard L-function

In this section, we review a result in [5] in a modified way. Let (M,q) be a quadratic module over a ring R with a quadratic form q, and b the associated symmetric bilinear form defined by $b(x,y)=q(x+y)-q(x)-q(y)$ for $x,y\in M$. In particular, if R is an integral domain of characteristic different from 2, we define a bilinear form B on M by $B(x,y)=\frac{1}{2}b(x,y)$. We define submodules $M^{\perp}$ and Rad M by

$$M^{\perp}=\{x\in M; b(x,y)=0 \text{ for any } y\in M,\}$$

and

$$\mathrm{Rad}\ M=\{x\in M^{\perp}; q(x)=0 \text{ for any } x\in M\}.$$

We note that $M^{\perp}=\mathrm{Rad}\ M$ if the characteristic of R is not 2. We say that q is non-degenerate if Rad $M=\{0\}$. For a half-integral matrix A over a ring R of degree n, let q_A be the quadratic form of $M_{n1}(R)$ over R defined by $q_A(\mathbf{x})=A[\mathbf{x}]$ for $\mathbf{x}\in M_{n1}(R)$. Then the quadratic space $M_A=(M_{n1}(R),q_A)$ is called the quadratic space associated with A.

Let F be a field of characteristic different from 2, and R a subring of F. Let (V,q) be a quadratic space over F, and L a quadratic R-lattice in V, that is, a finitely generated R-module such that $L\otimes_R F=V$. For a symmetric matrix A with entries in F, let $M_A=(M_{n1}(F),q_A)$ be the quadratic space associated with A as above. Then $M_{n1}(R)$ can be regarded as a quadratic R-lattice of M_A in a natural manner, which we write as L_A.

A half-integral matrix A over $\mathbf{Z}_p$ is called non-degenerate modulo p if the reduction $M_A\otimes_{\mathbf{Z}_p}\mathbf{Z}_p/p\mathbf{Z}_p$ of the quadratic space M_A is non-degenerate. Define a subset $\mathcal{K}'_n(\mathbf{Z}_p)$ of $\mathcal{H}_n(\mathbf{Z}_p)$ by

$$\mathcal{K}'_n(\mathbf{Z}_p)=\{A\in\mathcal{H}_n(\mathbf{Z}_p); A\sim V_0\perp pV_1 \text{ with } V_0,V_1 \text{ non}-\text{degenerate modulo } p\}.$$

Further define a subset $\mathcal{K}''_n(\mathbf{Z}_2)$ of $\mathcal{H}_n(\mathbf{Z}_2)$ by

$$\mathcal{K}''_n(\mathbf{Z}_2)=\{A\in\mathcal{H}_n(\mathbf{Z}_2); A\sim\frac{1}{2}V_0\perp V\perp V_1 \text{ with } V_0,V_1 \text{ even}-\text{integral unimodular}$$

$$\text{and } V \text{ a diagonal unimodular matrix of degree 2 such that } \det V\equiv 1 \bmod 4\}.$$

Put $\mathcal{K}_n(\mathbf{Z}_p)=\mathcal{K}'_n(\mathbf{Z}_2)\cup\mathcal{K}''_n(\mathbf{Z}_2)$ or $\mathcal{K}'_n(\mathbf{Z}_p)$ according as $p=2$ or not. For a p-adic number c put

$$\chi_p(c)=1,-1 \text{ or } 0$$

according as $\mathbf{Q}_p(\sqrt{c})=\mathbf{Q}_p$, $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic unramified, or $\mathbf{Q}_p(\sqrt{c})/\mathbf{Q}_p$ is quadratic ramified. Further for a symmetric matrix A of even degree n with entries in $\mathbf{Q}_p$ we put

$$\xi_p(A)=\chi_p((-1)^{n/2}\det A).$$

For a non-degenerate half-integral matrix A we define $\sigma_p(A)$ as follows; first assume that A belongs to $\mathcal{K}_n'(\mathbf{Z}_p)$. Then we have $A \sim \frac{1}{2}V_0 \perp \frac{p}{2}V_1$ with V_0, V_1 non-degenerate matrices modulo p of degree n_0 and n_1, respectively. Then we put

$$\sigma_p(A) = \begin{cases} (-1)^{n_1/2}\xi_p(V_1)p^{(n_1^2-2n_1)/4} & \text{if } n_1 \text{ is even} \\ (-1)^{(n_1-1)/2}p^{(n_1-1)^2/4} & \text{if } n_1 \text{ is odd.} \end{cases}$$

Next let $p = 2$ and assume that A belongs to $\mathcal{K}_n''(\mathbf{Z}_2)$. Then we have $A \sim \frac{1}{2}V_0 \perp V \perp V_1$ with V_0, V_1 even-integral unimodular matrices of degree n_0 and n_1, respectively, and V a unimodular diagonal matrix of degree 2 such that $\det V \equiv 1 \bmod 4$. Then we put

$$\sigma_p(A) = (-1)^{n_1/2}p^{n_1^2/4}.$$

Finally if A does not belong to $\mathcal{K}_n(\mathbf{Z}_p)$ we put $\sigma_p(A) = 0$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$\sigma(A) = \prod_p \sigma_p(A).$$

By definition $\sigma(A)$ depends only on the genus of A. Put $\mathcal{K}_n(\mathbf{Z}) = \mathcal{H}_n(\mathbf{Z})^\times \cap \prod_p \mathcal{K}_n(\mathbf{Z}_p)$. Then by definition we have $\sigma(A) = 0$ for $A \notin \mathcal{K}_n(\mathbf{Z})$. We call σ the squared Möbius function over $\mathcal{H}_n(\mathbf{Z})$. Now for a non-degenerate positive definite half-integral matrices A and B of degree n over $\mathbf{Z}$ put

$$G(A, B) = \sum_{A' \in \mathcal{G}(A)} \frac{a(A', B)}{a(A', A')},$$

where $\mathcal{G}(A)$ denotes the set of equivalence classes belonging to the genus of A, and $a(A, B)$ the representation number of B by A. As is well-known $G(A, B)$ is determined by $\mathcal{G}(A)$ and $\mathcal{G}(B)$.

From now on for a p-adic number c let $\nu(c) = \nu_p(c)$ denote the normalized additive valuation on $\mathbf{Q}_p$. Now for a half-integral matrix A over $\mathbf{Z}_p$ put $\bar{M}_A = M_A \otimes \mathbf{Z}_p/p\mathbf{Z}_p$. Then we have

$$\bar{M}_A = U \perp \mathrm{Rad}\ \bar{M}_A$$

with U a non-degenerate quadratic submodule of $\bar{M}_A$. Put $l = l(A) = \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p} U$. If $l(A)$ is even, put $\bar{\xi}_p(A) = 1$ or -1 according as U is hyperbolic space or not. Here we make the convention that $\bar{\xi}_p(A) = 1$ if $l(A) = 0$. Then we define Andrianov's polynomial $B_p(v; A)$ as follows:

$$B_p(v, A) = \begin{cases} (1+v)(1-\bar{\xi}_p(A)p^{-l/2}v)\prod_{i=1}^{l/2-1}(1-p^{-2i}v^2) & \text{if } l \text{ is even} \\ (1+v)\prod_{i=1}^{(l-1)/2}(1-p^{-2i}v^2) & \text{if } l \text{ is odd.} \end{cases}$$

Here we understand that we have $B_p(v, A) = 1$ if $l = 0$. For a non-degenerate half-integral matrix A over $\mathbf{Z}$ put

$$B(s; A) = \prod_p B_p(p^{-s}; A).$$

For a positive definite half-integral matrix A of degree n over $\mathbf{Z}$, put

$$M(A) = \sum_{A' \in \mathcal{G}(A)} \frac{1}{a(A', A')}.$$

Now let $\mathbf{H}_n$ be the Siegel's upper half-space. A holomorphic function F on $\mathbf{H}_n$ is called a modular form of weight k with respect to Γ_n, or simply a Siegel modular form of degree n, if it satisfies the following conditions:
(i) $F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z)$ for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$;
(ii) if $n = 1$, for any $\alpha > 0$, $F(z)$ is bounded on each set $\{x + iy; y \geq \alpha\}$.
Assume that F is a Hecke eigenform. For each prime number p, let $\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p}$ be the Satake p-parameters of the local Hecke algebra $\mathbf{L}(GSp_n(\mathbf{Q}_p), GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p))$ determined by F (cf. [5]). We then define the standard zeta function $\zeta^{st}(F, s)$ of F by

$$\zeta^{st}(F, s) = \prod_p \{(1 - p^{-s}) \prod_{i=1}^{n} (1 - \alpha_{i,p} p^{-s})(1 - \alpha_{i,p}^{-1} p^{-s})\}^{-1}.$$

For a modular form $F(Z)$ of weight k with respect to Γ_n, let

$$F(Z) = \sum_A a_F(A) \exp(2\pi i \ \mathrm{tr}(AZ))$$

be the Fourier expansion of $F(Z)$ as in Introduction. We then define the Koecher-Maaß Dirichlet series $L(F, s)$ for F by

$$L(F, s) = \sum_A \frac{a_F(A)}{e(A)(\det A)^s}$$

as in Introduction. We note that in [5], we defined the Kocher-Maaß Dirichlet series in a slightly different way, but it coincides with half the above one.

Now for an element D of $M_n(\mathbf{Z}_p)^\times$, we put $\pi_p(D) = (-1)^i p^{<i-1>}$ or 0 according as D belongs to $\Lambda_{np}(E_{n-i} \perp pE_i)\Lambda_{np}$ for some $0 \leq i \leq n$, or not. Here we write $< j >= j(j+1)/2$ for an integer j. Furthermore, for an element D of $M_n(\mathbf{Z})^\times$ put $\pi(D) = \prod_p \pi_p(D)$. This is a certain generalization of the Möbius function and we call this the generalized Möbius function (cf. [5]). Now assume that n is even. Let $\hat{\mathcal{H}}_n(\mathbf{Z}_p)$ be the subset of $\mathcal{H}_n(\mathbf{Z}_p)^\times$ consisiting of matrices equivalent, over $\mathbf{Z}_p$, to some matrix of $\mathcal{H}_n(\mathbf{Z})_{>0}$. Let A be an element of $\hat{\mathcal{H}}_n(\mathbf{Z}_p)$. Let p be an odd prime. Then $(-1)^{n/2}2^n \det A$ can be expressed as $d_A p^{2b_A}$ with d_A a p-adic integer such that $\nu_p(d_A) \leq 1$ and b_A a non-negative integer. Let $p = 2$. Then $(-1)^{n/2}2^n \det A$ can be expressed as $d_A 2^{2b_A}$ with d_A a p-adic integer such that $d_A \equiv 1 \bmod 4$, or $4^{-1}d_A \equiv -1 \bmod 4$, or $8^{-1}d_A \in \mathbf{Z}_2^*$, and b_A a non-negative integer. For each prime number p, fix a non-zero complex number c_p and put $\gamma_p(A) = c_p^{b_A}$ for $A \in \hat{\mathcal{H}}_n(\mathbf{Z}_p)$. Furthermore, we put $\gamma(A) = \prod_p \gamma_p(A)$ for $A \in \mathcal{H}_n(\mathbf{Z})_{>0}$. For a positive definite half-integral matrix A of even degree n and the function γ define above put

$$a_{F,\gamma}(A) = \gamma(A)a_F(A),$$

where $a_F(A)$ is the A-th Fourier coefficient of F. Then for an element $A \in \mathcal{H}_n(\mathbf{Z})_{>0}$, we define the *modified* A-th *primitive Fourier coefficient* $a^*_{F,\gamma}(A)$ with respect to γ of F by

$$a^*_{F,\gamma}(A) = \sum_D \pi(D)a_{F,\gamma}(A[D^{-1}]),$$

where D runs over a complete set of representatives of left $GL_n(\mathbf{Z})$-equivalence classes of non-degenerate square matrices of degree n with entries in $\mathbf{Z}$. By the inversion formula for the generalized Möbius function we have

$$a_{F,\gamma}(A) = \sum_D a^*_{F,\gamma}(A[D^{-1}]),$$

where D runs over a complete set of representatives of left $GL_n(\mathbf{Z})$-equivalence classes of non-degenerate square matrices of degree n with entries in $\mathbf{Z}$ (cf. [5]). We note that $a^*_{F,\gamma}(A)$ is the primitive Fourier coefficient introduced by Böcherer and Raghavan [2] if γ is the constant function taking the value 1.

Put

$$G^*_{F,\gamma}(A) = \sum_{C \in \mathcal{G}(A)} \frac{a^*_{F,\gamma}(C)}{a(C,C)}.$$

Now let A and B be non-degenerate half-integral matrices of degree n over $\mathbf{Z}_p$. We say A dominates B over $\mathbf{Z}_p$ if there is a square matrix D with entries in $\mathbf{Z}_p$ such that $B = A[D]$, and define a polynomial $T_p(u; A, B)$ in u by

$$T_p(u; A, B) = \prod_{i=1}^{m_p}(1 - p^{-n+i}u),$$

where $m_p = 1/2(\nu_p(\det B) - \nu_p(\det A))$. We also put $T_p(s; A, B) = 1$ if A does not dominate B over $\mathbf{Z}_p$. For positive definite half-integral matrices A and B of degree n over $\mathbf{Z}$ and the γ we define a finite Euler product $T(s; A, B, \gamma)$ by

$$T(s; A, B, \gamma) = \prod_p T_p(p^{-s}c_p; A, B).$$

Then modifying [5], Theorem 3.3, we obtain:

Theorem 2. *In addition to the above notation and the assumption put*

$$K(F,\gamma,s)=\sum_{A_0}\frac{\sigma(A_0)B(2s+1-k,A_0)}{(\det A_0)^s\gamma(A)}$$

$$\times M(A_0)\sum_{C_0}\frac{G(C_0,A_0)G^*_{F,\gamma}(C_0)}{M(C_0)}T(2s+2-2k;\gamma,C_0,A_0),$$

where A_0 and C_0 run over all genera of positive definite half-integral matrices of degree n. Then we have

$$L(F,s)=\frac{2L^{st}(F,2s-k+1)}{\zeta(2s-k+1)}K(F,\gamma,s)$$

3 Koecher-Maaß Dirichlet series for the Ikeda lifting

Throughout this section and the next, we assume that n and k are even positive integers. Let

$$f(z)=\sum_{m=1}^{\infty}a(m)\exp(2\pi imz)$$

be a normalized Hecke eigenform of weight $2k-n$ with respect to $SL_2(\mathbf{Z})$. Furthermore, let $\tilde{f}$ be the cusp form of weight $k-n/2+1/2$ belonging to the Kohnen plus space corresponding to f via the Shimura correspondence (cf. [10]). Then $\tilde{f}$ has the following Fourier expansion:

$$\tilde{f}(z)=\sum_{e}c(e)\exp(2\pi iez),$$

where e runs over all positive integers such that $(-1)^{k-n/2}e\equiv 0,1 \bmod 4$. For a prime number p let β_p be a non-zero complex number such that $\beta_p+\beta_p^{-1}=p^{-k+n/2+1/2}a(p)$. In [4], these $\beta_p^{\pm}$ is called "the Satake parameter" of f. We note that the definition of "the Satake parameter" is different from the one in Section 2 of this paper. Let l be a positive integer. For a nonnegative integer e such that we define the Cohen function $H(l.e)$ as

$$H(l,e)=\begin{cases}\zeta(1-2l) & e=0\\ L(1-l,\chi_{(-1)^le}) & e>0,\ \text{and } (-1)^le\equiv 0 \text{ or } 1 \bmod 4\\ 0 & \text{otherwise,}\end{cases}$$

where $\chi_{(-1)^l e}$ is the Dirichlet character corresponding to the extension $\mathbf{Q}(\sqrt{(-1)^l e}/\mathbf{Q})$, and $L(s, \chi_{(-1)^l e})$ is the Dirichlet L function associated to $\chi_{(-1)^l e}$. Furthermore we define the Cohen Eisenstein series $E_{l+1/2}(z)$ by

$$E_{l+1/2}(z) = \sum_{e=0}^{\infty} H(l, e) \exp(2\pi i e z).$$

It is known that $E_{l+1/2}(z)$ is a modular form of weight $l + 1/2$ belonging to the Kohnen plus space.

For a positive definite half integral matrix T of degree n write $(-1)^{n/2} \det T$ as $(-1)^{n/2} \det T = \mathfrak{d}_T \mathfrak{f}_T^2$ with $\mathfrak{d}_T$ a fundamental discriminant and $\mathfrak{f}_T$ a positive integer. For each prime number p let

$$b_p(T, s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \exp(2\pi i \mathrm{tr}(TR)') p^{-ord(\mu(R))s}$$

be the local Siegel series, where $\mathrm{tr}(TR)'$ is a rational number such that $\mathrm{tr}(TR)' - \mathrm{tr}(TR) \in \mathbf{Z}_p$, and $\mu(R)$ denotes the product of denominators of elementary divisors of R. Then there exists a polynomial $F_p(T, X)$ in X such that

$$b_p(T, s) = F_p(T, p^{-s})(1 - p^{-s})(1 - \xi_p(T) p^{n/2-s})^{-1} \prod_{i=1}^{n/2} (1 - p^{2i-2s})$$

(cf. [8].) We then put

$$a_{I(f)^n}(T) = c(\mathfrak{d}_T) \prod_p (p^{k-n/2-1/2}\beta_p)^{\nu_p(\mathfrak{f}_T)} F_p(T, p^{-(n+1)/2}\beta_p^{-1}).$$

We note that $a_{I(f)^n}(T)$ does not depend on the choice of β_p. Define a Fourier series $I(f)^n(Z)$ by

$$I(f)^n(Z) = \sum_{T \in \mathcal{H}_n(\mathbf{Z})_{>0}} a_{I(f)^n}(T) \exp(2\pi i\ \mathrm{tr}(TZ)).$$

In [4] Ikeda showed that $I(f)^n(Z)$ is a cuspidal Hecke eigenform of weight k with respect to Γ_n and that its standard L-function satisfies the properties in Introduction. Now put

$$F_p^*(B, X) = \sum_D \pi_p(D)(X^2 p^{n+1})^{\nu_p(\det D)} F_p(B[D^{-1}], X),$$

where D runs over all $GL_n(\mathbf{Z}_p)$-equivalence classes of non-degenerate square matrices of degree n with entries in $\mathbf{Z}_p$. An explicit form of $F_p^*(B, X)$ will be given in Proposition 4.1.

Proposition 3.1. *Let the notation and the assumption be as above. Let* $\gamma_p(B) = (p^{-k+n/2+1/2}\beta_p)^{\nu_p(\mathfrak{f}_B)}$, *and* $\gamma(B) = \prod_p \gamma_p(B)$ *for* $B \in \mathcal{H}_n(\mathbf{Z})_{>0}$. *Let* $a^*_{I(f)^n,\gamma}(B)$ *be the primitive Fourier coefficient of* $I(f)^n$ *with respect to* γ. *Then we have*

$$a^*_{I(f)^n;\gamma}(B) = c(\mathfrak{d}_B)\prod_p \beta_p^{2\nu_p(\mathfrak{f}_B)}F_p^*(B, p^{-n/2-1/2}\beta_p^{-1}).$$

Proof. We have

$$a^*_{I(f)^n;\gamma}(B) = c(\mathfrak{d}_B)$$

$$\times \sum_{D\in GL_n(\mathbf{Z})\backslash M_n(\mathbf{Z})^\times} \pi(D)\prod_p (p^{k-n/2-1/2}\beta_p)^{\nu_p(\mathfrak{f}_{B[D^{-1}]})}F_p(B[D^{-1}],\beta_p^{-1})\gamma_p(B[D^{-1}])$$

$$= c(\mathfrak{d}_B)\prod_p \beta_p^{2\nu_p(\mathfrak{f}_B)}$$

$$\times \sum_{D\in GL_n(\mathbf{Z}_p)\backslash M_n(\mathbf{Z}_p)^\times} \pi_p(D)((p^{-n/2-1/2}\beta_p^{-1})^2p^{n+1})^{\nu_p(\det D)}F_p(B[D^{-1}], p^{-n/2-1/2}\beta_p^{-1}).$$

Thus the assertion holds. □

Now for a non-degenerate half-integral matrix A of degree m over $\mathbf{Z}_p$ and a non-degenerate symmetric matrix B of degree l with entries in $\mathbf{Q}_p$, we define the local density $\alpha_p(A,B)$ representing B by A as

$$\alpha_p(A,B) = 2^{-\delta_{ml}}\lim_{e\to\infty} p^{e(-ml+l(l+1)/2)}\#\mathcal{A}_e(A,B),$$

where

$$\mathcal{A}_e(A,B) = \{X \in M_{ml}(\mathbf{Z}_p)/p^eM_{ml}(\mathbf{Z}_p);\, A[X]-B \in p^e\mathcal{H}_l(\mathbf{Z}_p)\}.$$

Further put

$$G_p(A,B) = \frac{\alpha_p(A,B)}{\alpha_p(A,A)}p^{\nu_p(\det B)(m-l-1)/2+\nu_p(\det A)(m-l+1)/2}.$$

Let γ_p and γ be as above. By Siegel's main theorem on the quadratic forms, we have

$$G(A,B) = \prod_p G_p(A,B),$$

and

$$M(A) = 2\kappa_n\prod_p \alpha_p(A,A)^{-1}$$

for a positive definite half-integral matrices A and B of degree n, where $\kappa_n = 2^{n(n-1)}\pi^{-n(n+1)/4}\prod_{i=1}^n\Gamma(i/2)$. Thus we have

$$K(I(f^n),\gamma,s) = \sum_A \frac{M(A)B(2s-k+1,A)\sigma(A)}{\det A^s\gamma(A)}$$

$$\times \sum_{C_0} \frac{G(C_0,A)G^*_{I(f)^n,\gamma}(C_0)}{M(C_0)} T(2s-2k+2;\gamma,C_0,A)$$

$$= \sum_{A} \frac{M(A)B(2s-k+1,A)\sigma(A)}{\det A^s \gamma(A)}$$

$$\times \sum_{C_0} G(C_0,A)T(2s-2k+2;\gamma,C_0,A)c(\mathfrak{d}_{C_0}) \prod_p F_p^*(C_0,p^{-1/2-n/2}\beta_p^{-1})$$

$$= 2\kappa_n \sum_A (\det A)^{-s+(n+1)/2} \prod_p (\beta_p^{-1}p^{k-n/2-1/2})^{\nu_p(\mathfrak{f}_A)} \alpha_p(A,A)^{-1} B_p(p^{-2s+k-1},A)\sigma_p(A)$$

$$\times \sum_{C_0} c(\mathfrak{d}_{C_0}) \prod_p G_p(C_0,A) T_p(p^{-2s+k+(n-3)/2}\beta_p;C_0,A)\beta_p^{2\nu_p(\mathfrak{f}_{C_0})} F_p^*(C_0,\beta_p^{-1}p^{-(n+1)/2}),$$

where A and C_0 run over all genera of positive definite half-integral matrices of degree n.

For an element A of $\hat{\mathcal{H}}_n(\mathbf{Z}_p)$ put

$$H_p(s;A,\gamma_p)$$

$$= \frac{p^{((n+1)/2-s)\nu_p(\det A)+(k-n/2-1/2)b_A}\sigma_p(A)B_p(p^{-2s+k-1};A)}{\alpha_p(A,A)} \sum_{C_0\in\hat{\mathcal{H}}_n(\mathbf{Z}_p)} \beta_p^{2b_{C_0}-b_A} G_p(C_0,A)$$

$$\times T_p(p^{-2s+k+n/2-3/2}\beta_p;C_0,A)F_p^*(C_0,\beta_p^{-1}p^{-(n+1)/2}),$$

where b_A and b_{C_0} are non-negative integer defined in Section 2. Furthermore, for a non-zero p-adic number d_0 such that $(-1)^{n/2}d_0 \equiv 1$ or $0 \bmod 4$, and a $GL_n(\mathbf{Z})_p$-invariant function ω_p on $\mathcal{H}_n(\mathbf{Z}_p)^\times$ put

$$H_p(s;d_0;\omega_p,\gamma_p) = \sum_{l=0}^{\infty} \sum_{A\in\mathcal{A}(d_0,l)} \omega_p(A)H_p(s;A;\gamma_p),$$

where C_0 runs over a complete set of representatives of $GL_n(\mathbf{Z}_p)$-equivalence classes of half-integral matrices of degree n over $\mathbf{Z}_p$, and $\mathcal{A}(d_0,l)$ denotes a complete set of representatives of $GL_n(\mathbf{Z}_p)$-equivalence classes of half-integral matrices of degree n over $\mathbf{Z}_p$ whose determinant is $p^{2l-2[n/2]\delta_{2p}}d_0$. We note that $H_p(s;d_0;\omega_p,e)$ is a polynomial in p^{-s}. Let ι_p be the constant function of $\mathcal{H}_n(\mathbf{Z}_p)^\times$ taking the value 1, and h_p the function of $\mathcal{H}_n(\mathbf{Z}_p)^\times$ assigning the Hasse invariant of A for $A \in \mathcal{H}_n(\mathbf{Z}_p)^\times$. For $l = \pm 1$, put

$$\mathcal{F}_l = \{D_0 \in \mathbf{Z}_{>0}; lD_0 \text{ is the fundamental discriminant of a quadratic field or } 1\}.$$

Then, similarly to [6], Theorem 3.3, (1), we have

Theorem 3.2. *Under the above notation and the assumption, we have*

$$K(I(f)^n, \gamma, s) = \kappa_n \sum_{d_0 \in \mathcal{F}_{(-1)^{n/2}}} c(d_0)(\prod_p H_p(s; d_0; \iota_p; \gamma_p) + \prod_p H_p(s; d_0; h_p; \gamma_p)).$$

4 Proof of the main result

In this section we prove the main result and its corollaries. Let $p \neq 2$. Let C be an element of $\mathcal{K}_n(\mathbf{Z}_p)$. Then C is equivalent to the following form:

$$U_0 \perp pU_1,$$

where U_0 and U_1 are symmetric unimodular matrices of degree n_0 and n_1, respectively. Here we understand that U_0 or U_1 is the *empty matrix* according as $n_0 = 0$ or $n_1 = 0$. We say that C is of type (i, j) if $(n_0, n_1) \equiv (i, j) \bmod 2$. Then B is of type $(0, 0)$ or $(1, 1)$ since n is even. If C is of type $(0, 0)$, we put $\Xi^{(i)}(C) = (-1)^{n_i/2} \det U_i$, $\xi^{(i)}(C) = \xi_p^{(i)}(C) = \chi_p((-1)^{n_i/2} \det U_i)$ for $i = 0, 1$. Here we understand that we have $\Xi^{(i)}(C) = 1$ and $\xi^{(i)}(C) = 1$ if $n_i = 0$. Then $\Xi^{(i)}(C)$ is uniquely determined, up to $\mathbf{Z}_p^{*\square}$, by C, and thus $\xi^{(i)}(C)$ is uniquely determined by C.

Let C be an element of $\mathcal{K}_n(\mathbf{Z}_2)$. Then C is exactly one of the following forms:

(0, 0) $\frac{1}{2}U_0 \perp U_1$,

(1, 1) $H_{n_0/2} \perp c_0 \perp 2H_{n_1/2} \perp 2c_1$,

(2, 0) $H_{n_0/2} \perp V \perp 2H_{n_1/2}$,

where U_0 and U_1 are even-integral unimodular matrices of degree n_0 and n_1, respectively, V is a diagonal unimodular matrix of degree 2 whose determinant is congruent to 1 modulo 4, and c_0, c_1 are 2-adic units. Here we understand that U_0 is the *empty matrix* if $n_0 = 0$, and the others. If C is of type (0,0), put $\Xi^{(i)}(C) = (-1)^{n_i/2} \det U_i$ and $\xi^{(i)}(C) = \xi_2^{(i)}(C) = \chi_2((-1)^{n_i/2} \det U_i)$ for $i = 0, 1$. We make the convention that we have $\Xi^{(i)}(C) = 1$ and $\xi^{(i)}(C) = 1$ if $n_i = 0$. The quantity $\Xi^{(i)}(C)$ is uniquely determined, up to $\mathbf{Z}_2^{*\square}$, by C, and $\xi^{(i)}(C)$ is uniquely determined by C.

We denote by $\mathcal{K}_n(\mathbf{Z}_p)^{(i,j)}$ the subset of $\mathcal{K}_n(\mathbf{Z}_p)$ consisting of all matrices of type (i, j).

From now on for two elements a and b of $\mathbf{Z}_p^*$ we write $a \equiv b$ if $ab^{-1} \in \mathbf{Z}_p^{*\square}$. Further we use the same symbol a to denote the equivalence class of a in $\mathbf{Z}_p^*/\mathbf{Z}_p^{*\square}$.

We denote by $\mathcal{D} = \mathcal{D}_p$ the set $\{1, 5\}$ or the complete set of representatives of $\mathbf{Z}_p^*/\mathbf{Z}_p^{*\square}$ according as $p = 2$ or not.

The following proposition is well known (e.g. cf. [9], Lemma 9)

Proposition 4.1. *Let C be an element of $\mathcal{K}_n(\mathbf{Z}_p)$. Put $l = [\nu(2^n \det C)/2]$.*
(1) *Let $p \neq 2$. Then we have*

$$F_p^*(C, X)$$
$$= \begin{cases} \prod_{i=n/2+1}^{n/2+l}(1 - p^{2i}X^2)\dfrac{1 - \xi^{(0)}(C)p^{n/2}X}{1 - \xi^{(0)}(C)p^{n/2+l}X} & \text{if } C \text{ is of type } (0,0) \\ \prod_{i=n/2+1}^{n/2+l}(1 - p^{2i}X^2) & \text{if } C \text{ is of type } (1,1). \end{cases}$$

(2) *Let $p = 2$. Then we have*

$$F_p^*(C, X)$$
$$= \begin{cases} \prod_{i=n/2+1}^{n/2+l}(1 - 2^{2i}X^2)\dfrac{1 - \xi^{(0)}(C)2^{n/2}X}{1 - \xi^{(0)}(C)2^{n/2+l}X} & \text{if } C \text{ is of type } (0,0) \\ \prod_{i=n/1+1}^{n/2+l}(1 - 2^{2i}X^2) & \text{if } C \text{ is of type } (1,1) \\ \prod_{i=n/2+1}^{n/2+l-1}(1 - 2^{2i}X^2) & \text{if } C \text{ is of type } (2,0). \end{cases}$$

Now for a non-negative integer m we define a polynomial $\varphi_m(x)$ in x by $\varphi_m(x) = \prod_{i=1}^{m}(1 - x^i)$. Let X, Y, w be variables, and N be a positive integer. Then for integers l, m put

$$Q(X, Y, w; N, l, m) = \prod_{i=1}^{l}(1 - X^{-1}Yw^{2i-2}) \prod_{i=1}^{N-l-m}(1 - XYw^{2-2i-2N}) \prod_{i=1}^{m}(1 - Xw^{-2N+i})$$
$$\times \frac{\varphi_N(w^{-2})(-1)^{l+m}w^{-l^2+l}w^{-m^2/2+m/2}X^lY^m}{\varphi_l(w^{-2})\varphi_{N-l-m}(w^{-2})\varphi_m(w^{-1})}.$$

For integers i and j, put

$$Q_{ij}(X, Y, w; N, l, m) = Q(X, Y, w; N, l, m)w^{li+mj},$$

and

$$Q_{ij}(X, Y, w; N) = \sum_{l=0}^{N}\sum_{m=0}^{N-l} Q_{ij}(X, Y, w; N, l, m).$$

Further put

$$Q(X, Y, w; N) = \prod_{i=0}^{N-1}(1 - w^{2i-2N+2}X)(1 - w^{2i-2N+1}Y).$$

Theorem 4.2. *Let* $D_0 \in \mathbf{Z}_p^*$ *with* p *odd, or* $D_0 \in \mathbf{Z}_2^*$ *such that* $(-1)^{n/2}D_0 \equiv 1 \bmod 4$. *Then we have*

(1)

$$H_p(s; D_0; \iota_p, \gamma_p) = 2^{\delta_{2,p}n(s-(n+1)/2)}\varphi_{n/2-1}(p^{-2})^{-1}(1-p^{-n/2}\delta)^{-1}$$
$$\times[\{(1+p^{-2s+k-1})(1+p^{-2s+2k-2})-\delta p^{-2s+k-3/2}\beta_p(1+p^{1/2-n/2}\beta_p^{-1})(1+p^{-1/2+n/2}\beta_p^{-1})\}$$
$$\times\prod_{i=0}^{n/2-2}(1-p^{2i+1-2s}p^{k-n/2-1/2}\beta_p)(1-p^{2i+1-2s}p^{k-n/2-1/2}\beta_p^{-1})].$$

(2)

$$H_p(s; D_0; h_p, e) = (-1,-1)_p^{n(n+2)/8}2^{\delta_{2,p}n(s-(n+1)/2)}\varphi_{n/2-1}(p^{-2})^{-1}(1-p^{-n/2}\delta)^{-1}$$
$$\times\prod_{i=0}^{n/2-1}(1-p^{2i-n/2+k-2s}p^{k-1/2}\beta_p)(1-p^{2i-n/2+k-2s}p^{k-1/2}\beta_p^{-1}).$$

Proof. The assertion can be proved by the same method as [6], Theorem 5.2. Put $D_0 = 2^n d_0$ or d_0 according as $p=2$ or not. Now let A be an element of $\mathcal{K}_n(\mathbf{Z}_p)$ and Put $\Xi = \Xi^{(0)}(A)$ Then we have $G_p(C_0, A) \neq 0$ if and only if $C_0 \in \mathcal{K}_n(\mathbf{Z}_p), (\det A)(\det C_0)^{-1} \in \mathbf{Z}_p^{\square}$, and $\Xi^{(0)}(C_0) \equiv \Xi$. Further the $GL_n(\mathbf{Z}_p)$-equivalence class of $A \in \mathcal{K}_n(\mathbf{Z}_p)$ is uniquely determined by its determinant d and Ξ. Thus we denote by $A(d;\Xi)$ its representative. We note that the p-adic order of the determinant of $A \in \mathcal{K}_n(\mathbf{Z}_p)$ is at most $n/2$. Thus for $\omega_p = \iota_p$ or h_p we have

$$H_p(s; D_0; \omega_p, \gamma_p) = 2^{\delta_{2,p}n(s-(n+1)/2)}$$
$$\times[F_p^*(A(d_0;\Delta), \beta_p^{-1}p^{-(n+1)/2})$$
$$\times T_p(p^{-2s+k+(n-3)/2}; A(d_0;\Delta), A(d_0;\Delta))G_p(A(d_0;\Delta), A(d_0;\Delta))$$
$$\times\frac{\omega_p(A(d_0;\Delta)\sigma_p(A(d_0;\Delta))B_p(p^{-2s+k-1}; A(d_0;\Delta))}{\alpha_p(A(d_0;\Delta), A(d_0;\Delta))}$$
$$+\sum_{\Xi\in\mathcal{D}}\sum_{1\le l+m\le n/2-1}p^{(-2s+k+(n+1)/2)(l+m)}\beta_p^{l-m}$$
$$\times F_p^*(A(p^{2l}d_0;\Xi), \beta_p^{-1}p^{-(n+1)/2})T_p(p^{-2s+k+(n-3)/2}\beta_p; A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))$$
$$\times G_p(A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))$$
$$\times\frac{\omega_p(A(p^{2l+2m}d_0;\Xi))\sigma_p(A(p^{2l+2m}d_0;\Xi))B_p(p^{-2s+k-1}; A(p^{2l+2m}d_0;\Xi))}{\alpha_p(A(p^{2l+2m}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))}$$
$$+\sum_{l=0}^{n/2}p^{(-2s+k+(n+1)/2)n/2}\beta_p^{2l-n/2}F_p^*(A(p^{2l}d_0;1), \beta_p^{-1}p^{-(n+1)/2})$$

$$\times T_p(p^{-2s+2k-2}\beta_p p^{-k-n/2-1}; A(p^{2l}d_0;1), A(p^n d_0;1))G_p(A(p^{2l}d_0;1), A(p^n d_0;1))$$
$$\times \frac{\omega_p(A(p^n d_0;1))\sigma_p(A(p^n d_0;1))B_p(p^{-2s+k-1}; A(p^n d_0;1))}{\alpha_p(A(p^n d_0;1), A(p^n d_0;1))}],$$

where $\Delta = (-1)^{n/2}d_0$. For integers l, m let

$$Q(X,Y,w;n/2,l,m) = \prod_{i=1}^{l}(1-X^{-1}Yw^{2i-2})\prod_{i=1}^{n/2-l-m}(1-XYw^{2-2i-n})\prod_{i=1}^{m}(1-Xw^{-n+i})$$
$$\times \frac{\varphi_{n/2}(w^{-2})(-1)^{l+m}w^{-l^2+l}w^{-m^2/2+m/2}X^lY^m}{\varphi_l(w^{-2})\varphi_{n/2-l-m}(w^{-2})\varphi_m(w^{-1})}$$

be the rational function in X, Y and w defined above, and put

$$\tilde{Q}(X,Y,w;n/2,l,m) = \prod_{i=1}^{l}(1-X^{-1}Yw^{2i-2})\prod_{i=1}^{n/2-l-m-1}(1-XYw^{2-2i-n})\prod_{i=1}^{m}(1-Xw^{-n+i})$$
$$\times \frac{\varphi_{n/2}(w^{-2})(-1)^{l+m}w^{-l^2+l}w^{-m^2/2+m/2}X^lY^m}{\varphi_l(w^{-2})\varphi_{n/2-l-m}(w^{-2})\varphi_m(w^{-1})}.$$

Fix a square root $\sqrt{\beta_p}$ of β_p and put $u = p^{-s+k/2+(n-3)/4}\sqrt{\beta_p}$ and $v = p^{-s+k/2+(n-5)/4}\sqrt{\beta_p}^{-1}$, and

$$\Phi_n = \frac{1}{\varphi_{n/2}(p^{-2})}.$$

Then by Propositions 4.3, 4.4 of [6], and Proposition 4.1, we have

$$p^{(-2s+k+(n+1)/2)(l+m)}\beta_p^{l-m}F_p^*(A(p^{2l}d_0;\Xi), \beta_p^{-1}p^{-(n+1)/2})$$
$$\times T_p(p^{-2s+k+(n-3)/2}\beta_p; A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))G_p(A(p^{2l}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))$$
$$\times \frac{\sigma_p(A(p^{2l+2m}d_0;\Xi))B_p(p^{-2s+k-1}; A(p^{2l+2m}d_0;\Xi))}{\alpha_p(A(p^{2l+2m}d_0;\Xi), A(p^{2l+2m}d_0;\Xi))}$$
$$= \frac{\Phi_n}{2}p^{-l}\tilde{Q}(u^2, v^2, p; n/2, l, m)\delta\xi(1+\delta\xi p^{-l})(1+\xi p^{-n/2+l+m})$$
$$\times(1+\xi p^l u^{-1}v)(1+\delta u^{-1}v)^{-1}(1-\xi p^{-n+m+l+1}uv)(1+p^{-n/2+1}uv)$$

or

$$= \Phi_n p^{-l}Q(u^2, v^2, p; n/2, l, n/2-l)\delta(1+\delta p^{-l})(1+p^l u^{-1}v)(1+\delta u^{-1}v)^{-1}$$

according as $l+m \le n/2-1$ or $l+m = n/2$, where $\xi = \chi_p(\Xi)$. Here we understand $\xi = \delta$ or 1 if $l = m = 0$ or if $l+m = n/2$. Thus we have

$$H_p(s; D_0; \omega_p, \gamma_p) = 2^{\delta_{2,p}n(s-(n+1)/2)}\Phi_n$$

$$\times[\tilde{Q}(u^2,v^2,p;n/2,0,0)(1-p^{-n})(1-\delta p^{-n+1}uv)(1+p^{-n/2+1}uv)$$
$$\times\omega_p(A(p^{2l+2m}d_0,\Delta))$$
$$+1/2\sum_{\Xi\in\mathcal{D}}\sum_{1\le l+m\le n/2-1} p^{-l}\tilde{Q}(u^2,v^2,p;n/2,l,m)\delta\xi(1+\delta\xi p^{-l})$$
$$\times(1+\xi p^{-n/2+l+m})(1+\xi p^l u^{-1}v)(1+\delta u^{-1}v)^{-1}(1-\xi p^{-n+m+l+1}uv)(1+p^{-n/2+1}uv)$$
$$\times\omega_p(A(p^{2l+2m}d_0,\Xi))$$
$$+\sum_{l=0}^{n/2} p^{-l}Q(u^2,v^2,p;n/2,l,n/2-l)\delta(1+\delta p^{-l})(1+p^l u^{-1}v)(1+\delta u^{-1}v)^{-1}$$
$$\times\omega_p(A(p^n d_0,1))].$$

For any $0\le l+m\le n/2-1$ we have

$$(1+\xi p^{-n/2+l+m})(1+\xi p^l u^{-1}v)(1-\xi p^{-n+m+l+1}uv)(1+p^{-n/2+1}uv)$$
$$=1-p^{-2n+2l+2m+2}u^2v^2+p^{n/2-m}u^{-1}v(1-p^{-2n+2l+2m+2}u^2v^2)$$
$$-p^{n/2-m}u^{-1}v(1-p^{-n+m+1}u^2)(1-p^{-n+2l+2m})$$
$$+\xi[u^{-1}vp^l(1-p^{-2n+2l+2m+2}u^2v^2)+p^{n/2-l-m}(1-p^{-2n+2l+2m+2}u^2v^2)$$
$$-p^{n/2-l-m}\{(1-p^{-n+m+1}u^2)+p^{-n+m+1}u^2(1-p^{2l}u^{-2}v^2)\}(1-p^{-n+2l+2m})].$$

We note that

$$\tilde{Q}(u^2,v^2,p;n/2,l,m)(1-p^{-2n+2l+2m+2}u^2v^2)=Q(u^2,v^2,p;n/2,l,m),$$
$$\tilde{Q}(u^2,v^2,p;n/2,l,m)(1-p^{-n+m+1}u^2)(1-p^{-n+2l+2m})$$
$$=(1-p^{-n})(1-p^{-n+1}u^2)p^lp^mQ(p^{-1}u^2,p^{-1}v^2,p;n/2-1,l,m),$$

and

$$\tilde{Q}(u^2,v^2,p;n/2,l,m)(1-p^{2l}u^{-2}v^2)(1-p^{-n+2l+2m})$$
$$=(1-p^{-n})(1-u^{-2}v^2)p^{2l}Q(p^{-2}u^2,v^2,p;n/2-1,l,m).$$

Thus we have

$$H_p(s;D_0;\iota,\gamma_p)=2^{\delta_{2,p}n(s-(n+1)/2)}\Phi_n(1+u^{-1}v\delta)^{-1}$$
$$\times[\{(Q_{-2,0}(u^2,v^2,p;n/2)$$
$$+p^{n/2}u^{-1}v(Q_{-2,-1}(u^2,v^2,p;n/2)$$
$$-p^{n/2}u^{-1}v(1-p^{-n})(1-p^{-n+1}u^2)Q_{-1,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1)$$
$$+\delta\{u^{-1}vQ_{0,0}(u^2,v^2,p;n/2)+p^{n/2}Q_{-2,-1}(u^2,v^2,p;n/2)$$
$$-p^{n/2}(1-p^{-n})(1-p^{-n+1}u^2)Q_{-1,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1)$$

$$-p^{-n/2+1}u^2(1-p^{-n})(1-u^{-2}v^2)Q_{0,0}(p^{-2}u^2,v^2,p;n/2-1)\}].$$

Thus the assertion (1) follows from (1.2) and (2.2) of Lemma 4.6, and Lemma 4.7 of [6].

Next let $\omega_p = h_p$. Note that we have $h_p(A(p^{2l}d_0,\Xi)) = (-1,-1)_p^{n(n+2)/8}\delta\xi$ for any $l \geq 0$. Thus similarly to the case of $\omega_p = \iota_p$, we have

$$H_p(s;D_0;h_p,\gamma_p) = (-1,-1)_p^{n(n+2)/8}2^{\delta_{2,p}n(s-(n+1)/2)}\Phi_n(1-u^{-1}v)^{-1}$$

$$\times[\{Q_{-1,0}(u^2,v^2,p;n/2)+p^{n/2}u^{-1}vQ_{-1,-1}(u^2,v^2,p;n/2)$$

$$-p^{n/2}(1-p^{-n})(1-p^{-n+1}u^2)u^{-1}vQ_{0,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1)\}$$

$$+\delta\{u^{-1}vQ_{-1,0}(u^2,v^2,p;n/2)+p^{n/2}Q_{-2,-1}(u^2,v^2,p;n/2)$$

$$-p^{n/2}(1-p^{-n})(1-p^{-n+1}u^2)Q_{-1,0}(p^{-1}u^2,p^{-1}v^2,p;n/2-1)$$

$$-p^{-n/2+1}u^2(1-p^{-n})(1-u^{-2}v^2)Q_{-1,0}(p^{-2}u^2,v^2,p;n/2-1)\}].$$

Thus the assertion (2) can also be proved by (1.2) and (2.1) of Lemma 4.6, and Lemma 4.7 of [6]. □

Theorem 4.3. *Let $D_0 \in p\mathbf{Z}_p^*$ with p odd, or $D_0 \in 4\mathbf{Z}_2^*$ such that $(-1)^{n/2}4^{-1}D_0 \equiv$ 3 mod 4 or $D_0 \in 8\mathbf{Z}_2^*$. Put $l_0 = \nu_p(D_0)$.*
(1) *We have*

$$H_p(s;D_0;\iota_p,\gamma_p) = 2^{\delta_{2,p}n(s-(n+1)/2)}\phi_{n/2-1}(p^{-2})^{-1}p^{(-s+(n-1)/2)l_0}(1-p^{n-2k})$$

$$\times(1+p^{-2s+k-1})\prod_{i=0}^{n/2-2}(1-p^{2i+1-2s}p^{k-n/2-1/2}\beta_p)(1-p^{2i+1-2s}p^{k-n/2-1/2}\beta_p^{-1}).$$

(2)

$$H_p(s;D_0;h_p,\gamma_p) = 0.$$

Proof. The assertion can be proved by the same method as [6], Theorem 5.3. Put $d_0 = 2^{-\delta_{2,p}n}p^{-l_0}D_0$. First let $p \neq 2$. Then $l_0 = 1$ and $D_0 = pd_0$. If an element A of $\mathcal{K}_n(\mathbf{Z}_p)$ is of type (1,1), A can be expressed as

$$A \sim U_0 \perp pU_1$$

where U_0 (resp. U_1) is a unimodular symmetric matrix of odd degree n_0 (resp. n_1). Put $\Xi = (-1)^{n_1-1}\det U_0$. Then the $GL_n(\mathbf{Z}_p)$- equivalence class of $A \in \mathcal{K}_n(\mathbf{Z}_p)$ is uniquely determined by its determinant d and Ξ. Thus we denote by $A(d;\Xi)$ its representative. Then similarly to Theorem 4.2 we have

$$H_p(s;D_0;\omega_p,\gamma_p)$$

$$= \sum_{\Xi \in \mathcal{D}} \sum_{l=0}^{n/2-1} \sum_{m=0}^{n/2-l} \beta_p^{2l} (\beta_p p^{-k-n/2-1})^{-l-m} p^{(l+m+1/2)(-2s+n+1)}$$

$$\times F_p^*(A(p^{2l+1}d_0;\Xi), \beta_p^{-1} p^{-(n+1)/2}) \omega_p(A(p^{2l+2m+1}d_0;\Xi))\sigma_p(A(p^{2l+2m+1}d_0;\Xi))$$

$$\times \frac{G_p(A(p^{2l+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))}{\alpha_p(A(p^{2l+2m+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))}$$

$$\times T_p(p^{-2s+k+(n-3)/2}\beta_p; A(p^{2l+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi)) B_p(p^{-2s+k-1}; A(p^{2l+2m+1}d_0;\Xi)).$$

By Propositions 4.3 and 4.4 of [6], and Proposition 4.1, the quantity

$$\beta_p^{2l} (\beta_p p^{-k-n/2-1})^{-l-m} p^{(l+m+1/2)(-2s+n+1)} F_p^*(A(p^{2l+1}d_0;\Xi), \beta_p^{-1} p^{-(n+1)/2})$$

$$\times \sigma_p(A(p^{2l+2m+1}d_0;\Xi)) \frac{G_p(A(p^{2l+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))}{\alpha_p(A(p^{2l+2m+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi))}$$

$$\times T_p(p^{-2s+k+(n-3)/2}\beta_p; A(p^{2l+1}d_0;\Xi), A(p^{2l+2m+1}d_0;\Xi)) B_p(p^{-2s+k-1}; A(p^{2l+2m+1}d_0;\Xi))$$

is independent of d_0 and Ξ, and it is equal to

$$\Psi_n(s)(1 + p^{-2s+k-1}) Q(p^{-2}u^2, v^2, p; n/2-1, l, m),$$

where u and v are those in the proof of Theorem 4.2,

$$\Psi_n(s) = \frac{2^{\delta_{2,p} n(s-(n+1)/2)} p^{-s+(n-1)/2}}{2\varphi_{n/2-1}(p^{-2})}.$$

Thus we have

$$H_p(s; D_0; \omega_p, \gamma_p) = \Psi_n(s)(1 + p^{-2s+k-1}) \sum_{l=0}^{n/2-1} \sum_{m=0}^{n/2-1-l} Q(p^{-2}u^2, v^2, p; n/2-1, l, m)$$

$$\times \sum_{\Xi \in \mathcal{D}} \omega_p(A(p^{2l+2m+1}d_0;\Xi)).$$

By [6], Lemma 5.1 we have

$$\sum_{\Xi \in \mathcal{D}} \omega_p(A(p^{2l+2m+1}d_0;\Xi)) = 2 \text{ or } 0$$

according as $\omega_p = \iota_p$ or h_p, and therefore, the assertion follows from (2.1) of [6], Lemma 4.6. Similarly the assertion holds for $p = 2$. $\square$

Proof of Theorem 1. Put $\Omega = \{\omega_p\}$, and let $d_0 \in \mathcal{F}_{(-1)^{n/2}}$. Put

$$H(s, d_0, \Omega) = \prod_p H_p(s, d_0, \omega_p, \gamma_p).$$

Then by Theorems 4.2 and 4.3, we have

$$H(s, d_0, \{\iota_p\}) = 2^{n(s-(n+1)/2)} \prod_{i=1}^{n/2-1} \zeta(2i) \frac{d_0^{-s+(n-1)/2} c(d_0) L(n/2, \chi_{(-1)^{n/2}d_0})}{\prod_{i=0}^{n/2-2} L(f, 2s-1-2i)}$$

$$\times \{(1+p^{-2s+k-1})(1+\chi_{(-1)^{n/2}d_0}(p)^2 p^{-2s+2k-2})$$

$$-\chi_{(-1)^{n/2}d_0}(p) p^{-2s+k-3/2} \beta_p (1+p^{1/2-n/2}\beta_p^{-1})(1+p^{-1/2+n/2}\beta_p^{-1})\}.$$

We note that $L(\tilde{f}, s)$ and $L(E_{n/2+1}, s)$ can be expressed as

$$L(\tilde{f}, s) = L(f, 2s) \sum_{(-1)^{k-n/2} d_0 \equiv 0,1 \bmod 4} c(d_0) d_0^{-s} \prod_p (1 - \chi_{(-1)^{k-n/2}d_0}(p) p^{k-n/2-1-2s}),$$

and

$$L(E_{n/2+1}, s) = \zeta(2s)\zeta(2s-n+1)$$

$$\times \sum_{(-1)^{n/2} d_0 \equiv 0,1 \bmod 4} L(1-n/2, \chi_{(-1)^{n/2}d_0}) d_0^{-s} \prod_p (1 - \chi_{(-1)^{n/2}d_0}(p) p^{n/2-1-2s}),$$

and therefore, by remarking $(-1)^{k-n/2} = (-1)^{n/2}$, we easily see that $L(\tilde{f}, E_{n/2+1/2}; s)$ can be expressed as

$$L(\tilde{f}, E_{n/2+1/2}; s) = L(f, 2s) L(f, 2s-n+1)$$

$$\times \sum_{d_0} d_0^{-s} c(d_0) L(1-n/2, \chi_{(-1)^{n/2}d_0}) \prod_p \{(1+p^{-2s+k-1})(1+\chi_{(-1)^{n/2}d_0}(p)^2 p^{-2s+k-2})$$

$$-\chi_{(-1)^{n/2}d_0}(p) p^{-2s+k-3/2} \beta_p (1+p^{1/2-n/2}\beta_p^{-1})(1+p^{-1/2+n/2}\beta_p^{-1})\}$$

(cf. [13], Lemma 1.) Thus, by remarking the functional equation

$$L(1-n/2, \chi_{(-1)^{n/2}d_0}) = 2^{1-n/2} \pi^{-n/2} \Gamma(n/2) d_0^{(n-1)/2} L(n/2, \chi_{(-1)^{n/2}d_0}),$$

we have

$$\sum_{d_0 \in \mathcal{F}_{(-1)^{n/2}}} c(d_0) H(s; d_0; \iota_p)$$

$$= 2^{n(s-(n+1)2)} \prod_{i=1}^{n/2-1} \zeta(2i) \frac{2^{n/2-1}\pi^{n/2}}{\Gamma(n/2)} \frac{L(\tilde{f}, E_{n/2+1/2}; s)}{L(f, 2s) \prod_{i=0}^{n/2-1} L(f, 2s-1-2i)}.$$

On the other hand, if $d_0 \neq 1$, by (2) of Theorem 4.2, we have

$$H(s, d_0, \{h_p\}) = 0.$$

Thus if $n \equiv 2 \bmod 4$, for any $d_0 \in \mathcal{F}_{(-1)^{n/2}}$,

$$H(s, d_0, \{h_p\}) = 0.$$

If $n \equiv 0 \bmod 4$, by (2) of Theorem 4.2, we have

$$H(s, d_0, \{h_p\}) = 2^{n(s-(n+1)/2)} \prod_{i=1}^{n/2-1} \zeta(2i) \prod_{i=0}^{n/2-1} L(f, 2s-2i)^{-1}.$$

As stated in Introduction we have

$$L^{st}(I(f)^n; 2s-k+1) = \zeta(2s-k+1) \prod_{i=0}^{n-1} L(f, 2s-i).$$

Thus the assertion follows from Theorem 2 and Theorem 3.2. □

Proof of Corollary 1. The assertion follows from the meromorphy and the functional equation of $L(I(f)^n, s)$ and $L(f, s)$ (cf. [12]).

Proof of Corollary 2. By [14], Theorem 1, for $j = 1, ..., n/2-1$, the values $L(f, l-2j)/((\pi i)^{l-2j} u_-(f))$ and $L(f, l-2j+1)/((\pi i)^{l-2j+1} u_+(f))$ belong to K_f, and by [15], Theorem 2, the value $L(\tilde{f}, E_{n/2+1}; l/2)/(u_-(f) i \pi^{n/2-l})$ belongs to K_f for an odd integer l such that $n-1 \leq l \leq 2k-n-1$. Thus, by remarking that $\lambda_n = \pi^{-n/2} \mu_n$ with μ_n a rational number, we have proved the assertion.

Remark. By using the same method, we can give an explicit form of the Koecher-Maaß Dirichlet series for the Sigel Eisenstein series (cf. [7]).

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