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The Stokes operator with Robin boundary conditions in solenoidal subspaces of $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$

Jürgen Saal

Abstract

We prove that the Stokes operator with Robin boundary conditions is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on the space $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. Contrary to that result it is also proved that there exists no Stokes semigroup on $L^1_\sigma(\mathbb{R}^n_+)$, except if we assume the special case of Neumann boundary conditions. Nevertheless, we also obtain gradient estimates for the solution of the Stokes equations in $L^1_\sigma(\mathbb{R}^n_+)$ for all types of Robin boundary conditions.

1 Introduction

Here we consider the Stokes equations with Robin boundary conditions

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla p &=& 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \operatorname{div} u &=& 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ u(0) &=& u_0 & \text{in } \mathbb{R}^n_+, \\ T_\alpha u &=& 0 & \text{in } \partial\mathbb{R}^n_+ \times (0,\infty), \end{array}\right. \tag{1}$$

i.e. the trace operator T_α is given by

$$T_\alpha u := \left(\begin{array}{c} \alpha u' - \partial_n u' \\ u^n \end{array} \right)\Big|_{\partial\mathbb{R}^n_+}, \tag{2}$$

where u' denotes the tangential part of u and $\alpha \in [0,\infty]$[1]. Observe, that the case $\alpha = 0$ or $\alpha = \infty$ corresponds to the classical Neumann or Dirichlet boundary conditions respectively.

The particular matter in our investigation of (1) are not only the Robin boundary conditions, but also the function spaces for the initial value u_0. The most available literature, which deals with the Stokes equations in the L^q-framework, only includes the case where $1 < q < \infty$. However, in this note we examine system (1) for initial values u_0 in $L^1(\mathbb{R}^n_+)$ or $L^\infty(\mathbb{R}^n_+)$. In particular we will prove resolvent estimates for the solution of the associated resolvent problem to (1) in solenoidal subspaces of $L^\infty(\mathbb{R}^n_+)$. This leads to a generation result for the Stokes operator in certain spaces $L^\infty_\sigma(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. Note that we can not define the Stokes operator as it is usually done in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 < q < \infty$. This is due to the fact that the Helmholtz projection, associated to the Helmholtz decomposition $L^q(\mathbb{R}^n_+) = L^q_\sigma(\mathbb{R}^n_+) \oplus G_q(\mathbb{R}^n_+)$ for $1 < q < \infty$ (see [Sol77], [McC81], [BM88]), does not act as

[1] The case $\alpha = \infty$ is to understand in the following sense: divide the first line in (2) by α and let $\alpha \to \infty$.

a bounded operator on $L^\infty(\mathbb{R}^n_+)$ or $L^1(\mathbb{R}^n_+)$. Therefore we will give a definition of the Stokes operator with Robin boundary conditions through its resolvent.

The situation in $L^1(\mathbb{R}^n_+)$ is different. According to a result in [DHP01] it is known that there exists no Stokes semigroup in $L^1_\sigma(\mathbb{R}^n_+)$. On the other hand, by a reflection argument it can be easily seen that the Stokes operator with Neumann boundary conditions is the generater of a bounded holomorphic C_0-semigroup on $L^1_\sigma(\mathbb{R}^n_+)$. Thus, the natural question arises, what happens between, i.e. if one considers a mixture if these two special types of Robin boundary conditions. In this paper we give an answer to this question. It turns out, that the generation result holds if and only if we assume Neumann boundary conditions, i.e. in the case $\alpha = 0$. In other words, whenever we add an arbitrary small Dirichlet part to the Neumann boundary conditions, the generation result in $L^1_\sigma(\mathbb{R}^n_+)$ fails. Let us remark that the non generation result in $L^1(\mathbb{R}^n_+)$ for Dirichlet boundary conditions is also physically motivated. Indeed, in [Koz98] it is proved that the existence of a local strong solution of the Navier-Stokes equations with Dirichlet boundary conditions in $L^1(\Omega)$ for an exterior domain $\Omega \subseteq \mathbb{R}^n$, implies that no force could act on the boundary. This would mean that the Navier-Stokes equations are physically meaningless, hence one does not expect a generation result in L^1 to be valid. Nevertheless, we can prove gradient estimates $\|\nabla u\|_1 \leq C\|u_0\|_1$ for all types of Robin boundary conditions and initial values in $L^1_\sigma(\mathbb{R}^n_+)$.

In the previous literature problem (1) with initial values in $L^\infty_\sigma(\mathbb{R}^n_+)$ and $L^1_\sigma(\mathbb{R}^n_+)$ was only investigated for the special case of Dirichlet boundary conditions. As well as the couterexample in $L^1_\sigma(\mathbb{R}^n_+)$, the generation result for the Stokes operator with Dirichlet boundary conditions in $L^\infty_\sigma(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$ is also contained in [DHP01]. Based on the Green matrix for the Stokes system in $\mathbb{R}^n_+$, a similar result is proved in a space of bounded and continuous solenoidal fields in [Sol03]. There the author also applied his result for the linear problem to the nonlinear Navier-Stokes equations and proved a local existence result for nondecaying initial values. Also a local existence result for the Navier-Stokes equations is proved in [IM] for initial values in $L^\infty_\sigma(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$. Their proof relies on the generation results in [DHP01]. Gradient estimates for Dirichlet boundary conditions in $L^1_\sigma(\mathbb{R}^n_+)$ are proved in [GMS99], [SS01], and in $L^\infty_\sigma(\mathbb{R}^n_+)$ in [Shi99], [SS01].

The content of this article can be regarded as a generalization to Robin boundary conditions of the above mentioned results in [DHP01], [GMS99], [Shi99], and [SS01] for Dirichlet boundary conditions. Our results are based on an explicit solution formula provided in [Saa04]. The construction of this formula is similar to the one in [DHP01]. However, the method for estimating it differs from their method. In [DHP01] the authors provide pointwise kernel estimates, whereas our proofs rely on a multiplier result for rotation invariant multipliers. The mentioned results in [GMS99], [Shi99], and [SS01] are based on Ukai's formula for the solution of the Stokes equations with Dirichlet boundary conditions (see [Uka87]).

We want to remark that the content of this paper is included in [Saa03] and it extends the previous work [Saa04] which deals with (1) in $L^q(\mathbb{R}^n_+)$ for $1 < q < \infty$. In that work, for proving resolvent estimates we made use of the rotation invariance in $n-1$ dimensions of large parts of the constucted solution formula. This enabled us to apply the bounded H^∞-calculus of the Poisson operator $(-\Delta_{\mathbb{R}^{n-1}})^{1/2}$ on $L^q(\mathbb{R}^{n-1})$. This means we regarded the computed representation of the solution as a function of the Poisson operator $g((-\Delta_{\mathbb{R}^{n-1}})^{1/2})$, which can be estimated by

$$\|g((-\Delta_{\mathbb{R}^{n-1}})^{1/2})\|_{\mathcal{L}(L^q(\mathbb{R}^{n-1}))} \leq C\|g\|_{H^\infty(\Sigma)} \tag{3}$$

where $H^\infty(\Sigma)$ denotes the space of all bounded holomorphic functions on a certain complex sector Σ, equipped with the infinity norm. Thus, besides the holomorphy of g, it was sufficient to verify pointwise estimates on a complex sector for the terms in the representation of the Stokes resolvent, regarded as functions of $(-\Delta_{\mathbb{R}^{n-1}})^{1/2}$. These estimates also provided a sufficient decay in the normal component x_n, such that the q-integration over x_n afterwards was feasible for $1 < q < \infty$. In [Saa04] this method leads to resolvent estimates and a bounded H^∞-calculus for the Stokes operator with Robin boundary conditions in $L^q_\sigma(\mathbb{R}^n_+)$ for $1 < q < \infty$.

For estimating the solution formula in $L^1(\mathbb{R}^n_+)$ and $L^\infty(\mathbb{R}^n_+)$ we intend to adapt the methods in [Saa04]. However, for this purpose we have to circumvent the following two main difficulties. First one, the bounded H^∞-calculus for the Poisson operator $|\nabla'| = (-\Delta_{\mathbb{R}^{n-1}})^{1/2}$ on $L^q(\mathbb{R}^{n-1})$, which is the main ingredient for the proof of the estimates in [Saa04], is neither valid in $L^1(\mathbb{R}^{n-1})$ nor in $L^\infty(\mathbb{R}^{n-1})$. Here we have to provide an appropriate substitute which is based on the above mentioned classical result for rotation invariant multipliers. This result enables us to obtain estimates as (3), which are now also valid in $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$. Besides the holomorphy and the boundedness on a complex sector, the functions here also have to be holomorphic in 0 and have to satisfy a certain decay at infinity. But we will show that the terms in our solution formula still satisfy these two additional conditions. The second problem, we have to deal with, is the unboundedness of the Riesz operator $R' := \mathcal{F}^{-1}\frac{i\xi'}{|\xi'|}\mathcal{F}$ in $L^1(\mathbb{R}^{n-1})$ and $L^\infty(\mathbb{R}^{n-1})$. Roughly speaking, we will overcome this problem by rephrasing the solution formula in a way such that no Riesz operator appears.

We organized this article as follows. In Section 2 we state the necessary basics. After introducing the notation we state our mentioned substitute (Corollary 2.3) for the bounded H^∞-calculus of the Poisson operator which is only valid in $L^q(\mathbb{R}^{n-1})$ for $1 < q < \infty$. We also recall some known properties of the Laplacian in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ and the explicit solution formula for the system (1) provided in [Saa04]. With the aid of these preparations, in Section 3 we will first prove that the Stokes operator with Neumann boundary conditions is the generator of a bounded holomorphic C_0-semigroup on $L^1_\sigma(\mathbb{R}^n_+)$. Furthermore, we state the counterexample for the other types of mixed boundary conditions, as well as the mentioned gradient estimates for the Stokes semigroup on $L^1(\mathbb{R}^n_+)$, valid for all considered boundary conditions. In Section 4 then we will verify resolvent estimates in $L^\infty(\mathbb{R}^n_+)$. This leads to the result that the Stokes operator is the generator of a bounded holomorphic semigroup on the space $L^\infty_\sigma(\mathbb{R}^n_+)$, which is even strongly continuous on the spaces $C_{0,\sigma}(\mathbb{R}^n_+)$ and $\mathrm{BUC}_\sigma(\mathbb{R}^n_+)$.

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2 Preliminaries

2.1 Notations

In most parts of this note we use standard notation. For $m \in \{0, 1, \ldots, \infty\}$ and a domain $\Omega \subseteq \mathbb{R}^n$, by $C^m(\Omega)$ we denote the space of all m-times continuously differentiable functions and by $C^m_c(\Omega)$ its subspace consisting of all functions in $C^m(\Omega)$ which are compactly supported. Further, let $C^m_c(\overline{\Omega}) := \{u \restriction_\Omega \colon u \in C^m_c(\mathbb{R}^n)\}$ and $C^m_b(\Omega)$ the Banach space of all

m-times continuously differentiable functions whose derivatives up to order m are bounded. Furthermore, we write $\mathrm{BUC}(\Omega)$ for the space of all bounded and uniformly continuous functions in Ω.

The Fourier transform defined on $\mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions, we denote by

$$\hat{u}(\xi) := \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi} u(x)\mathrm{d}x, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

For $q \in [1,\infty]$, $L^q(\Omega)$ denotes the Lebesgue space, which consists of all q-integrable functions if $1 \le q < \infty$ and $L^\infty(\Omega)$ is the space of all functions u that satisfy $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega}|u(x)| < \infty$. The space of solenoidal functions in $L^q(\Omega)$ for $1 < q < \infty$ we define by $L^q_\sigma(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_q}$, where $C^\infty_{c,\sigma}(\Omega)$ denotes all $C^\infty_c(\Omega)$-functions with vanishing divergence, i.e. $\operatorname{div} u = \nabla\cdot u = 0$. Further $L^q_{\mathrm{loc}}(\Omega) := \{u \in \mathcal{S}'(\Omega) : u \in L^q(K) \text{ for each compact } K \subseteq \Omega\}$. $W^{m,q}(\Omega)$ denotes the Sobolev space of order $m \in \mathbb{N}_0$. Its norm is given by

$$\|u\|_{W^{m,q}(\Omega)} := \left(\sum_{j=0}^{m} \|\nabla^j u\|^q_{L^q(\Omega)}\right)^{1/q},$$

where ∇^j is the tensor of all possible j-th order differentials. Moreover, $W^{m,q}_0(\Omega)$ denotes the closure of $C^\infty_c(\Omega)$ in $W^{m,q}(\Omega)$. If no confusion seems likely, we also write $\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{m,q} := \|\cdot\|_{W^{m,q}(\Omega)}$.

We also make use of the homogeneous Sobolev space $\widehat{W}^{1,q}(\Omega)$ consisting of all $L^1_{\mathrm{loc}}(\Omega)$-functions u having finite Dirichlet energy $\int_\Omega |\nabla u|^q \mathrm{d}x$, modulo constants. It becomes a Banach space when equipped with the norm

$$\|u\|_{\widehat{W}^{1,q}(\Omega)} := \left(\int_\Omega |\nabla u|^q \mathrm{d}x\right)^{1/q}.$$

We will use the same notation for the corresponding spaces of vector fields on Ω, i.e. $(L^q(\Omega))^n = L^q(\Omega)$, $(W^{k,q}(\Omega))^n = W^{k,q}(\Omega)$, etc. Denote by q' the Hölder conjugated exponent, i.e. $\frac{1}{q'} + \frac{1}{q} = 1$. If $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$ we use the notation $(u,v) := (u,v)_\Omega := \int_\Omega uv\mathrm{d}x$ for the dual pairing.

If X and Y are Banach spaces, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$, and $\mathcal{L}(X)$ is the abbreviation for $\mathcal{L}(X,X)$. For any closed operator A in X, its domain and range are denoted by $D(A)$ and $R(A)$, respectively. Its resolvent set is denoted by $\rho(A)$ and its spectrum by $\sigma(A)$. Furthermore, we call A a generator, if $(\mathrm{e}^{tA})_{t\ge 0}$ satisfies the semigroup properties.

As usual C, M, ... denote constants that may change from line to line. Sometimes we would like to express a special dependence on some parameter s. Then we use either the subscript notation C_s, M_s, ... or we write it as an argument $C(s)$, $M(s)$, ...

2.2 Rotation Invariant Multipliers

For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we denote by BV_{k+1} the normed space of all functions $m \in C_0([0,\infty),\mathbb{C}) := \{v \in C([0,\infty),\mathbb{C}) : \lim_{t\to\infty} v(t) = 0\}$ with $m, m', \ldots, m^{(k)}$ being locally

absolutely continuous on $(0,\infty)$ and satisfying $\lim_{t\to\infty} m^{(j)}(t) = 0$ for $j = 0,\dots,k$ and

$$\|m\|_{BV_{k+1}} = \frac{1}{\Gamma(k+1)} \int_0^\infty t^k |m^{(k+1)}(t)| \mathrm{d}t < \infty.$$

By a simple calculation we can see that $BV_{k+1} \hookrightarrow BV_k$. The introduction of this space allows us to formulate the next lemma about rotation invariant multipliers. A proof can be found in [Tre73].

Lemma 2.1 *Let $n, k \in \mathbb{N}$ satisfying $k > n/2$ and let $m \in BV_{k+1}$. Then the function $m(|\cdot|) : \mathbb{R}^n \to \mathbb{C}$, $\xi \mapsto m(|\xi|)$ belongs to the space $\mathcal{F}L^1(\mathbb{R}^n) := \{\mathcal{F}f : f \in L^1(\mathbb{R}^n)\}$ and there is a constant $C = C(n,k) > 0$ such that*

$$\|\mathcal{F}^{-1} m(|\cdot|)\|_{L^1(\mathbb{R}^n)} \le C \|m\|_{BV_{k+1}}.$$

As a consequence we obtain the following proposition for bounded holomorphic functions m, where we denote the space of bounded holomorphic functions on a domain $G \subseteq \mathbb{C}$ by $H^\infty(G)$.

Proposition 2.2 *Let $m : [0,\infty) \to \mathbb{C}$. Assume there exist $\phi \in (0,\pi)$, $\varepsilon \in (0,1)$ and $C_0 > 0$, such that $m \in H^\infty(\Sigma_\phi \cup \{0\})$ and*

$$|z^\varepsilon m(z)| \le K, \quad z \in \Sigma_\phi. \tag{4}$$

Then $[\xi \mapsto m(|\xi|)] \in \mathcal{F}L^1(\mathbb{R}^n)$ and there is a constant $C > 0$ such that

$$\|\mathcal{F}^{-1} m(|\cdot|)\|_{L^1(\mathbb{R}^n)} \le CK.$$

Proof. In view of Lemma 2.1 we have to prove that $m \in BV_{k+1}$ for some $k > n/2$. Since $m \in \mathcal{H}^\infty(\Sigma_\phi \cup \{0\}) \subseteq C^\infty([0,\infty))$, by (4) it immediately follows that $m \in C_0([0,\infty))$ and all derivatives $m^{(j)}$ with $j \le k$ are locally absolutely continuous functions on $(0,\infty)$. By the holomorphy of m in 0 there is a $\delta_0 > 0$ such that m is holomorphic in $B_{\delta_0}(0) \cup \Sigma_\phi$, where $B_r(w) := \{z \in \mathbb{C} : |z - w| < r\}$. Thus we may choose $\delta_i = \delta_i(\delta_0) \in (0,1)$, $i = 1,2$, in a way such that for

$$r : \mathbb{C} \to (0,\infty), \quad z \mapsto r(z) := \delta_1(\mathrm{Re}\ z + \delta_2),$$

the ball $B_{r(z)}(z)$ lies completely in the domain $B_{\delta_0}(0) \cup \Sigma_\phi$ for all $z \in [0,\infty)$ (see Figure 1). For $t \in [0,\infty)$ we therefore get by Cauchy's formula and by our assumption (4) on m

$$\begin{aligned} |m^{(j)}(t)| &\le \frac{j!}{r(t)^j} \max_{|t-z|=r(t)} |m(z)| \le C(j,\delta_0) \frac{1}{(t+\delta_2)^j} \max_{|t-z|=r(t)} \frac{K}{|z|^\varepsilon} \\ &\le \frac{C(j,\delta_0)}{(t+\delta_2)^j} \frac{K}{|t - r(t)|^\varepsilon} \le \frac{C(j,\delta_0) K}{(t+\delta_2)^j |(1-\delta_1)t - \delta_1\delta_2|^\varepsilon} \end{aligned}$$

for $j \in \mathbb{N} \cup \{0\}$. This implies $\lim_{t\to\infty} m^{(j)}(t) = 0$ for $j \in \mathbb{N} \cup \{0\}$ and if we set $j = k+1$

$$\begin{aligned} \|m\|_{BV_{k+1}} &= \frac{1}{\Gamma(k+1)} \int_0^\infty t^k |m^{(k+1)}(t)| \mathrm{d}t \\ &\le C(k,\delta_0) K \int_0^\infty t^k \frac{1}{(t+\delta_2)^{k+1} |(1-\delta_1)t - \delta_1\delta_2|^\varepsilon} \mathrm{d}t \\ &\le C(k,\delta_0,\varepsilon) K, \end{aligned} \tag{5}$$

which yields the assertion. □

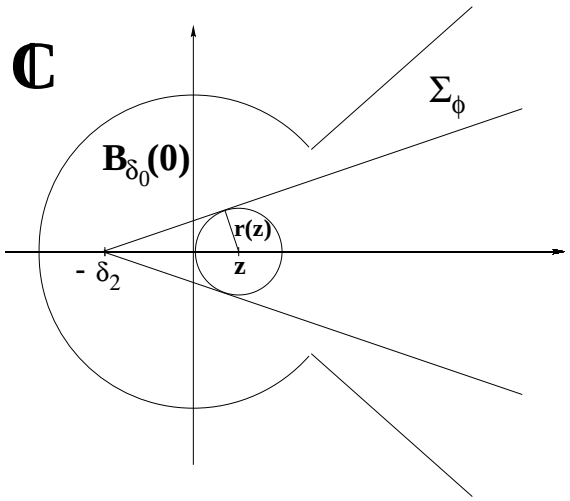

Figure 1: The ball with center $z \in [0,\infty)$ and radius $r(z)$ is contained in $B_{\delta_0}(0) \cup \Sigma_\phi$.

The following corollary about multipliers which depend on parameters then applies to the spaces $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ as our mentioned substitute for the bounded H^∞-calculus of the Poisson operator on $L^q(\mathbb{R}^n)$, $1 < q < \infty$. It is an immediate consequence of estimate (5).

Corollary 2.3 *Let $n \in \mathbb{N}$, $I \subseteq \mathbb{R}^n$, and $b : I \to [0,\infty)$, $s \mapsto b(s)$. Assume there exist $\phi \in (0,\pi)$, $\varepsilon \in (0,1)$, and $\delta_0 > 0$, such that $m : [0,\infty) \times I \to \mathbb{C}$ satisfies $m(\cdot,s) \in H^\infty(\Sigma_\phi \cup B_{\delta_0}(0))$ for all $s \in I$ and*

$$|z^\varepsilon m(z,s)| \le b(s), \quad z \in \Sigma_\phi, \quad s \in I.$$

Then $[\xi \mapsto m(|\xi|,s)] \in \mathcal{F}L^1(\mathbb{R}^n)$ and there is a constant $C > 0$ such that

$$\|\mathcal{F}^{-1} m(|\cdot|,s)\|_{L^1(\mathbb{R}^n)} \le C b(s), \quad s \in I. \tag{6}$$

2.3 Some known results for the Laplacian

Here we recall some well known results for the Laplace and Poisson operator in L^1- and L^∞-spacs which we will use in the sequel. Let $n \in \mathbb{N}$ and $|\nabla| = (-\Delta_{\mathbb{R}^n})^{1/2}$ the Poisson operator in $L^q(\mathbb{R}^n)$ for $1 \le q \le \infty$. Then we have the estimates

$$\||\nabla|^s \mathrm{e}^{-|\nabla|^{2m} t}\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \le \frac{C}{t^{s/2m}}, \quad t > 0,\ m \in \mathbb{N},\ s \ge 0,\ 1 \le q \le \infty, \tag{7}$$

$$\|\nabla^j \mathrm{e}^{-|\nabla|^{2m} t}\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \le \frac{C}{t^{j/2m}}, \quad t > 0,\ m \in \mathbb{N},\ j \in \mathbb{N}_0,\ 1 \le q \le \infty. \tag{8}$$

Let Δ_D and Δ_N be the Dirichlet Laplacian and the Neumann Laplacian on $L^q(\mathbb{R}^n_+)$ for $1 \le q \le \infty$ respectively, defined by their L^q-realizations

$$\Delta_D u := \Delta u, \qquad u \in D(\Delta_D) := \{v \in L^q(\mathbb{R}^n_+) : \Delta v \in L^q(\mathbb{R}^n_+), v|_{\partial\mathbb{R}^n_+} = 0\},$$
$$\Delta_N u := \Delta u, \qquad u \in D(\Delta_N) := \{v \in L^q(\mathbb{R}^n_+) : \Delta v \in L^q(\mathbb{R}^n_+), \partial_n v|_{\partial\mathbb{R}^n_+} = 0\}.$$

Then for each $\varphi_0 \in (0,\pi)$ and $1 \leq q \leq \infty$ there is a $C = C(\varphi_0) > 0$ such that the resolvent estimates

$$|\lambda|\|(\lambda - \Delta_D)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} + \sqrt{|\lambda|}\|\nabla(\lambda - \Delta_D)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} \leq C, \tag{9}$$

$$|\lambda|\|(\lambda - \Delta_N)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} + \sqrt{|\lambda|}\|\nabla(\lambda - \Delta_N)^{-1}\|_{\mathcal{L}(L^q(\mathbb{R}^n_+))} \leq C \tag{10}$$

are valid for all $\lambda \in \Sigma_{\pi-\varphi_0}$. Observe that the resolvents of Δ_D and Δ_N are given by

$$(\lambda - \Delta_D)^{-1} f = (\lambda - \Delta_{\mathbb{R}^n})^{-1} E^+ f \restriction_{\mathbb{R}^n_+}, \quad f \in L^q(\mathbb{R}^n_+), \tag{11}$$

and

$$(\lambda - \Delta_N)^{-1} f = (\lambda - \Delta_{\mathbb{R}^n})^{-1} E^- f \restriction_{\mathbb{R}^n_+}, \quad f \in L^q(\mathbb{R}^n_+), \tag{12}$$

respectively, where

$$(E^{\pm} f)(x', x_n) = \begin{cases} f(x', x_n) & : \quad x_n > 0, \\ \pm f(x', -x_n) & : \quad x_n < 0, \end{cases} \tag{13}$$

(see [Saa04]). Thus, the estimates (9) and (10) immediately follow from the estimates for the Laplacian in the whole space $\Delta_{\mathbb{R}^n}$. However, the resolvent estimates for $\Delta_{\mathbb{R}^n}$ as well as (7) and (8) can be obtained by an application of Proposition 2.2 and the Bernstein inequality, for instance (see [Saa03] for the details).

2.4 A solution formula for the Stokes resolvent problem

In the sequel we will use the same the notation as in [Saa04], i.e. $x' := (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ always denotes the first $n-1$ components of the variable $x \in \mathbb{R}^n_+$. Also for vector fields and operators we write $u = (u', u^n) = (u^1, \ldots, u^{n-1}, u^n)$ and $R = (R', R_n) = (R_1, \ldots, R_{n-1}, R_n)$.

We recall the representation for the Stokes flow u of the Stokes resolvent problem

$$(SRP)_{f,\lambda,\alpha} \begin{cases} (\lambda - \Delta) u + \nabla p &= f & \text{in} & \mathbb{R}^n_+, \\ \text{div } u &= 0 & \text{in} & \mathbb{R}^n_+, \\ T_\alpha u &= 0 & \text{in} & \mathbb{R}^{n-1}, \end{cases}$$

calculated in [Saa04]. Here $\lambda \in \Sigma_{\pi-\varphi_0}$ for some $\varphi_0 \in (0,\pi)$ and f satisfies the compatibility conditions $\text{div } f = 0$ and $f^n|_{\partial\mathbb{R}^n_+} = 0$. By applying partial Fourier transform, denoted by $\mathcal{F}$, i.e. Fourier transform with respect to x' in [Saa04, section 6] we stated a representation for u as

$$u' = (\lambda - \Delta_D)^{-1} f' - R' v^n + \mathrm{e}^{-\omega(|\nabla'|)\cdot} \phi', \tag{14}$$

$$u^n = (\lambda - \Delta_D)^{-1} f^n + v^n, \tag{15}$$

where $R' = \mathcal{F}^{-1} \frac{i\xi'}{|\xi'|} \mathcal{F}$ denotes the Riesz operator,

$$\hat{v}^n(\xi', x_n) := \mathcal{F} v^n(\xi', x_n) := M_{x_n,\lambda}(|\xi'|) \left[1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)\right] \hat{h}^n(\xi') \tag{16}$$

for $(\xi', x_n) \in \mathbb{R}^n_+$, and

$$\hat{\phi}'(\xi') = \frac{1}{\omega(|\xi'|) + \alpha} \left(\hat{h}'(\xi') + \alpha \frac{i\xi'}{|\xi'|} m_\lambda(|\xi'|) \hat{h}^n(\xi') \right), \quad \xi' \in \mathbb{R}^{n-1}. \tag{17}$$

Here, the functions ω, $M_{x_n,\lambda}$, and m_λ are given by

$$\omega(z) := \sqrt{\lambda + z^2}, \tag{18}$$

$$M_{x_n,\lambda}(z) := \frac{e^{-\omega(z)x_n} - e^{-zx_n}}{\omega(z) - z}, \tag{19}$$

$$m_\lambda(z) := \frac{1}{\omega(z) + z + \alpha}, \tag{20}$$

whereas $h = (h', h^n)$ is defined by

$$\hat{h}^n(\xi') := \int_0^\infty e^{-\omega(|\xi'|)s} \hat{f}^n(\xi', s) \mathrm{d}s, \qquad \xi' \in \mathbb{R}^{n-1},$$

$$\hat{h}'(\xi') := \int_0^\infty e^{-\omega(|\xi'|)s} \hat{f}'(\xi', s) \mathrm{d}s, \qquad \xi' \in \mathbb{R}^{n-1}.$$

Applying once integration by parts and using the compatibility conditions for f we easily observe the relation

$$\omega(|\xi'|)\hat{h}^n(\xi') = -i\xi' \cdot \hat{h}'(\xi'), \quad \xi' \in \mathbb{R}^{n-1}. \tag{21}$$

We also define

$$G_\lambda(z) := \frac{z}{\omega(z)}. \tag{22}$$

By elementary calculations we can obtain

Lemma 2.4 *Let $\varphi_0 \in (0, \pi/2)$ and $\varphi \in (0, \varphi_0/2)$. Then there is a constant $C = C(\varphi_0, \varphi)$, such that*

(a) $|\arg \omega(z)| \leq \frac{\pi - \varphi_0}{2}$,

(b) $\mathrm{Re}\, \omega(z) \geq C\sqrt{|\lambda|}$,

(c) $\mathrm{Re}\, \omega(z) \geq C|z|$

for all $\lambda \in \Sigma_{\pi-\varphi_0}$ and $z \in \Sigma_\varphi$.

Lemma 2.5 *Let $\sigma, \rho \geq 0$, $\varphi_0 \in (0, \pi/2)$ and $\varphi \in (0, \varphi_0/4)$. Then there are constants $C, \delta > 0$ such that*

(a) $|G_\lambda(z)| \leq C$,

(b) $|z^{1+\rho} M_{x_n,\lambda}(z)| \leq C \dfrac{e^{-\delta|z|x_n}}{x_n^\rho(1 + \sqrt{|\lambda|}x_n)}$,

(c) $|\omega(z) m_\lambda(z)| \leq C$,

(d) $|\alpha m_\lambda(z)| \leq C \dfrac{\alpha}{\sqrt{|\lambda|} + \alpha}$,

(e) $|z m_\lambda(z)| \leq C$,

(f) $|\omega(z)^\sigma \mathrm{e}^{-\omega(z)x_n}| \leq C \dfrac{\mathrm{e}^{-\delta\sqrt{|\lambda|}x_n}}{x_n^\sigma}$,

for all $z \in \Sigma_\varphi$, $x_n > 0$ and $\lambda \in \Sigma_{\pi-\varphi_0}$.

The above two lemmata will turn out to be the key-lemmata for estimating the Stokes flow u in $L^\infty(\mathbb{R}^n_+)$ and $L^1(\mathbb{R}^n_+)$. For a detailed proof of the lemmata we refer to [Saa04] and [Saa03].

3 The L^1-case

We start our discussion with the case of Neumann boundary conditions. Setting $\alpha = 0$ by (16) and (17) we see that

$$v^n = 0 \quad \text{and} \quad \hat{\phi}' = \frac{1}{\omega}\hat{h}'.$$

Inserting this in (14) and (15), the formula for $u = (u', u^n)$ simplifies considerably. Namely in this case we have

$$u' = (\lambda - \Delta_N)^{-1} f', \tag{23}$$
$$u^n = (\lambda - \Delta_D)^{-1} f^n, \tag{24}$$

since

$$\mathcal{F}[(\lambda - \Delta_D)^{-1} f^n](\xi', x_n) = \int_0^\infty k_-(\xi', x_n, s, \lambda)\hat{f}^n(\xi', s)\mathrm{d}s \quad (\xi', x_n) \in \mathbb{R}^n_+,$$

and

$$\mathcal{F}[(\lambda - \Delta_N)^{-1} f'](\xi', x_n) = \int_0^\infty k_+(\xi', x_n, s, \lambda)\hat{f}'(\xi', s)\mathrm{d}s \quad (\xi', x_n) \in \mathbb{R}^n_+,$$

with

$$k_\pm(\xi', x_n, s, \lambda) := \frac{e^{-\omega(|\xi'|)|x_n - s|} \pm e^{-\omega(|\xi'|)(x_n + s)}}{2\omega(|\xi'|)}$$

(see [Saa04],[Saa03] for the details). In view of (9) and (10) it immediately follows

Corollary 3.1 *Let $f \in L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$ for some $q \in (1, \infty)$. For $\alpha = 0$, i.e. in the case of Neumann boundary conditions, there is a constant $C = C(\varphi_0) > 0$, such that the solution u of the Stokes resolvent problem $(SRP)_{f,\lambda,\alpha}$ satisfies*

$$|\lambda|\|u\|_1 + \sqrt{|\lambda|}\|\nabla u\|_1 \leq C\|f\|_1, \quad \lambda \in \Sigma_{\pi - \varphi_0}.$$

Denote by $\Delta_{N,1}$ or $\Delta_{D,1}$ the Neumann Laplace or the Dirichlet Laplacian in $L^1(\mathbb{R}^n_+)$, respectively. The formulas (23) and (24) motivate

Definition 3.2 Let $L^1_\sigma(\mathbb{R}^n_+) := \overline{L^1(\mathbb{R}^n_+) \cap D}^{\|\cdot\|_1}$, where $D = \bigcap_{q \in (1,\infty)} L^q_\sigma(\mathbb{R}^n_+)$. We call

$$A_N := A_{N,1} := \begin{pmatrix} -\Delta_{N,1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\Delta_{N,1} & 0 \\ 0 & \dots & 0 & -\Delta_{D,1} \end{pmatrix}$$

defined in $L^1_\sigma(\mathbb{R}^n_+)$ with domain

$$D(A_N) := \big(D(\Delta_{N,1})^{n-1} \times D(\Delta_{D,1})\big) \cap L^1_\sigma(\mathbb{R}^n_+)$$

the *Stokes operator with Neumann boundary conditions* in $L^1_\sigma(\mathbb{R}^n_+)$.

Theorem 3.3 *The operator $-A_N$ is the generator of a bounded holomorphic C_0-semigroup on $L^1_\sigma(\mathbb{R}^n_+)$. Moreover, $\rho(A_N) = \mathbb{C} \setminus [0,\infty)$, and if $u_f(\lambda)$ is the unique solution of $(SRP)_{f,\lambda,0}$ for $f \in L^1_\sigma(\mathbb{R}^n_+) \cap D$, then $u_f(\lambda) = (\lambda + A_N)^{-1} f$ for $-\lambda \in \rho(A_N)$. The semigroup $(\mathrm{e}^{-tA_N})_{t\geq 0}$ also satisfies the gradient estimate*

$$\|\nabla \mathrm{e}^{-tA_N} f\|_1 \leq C t^{-1/2} \|f\|_1, \quad t > 0, \ f \in L^1_\sigma(\mathbb{R}^n_+). \tag{25}$$

Proof. Since

$$B := \begin{pmatrix} -\Delta_{N,1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\Delta_{N,1} & 0 \\ 0 & \dots & 0 & -\Delta_{D,1} \end{pmatrix}$$

with domain

$$D(B) := D(\Delta_{N,1})^{n-1} \times D(\Delta_{D,1})$$

generates a bounded holomorphic C_0-semigroup on $L^1(\mathbb{R}^n_+)$, it remains to verify the invariance of the subspace $L^1_\sigma(\mathbb{R}^n_+)$ under $(\lambda+B)^{-1}$ for proving the generation result for A_N. To this end, let $\lambda \in \mathbb{C}\setminus[0,\infty)$ and $f \in L^1(\mathbb{R}^n_+)\cap D$. According to [Saa04, Theorem 5.7] and representation (23) and (24) we have

$$(\lambda + B)^{-1} f = u_f(\lambda) = (\lambda + A_{N,q})^{-1} f \in L^q_\sigma(\mathbb{R}^n_+)$$

for all $q \in (1,\infty)$, where $A_{N,q} := A_{0,q}$ for $q \in (1,\infty)$. Hence the continuity of $(\lambda + B)^{-1}$ in $L^1(\mathbb{R}^n_+)$ implies $(\lambda + B)^{-1}(L^1_\sigma(\mathbb{R}^n_+)) \subseteq L^1_\sigma(\mathbb{R}^n_+)$.

Clearly, $(\lambda + A_N)^{-1} f = u_f(\lambda)$ for $f \in L^1(\mathbb{R}^n_+) \cap D$ and $-\lambda \in \rho(A_N) = \mathbb{C} \setminus [0,\infty)$, which implies that $(A_{N,q})_{q\in[1,\infty)}$ is a compatible family. Furthermore, for $f \in L^1_\sigma(\mathbb{R}^n_+)$ we may write

$$\mathrm{e}^{-tA_N} f = \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{-\lambda t} (\lambda - A_N)^{-1} f \mathrm{d}\lambda$$

with $\Gamma = \{t\mathrm{e}^{i\theta} : \infty > t > \varepsilon\} \cup \{\varepsilon \mathrm{e}^{it} : \theta \leq t \leq 2\pi - \theta\} \cup \{t\mathrm{e}^{-i\theta} : \varepsilon < t < \infty\}$ for $\varepsilon > 0$ and $\theta \in (0, \pi/2)$. This results by Corollary 3.1 and a density argument

$$\begin{aligned} \|\nabla \mathrm{e}^{-tA_N} f\|_1 &= \|\frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{-\lambda t} \nabla (\lambda - A_N)^{-1} f \mathrm{d}\lambda\|_1 \\ &\leq C \left(\varepsilon^{1/2} \int_\theta^{2\pi-\theta} \mathrm{e}^{\varepsilon t \cos\theta} \mathrm{d}s + \int_\varepsilon^\infty \frac{\mathrm{e}^{-st\cos\theta}}{\sqrt{s}} \mathrm{d}s \right) \|f\|_1 \end{aligned}$$

for all $f \in L^1_\sigma(\mathbb{R}^n_+)$. Letting $\varepsilon \to 0$ yields (25). □

Remark 3.4 (a) The operator $A_N = A_{N,1}$ can be regarded as the restriction of B to $L^1_\sigma(\mathbb{R}^n_+)$, i.e.

$$A_N = B \restriction_{L^1_\sigma(\mathbb{R}^n_+)}.$$

Actually we have $A_{N,q} = B \restriction_{L^q_\sigma(\mathbb{R}^n_+)}$ for all $1 \leq q < \infty$[2], since the arguments in the proof of Theorem 3.3 hold for all those q.

[2] Actually for all $1 \leq q \leq \infty$, where the case $q = \infty$ follows from the results in Section 4

(b) Note that $L^1_\sigma(\mathbb{R}^n_+)$ equals the space $\overline{L^1(\mathbb{R}^n_+)\cap L^q_\sigma(\mathbb{R}^n_+)}^{\|\cdot\|_1}$ for each $q\in(1,\infty)$. In fact, let $f\in \overline{L^1(\mathbb{R}^n_+)\cap L^q_\sigma(\mathbb{R}^n_+)}^{\|\cdot\|_1}$, $(v_k)\subseteq L^1(\mathbb{R}^n_+)\cap L^q_\sigma(\mathbb{R}^n_+)$ such that $v_k\to f$ in $L^1(\mathbb{R}^n_+)$, and consider the sequence

$$u_k := u_{k,j_k} := \exp(-\frac{1}{j_k}B)v_k,\quad k,j_k\in\mathbb{N},$$

with B defined as above. Choosing j_k for each $k\in\mathbb{N}$ and $\varepsilon>0$ in a way, such that $\|\exp(-\frac{1}{j_k}B)v_k - v_k\|_1<\varepsilon/2$, it easily follows that $u_k\to f$ in $L^1(\mathbb{R}^n_+)$. Moreover, since $\exp(-\frac{1}{j_k}B)g=\exp(-\frac{1}{j_k}A_{N,q})g$ for $g\in L^q_\sigma(\mathbb{R}^n_+)$, $1<q<\infty$, by the L^p-L^q-estimates in [Saa04, Corollary 5.8] for $\exp(-\frac{1}{j_k}A_{N,q})$ we see that $(u_k)\subseteq L^p_\sigma(\mathbb{R}^n_+)$ for each $p\in[q,\infty)$. In view of the inclusion $L^q_\sigma(\mathbb{R}^n_+)\cap L^1(\mathbb{R}^n_+)\subseteq L^p_\sigma(\mathbb{R}^n_+)$ for $p\in(1,q]$, this implies $u_k\in L^1(\mathbb{R}^n_+)\cap D$, hence $f\in L^1_\sigma(\mathbb{R}^n_+)$.

As we could see in the case of Neumann boundary conditions the crucial terms in the solution formula for $u=(u',u^n)$ vanish. That is not the case if $\alpha\in(0,\infty]$. Here the corresponding result to Corollary 3.1 reads as follows

Theorem 3.5 *Let $\alpha\in(0,\infty]$. For $\lambda\in\mathbb{R}_+$ there is an $f\in L^1(\mathbb{R}^n_+)\cap L^2_\sigma(\mathbb{R}^n_+)$ such that $u\notin L^1(\mathbb{R}^n_+)$, where (u,p) is the solution of problem $(SRP)_{f,\lambda,\alpha}$.*

Proof. For the proof we consider representation (15) of the n-th component of the Stokes flow u and denote the two addends of u^n by u^n_1 and u^n_2. Next we define a special function f satisfying the assumptions of the theorem such that the $L^1(\mathbb{R}^n_+)$-norm of the second addend of u^n is infinite. Since in virtue of (9) u^n_1 for each $f\in L^1_\sigma(\mathbb{R}^n_+)$ is an L^1-function we deduce

$$\|u\|_{L^1(\mathbb{R}^n_+)}\geq\|u^n\|_{L^1(\mathbb{R}^n_+)}=\infty$$

for this special f.

The construction of our counterexample is similar to [DHP01]. We also remind the reader to well known properties of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. It is defined as

$$\mathcal{H}^1(\mathbb{R}^n):=\{f\in L^1(\mathbb{R}^n): f^*\in L^1(\mathbb{R}^n)\},$$

where $f^*(x):=\sup_{t>0}|(G_t*f)(x))|$, $x\in\mathbb{R}^n$, and G_t denotes the heat kernel given by $G_t(x)=\frac{1}{(4\pi t)^{n/2}}\mathrm{e}^{-|x|^2/4t}$, $x\in\mathbb{R}^n$, $t>0$. Equipped with the norm $\|f\|_{\mathcal{H}^1}:=\|f^*\|_1$ the space $\mathcal{H}^1(\mathbb{R}^n)$ becomes a Banach space (see [Ste93] or [FS72] for more details). It is well known that a $L^1(\mathbb{R}^n)$-function f belongs to $\mathcal{H}^1(\mathbb{R}^n)$ if and only if its Riesz transforms R_jf belong to $L^1(\mathbb{R}^n)$ for $j=1,\dots,n$. This equivalent definition will be of crucial importance in what follows.

Now let $\lambda\in\mathbb{R}_+$. We define γ by

$$\gamma(|\xi'|):=1-(\omega(|\xi'|)+|\xi'|)m_\lambda(|\xi'|)=\frac{\alpha}{\omega(|\xi'|)+|\xi'|+\alpha}\tag{26}$$

and consider the function

$$\hat f^n(\xi',x_n):=\frac{i\xi_j}{\omega(|\xi'|)^4\gamma(|\xi'|)}x_n^2\mathrm{e}^{-\omega(|\xi'|)x_n}\hat g_r(\xi'),\quad(\xi',x_n)\in\mathbb{R}^n_+,$$

where $j \in \{1, \dots, n-1\}$ will be fixed later in the sequel and g_r is defined as

$$g_r := (\lambda - \Delta')^4 G_r \in \mathcal{S}(\mathbb{R}^{n-1}) \subseteq L^q(\mathbb{R}^{n-1}), \quad 1 \leq q \leq \infty,$$

for some fixed $r \in (0, \infty)$. Setting

$$f := (0, \dots, 0, f^j, 0, \dots, 0, f^n)$$

with

$$\hat{f}^j(\xi', x_n) := -\frac{1}{\omega(|\xi'|)^4 \gamma(|\xi'|)} \left(2x_n - \omega(|\xi'|) x_n^2\right) \mathrm{e}^{-\omega(|\xi'|)x_n} \hat{g}_r(\xi'), \quad (\xi', x_n) \in \mathbb{R}^n_+,$$

it follows $\operatorname{div} f = 0$ and $f \restriction_{\partial \mathbb{R}^n_+} = 0$. To see that f fulfills our assumptions it remains to show $f \in L^1(\mathbb{R}^n_+) \cap L^2(\mathbb{R}^n_+)$. To this end observe that

$$\begin{aligned} \frac{1}{\omega(z)^4 \gamma(z)} &= \frac{1}{\omega(z)^4} \frac{1}{\alpha} (\omega(z) + z + \alpha) \\ &= \frac{1}{\omega(z)^3} \left[\frac{1}{\alpha}(1 + G(z)) + \frac{1}{\omega(z)}\right], \quad z \in \Sigma_\varphi. \end{aligned}$$

Hence, for $\varepsilon \in (0, 1)$ we may conclude by Lemma 2.5 (a)

$$\begin{aligned} \left|\frac{z^\varepsilon}{\omega(z)^4 \gamma(z)}\right| &= \left| G(z)^\varepsilon \frac{1}{\omega(z)^{3-\varepsilon}} \left[\frac{1}{\alpha}(1 + G(z)) + \frac{1}{\omega(z)}\right]\right| \\ &\leq C(\lambda) \left|\frac{1}{\alpha}(1 + G(z)) + \frac{1}{\omega(z)}\right| \\ &\leq C(\lambda) \left(\frac{1}{\alpha} + \frac{1}{\sqrt{\lambda}}\right) \end{aligned}$$

for $z \in \Sigma_\varphi$. Hence $\frac{1}{\omega^4 \gamma}$ is a Fourier multiplier on $L^q(\mathbb{R}^{n-1})$, $1 \leq q \leq \infty$, by Proposition 2.2 for $\alpha \in (0, \infty]$. In virtue of Lemma 2.5 (f) this leads to

$$\begin{aligned} \|f^n\|_{L^q(\mathbb{R}^n_+)} &= \left(\int_0^\infty \|\mathcal{F}^{-1} \frac{1}{\omega(|\cdot|)^4 \gamma(|\cdot|)} x_n^2 \mathrm{e}^{-\omega(|\cdot|)x_n} \mathcal{F} \nabla' g_r\|^q_{L^q(\mathbb{R}^{n-1})} \mathrm{d}x_n\right)^{1/q} \\ &\leq C(\lambda) \left(\int_0^\infty x_n^{2q} \mathrm{e}^{-c_1 \sqrt{\lambda} q x_n} \mathrm{d}x_n\right)^{1/q} \|\nabla' g_r\|_{L^q(\mathbb{R}^{n-1})} \\ &\leq C(\lambda) \|\nabla' g_r\|_{L^q(\mathbb{R}^{n-1})} \end{aligned}$$

for $1 \leq q < \infty$. Analogously we see $f^j \in L^q(\mathbb{R}^n_+)$, which implies $f \in L^q(\mathbb{R}^n_+)$ for $1 \leq q < \infty$.

Now let us calculate the $L^1(\mathbb{R}^n_+)$-norm of u_2^n, the second addend of u^n. Note that

$$\begin{aligned} \left[1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)\right] \hat{h}^n(\xi') &= \gamma(|\xi'|) \int_0^\infty \mathrm{e}^{-\omega(|\xi'|)s} \hat{f}^n(\xi', s) \mathrm{d}s \\ &= \gamma(|\xi'|) \frac{i\xi_j \hat{g}_r(\xi')}{\omega(|\xi'|)^4 \gamma(|\xi'|)} \int_0^\infty s^2 \mathrm{e}^{-2\omega(|\xi'|)s} \mathrm{d}s \\ &= \frac{i\xi_j}{4\omega(|\xi'|)^7} \hat{g}_r(\xi') \end{aligned}$$

and

$$-\int_0^\infty M_{x_n,\lambda}(|\xi'|)\mathrm{d}x_n = \int_0^\infty \frac{\mathrm{e}^{-|\xi'|x_n} - \mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|) - |\xi'|}\mathrm{d}x_n = \frac{1}{\omega(|\xi'|)|\xi'|}.$$

Hence, by Lemma 2.5 and Corollary 2.3 the function

$$\xi' \mapsto \hat{u}_2^n(\xi', x_n) = -M_{x_n,\lambda}(|\xi'|)\frac{i\xi_j}{4\omega(|\xi'|)^7}\hat{g}_r(\xi'), \quad x_n > 0,$$

belongs to $L^1(\mathbb{R}_+; L^2(\mathbb{R}^{n-1}))$ and the function

$$\xi' \mapsto \int_0^\infty \hat{u}_2^n(\xi', x_n)\mathrm{d}x_n = \frac{1}{4\omega(|\xi'|)^8}\frac{i\xi_j}{|\xi'|}\hat{g}_r(\xi') = \frac{1}{4}\frac{i\xi_j}{|\xi'|}\mathrm{e}^{-|\xi'|^2 r}$$

belongs to $L^2(\mathbb{R}^{n-1})$. The continuity of $\mathcal{F}$ on $L^2(\mathbb{R}^{n-1})$ then implies

$$\int_0^\infty u_2^n(x', x_n)\mathrm{d}x_n = \mathcal{F}^{-1}\left(\int_0^\infty \hat{u}_2^n(\cdot, x_n)\mathrm{d}x_n\right)(x') = \frac{1}{4}R_j G_r(x'), \quad x' \in \mathbb{R}^{n-1}.$$

The definition of $\mathcal{H}^1(\mathbb{R}^{n-1})$ and a simple calculation show that $G_r \notin \mathcal{H}^1(\mathbb{R}^{n-1})$, since $G_r^* = \sup_{t>0}|G_t * G_r| \notin L^1(\mathbb{R}^{n-1})$. By the equivalent definition of $\mathcal{H}^1(\mathbb{R}^{n-1})$ we thus may choose $j \in \{1, \dots, n-1\}$, such that $R_j G_r \notin L^1(\mathbb{R}^{n-1})$. This leads to

$$\begin{aligned}\|u_2^n\|_{L^1(\mathbb{R}^n_+)} &\geq \int_{\mathbb{R}^{n-1}} |\int_0^\infty u_2^n(x', x_n)\mathrm{d}x_n|\mathrm{d}x' = \frac{1}{4}\int_{\mathbb{R}^{n-1}} |R_j G_r(x')|\mathrm{d}x' \\ &= \frac{1}{4}\|R_j G_r\|_{L^1(\mathbb{R}^{n-1})} = \infty\end{aligned}$$

and the claim is proved. □

From Theorem 3.3 and Theorem 3.5 we now obtain the somehow surprising result

Corollary 3.6 *The Stokes operator with Robin boundary conditions A_α is the the generator of a semigroup on $L^1_\sigma(\mathbb{R}^n_+)$ if and only if $\alpha = 0$, i.e. in the case of Neumann boundary conditions.*

Next we recall a scaling argument, which allows us to confine ourselves to the case $|\lambda| = 1$, $\lambda \in \Sigma_{\pi-\varphi_0}$. For the time being, we set

$$\omega(\lambda, \xi') := \omega(|\xi'|), \quad \hat{h}_{\hat{f}}(\lambda, \xi') := \hat{h}(\xi'),$$

and

$$m_\alpha(\lambda, \xi') := m_\lambda(|\xi'|), \quad \hat{\phi}_{\alpha,\hat{f}}(\lambda, \xi') := \hat{\phi}(\xi'), \quad \text{and} \quad \hat{u}_{\alpha,\hat{f}}(\lambda, \xi', x_n) := \hat{u}(\xi', x_n)$$

for $(\xi', x_n) \in \mathbb{R}^n_+$ and $\lambda \in \Sigma_{\pi-\varphi_0}$, where f is the right hand side of $(SRP)_{f,\lambda,\alpha}$ and ω, $h = (h', h^n)$, $\phi = (\phi', \phi^n)$, and u are defined as above. Further we define

$$\hat{f}_D(\xi', x_n) := \hat{f}(|\lambda|^{1/2}\xi', |\lambda|^{-1/2}x_n), \quad \lambda \in \Sigma_{\pi-\varphi_0},$$

for a function $f \in L^q(\mathbb{R}^n_+)$. We may check that

$$\omega(\lambda, \xi') = |\lambda|^{1/2}\omega\left(\frac{\lambda}{|\lambda|}, |\lambda|^{-1/2}\xi'\right)$$

and

$$\begin{aligned}\hat{h}_{\hat{f}}(\lambda,\xi') &= \int_0^\infty \mathrm{e}^{-|\lambda|^{1/2}\omega\left(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi'\right)s}\hat{f}(\xi',s)\mathrm{d}s \\ &= |\lambda|^{-1/2}\int_0^\infty \mathrm{e}^{-\omega\left(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi'\right)\tilde{s}}\hat{f}(\xi',|\lambda|^{-1/2}\tilde{s})\mathrm{d}\tilde{s} \\ &= |\lambda|^{-1/2}\int_0^\infty \mathrm{e}^{-\omega\left(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi'\right)\tilde{s}}\hat{f}_D(|\lambda|^{-1/2}\xi',\tilde{s})\mathrm{d}\tilde{s} \\ &= |\lambda|^{-1/2}\hat{h}_{\hat{f}_D}\left(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi'\right),\end{aligned}$$

as well as

$$m_\alpha(\lambda,\xi') = |\lambda|^{-1/2}m_\beta(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi')$$

with $\beta = \alpha|\lambda|^{-1/2}$, which implies

$$\hat{\phi}_{\alpha,\hat{f}}(\lambda,\xi') = |\lambda|^{-1}\hat{\phi}_{\beta,\hat{f}_D}\left(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi'\right)$$

and

$$\hat{u}_{\alpha,\hat{f}}(\lambda,\xi',x_n) = |\lambda|^{-1}\hat{u}_{\beta,\hat{f}_D}\left(\frac{\lambda}{|\lambda|},|\lambda|^{-1/2}\xi',|\lambda|^{1/2}x_n\right).$$

So, since $\mathbb{R}^{n-1}$ and $\mathbb{R}_+$ are invariant under dilations, by the change of coordinates

$$\xi' \to |\lambda|^{1/2}\xi', \qquad x' \to |\lambda|^{-1/2}x', \qquad x^n \to |\lambda|^{-1/2}x^n,$$

we may suppose $|\lambda| = 1$, $\lambda \in \Sigma_{\pi-\varphi_0}$. Let us mention that the dilated function u_{β,f_D} now satisfies the boundary condition $\beta u'_{\beta,f_D} - \partial_n u'_{\beta,f_D} = 0$ on $\mathbb{R}^{n-1}$, i.e. actually we reduced $(SRP)_{f,\lambda,\alpha}$ with arbitrary $\lambda \in \Sigma_{\pi-\varphi_0}$ to $(SRP)_{f_D,\mu,\beta}$ with $|\mu| = 1$ and $\beta = \alpha|\lambda|^{-1/2}$. Nevertheless, this causes no further problems since all estimates we prove in the sequel do not depend on the respective regarded boundary parameter, therefore they also do not depend on λ if for instance we have $\beta = \alpha|\lambda|^{-1/2}$.

In the next lemma we provide estimates for the function v which was defined by

$$\hat{v}(\xi',x_n) := M_{x_n,\lambda}(|\xi'|)\left[1-(\omega(|\xi'|)+|\xi'|)m_\lambda(|\xi'|)\right]\hat{h}(\xi'), \quad (\xi',x_n)\in\mathbb{R}^n_+. \tag{27}$$

Lemma 3.7 *Let $p \in \{1,\infty\}$, $\varphi_0 \in (0,\pi)$, and $\lambda \in \Sigma_{\pi-\varphi_0}$ with $|\lambda| = 1$. Further let $f \in L^p(\mathbb{R}^n_+)$ satisfying* div $f = 0$ *and $f^n \restriction_{\mathbb{R}^{n-1}} = 0$ in the sense, such that relation (21) for h stays valid* [3].
For $\delta \in [0,1)$ there are constants $C = C(\delta) > 0$ and $\sigma = \sigma(\delta) \in (0,1)$ such that

(i) $\||\nabla'|^\delta v(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \le C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad x_n > 0,$

(ii) $\||\nabla'|^{1+\delta} v(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} \le \dfrac{C}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,$

(iii) $\||\nabla'|^\delta R_j v^n(\cdot,x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \le C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad x_n > 0,\ j = 1,\dots,n-1,$

[3] If $p = 1$, we assume $f \in L^1(\mathbb{R}^n_+) \cap L^2_\sigma(\mathbb{R}^n_+)$. The sense in the case $p = \infty$ will be given in Section 4.

(iv) $\||\nabla'|^{1+\delta} R_j v^n(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)} \|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,\ j = 1, \ldots, n-1,$

(v) $\||\nabla'|^{\delta} \partial_k v^n(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)} \|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,\ k = 1, \ldots, n,$

(vi) $\||\nabla'|^{\delta} R_j \partial_k v^n(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} \leq \dfrac{C}{x_n^{\sigma/p}(1+x_n)} \|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0,\ j = 1, \ldots, n-1,$
$k = 1, \ldots, n.$

Proof. We start with (i). Note that $M_{x_n,\lambda}$ can be represented as

$$M_{x_n,\lambda}(z) = -\int_0^{x_n} e^{-\omega(z)(x_n - s)} e^{-zs} \mathrm{d}s.$$

Hence, for $\delta \in [0,1)$ and $\varepsilon = \varepsilon(\delta) \in (0, 1-\delta)$ we have

$$\begin{aligned} |\xi'|^{\delta} \hat{v}(\xi', x_n) &= -\int_0^{\infty} \int_0^{x_n} \frac{|\xi'|^{\delta}}{\omega(|\xi'|)^{\delta+\varepsilon}} \mathrm{e}^{-\omega(|\xi'|)(x_n-\rho)} \mathrm{e}^{-|\xi'|\rho} \\ &\quad \cdot \left[1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)\right] \omega(|\xi'|)^{\delta+\varepsilon} \mathrm{e}^{-\omega(|\xi'|)s} \hat{f}(\xi', s) \mathrm{d}\rho \mathrm{d}s. \end{aligned}$$

for $(\xi', x_n) \in \mathbb{R}^n_+$. Lemma 2.5 implies for the kernel of the integral

$$|z^{\varepsilon} \frac{z^{\delta}}{\omega(z)^{\delta+\varepsilon}} \mathrm{e}^{-\omega(z)(x_n-\rho)} \mathrm{e}^{-z\rho} \left[1 - (\omega(z) + z) m_\lambda(z)\right] \omega(z)^{\delta+\varepsilon} \mathrm{e}^{-\omega(z)s}| \leq C \mathrm{e}^{-c_1(x_n-\rho)} \frac{\mathrm{e}^{-c_1 s}}{s^{\delta+\varepsilon}}$$

for all $z \in \Sigma_\varphi$, where $\varphi \in (0, \varphi_0/4)$. Hence we may conclude by Corollary 2.3

$$\begin{aligned} \||\nabla'|^{\delta} v(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} &\leq C \int_0^{x_n} \mathrm{e}^{-c_1(x_n-\rho)} \mathrm{d}\rho \int_0^{\infty} \frac{\mathrm{e}^{-c_1 s}}{s^{\delta+\varepsilon}} \|f(\cdot, s)\|_{L^\infty(\mathbb{R}^{n-1})} \mathrm{d}s \\ &\leq C(\delta) \|f\|_{L^\infty(\mathbb{R}^n_+)} \end{aligned}$$

for all $x_n > 0$.

Let p' be the Hölder conjugated exponent to $p \in \{1, \infty\}$. After Fourier transform the term in (ii) can be expressed as

$$\begin{aligned} |\xi'|^{1+\delta} \hat{v}(\xi', x_n) &= \int_0^{\infty} \frac{|\xi'|^{\delta}}{\omega(|\xi'|)^{\frac{\delta+\varepsilon}{p'}}} |\xi'| M_{x_n,\lambda}(|\xi'|) \\ &\quad \cdot \left[1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)\right] \omega(|\xi'|)^{\frac{\delta+\varepsilon}{p'}} \mathrm{e}^{-\omega(|\xi'|)s} \hat{f}(\xi', s) \mathrm{d}s. \end{aligned} \tag{28}$$

for $(\xi', x_n) \in \mathbb{R}^n_+$. Again Lemma 2.5 implies

$$\left| z^{\varepsilon} \frac{z^{\delta}}{\omega(z)^{\frac{\delta+\varepsilon}{p'}}} z M_{x_n,\lambda}(z) \left[1 - (\omega(z) + z) m_\lambda(z)\right] \omega(z)^{\frac{\delta+\varepsilon}{p'}} \mathrm{e}^{-\omega(z)s} \right| \leq \frac{C}{x_n^{(\delta+\varepsilon)/p}(1+x_n)} \frac{\mathrm{e}^{-c_1 s}}{s^{(\delta+\varepsilon)/p'}}$$

for all $z \in \Sigma_\varphi$. Here we applied Lemma 2.5 (b) with $\rho = \delta + \varepsilon$ if $p = 1$ (observe that $\omega(z)^{\frac{\delta+\varepsilon}{p'}} \equiv 1$ in this case!), and if $p = \infty$ we applied Lemma 2.5 (a) on $\frac{z^{\delta+\varepsilon}}{\omega(z)^{\delta+\varepsilon}} = G_\lambda(z)^{\delta+\varepsilon}$

and Lemma 2.5 (b) with $\rho = 0$. Corollary 2.3 results

$$\begin{aligned} \||\nabla'|^{\delta} v(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} &\leq \frac{C(\delta)}{x_n^{\sigma/p}(1+x_n)} \int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\sigma/p'}} \|f(\cdot, s)\|_{L^p(\mathbb{R}^{n-1})} \mathrm{d}s \\ &\leq \frac{C(\delta)}{x_n^{\sigma/p}(1+x_n)} \|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n > 0, \end{aligned}$$

where $\sigma := \delta + \varepsilon \in (0,1)$, and (ii) is proved.

Let $\delta_1 \in [0,1)$. To see (iii) observe that $\omega \hat{v}^n = -i\xi' \cdot \hat{v}'$ by (21). Inserting $1 = |\xi'|^4 \int_0^\infty \mathrm{e}^{-|\xi'|^4 r} \mathrm{d}r$ in the formula of $\mathcal{F}|\nabla'|^{\delta_1} R_j v^n$ we obtain for $\gamma \in (0, 1-\delta_1)$

$$\begin{aligned} |\xi'|^{\delta_1} \frac{i\xi_j}{|\xi'|} \hat{v}^n(\xi', x_n) &= i\xi_j |\xi'|^{3+\delta_1} \int_0^\infty \mathrm{e}^{-|\xi'|^4 r} \hat{v}^n(\xi', x_n) \mathrm{d}r \\ &= \int_0^1 i\xi_j |\xi'|^{3-\gamma} \mathrm{e}^{-|\xi'|^4 r} |\xi'|^{\delta_1+\gamma} \hat{v}^n(\xi', x_n) \mathrm{d}r \\ &\quad - \sum_{k=1}^{n-1} \int_1^\infty i\xi_j i\xi_k |\xi'|^3 \mathrm{e}^{-|\xi'|^4 r} \frac{|\xi'|^{\delta_1}}{\omega(|\xi'|)} \hat{v}^k(\xi', x_n) \mathrm{d}r. \end{aligned}$$

Now (7) and (8) (with $m = 2$) imply

$$\begin{aligned} \||\nabla'|^{\delta_1} R_j v^n(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} &\leq C \int_0^1 r^{\frac{\gamma}{4}-1} \||\nabla'|^{\delta_1+\gamma} v^n(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \mathrm{d}r \\ &\quad + \sum_{k=1}^{n-1} \int_1^\infty r^{-5/4} \|\omega(|\nabla'|)^{-1} |\nabla'|^{\delta_1} v^k(\cdot, x_n)\|_{L^\infty(\mathbb{R}^{n-1})} \mathrm{d}r \\ &\leq C(\delta) \|f\|_{L^\infty(\mathbb{R}^n_+)} \end{aligned}$$

for $x_n > 0$, where we used the boundedness of the operator $\omega(|\nabla'|)^{-1}$ and applied (i) once with $\delta = \delta_1 + \gamma$ and once with $\delta = \delta_1$.

We will not carry out the proof of (iv), since the term in (iv) can be reduced to the one in (ii) exactly in the same way as (iii) is reduced to (i) .

For $k = 1, \ldots, n-1$ assertion (v) is an immediate consequence of (iv), because $|\nabla'|^{\delta} \partial_k v^n = |\nabla'|^{1+\delta} R_k v^n$. If $k = n$, note that

$$\partial_n M_{x_n,\lambda}(|\xi'|) = -|\xi'| M_{x_n,\lambda}(|\xi'|) - \mathrm{e}^{-\omega(|\xi'|)x_n}, \quad (\xi', x_n) \in \mathbb{R}^n_+.$$

Hence, with $w(\cdot, x_n) := |\nabla'|^{\delta} \mathrm{e}^{-\omega(|\nabla'|)x_n} \left[1 - (\omega(|\nabla'|) + |\nabla'|) m_\lambda(|\nabla'|)\right] h^n$ we get

$$|\nabla'|^{\delta} \partial_n v^n(\cdot, x_n) = -w(\cdot, x_n) - |\nabla'|^{1+\delta} v^n(\cdot, x_n), \ x_n > 0. \tag{29}$$

Similarly to (28) we write the first addend w in the form

$$\begin{aligned} \hat{w}(\xi', x_n) &= \int_0^\infty \frac{|\xi'|^{\delta}}{\omega(|\xi'|)^{\delta+\varepsilon}} \omega(|\xi'|)^{\frac{\delta+\varepsilon}{p}} \mathrm{e}^{-\omega(|\xi'|)x_n/2} \mathrm{e}^{-\omega(|\xi'|)x_n/2} \\ &\quad \cdot \left[1 - (\omega(|\xi'|) + |\xi'|) m_\lambda(|\xi'|)\right] \omega(|\xi'|)^{\frac{\delta+\varepsilon}{p'}} \mathrm{e}^{-\omega(|\xi'|)s} \hat{f}^n(\xi', s) \mathrm{d}s \end{aligned}$$

and obtain analogously as in the proof of (ii)

$$\begin{aligned}\|w(\cdot,x_n)\|_{L^p(\mathbb{R}^{n-1})} &\leq \frac{C(\delta)\mathrm{e}^{-c_1 x_n/2}}{x_n^{\sigma/p}}\int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\sigma/p'}}\|f^n(\cdot,s)\|_{L^p(\mathbb{R}^{n-1})}\mathrm{d}s \\ &\leq \frac{C(\delta)}{x_n^{\sigma/p}(1+x_n)}\|f\|_{L^p(\mathbb{R}^n_+)}, \quad x_n>0,\end{aligned}$$

where $\sigma=\delta+\varepsilon$. Relation (v) now follows, if we apply (ii) on the second addend of (29).

We shall also omit the proof of (vi) since the same method for proving (iii) applies to (vi). □

Although the Stokes flow u of $(SRP)_{f,\lambda,\alpha}$ does not belong to $L^1(\mathbb{R}^n_+)$, it is known that in the case of Dirichlet boundary conditions the gradient of the velocity of the instationary Stokes equations does, according to a result of Giga, Matsui, and Shimizu (see [GMS99]). As a consequence of the theorem below we will see that this result generalizes to the case of Robin boundary conditions.

Theorem 3.8 *Let $\alpha\in[0,\infty]$ and $\varphi_0\in(0,\pi)$. Then there exists a constant $C=C(\varphi_0)>0$ such that the Stokes flow u of $(SRP)_{f,\lambda,\alpha}$ satisfies*

$$\|\nabla u\|_1 \leq \frac{C}{\sqrt{|\lambda|}}\|f\|_1 \tag{30}$$

for $\lambda\in\Sigma_{\pi-\varphi_0}$ and $f\in L^1_\sigma(\mathbb{R}^n_+)$.

Proof. By definition we may assume $f\in L^1(\mathbb{R}^n_+)\cap D$. Again we consider representation (14) and (15) for the components of the Stokes flow u. The estimate for $\nabla(\lambda-\Delta_D)^{-1}f$ is a consequence of (9). Due to Lemma 3.7 (v) and (vi) with $\delta=0$ we further have

$$\|\nabla v\|_{L^1(\mathbb{R}^n_+)} \leq C\int_0^\infty \frac{1}{x_n^\sigma(1+x_n)}\|f\|_{L^1(\mathbb{R}^n_+)}\mathrm{d}x_n \leq C\|f\|_{L^1(\mathbb{R}^n_+)}$$

and

$$\|\nabla R'v\|_{L^1(\mathbb{R}^n_+)} \leq C\|f\|_{L^1(\mathbb{R}^n_+)}$$

for $\lambda\in\Sigma_{\pi-\varphi_0}$, $|\lambda|=1$. So it remains to prove the corresponding estimate for the term $u_2'=\mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi'$.

Let us remark that if $\alpha=\infty$, then m_λ vanishes, which implies that the term $\mathrm{e}^{-\omega(|\nabla'|)(\cdot)}\phi'$ does not occur in the formula for u. Thus, by rescaling, the claim follows in the case of Dirichlet boundary conditions, which follows as mentioned also from the results in [GMS99] (see also [SS01]).

If $\alpha\in[0,\infty)$, then in view of (17) $i\xi_j\hat{u}_2'$ is represented as

$$i\xi_j\hat{u}_2'(\xi',x_n)=\frac{i\xi_j\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)+\alpha}\left(\hat{h}'(\xi')+\alpha\frac{i\xi'}{|\xi'|}m_\lambda(|\xi'|)\hat{h}^n(\xi')\right), \quad \xi'\in\mathbb{R}^{n-1},\ x_n>0,$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| = 1$, and $j = 1, \dots, n-1$. To evaluate this term we use the same methods as in the proof of Lemma 3.7. By inserting $1 = |\xi'|^4 \int_0^\infty \mathrm{e}^{-|\xi'|^4 r} \mathrm{d}r$ we obtain for the first addend of $i\xi_j \hat{u}_2'$

$$\begin{aligned}\frac{i\xi_j \mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|)+\alpha}\hat{h}'(\xi') &= \int_0^\infty \int_0^1 |\xi'|^{5/2} i\xi_j \mathrm{e}^{-|\xi'|^4 r} \frac{|\xi'|^{3/2}}{\omega(|\xi'|)+\alpha} \mathrm{e}^{-\omega(|\xi'|)(x_n+s)} \hat{f}'(\xi', s) \mathrm{d}r\mathrm{d}s \\ &\quad + \int_0^\infty \int_1^\infty |\xi'|^4 i\xi_j \mathrm{e}^{-|\xi'|^4 r} \frac{1}{\omega(|\xi'|)+\alpha} \mathrm{e}^{-\omega(|\xi'|)(x_n+s)} \hat{f}'(\xi', s) \mathrm{d}r\mathrm{d}s.\end{aligned}$$

Let $\varphi \in (0, \varphi_0/4)$. According to Lemma 2.5 we get

$$\begin{aligned}&\left| z^{1/4} \frac{z^{3/2}}{(\omega(z)+\alpha)} \mathrm{e}^{-\omega(z)(x_n+s)} \right| = \\ &= \left| \left(z^{1/4} \frac{z^{3/2}}{(\omega(z)+\alpha)\omega(z)^{3/4}} \right) \left(\omega(z)^{3/4} \mathrm{e}^{-\omega(z)x_n} \right) \mathrm{e}^{-\omega(z)s} \right| \\ &\leq \frac{C}{(1+\alpha)x_n^{3/4}} \mathrm{e}^{-c_1(x_n+s)}, \quad x_n, s > 0,\ z \in \Sigma_\varphi,\end{aligned}$$

where we estimated the terms in brackets separately, and

$$\left| \frac{z^{3/4}}{(\omega(z)+\alpha)} \mathrm{e}^{-\omega(z)(x_n+s)} \right| \leq \frac{C}{(1+\alpha)x_n^{3/4}} \mathrm{e}^{-c_1(x_n+s)}, \quad x_n, s > 0,\ z \in \Sigma_\varphi.$$

Then Corollary 2.3 in combination with (7) and (8) leads to

$$\begin{aligned}&\|\partial_j (\omega(|\nabla'|)+\alpha)^{-1} \mathrm{e}^{-\omega(|\nabla'|)x_n} h'\|_{L^1(\mathbb{R}^{n-1})} \leq \\ &\leq \frac{C}{(1+\alpha)} \int_0^\infty \int_0^1 r^{-7/8} x_n^{-3/4} \mathrm{e}^{-c_1(x_n+s)} \|f'(\cdot, s)\|_{L^1(\mathbb{R}^{n-1})} \mathrm{d}r\mathrm{d}s \\ &\quad + \frac{C}{(1+\alpha)} \int_0^\infty \int_1^\infty r^{-5/4} \mathrm{e}^{-c_1(x_n+s)} \|f'(\cdot, s)\|_{L^1(\mathbb{R}^{n-1})} \mathrm{d}r\mathrm{d}s \\ &\leq C(x_n^{-3/4}+1) \mathrm{e}^{-c_1 x_n} \|f\|_{L^1(\mathbb{R}^n_+)}, \quad x_n > 0,\end{aligned}$$

for $j = 1, \dots, n-1$. Consequently we get for the first addend of $\nabla' u_2'$

$$\|\nabla' (\omega(|\nabla'|)+\alpha)^{-1} \mathrm{e}^{-\omega(|\nabla'|)\cdot} h'\|_{L^1(\mathbb{R}^n_+)} \leq C\|f\|_{L^1(\mathbb{R}^n_+)}$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| = 1$. Completely analogous calculations lead to the same estimate for the second addend of $\nabla' u_2'$, i.e.

$$\|\nabla' \mathrm{e}^{-\omega(|\nabla'|)\cdot} (\omega(|\nabla'|)+\alpha)^{-1} \alpha R' m_\lambda h^n\|_{L^1(\mathbb{R}^n_+)} \leq C\|f\|_{L^1(\mathbb{R}^n_+)}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ |\lambda| = 1.$$

We also shall omit the details of the estimate for the n-th order derivative $\partial_n u_2'$ since this is very similar to the one of $\nabla' u_2'$. Summarizing, we have

$$\|\nabla u\|_1 \leq C\|f\|_1, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ |\lambda| = 1, \tag{31}$$

Now, if $\lambda \in \Sigma_{\pi-\varphi_0}$ the claim is proved for the scaled flow $u_{\beta, f_D}(\frac{\lambda}{|\lambda|}, \cdot)$, with $\beta = \alpha|\lambda|^{-1/2}$. Because the constant C in (31) does not depend on β (therefore also not on λ) rescaling yields the assertion. □

Remark 3.9 The proof of the above theorem shows, that by our methods we even can obtain more regularity for u. Indeed, for $\delta_1 \in (0,2), \delta_2 \in [0,1)$ we can prove an estimate as

$$|\lambda|^{(2-\delta_1)/2}\||\nabla'|^{\delta_1}u\|_1 + |\lambda|^{(1-\delta_2)/2}\||\nabla'|^{\delta_2}\nabla u\|_1 \leq C(\delta_1,\delta_2)\|f\|_1$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$ and $f \in L^1_\sigma(\mathbb{R}^n_+)$. But, of course the constant $C(\delta_1,\delta_2)$ blows up if $\delta_1 \to 0$, $\delta_1 \to 2$, or $\delta_2 \to 1$.

Corollary 3.10 *Let $\alpha \in [0,\infty]$. The Stokes semigroup $(\mathrm{e}^{-tA_\alpha})_{t\geq 0}$ on $L^q_\sigma(\mathbb{R}^n_+)$, $1 < q < \infty$, satisfies the estimate*

$$\|\nabla \mathrm{e}^{-tA_\alpha}f\|_1 \leq Ct^{-1/2}\|f\|_1, \quad t > 0,\ f \in L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+).$$

Proof. We can argue analogously as in the proof of (25). For $f \in L^1(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$ and $\lambda \in \rho(A_\alpha)$ the Stokes flow u is given by $u_f(-\lambda) = -(\lambda - A_\alpha)^{-1}f$. Hence we have $\mathrm{e}^{-tA_\alpha}f = -\int_\Gamma \mathrm{e}^{-\lambda t}u_f(-\lambda)\mathrm{d}\lambda$, and the assertion follows from estimate (30). □

4 The Stokes operator in solenoidal L^∞-spaces

In order to prove our generation results for the Stokes operator with Robin boundary conditions on spaces of bounded functions we define appropriate subspaces of solenoidal vector fields. It is well known (see e.g. [Gal98]) that we have the following characterizations for $L^q_\sigma(\mathbb{R}^n_+)$ if $1 < q < \infty$:

$$\begin{aligned} L^q_\sigma(\mathbb{R}^n_+) &= \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_q} \\ &= \{u \in L^q(\mathbb{R}^n_+) :\ \operatorname{div} u = 0,\ \nu \cdot u \restriction_{\partial\mathbb{R}^n_+} = 0\} \\ &= \{u \in L^q(\mathbb{R}^n_+) :\ (u, \nabla p) = 0,\ p \in \widehat{W}^{1,q'}(\mathbb{R}^n_+)\}. \end{aligned}$$

For the author the third one seems to be an appropriate characterization for defining $L^\infty_\sigma(\mathbb{R}^n_+)$, since the first one can only be used to define a subspace of continuous functions, whereas the problem with the third one is to give a sense to the trace $u^n \restriction_{\mathbb{R}^{n-1}}$. Thus we set

$$L^\infty_\sigma(\mathbb{R}^n_+) := \{u \in L^\infty(\mathbb{R}^n_+) :\ (u, \nabla p) = 0,\ p \in \widehat{W}^{1,1}(\mathbb{R}^n_+)\}.$$

Theorem 4.1 *Let $\alpha \in [0,\infty]$ and $\varphi_0 \in (0,\pi)$. For each $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ and $\lambda \in \Sigma_{\pi-\varphi_0}$ there is a unique solution (u,p) of $(SRP)_{f,\lambda,\alpha}$ such that $u = u_f(\lambda) \in C^1_b(\mathbb{R}^n_+)$. Moreover, there is a constant $C = C(\varphi_0) > 0$ such that u satisfies*

$$|\lambda|\|u\|_\infty + \sqrt{|\lambda|}\|\nabla u\|_\infty \leq C\|f\|_\infty, \quad f \in L^\infty_\sigma(\mathbb{R}^n_+),\ \lambda \in \Sigma_{\pi-\varphi_0}. \tag{32}$$

Proof. For the proof we also intend to use representation (14), (15) for u. Therefore we first have to show the validity of that formula for all $f \in L^\infty_\sigma(\mathbb{R}^n_+)$. Since $\operatorname{div} f = 0$ is still valid in the sense of distributions, we realize that the only crucial step in the derivation of the formula in [Saa04, section 4] is the calculation of relation (21) for the function h. So, we have to justify $\omega(|\nabla'|)h^n = -\nabla' \cdot h'$ (in $\mathcal{S}'(\mathbb{R}^{n-1})$) for $f \in L^\infty_\sigma(\mathbb{R}^n_+)$. To this end, let $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$

and observe that the function $s \mapsto \mathrm{e}^{-\omega(|\nabla'|)s}\varphi$ belongs to $W^{1,1}(\mathbb{R}^n_+)$ in view of Corollary 2.3, since $\nabla'\varphi, \omega(|\nabla'|)\varphi \in S(\mathbb{R}^{n-1})$ and according to Lemma 2.5 (a) and (f) we have

$$|z^{1/2}\mathrm{e}^{-\omega(z)s}| = |\frac{z^{1/2}}{\omega(z)^{1/2}}\omega(z)^{1/2}\mathrm{e}^{-\omega(z)s}| \leq C\frac{\mathrm{e}^{-c_1 s}}{s^{1/2}}$$

for all $s > 0$, $z \in \Sigma_\varphi$ and some $\varphi \in (0, \varphi_0/4)$. Thus we may conclude

$$\begin{aligned}
&\langle \omega(|\nabla'|)h^n + \nabla' \cdot h', \varphi\rangle = \\
= \quad & \int_0^\infty \langle \omega(|\nabla'|)\mathrm{e}^{-\omega(|\nabla'|)s} f^n(\cdot, s), \varphi\rangle \mathrm{d}s + \int_0^\infty \langle \nabla' \cdot \mathrm{e}^{-\omega(|\nabla'|)s} f'(\cdot, s), \varphi\rangle \mathrm{d}s \\
= \quad & -\int_0^\infty (f^n(\cdot, s), \partial_s \mathrm{e}^{-\omega(|\nabla'|)s}\varphi)_{\mathbb{R}^{n-1}} \mathrm{d}s - \int_0^\infty (f'(\cdot, s), \nabla' \mathrm{e}^{-\omega(|\nabla'|)s}\varphi)_{\mathbb{R}^{n-1}} \mathrm{d}s \\
= \quad & -(f, \nabla \mathrm{e}^{\omega(|\nabla'|)s}\varphi)_{\mathbb{R}^n_+} = 0
\end{aligned}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$ and $f \in L^\infty_\sigma(\mathbb{R}^n_+)$, which allows us to apply formulas (14) and (15) for the proof of Theorem 4.1.

Here it also suffices to focus on the second addend u_2' in (14) since (9) implies (32) for the term $(\lambda - \Delta_D)^{-1} f$, whereas for v^n and $R'v^n$ estimate (32) follows from Lemma 3.7 (i), (iii), (v), and (vi) by setting $\delta = 0$ and $p = \infty$. To obtain (32) for u_2' we will be brief in details, since the calculations are very similar to that one in the proof of Theorem 3.8.

Observe that at this point the proof is finished in the case of Dirchlet boundary conditions, because u_2' vanishes for $\alpha = \infty$. As mentioned in the introduction the inequality $|\lambda|\|u\|_\infty \leq \|f\|_\infty$ is already proved in [DHP01]. However, their proof is based on kernel estimates and their result does not include higher regularity. Estimates for Dirichlet boundary conditions and first order derivatives are given in [Shi99] and [SS01] based on the formula of Ukai [Uka87].

By (17) we have, if $\alpha \in [0, \infty)$ and $f \in L^\infty_\sigma(\mathbb{R}^n_+)$,

$$\hat{u}_2'(\xi', x_n) = \frac{\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|) + \alpha}\left(\hat{h}'(\xi') + \alpha\frac{i\xi'}{|\xi'|}m_\lambda(|\xi'|)\hat{h}^n(\xi')\right), \quad \xi' \in \mathbb{R}^{n-1},\ x_n > 0,$$

for $\lambda \in \Sigma_{\pi-\varphi_0}$, $|\lambda| = 1$. For the first term of this formula we get according to Lemma 2.5 and Corollary 2.3

$$\begin{aligned}
&\|\mathrm{e}^{-\omega(|\nabla'|)\cdot}(\omega(|\nabla'|) + \alpha)^{-1} h'\|_{L^\infty(\mathbb{R}^n_+)} \leq \\
\leq \quad & \sup_{x_n > 0} \int_0^\infty \|\mathrm{e}^{-\omega(|\nabla'|)(x_n+s)}(\omega(|\nabla'|) + \alpha)^{-1} f'(\cdot, s)\|_{L^\infty(\mathbb{R}^{n-1})} \mathrm{d}s \\
\leq \quad & \frac{C}{1+\alpha} \sup_{x_n > 0} \mathrm{e}^{-c_1 x_n} \int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{\sqrt{s}} \|f'(\cdot, s)\|_{L^\infty(\mathbb{R}^{n-1})} \mathrm{d}s \\
\leq \quad & C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad \lambda \in \Sigma_{\pi-\varphi_0},\ |\lambda| = 1.
\end{aligned}$$

The second addend we write with the help of (21) as

$$\begin{aligned}
&\frac{\mathrm{e}^{-\omega(|\xi'|)x_n}}{\omega(|\xi'|) + \alpha}\alpha m_\lambda(|\xi'|)\frac{i\xi'}{|\xi'|}\hat{h}^n(\xi') = \\
= \quad & \int_0^\infty \int_0^1 |\xi'|^{5/2} i\xi' \mathrm{e}^{-|\xi'|^4 r} \frac{|\xi'|^{1/2}}{\omega(|\xi'|) + \alpha}\alpha m_\lambda(|\xi'|)\mathrm{e}^{-\omega(|\xi'|)(x_n+s)} \hat{f}^n(\xi', s)\mathrm{d}r\mathrm{d}s \\
& - \int_0^\infty \int_1^\infty |\xi'|^3 i\xi' i\xi' \cdot \mathrm{e}^{-|\xi'|^4 r}\alpha m_\lambda(|\xi'|)\frac{\mathrm{e}^{-\omega(|\xi'|)(x_n+s)}}{\omega(|\xi'|)(\omega(|\xi'|) + \alpha)} \hat{f}'(\xi', s)\mathrm{d}r\mathrm{d}s.
\end{aligned}$$

Relations (7), (8), Lemma 2.5, and Corollary 2.3 then lead to the estimate

$$\begin{aligned}
&\|\mathrm{e}^{-\omega(|\nabla'|)\cdot}(\omega(|\nabla'|)+\alpha)^{-1}\alpha m_\lambda(|\nabla'|)R'h^n\|_{L^\infty(\mathbb{R}^n_+)} \leq \\
\leq\ & \frac{C}{1+\alpha}\sup_{x_n>0}\left[\int_0^\infty\int_0^1 r^{-7/8}\mathrm{e}^{-c_1(x_n+s)}\|f^n(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\mathrm{d}s\right. \\
&\left.+\int_0^\infty\int_1^\infty r^{-5/4}\mathrm{e}^{-c_1(x_n+s)}\|f'(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\mathrm{d}s\right] \\
\leq\ & C\|f\|_{L^\infty(\mathbb{R}^n_+)}, \quad \lambda\in\Sigma_{\pi-\varphi_0},\ |\lambda|=1.
\end{aligned}$$

The estimate for $\nabla u_2'$ in $L^\infty(\mathbb{R}^n_+)$ is analogous to the one in the proof of Theorem 3.8 in the $L^1(\mathbb{R}^n_+)$-norm with the only difference that the roles of the terms $\mathrm{e}^{-\omega x_n}$ and $\mathrm{e}^{-\omega s}$ interchange. In other words, for some $\delta\in(0,1)$ we have to estimate in the present situation expressions of the form $\omega^\delta\mathrm{e}^{-\omega(x_n+s)}$ by

$$|\omega(z)^\delta\mathrm{e}^{-\omega(z)(x_n+s)}|\leq C\frac{\mathrm{e}^{-c_1(x_n+s)}}{s^\delta}\quad x_n,s>0,\ z\in\Sigma_\varphi.$$

This is due to the fact that we may have a singularity in s but not in x_n, since we estimate in the L^∞-norm. So, (32) is proved for all $\lambda\in\Sigma_{\pi-\varphi_0}$ with $|\lambda|=1$. Then rescaling yields (32) for all $\lambda\in\Sigma_{\pi-\varphi_0}$.

It remains to prove $u\in C^1_b(\mathbb{R}^n_+)$. To this end, observe that

$$\Delta p=\mathrm{div}\,(f-(\lambda-\Delta)u)=0$$

implies $p\in C^\infty(\mathbb{R}^n_+)$. Then, since $u,f\in L^\infty(\mathbb{R}^n_+)$, we obtain

$$\Delta u=\lambda u+\nabla p-f\in L^q(K)$$

for some $q>n$ and each smooth compact $K\subseteq\mathbb{R}^n_+$. By elliptic regularity we deduce $u\in W^{2,q}(K)$ and thanks to Sobolev's embedding $u\in C^1_b(K)$ for each smooth compact $K\subseteq\mathbb{R}^n_+$, consequently $u\in C^1(\mathbb{R}^n_+)$. In view of (32) this yields $u\in C^1_b(\mathbb{R}^n_+)$ and the proof is complete. □

Remark 4.2 A similar statement as in Remark 3.9 is valid in $L^\infty_\sigma(\mathbb{R}^n_+)$. By checking the details of the above proof, one will realize that by our methods for each $\delta_1\in[0,2),\delta_2\in[0,1)$ we also can get an estimate as

$$|\lambda|^{(2-\delta_1)/2}\||\nabla'|^{\delta_1}u\|_\infty+|\lambda|^{(1-\delta_2)/2}\||\nabla'|^{\delta_2}\nabla u\|_\infty\leq C(\delta_1,\delta_2)\|f\|_\infty$$

for $\lambda\in\Sigma_{\pi-\varphi_0}$ and $f\in L^\infty_\sigma(\mathbb{R}^n_+)$. However, the constant $C(\delta_1,\delta_2)$ blows up if $\delta_1\to 2$ or $\delta_2\to 1$. But this is reasonable, since we do not expect $u\in W^{2,\infty}(\mathbb{R}^n_+)$.

Due to Theorem 4.1 above, for each $\lambda\in\mathbb{C}\setminus(-\infty,0]$ the mapping $R(\lambda):L^\infty_\sigma(\mathbb{R}^n_+)\to C^1_b(\mathbb{R}^n_+)$, $R(\lambda)f:=u_f(\lambda)$ is well-defined and continuous. Since $C^1_b(\mathbb{R}^n_+)\hookrightarrow C_b(\overline{\mathbb{R}^n_+})$, we may apply Gauss' Theorem to obtain for $u=u_f(\lambda)$

$$(u,\nabla\phi)=\int_{\mathbb{R}^{n-1}}\phi u^n\mathrm{d}x'-(\mathrm{div}\,u,\phi)=0$$

for all $\phi \in C_c^\infty(\overline{\mathbb{R}^n_+})$, which shows that we even have $R(\lambda) : L^\infty_\sigma(\mathbb{R}^n_+) \to C^1_b(\mathbb{R}^n_+) \cap L^\infty_\sigma(\mathbb{R}^n_+)$. In order to establish a Stokes operator in $L^\infty_\sigma(\mathbb{R}^n_+)$ we now intend to prove $R(\lambda)$ to be a resolvent. In view of [Saa04, Theorem 5.7] this is true if $f \in L^q_\sigma(\mathbb{R}^n_+)$, $1 < q < \infty$. Thus, also $\hat{R}(\lambda)\hat{f} = \mathcal{F}R(\lambda)f$ satisfies the resolvent identity. On the other hand (14), (15) is an explicit representation of $\hat{R}(\lambda)\hat{f}$, which implies that the resolvent identity could also be verified by a direct calculation. Since a direct calculation obviously does not depend on the data f, but only on the formula itself, it must be true whenever representation (14), (15) for $\hat{R}(\lambda)\hat{f}$ holds. The proof of Theorem 4.1 shows that this is the case for $f \in L^\infty_\sigma(\mathbb{R}^n_+)$, hence $R(\lambda) : L^\infty_\sigma(\mathbb{R}^n_+) \to L^\infty_\sigma(\mathbb{R}^n_+)$ is a pseudo resolvent. To see the injectivity of $R(\lambda)$ let $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ with $R(\lambda)f = 0$. Since $R(\lambda)f = u_f(\lambda)$ solves $(SRP)_{f,\lambda,\alpha}$ we deduce

$$\nabla p = (\lambda - \Delta)u_f(\lambda) + \nabla p = f.$$

Moreover, by $\Delta p = \operatorname{div} f = 0$ in the sense of distributions, p and also $f = \nabla p$ are harmonic in $\mathbb{R}^n_+$. By the Schwarz reflection principle the odd extension of $\partial_n p$ is a bounded and harmonic function on $\mathbb{R}^n$, hence $\partial_n p = 0$ in view of $\partial_n p \restriction_{\partial\mathbb{R}^n_+} = f^n \restriction_{\partial\mathbb{R}^n_+} = 0$. So, p depends only on x' and we obtain $\Delta' p = \Delta p = 0$. Consequently $\nabla' p$ is bounded and harmonic in $\mathbb{R}^{n-1}$, which yields $f = \nabla p = (c', 0)$ with $c' \in \mathbb{R}^{n-1}$. By a special choice of a function $\phi \in \widehat{W}^{1,1}(\mathbb{R}^n_+)$ we will show that $(f, \nabla\phi) = 0$ implies $c' = 0$. Indeed, let $\psi \in C^\infty(\mathbb{R})$ satisfying $\psi(s) \equiv 0$ for $s \le -1$ and $\psi(s) \equiv 1$ for $s \ge 1$. For $j \in \{1, \dots, n-1\}$ put

$$\phi(x) := \psi(x_j)\mathrm{e}^{-|\bar{x}|^2} \in \widehat{W}^{1,1}(\mathbb{R}^n_+),$$

where $\bar{x} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$. Observe that $\int_{\mathbb{R}^n_+} \partial_k\phi(x)\mathrm{d}x = 0$ for $k \neq j$, since ϕ is even in x_k for $k \neq j$. This implies

$$\begin{aligned} 0 &= (f, \nabla\phi) = (c', \nabla'\phi) = c' \cdot \int_{\mathbb{R}^n_+} \nabla'\phi(x)\mathrm{d}x \\ &= c_j \int_{-\infty}^{\infty} \psi'(x_j)\mathrm{d}x_j \int_{\mathbb{R}^{n-1}_+} \mathrm{e}^{-|\bar{x}|^2}\mathrm{d}\bar{x} \\ &= c_j \int_{\mathbb{R}^{n-1}_+} \mathrm{e}^{-|\bar{x}|^2}\mathrm{d}\bar{x}, \end{aligned}$$

which yields $c_j = 0$ for $j \in \{1, \dots, n-1\}$. Consequently $f = 0$. This implies $R(\lambda)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ to be a resolvent, hence there exists a closed operator $A_{\alpha,\infty}$ in $L^\infty_\sigma(\mathbb{R}^n_+)$ such that

$$(\lambda + A_{\alpha,\infty})^{-1} = R(\lambda), \quad -\lambda \in \rho(A_{\alpha,\infty}) = \mathbb{C} \setminus [0, \infty).$$

We call $A_\alpha := A_{\alpha,\infty}$ the *Stokes operator in* $L^\infty_\sigma(\mathbb{R}^n_+)$. Theorem 4.1 now implies (where the proof of the gradient estimates is analogous to Theorem 3.3)

Theorem 4.3 *Let $\alpha \in [0, \infty]$. The operator $-A_\alpha$ is the generator of a bounded holomorphic semigroup on $L^\infty_\sigma(\mathbb{R}^n_+)$ (which is not strongly continuous). The semigroup $(\mathrm{e}^{-tA_\alpha})_{t\ge 0}$ also satisfies gradient estimates, i.e.*

$$\|\nabla\mathrm{e}^{-tA_\alpha} f\|_\infty \le Ct^{-1/2}\|f\|_\infty, \quad t > 0,\ f \in L^\infty_\sigma(\mathbb{R}^n_+). \tag{33}$$

Next we define corresponding Stokes operators in spaces of continuous functions. In the case of Dirichlet boundary conditions we set

$$C_{0,\sigma}(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_\infty}. \tag{34}$$

For $-\lambda \in \mathbb{C} \setminus [0,\infty)$ and $f \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ we have $R(\lambda)f = u_f(\lambda) \in W^{1,q}_0(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+)$, for each $1 < q < \infty$. Since

$$W^{1,q}_0(\mathbb{R}^n_+) \cap L^q_\sigma(\mathbb{R}^n_+) = W^{1,q}_{0,\sigma}(\mathbb{R}^n_+) := \overline{C^\infty_{c,\sigma}(\mathbb{R}^n_+)}^{\|\cdot\|_{1,q}}$$

for $1 < q < \infty$ (see [Gal98] Chapter III.4.) the function $u_f(\lambda)$ can be approximated by a sequence $(u_k) \subseteq C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ in $W^{1,q}(\mathbb{R}^n_+)$. The imbedding $W^{1,q}(\mathbb{R}^n_+) \hookrightarrow C_b(\overline{\mathbb{R}^n_+})$ for $q > n$ now implies

$$\|u_k - u_f\|_\infty \le C\|u_k - u_f\|_{1,q} \to 0, \quad \text{for} \quad k \to \infty,$$

consequently $R(\lambda)f \in C_{0,\sigma}(\mathbb{R}^n_+)$. By the continuity of $R(\lambda)$ we obtain $R(\lambda)(C_{0,\sigma}(\mathbb{R}^n_+)) \subseteq C_{0,\sigma}(\mathbb{R}^n_+)$, and since $C_{0,\sigma}(\mathbb{R}^n_+) \subseteq L^\infty_\sigma(\mathbb{R}^n_+)$, the mapping $R(\lambda) : C_{0,\sigma}(\mathbb{R}^n_+) \to C_{0,\sigma}(\mathbb{R}^n_+)$ is a resolvent. This results the existence of a closed operator $A_{C_{0,\sigma}}$ in $C_{0,\sigma}(\mathbb{R}^n_+)$ such that

$$(\lambda + A_{C_{0,\sigma}})^{-1} = R(\lambda), \quad -\lambda \in \rho(A_{C_{0,\sigma}}) = \mathbb{C} \setminus [0,\infty),$$

which we call *Stokes operator in* $C_{0,\sigma}(\mathbb{R}^n_+)$. Now $u \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$ yields $f := (1 + A_{\alpha,2})u \in C^\infty_{c,\sigma}(\mathbb{R}^n_+)$. Hence

$$u = (1 + A_{\alpha,2})^{-1} f = u_f(\lambda) = (1 + A_{C_{0,\sigma}})^{-1} f \in D(A_{C_{0,\sigma}}),$$

and we see, that $A_{C_{0,\sigma}}$ is densely defined. Thus, we have proved the following result, which partly can be found in [DHP01].

Theorem 4.4 *The Stokes operator* $-A_{C_{0,\sigma}}$ *is the generator of a bounded, holomorphic* C_0*-semigroup on* $C_{0,\sigma}(\mathbb{R}^n_+)$*. The gradient estimates (33) are also valid for the semigroup generated by* $-A_{C_{0,\sigma}}$*.*

A further space of continuous solenoidal functions in the case of Dirichlet boundary conditions is

$$\mathrm{BUC}_{\sigma,D}(\mathbb{R}^n_+) := \{u \in \mathrm{BUC}(\mathbb{R}^n_+) : \ \mathrm{div}\, u = 0, \ u \restriction_{\partial\mathbb{R}^n_+} = 0\},$$

where $\mathrm{BUC}(\mathbb{R}^n_+)$ denotes the space of all bounded uniformly continuous functions in $\mathbb{R}^n_+$. In the case that $\alpha \in [0,\infty)$ it suffices to demand the n-th component to be zero for the definition of a proper space, i.e.

$$\mathrm{BUC}_{\sigma}(\mathbb{R}^n_+) := \{u \in \mathrm{BUC}(\mathbb{R}^n_+) : \ \mathrm{div}\, u = 0, \ u^n \restriction_{\partial\mathbb{R}^n_+} = 0\}.$$

Now let $X_\sigma \in \{\mathrm{BUC}_{\sigma,D}(\mathbb{R}^n_+), \mathrm{BUC}_\sigma(\mathbb{R}^n_+)\}$. We will prove that the Stokes operator is the generator of a strongly continuous semigroup on X_σ. For this purpose we need

Lemma 4.5 *Let* $\alpha \in [0,\infty]$*,* $f \in X_\sigma$*, and* $v_{\alpha,f}(\lambda) = v$ *as in (27). Then for each* $\delta \in [0,1)$ *and* $j \in \{1,\dots,n-1\}$

$$\lambda^{1-\frac{\delta}{2}} |\nabla'|^\delta v^n_{\alpha,f}(\lambda) \to 0 \quad \textit{and} \quad \lambda R_j v^n_{\alpha,f}(\lambda) \to 0 \quad \textit{in} \quad L^\infty(\mathbb{R}^n_+)$$

if $\lambda \to \infty$*,* $\lambda > 0$*.*

Proof. By the scaling argument as described in the previous section and the proof of Lemma 3.7 (i) we have

$$\begin{aligned}
\|\lambda^{1-\frac{\delta}{2}}|\nabla'|^{\delta}v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty &= \lambda^{(n-1)/2}\sup_{x\in\mathbb{R}^n_+}||\nabla'|^{\delta}v^n_{\beta,f_D}(x',x_n,1)| \\
&\leq C(\delta)\lambda^{(n-1)/2}\sup_{x_n>0}(1-\mathrm{e}^{-c_1x_n})\int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}}\|f^n_D(\cdot,s)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}s \\
&= C(\delta)\int_0^\infty \frac{\mathrm{e}^{-c_1 s}}{s^{\frac{\delta}{2}+\varepsilon}}\|f^n(\cdot,s\lambda^{-1/2})\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}s.
\end{aligned}$$

Here $\delta,\varepsilon,\delta+\varepsilon\in(0,1)$, $\beta=\alpha\lambda^{-1/2}$ is the scaled boundary parameter, and f_D is the scaled function f. Now $f\in X_\sigma$ gives us

$$\sup_{x'\in\mathbb{R}^{n-1}}|f^n(x',s\lambda^{-1/2})| = \sup_{x'\in\mathbb{R}^{n-1}}|f^n(x',s\lambda^{-1/2})-f^n(x',0)| \longrightarrow 0,$$

if $\lambda\to\infty$. Furthermore, $\frac{\mathrm{e}^{-c_1 s}}{s^{\delta+\varepsilon}}\|f^n\|_\infty$ is an integrable majorant for the λ-dependent integrand of the latter integral. Lebesgue's dominated convergence Theorem results

$$\lim_{\lambda\to\infty}\|\lambda^{1-\frac{\delta}{2}}|\nabla'|^{\delta}v^n_{\alpha,f}(\lambda)\|_\infty = 0.$$

Similar calculations as in the proof of Lemma 3.7 (iii) lead to the estimate

$$\begin{aligned}
\|\lambda R_j v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty &= \lambda^{(n-1)/2}\sup_{x\in\mathbb{R}^n_+}|R_j v_{\beta,f_D}(x',x_n,1)| \\
&\leq C\lambda^{(n-1)/2}\sup_{x_n>0}\left(\int_0^R r^{\frac{\gamma}{4}-1}\||\nabla'|^{\gamma}v^n_{\beta,f_D}(\cdot,x_n,1)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\right. \\
&\quad \left.+\sum_{k=1}^{n-1}\int_R^\infty r^{-5/4}\|v^k_{\beta,f_D}(\cdot,x_n,1)\|_{L^\infty(\mathbb{R}^{n-1})}\mathrm{d}r\right)
\end{aligned}$$

for some $\gamma\in(0,1)$, where we splitted the integral at $r=R$, instead of $r=1$ as before. Applying Lemma 3.7 (i) to the second addend, we can continue the calculation obtaining

$$\begin{aligned}
\|\lambda R_j v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty &\leq C\lambda^{(n-1)/2}\left(R^{\gamma/4}\||\nabla'|^{\gamma}v^n_{\beta,f_D}(\cdot,x_n,1)\|_{L^\infty(\mathbb{R}^{n-1})}+R^{-1/4}\|f_D\|_\infty\right) \\
&= C\left(R^{\gamma/4}\|\lambda^{1-\frac{\gamma}{2}}|\nabla'|^{\gamma}v^n_{\alpha,f}(\cdot,x_n,\lambda)\|_{L^\infty(\mathbb{R}^{n-1})}+R^{-1/4}\|f\|_\infty\right).
\end{aligned}$$

Choosing $R>\left(\frac{\varepsilon}{2\|f\|_\infty}\right)^{-4}$ and afterwards λ big enough we achieve

$$\|\lambda R_j v^n_{\alpha,f}(\cdot,\cdot,\lambda)\|_\infty<\varepsilon$$

for each $\varepsilon>0$, which yields the assertion. □

In order to obtain densely defined generators we prove that for $f\in X_\sigma$

$$\lambda R(\lambda)f=\lambda u_{\alpha,f}(\lambda)\to f \tag{35}$$

in $L^\infty(\mathbb{R}^n_+)$ if $\lambda \to \infty$. We first consider the case $\alpha \in [0,\infty)$. Observe that $u_{\alpha,f}(\lambda)$ can also be represented by the formula

$$\begin{aligned} u'_{\alpha,f}(\lambda) &= (\lambda - \Delta_N)^{-1} f' - R' v^n_{\alpha,f}(\lambda) + \mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi' - \omega(|\nabla'|)^{-1}\mathrm{e}^{-\omega(|\nabla'|)\cdot}h', \\ u^n_{\alpha,f}(\lambda) &= (\lambda - \Delta_D)^{-1} f^n + v^n_{\alpha,f}(\lambda), \end{aligned}$$

which we will use to prove (35). Since $C^\infty_b(\mathbb{R}^n) \subseteq D(\Delta_{\mathbb{R}^n})$ lies dense in $\mathrm{BUC}(\mathbb{R}^n)$, the Laplacian $\Delta_{\mathbb{R}^n}$ is the generator of a bounded holomorphic C_0-semigroup on $\mathrm{BUC}(\mathbb{R}^n)$. This yields $\lambda(\lambda - \Delta_{\mathbb{R}^n})^{-1} f \to f$ for all $f \in \mathrm{BUC}(\mathbb{R}^n)$. For $f \in \mathrm{BUC}_\sigma(\mathbb{R}^n_+)$ we set $\tilde{f} := (E^+ f', E^- f^n)$, where $E^\pm$ is the extension operator as defined in (13). Then obviously $\tilde{f} \in \mathrm{BUC}(\mathbb{R}^n)$, hence the representations (12) and (11) imply

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda(\lambda - \Delta_N)^{-1} f' &= f' \quad \text{and} \\ \lim_{\lambda\to\infty} \lambda(\lambda - \Delta_D)^{-1} f^n &= f^n, \end{aligned}$$

respectively. Since $\lambda v^n_{\alpha,f}(\lambda)$ and $\lambda R' v^n_{\alpha,f}(\lambda)$ tend to zero if $\lambda \to \infty$ according to Lemma 4.5, it remains to show

$$\begin{aligned} & \lambda\left(\mathrm{e}^{-\omega(|\nabla'|)\cdot}\phi' - \omega(|\nabla'|)^{-1}\mathrm{e}^{-\omega(|\nabla'|)\cdot}h'\right) = \\ = \; & \lambda \mathrm{e}^{-\omega(|\nabla'|)\cdot}\alpha(\omega(|\nabla'|)+\alpha)^{-1}\left(-\omega(|\nabla'|)^{-1}h' + R' m_\lambda(|\nabla'|)h^n\right) \longrightarrow 0 \end{aligned}$$

as $\lambda \to \infty$, $\lambda > 0$. In virtue of Lemma 2.5, Corollary 2.3 and since $\alpha < \infty$, the latter term can be estimated by

$$\|\lambda \mathrm{e}^{-\omega(|\nabla'|)\cdot}\alpha(\omega(|\nabla'|)+\alpha)^{-1}\left(-\omega(|\nabla'|)^{-1}h' + R' m_\lambda(|\nabla'|)h^n\right)\|_\infty \le C\lambda^{(\varepsilon-1)/2}\|f\|_\infty$$

for some $\varepsilon \in (0,1)$, which proves (35) for $\alpha \in [0,\infty)$. In the case of Dirichlet boundary conditions the solution $u_{\infty,f}(\lambda)$ is given by

$$\begin{aligned} u'_{\infty,f}(\lambda) &= (\lambda - \Delta_D)^{-1} f' - R' v^n_{\infty,f}(\lambda), \\ u^n_{\infty,f}(\lambda) &= (\lambda - \Delta_D)^{-1} f^n + v^n_{\infty,f}(\lambda). \end{aligned}$$

Thus (35) follows directly from $\lambda(\lambda - \Delta_D) f \to f$, valid for all $f \in \mathrm{BUC}_{\sigma,D}(\mathbb{R}^n_+)$, and Lemma 4.5.

In view of the inclusions $X_\sigma \subseteq L^\infty_\sigma(\mathbb{R}^n_+)$ and $C^1_b(\mathbb{R}^n_+) \subseteq \mathrm{BUC}(\mathbb{R}^n_+)$ we have $R(\lambda)(X_\sigma) \subseteq X_\sigma$ for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. This implies $R(\lambda) : X_\sigma \to X_\sigma$ to be a resolvent, hence the existence of a closed operator A_{α,X_σ} such that

$$(\lambda + A_{\alpha,X_\sigma})^{-1} = R(\lambda), \quad -\lambda \in \rho(A_{\alpha,X_\sigma}) = \mathbb{C} \setminus [0,\infty).$$

A_{α,X_σ} is also densely defined, due to the validity of (35) for $f \in X_\sigma$. We call A_{α,X_σ} the Stokes operator in X_σ. Summarizing the just proved results gives us

Theorem 4.6 *The operator $-A_{\alpha,X_\sigma}$ is the generator of a bounded holomorphic C_0-semigroup on X_σ. The gradient estimates in (33) are also valid for the semigroup generated by $-A_{\alpha,X_\sigma}$.*

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