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On breakdown of solutions of a constrained gradient system of total variation

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Abstract

A gradient system of total variation is considered for a mapping from the unit disk to the unit sphere in $\mathbf{R}^3$. For a class of initial data it is shown that a solution of its Dirichlet problem loses its smoothness in finite time.

Ker words: total variation, breakdown, constrained gradient system, 1-harmonic flow

1 Introduction

We consider a gradient system of total variation for a mapping u from the unit disk $D^2 = \{(x_1, x_2) \in \mathbf{R}^2 | x_1^2 + x_2^2 < 1\}$ to the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | |x|^2 = x_1^2 + x_2^2 + x_3^2 = 1\}$. It is of the form

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| u, \tag{1.1}$$

where $u = (u^1, u^2, u^3)$, $|\nabla u|^2 = \sum_{j=1}^{2} \sum_{i=1}^{3} \left|\frac{\partial u^i}{\partial x_j}\right|^2$, $u_t = \partial u/\partial t$. We are concerned with the question whether or not a smooth solution breaks down in finite time for its Dirichlet problem.

We consider a rotationally symmetric solution of the form

$$u(x,t) = \left(\frac{x}{r}\sin h(r,t), \cos h(r,t)\right), \ r = |x|, \ x \in D^2. \tag{1.2}$$

Our goal in this paper is to prove that such a solution ceases to be smooth in finite time for a class of initial data.

Main Theorem. *Assume that the initial data* $h_0 = h(\cdot, 0) \in C^0[0,1] \cap C^1[0,1)$ *satisfies* $h_0(1) = \pi$, $0 < h_0(r) < \pi (0 < r < 1)$, $h_0'(0) > 0$. *Let* $T_0 \le \infty$ *be the maximal existence*

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time of a smooth solution (1.2) for the Dirichlet problem for (1.1) with $h(1,t) = \pi$. Then T_0 must be finite.

In fact, we have an estimate $T_0 < \sqrt{2}\lambda_0$ with $\lambda_0 > 0$ satisfying

$$\arccos\{(\lambda_0^2 - r^2)/(\lambda_0^2 + r^2)\} \le h_0(r) \quad \text{for} \quad r \in (0,1).$$

Of course, such a λ_0 exists (Lemma 3.2). The system has a very strong singularity at $\nabla u = 0$, so the meaning of a solution is not a priori clear even if we assume u is smooth. Although there are definitions of solutions given by [4] and [5], in this paper we do not touch this problem and assume that $\nabla u \neq 0$ if we say that u is a solutions; see §2 for definition. We do not know whether the derivative of h blows up at the maximal existence time. For our problem a local-in-time existence of a smooth solution seems to be an open problem although a local solution which is Lipschitz in spatial variable is constructed under periodic boundary condition allowing the place where $\nabla u = 0$ [5]. If the space dimension is one and initial data is piecewise constant, a unique global solvability is known in a class of spatially piecewise constant functions [4].

The gradient system of the total variation for $u : D^2 \to S^2$ is considered as a special example of a p-harmonic flow equation

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^p u. \tag{1.3}$$

This is the gradient system of energy $\frac{1}{p}\int_D |\nabla u|^p dx$ with constraint $|u| = 1$. The case $p = 2$ is called a harmonic flow equation. From this point of view (1.1) is interpreted as a 1-harmonic flow equation. For a harmonic flow equation complete results are known [1], [2]. In fact, if $|h_0| \le \pi$, then a unique global smooth solution exists for (1.3) with $p = 2$ under the Dirichlet condition on ∂D^2 of the form $h = h_0(1)$ [1]. However, if $|h_0(1)| > \pi$, a local solution breaks down in finite time [2]. Our result has a striking contrast to their results since our initial data provides a global smooth solution for $p = 2$. The problem for $p \neq 2$ is not well-studied compared with $p = 2$ especially for $p \in (1,2)$ since the equation (1.3) has a singularity at $\nabla u = 0$. We do not intend to exhaust references. Instead, we give two typical results for $p \in$ (1,2). In the case that source space is a compact manifold and the target manifold is S^{N-1} a global week solution is constructed by M. Misawa [8]. For general target (compact) manifold a global unique solvability is proved by A. Fardoun and R. Regbaoui [3] provided that the initial data is smooth with small energy $\frac{1}{p}\int |\nabla u_0|^p$ and small Lipschitz norm. For more results the reader is referred to [3] and papers cited there.

Our strategy is close to that of [2]. Namely, we shall construct a subsolution whose derivatives at the origin blows up in finite time. Since the equation is different, we cannot

argue in the same way. Moreover, we would like to avoid the situation that the jacobi matrix of a homotopy between of a solution and our subsolution vanishes somewhere. This forces us to assume that $0 < h_0 < \pi$. In fact, we are able to prove that if $0 < h_0 < \pi$ then $0 < h(r,t) < \pi$ for $r \in (0,1)$; see §4.

The problems (1.1) and (1.3) are proposed by [10] in image processing to denoise chromaticity ; see [9] for background. The total variation flow keeps edges but a harmonic map flow ((1.3) with $p = 2$) shade off edges. This is advantage of (1.1) over a harmonic map flow. The problem without constraint is well-studied since it can be formulated by a dissipative system. The bibliography of [4] included many references on this subject and related one. We do not repeat it here. Finally, we note that the constrainet gradient systems of total variation also naturally arise in multi-grain problems [6].

This paper is organized as follows. In §2 we derive an equation for h from (1.1) when it is given by (1.2). We give a definition of a solution and discuss the equivalence of u-formulation and h-formulation. In §3 we construct a subsolution whose derivative blows up in finite time. In §4 we compare with a subsolution and finally prove the Main Theorem - the breakdown of h in finite time.

2 Solutions with symmetry

We consider the initial-boundary value problem for one-harmonic flow equations from D^2 to S^2 of the form

$$u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u| u \quad \text{in} \quad D^2 \times (0,T), \tag{2.1}$$

$$u(x,0) = u_0(x) \quad \text{on} \quad \bar{D}^2, \tag{2.2}$$

$$u(\cdot,t) = u_0 \quad \text{on} \quad \partial D^2 \times (0,T) \tag{2.3}$$

The equation (2.1) has a very strong singularity at the place where $\nabla u = 0$ and it is nontrivial to handle this singularity. In this paper we do not touch this problem.

By a *solution* u of (2.1) -(2.3) we mean that $u \in C^{2,1}(D^2 \times (0,T)) \cap C^0(\bar{D}^2 \times [0,T))$ solves (2.1)-(2.3) and that $|\nabla u|$ is bounded away from zero on $D^2 \times (0,T)$. Here we implicitly assume that $u_0 \in C^0(\bar{D}^2)$. (By $C^{2,1}$ we mean that $u, u_t, \nabla u, \nabla^2 u$ are continuous.) If u_0 is $C^1(D^2)$ (with values in S^2), we further assume that $u \in C^{1,0}(D^2 \times [0,T))$ when we say that u is a solution.

We consider a rotationally symmetric initial data of the form

$$u_0(x) = \left(\frac{x}{|x|}\sin h_0(r), \cos h_0(r)\right) \text{ for } x = (x_1, x_2) \in \bar{D}^2 \setminus \{0\} \tag{2.4}$$

$$h_0 \in C^0[0,1] \text{ and } h_0(0) = 0, \tag{2.5}$$

where $r = |x|$ for $x = (x_1, x_2) \in D^2$ and we set $b = h_0(1)$. Since $h_0(0) = 0, u_0 \in C^0(\bar{D}^2)$ by assigning $u_0(0) = (0, 0, 1)$. Moreover, it is easy to see that if $h_0 \in C^k[0, 1)$, then $u_0 \in C^k(D), k = 1, 2$. For such an intial data we consider a special form of solution. We set

$$u(x,t) = \left(\frac{x}{|x|} \sin h(r,t), \cos h(r,t) \right) \quad \text{for} \quad x \in D^2,\ t \in (0,T) \tag{2.6}$$

with $h(0,t) = 0$ by assigning $u(0,t) = (0,0,1)$. Then, as we see later, (2.1) becomes

$$\begin{array}{ll} h_t & = J(h)^{-3}\tau(h) \quad \text{in} \quad (0,1)\times(0,T), \\ \tau(h) & = \frac{\sin^2 h}{r^2}\left(h_{rr} + \frac{2h_r}{r} - \frac{\sin 2h}{2r^2}\right) + \frac{h_r^2}{r}\left(h_r - \frac{\sin 2h}{r}\right), \\ J(h) & = \left(h_r^2 + \frac{\sin^2 h}{r^2}\right)^{1/2}. \end{array} \tag{2.7}$$

The initial condition (2.2) becomes

$$h(r,0) = h_0(r) \text{ in } [0,1] \tag{2.8}$$

and the condition (2.3) with the regularity condition at the origin becomes

$$h(0,t) = 0 \quad \text{and} \quad h(1,t) = b,\ t \in (0,T). \tag{2.9}$$

By a *solution* h of (2.7)-(2.9) we mean that $h \in C^{2,1}([0,1)\times(0,T)) \cap C^0([0,1]\times[0,T))$ solves (2.7)-(2.9) and that $h_r^2 + \frac{\sin^2 h}{r^2}$ is bounded away from zero on $[0,1)\times(0,T)$. If $h \in C^1[0,1)$, we further assume that $h \in C^{1,0}([0,1)\times[0,T))$ to say that h is a solution.

Lemma 2.1. *Assume that u_0 satisfies (2.4)-(2.5). Let u be a function defined by (2.6). Then u is a solution of (2.1)-(2.3) if and only if h is a solution of (2.7)-(2.9).*

Proof. The regularity conditions of u and h are equivalent up to the origin since $h(0,t) = 0$. The equivalence of initial and boundary conditions (except $h(0,t) = 0$) is clear.

By a direct calculation we observe that

$$\nabla u = \left(\frac{x^\perp \otimes x^\perp}{r^3} \sin h + \frac{x \otimes x}{r^2} h_r \cos h,\ \frac{-{}^t x}{r} h_r \sin h \right),\ x^\perp = (x_2, -x_1),$$

where u is interpreted as a row vector and $\otimes$ denotes the tensor product of vectors. This yields $|\nabla u| = J(h)$. Thus the condition that $h_r \neq 0$ or $\sin h \neq 0$ is equivalent to $\nabla u \neq 0$.

It remains to prove the equivalence of (2.1) and (2.7). By regularity of u we may assume that $x = re^{i\theta} \neq 0$. We further calculate

$$\Delta u = \left(e^{i\theta} \left\{ -\left(\frac{1}{r^2} + h_r^2\right) \sin h + \left(\frac{h_r}{r} + h_{rr}\right) \cos h \right\},\ -\left(\frac{h_r}{r} + h_{rr}\right) \sin h - h_r^2 \cos h \right).$$

Since

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \nabla \frac{1}{|\nabla u|} \cdot \nabla u + \frac{\Delta u}{|\nabla u|},$$

we calculate

$$\nabla\frac{1}{|\nabla u|} = -\frac{1}{r}|\nabla u|^{-3}\left(h_r h_{rr} - \frac{\sin^2 h}{r^3} + \frac{h_r \sin h\cos h}{r^2}\right)(x_1, x_2),$$
$$(x_1, x_2)\cdot\nabla u = r h_r(e^{i\theta}\cos h,\ -\sin h)$$

and conclude that

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) + |\nabla u|u = J(h)^{-3}\tau(h)(e^{i\theta}\cos h,\ -\sin h).$$

Since $u_t = h_t(e^{i\theta}\cos h, -\sin h)$, (2.1) is equivalent to

$$(h_t - J(h)^{-3}\tau(h))(e^{i\theta}\cos h, -\sin h) = 0. \tag{2.10}$$

We have thus proved that (2.1) is equivalent to (2.7). □

Remark. If one allows $\theta-$dependence in h, *i.e.*, $h = h(x_1, x_2, t)$, then we do not have a comon multiplier of $e^{i\theta}\cos h$ and $-\sin h$ in (2.10). So we cannot reduce (2.1) to the equation of h in that case.

3 Family of subsolutions

We shall construct a subsolution which forces a solution to break down in finite time.

For $\lambda > 0$ we set

$$\phi(r,\lambda) := \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right) \in [0,\pi)\ (0 \le r \le 1).$$

For $\varepsilon \in (0,1)$ and positive constants δ and λ_0 let $\lambda = \lambda(t)$ be the solution of

$$\frac{d\lambda}{dt} = -\delta\lambda^{\varepsilon},\ \lambda(0) = \lambda_0, \quad i.e., \quad \lambda(t) = (\lambda_0^{1-\varepsilon} - (1-\varepsilon)\delta t)^{1/(1-\varepsilon)}.$$

This function is a positive, smooth and monotone decreasing function in $t \in [0, T_{\lambda_0})$ with $T_{\lambda_0}(\delta,\varepsilon) := \lambda_0^{1-\varepsilon}/(1-\varepsilon)\delta$. We next set a function $f : [0,1]\times[0,T_{\lambda_0}) \to [0,\pi)$ by

$$f(r,t) := \phi(r,\lambda(t)).$$

We shall take paramaters so that f is a subusolution of (2.7).

Lemma 3.1. *There exists $\delta_0 = \delta_0(\lambda_0,\varepsilon) > 0$ such that $f_t \le J(f)^{-3}\tau(f)$ in $(0,1)\times(0,T_{\lambda_0})$ for $\delta \in (0,\delta_0)$.*

Proof. By a direct calculation

$$\phi_r(r,\lambda) = \frac{2\lambda}{r^2+\lambda^2},\ \phi_{rr}(r,\lambda) = \frac{-4\lambda r}{(\lambda^2+r^2)^2},\ \sin\phi = \frac{2\lambda r}{\lambda^2+r^2}$$

so that

$$\phi_{rr} + \frac{1}{r}\phi_r - \frac{\sin\phi\cos\phi}{r^2} = 0.$$

Thus we have

$$\tau(\phi) = \frac{32\lambda^3 r}{(\lambda^2 + r^2)^4},\ J(\phi)^2 = \frac{8\lambda^2}{(\lambda^2 + r^2)^2}$$

so that

$$J(f)^{-3}\tau(f) = \frac{\sqrt{2}r}{\lambda(t)^2 + r^2}.$$

Since

$$f_t(r,t) = \lambda'(t)\phi_\lambda(r,\lambda(t)) = -\delta\lambda(t)^\varepsilon \cdot \frac{-2r}{\lambda(t)^2 + r^2} = \frac{2\delta\lambda(t)^\varepsilon r}{\lambda(t)^2 + r^2},$$

we see that

$$f_t \le J(f)^{-3}\tau(f) \text{ in } (0,1)\times(0,T_{\lambda_0})$$

if and only if

$$\frac{2\delta\lambda(t)^\varepsilon r}{\lambda(t)^2 + r^2} \le \frac{\sqrt{2}r}{\lambda(t)^2 + r^2} \quad \text{for all} \quad t \in (0, T_{\lambda_0}).$$

This is equivalent to say that $\sqrt{2}\delta\lambda_0^\varepsilon \le 1$ since $0 < \lambda(t) \le \lambda_0$. The proof is now complete if we take $\delta_0 := (\sqrt{2}\lambda_0^\varepsilon)^{-1}$. □

Lemma 3.2. *Assume that the initial data h_0 in (2.5) fulfills*

$$h_0(r) > 0\ (0 < r \le 1),\ h_0'(0) > 0. \tag{3.1}$$

Then there exists a constant $\lambda_1 = \lambda_1(h_0)$ such that if $\lambda_0 > \lambda_1$, then $f(r,0) < h_0(r)$ $(0 < r \le 1)$

Proof. We set

$$\begin{aligned} \lambda_1(h_0) :\ &= \inf\{\lambda_0 > 0 \mid f(r,0) \le h_0(r) \text{ for } r \in (0,1)\} \\ &= \inf\{\lambda_0 > 0 \mid \arccos\left(\tfrac{\lambda_0^2 - r^2}{\lambda_0^2 + r^2}\right) \le h_0(r) \text{ for } r \in (0,1)\}. \end{aligned} \tag{3.2}$$

We shall prove that λ_1 is finite. Since $f_r(r,0) = 2\lambda_0/(\lambda_0^2 + r^2)$, we have $f_r(0,0) = 2/\lambda_0$. Since $h_0'(0) > 0$, we see that

$$\lambda_0 > 2h_0'(0)^{-1} \quad \text{implies} \quad f_r(0,0) < h_0'(0).$$

Since $h_0(0) = 0$, this implies

$$f(r,0) < h_0(r) \quad \text{for} \quad 0 < r \le \eta$$

for sufficiently small $\eta = \eta(\lambda_0) > 0$. By (3.1)

$$m := \min_{\eta \le r \le 1} h_0(r) > 0.$$

Since

$$f(r,0) \le \arccos\left(\frac{\lambda_0^2-1}{\lambda_0^2+1}\right) \to 0$$

as $\lambda_0 \to \infty$, we see that

$$f(r,0) < m \ (\eta \le r \le 1)$$

for sufficiently large λ_0 (fixing η). For such λ_0 we conclude $f(r,0) < h_0(r)$ $(0 < r \le 1)$ so that λ_1 is finite. □

For h_0 satisfying (2.5) and (3.1) we take $\lambda_0 > \lambda_1 = \lambda_1(h_0)$, where λ_1 is defined by (3.3). We take δ such that $\delta = \delta_0(\lambda_0, \varepsilon)$. Then f is a subsolution of (2.7)-(2.9) by Lemma 3.1. and 3.2, since $f(0,t) = 0$. Indeed, f solves

$$\begin{cases} f_t \le J(f)^{-3}\tau(f) & \text{in} \quad (0,1)\times(0,T(\varepsilon)), \\ f \le h_0 & \text{at} \quad t = 0 \\ f(0,t) = 0, \ f(1,t) < b = h_0(1) \end{cases}$$

with $T(\varepsilon) = T_{\lambda_0}(\delta_0, \varepsilon) = \sqrt{2}\lambda_0/(1-\varepsilon)$. provided that $b \ge \pi$. Moreover, the derivative at $r = 0$ blows up at $T(\varepsilon)$. Indeed,

$$\lim_{t\to T(\varepsilon)} f_r(0,t) = \lim_{t\to T(\varepsilon)} \frac{2}{\lambda(t)} = \infty. \tag{3.3}$$

Here and hereafter we fix λ_0 (and δ_0) so that f is a subsolution of (2.7)-(2.9).

4 Results by comparison

We begin by deriving an upper bound as well as the positivity preserving property.

Lemma 4.1. *Assume that h is a solution of (2.7)-(2.9). Assume that h_0 satisfies (2.5) and $h_0 \not\equiv 0$.*
(i) If $h_0 \ge 0$ on $[0,1]$, then $h(r,t) > 0$ for $r \in (0,1)\times(0,T)$
(ii) Let k be a positive integer. If $h_0 \le k\pi$ on $[0,1]$, then $h(r,t) < k\pi$ for $r \in (0,1)\times(0,T)$.

Proof. This follows from the maximum principle but we should be careful since there are apparently unbounded quatities near $r = 0$ in (2.7). We rewrite (2.7) and obtain

$$h_t = Ah_{rr} + Bh_r + Ch \tag{4.1}$$

with

$$\begin{aligned} A(r,t) &= J(h)^{-3}\sin^2 h/r^2 \\ B(r,t) &= J(h)^{-3}\left(h_r^2 + \tfrac{2\sin^2 h}{r^2}\right)\tfrac{1}{r} \\ C(r,t) &= -J(h)^{-3}\tfrac{\sin 2h}{2r^2 h}\left(2h_r^2 + \tfrac{\sin^2 h}{r^2}\right). \end{aligned}$$

Since $h(0,t)=0$ and $\lim_{r\to 0}\sin 2h/2h=1$, we see that for any $S\in(0,T)$ there is $\delta_S>0$ satisfying

$$C<0 \quad \text{in} \quad (0,\delta_S)\times(0,S].$$

By continuity of C on $(0,1]\times(0,T)$ the function C is bounded from above, *i.e.*,

$$C(r,t)\le M \quad \text{for} \quad (r,t)\in(0,1)\times(0,S)$$

with some constant M. As is standard, we consider $\omega=e^{-\lambda t}h$ with $\lambda>M$. Since ω satisfies

$$\omega_t - A\omega_{rr} - B\omega_r - (C-\lambda)\omega = 0 \quad \text{with} \quad A\ge 0,$$

ω cannot take a negative minimum on $(0,1)\times(0,S]$ by a weak maximum principle. Thus $h\ge 0$ on $(0,1)\times(0,T)$ since $S\in(0,T)$ is arbitrary. By our assumption $J(h)\neq 0$ on $(0,1)\times(0,T)$ the function h does not take zero at $(0,1)\times(0,T)$. Indeed, if h were equal to zero at some point of $(0,1)\times(0,T)$, then by $h\ge 0$ it is a minimum point so that h_r would vanish there. However, this would contradict our assumption $J(h)\neq 0$. This proves (i).

The proof of (ii) is similar if we replace h by $w=k\pi-h$ since $J(w)$ is bounded away from zero on $[0,1)\times(0,S]$. □

We shall give a comparison result for sub- and supersolutions of (2.7).

Lemma 4.2. *Assume that $h,g\in C^{2,1}([0,1]\times[0,T_1])$ satisfies $h(0,t)=g(0,t)=0$ for $t\in[0,T_1]$. For $v(r,t,\theta)=(1-\theta)g(r,t)+\theta h(r,t)$ assume that $J(v)>0$ for all $r\in[0,1]$, $t\in[0,T_1]$, $\theta\in[0,1]$. (The value at $r=0$ is given as the limit as $r\to 0$.) Assume that h and g are sub-and supersolutions of (2.7). In other words*

$$h_t\ge J(h)^{-3}\tau(h),\ g_t\le J(g)^{-3}\tau(g) \quad \text{in} \quad (0,1)\times(0,T_1).$$

If $h(1,t)\ge g(1,t)$ for all $t\in[0,T_1]$ and $h(r,0)\ge g(r,0)$ for all $r\in(0,1)$, then $h\ge g$ in $[0,1]\times[0,T_1]$.

Proof. The idea of the proof is standard. We shall apply a weak maximum principle for a linear parabolic equation. However, one should be careful about behavior near $r=0$ since coefficients include terms which looks singular at $r=0$.

We set

$$F(r,x,y,z)=\left(y^2+\frac{\sin^2 x}{r^2}\right)^{-\frac{3}{2}}\left\{\frac{\sin^2 x}{r^2}\left(z+\frac{2y}{r}-\frac{\sin 2x}{2r^2}\right)+\frac{y^2}{r}\left(y-\frac{\sin 2x}{r}\right)\right\}$$

and observe that

$$h_t\ge F(r,h,h_r,h_{rr}) \quad \text{and} \quad g_t\le F(r,g,g_r,g_{rr}).$$

By the mean value theorem the difference $w = h - g$ fulfills

$$w_t - A(r,t)w_{rr} - B(r,t)w_r - C(r,t)w \geq 0 \quad \text{in} \quad (0,1)\times(0,T_1)$$

with

$$A(r,t) = \int_0^1 F_z(r,v,v_r,v_{rr})d\theta,\ B(r,t) = \int_0^1 F_y(r,v,v_r,v_{rr})d\theta,\ C(r,t) = \int_0^1 F_x(r,v,v_r,v_{rr})d\theta.$$

By initial and boundary conditions we have

$$w(0,t) = 0,\ w(1,t) \geq 0 \quad \text{for} \quad t \in [0,T_1) \quad \text{and} \quad w(r,0) \geq 0 \quad \text{for} \quad r \in [0,1].$$

Moreover, $A \geq 0$ since

$$F_z(r,x,y,z) = \left(y^2 + \frac{\sin^2 x}{r^2}\right)^{-\frac{3}{2}} \frac{\sin^2 x}{r^2} \geq 0.$$

It suffices to prove that C is bounded from above in $(0,1)\times(0,T_1)$ to apply the maximum principle which yields $w \geq 0$ in $[0,1]\times[0,T_1]$. We calculate F_x to get

$$\begin{aligned}
F_x(r,x,y,z) &= \left(y^2 + \frac{\sin^2 x}{r^2}\right)^{-\frac{5}{2}} [P+Q+R] \\
P(r,x,y,z) &= \frac{\sin 2x}{r^2}\left(y^2 - \frac{\sin^2 x}{2r^2}\right) z \\
Q(r,x,y) &= -\frac{3\sin 2x}{2r^2}\left\{\frac{\sin^2 x}{r^2}\left(\frac{2}{r}y - \frac{\sin 2x}{2r^2}\right) + \frac{y^2}{r}\left(y - \frac{\sin 2x}{r}\right)\right\} \\
R(r,x,y) &= \left(y^2 + \frac{\sin^2 x}{r^2}\right)\left\{\frac{\sin 2x}{r^2}\left(\frac{2}{r}y - \frac{\sin 2x}{2r^2}\right) - \frac{\sin^2 x \cos 2x}{r^4} - \frac{2y^2\cos 2x}{r^2}\right\}.
\end{aligned}$$

We shall estimate P, Q, R with $x = v, y = v_r, z = v_{rr}$ near $r = 0$. Since $J(v) > 0$ up to $r = 0$, by regularity assumptions on v for each $a \in (0,1)$ there is $r_a > 0$ such that

$$y = v_r(r,t,\theta) > 0,\ a \leq \cos x < 1,\ a \leq \cos 2x < 1$$

for all $(r,t,\theta) \in W_a(t_0,\theta_0)$ with

$$W_a = W_a(t_0,\theta_0) = \{(r,t,\theta) | 0 < r < r_a,\ t \in [0,T_1],\ |t-t_0| < r_a,\ \theta \in [0,1],\ |\theta - \theta_0| < r_a\}.$$

for $(t_0,\theta_0) \in [0,T_1]\times[0,1]$.
We set

$$m = \min_{W_a} y, \quad M = \max_{W_a} y$$

and observe that $m > 0$. We may assume that $m \geq aM$ by taking r_a smaller if necessary ; r_a can be chosen independent of (t_0,θ_0). We have

$$am \leq \frac{\sin x}{r} \leq M, \quad 2a^2 m \leq \frac{\sin 2x}{r} \leq 2M \quad \text{in} \quad W_a$$

since

$$\frac{\sin x}{r} = \frac{\sin v(r,t,\theta)}{r} = (\cos\ v(r_c,t,\theta))v_r(r_c,t,\theta)$$

with some $r_c \in (0,r)$ (depending on t and θ). In the following calculations we shall use these estimates for $\sin x/r,\ \sin 2x/r$ as well as for $\cos x,\ \cos 2x$.

(i) Estimate for P. Since

$$m^2 - \frac{1}{2}M^2 \le y^2 - \frac{\sin^2 x}{2r^2} \le M^2 - \frac{1}{2}a^2m^2, 0 < \frac{\sin 2x}{r^2} \le 2M \quad \text{in} \quad W_a,$$

we conclude that

$$P(r,x,y,z) \le \frac{1}{r}M_1(a) \quad \text{in} \quad W_a$$

with

$$M_1(a) := 2MN \max\left\{ M^2 - \tfrac{1}{2}a^2m^2, \left(\tfrac{M^2}{2} - m^2\right)_+ \right\} > 0$$
$$N := \max_W |z| = \max_W |v_{rr}(r,t,\theta)|,\ W = (0,1)\times[0,T_1]\times[0,1].$$

(ii) Estimate for Q. Since

$$\begin{aligned} \frac{\sin^2 x}{r^2}\left(\frac{2}{r}y - \frac{\sin 2x}{2r^2}\right) &\ge a^2m^2\left(\frac{2}{r}m - \frac{M}{r}\right), \\ \frac{y^2}{r}\left(y - \frac{\sin 2x}{r}\right) &\ge \frac{m^2}{r}(m-2M) \quad \text{in } W_a, \end{aligned}$$

we observe that

$$\frac{-2r^2}{3\sin 2x}Q \ge \frac{m^2}{r}\{(2a^2+1)m - (a^2+2)M\} \quad \text{in} \quad W_a.$$

Since $a<1$, the right hand side is negative so that $Q \le M_2(a)/r^2 \quad \text{in} \quad W_a$ with $M_2(a) = 3m^2M\{(a^2+2)M - (2a^2+1)m\} > 0$.

(iii) Estimate for R. As in (i) and (ii) we estimate

$$(y^2 + \frac{\sin^2 x}{r^2})^{-1}R \le \frac{2M}{r^2}(2M - a^2m) - \frac{am^2}{r^2}a^2 - \frac{2m^2a}{r^2} = \frac{1}{r^2}M_3(a)$$

with $M_3(a) = 4M^2 - 2a^2mM - (a^3+2a)m^2$.
Since

$$M_3(a) \le M^2(4 - 2a^2a - (a^3+2a)a^2) = M^2(4 - 4a^3 - a^5),$$

we take $a<1$ close to 1 and observe that $M_3(a) < 0$. Thus we conclude that

$$R(r,x,y) \le \frac{1}{r^2}M_4(a) < 0 \quad \text{in} \quad W_a(t_0,\theta_0)$$

with $M_4(a) = m^2(1+a^2)M_3(a)$.

From (i)-(iii) we observe that

$$\begin{aligned} F_x(r,x,y,z) &\le \frac{1}{r^2}\left(y^2+\frac{\sin^2 x}{r^2}\right)^{-\frac{5}{2}}(rM_1(a)+M_5(a)), \\ M_5(a): &= M_2(a)+M_4(a) \\ &= 3m^2M\{(a^2+2)M-(2a^2+1)m\}+m^2(1+a^2)M_3(a). \end{aligned}$$

One is able to estimate

$$M_5(a) \le M^4[\{3(a^2+2)-3(2a^2+1)a\}+a^2(1+a^2)(4-4a^3-a^5)]$$

and conclude that $M_5(a)<0$ for $a<1$ sufficiently close to one. We shall take a close to 1 such that $M_3(a),\ M_5(a)<0$ and fix a. Then $F_x(r,x,y,z)<0 \quad \text{in} \quad W_a(t_0,\theta_0)$ provided that $r<-M_5(a)/M_1(a)$. Since $F_x(r,x,y,z)$ is continuous except the axis $r=0$, this implies that $F_x(r,x,y,z)$ is bounded in

$$U(t_0,\theta_0)=\{(r,t,\theta)|0<r<1,\ |t-t_0|<r_a,\ |\theta-\theta_0|<r_a,\ t\in[0,T_1],\ \theta\in[0,1]\}.$$

Since r_a can be taken indepedent of (t_0,θ_0), we conclude that $C(r,t)$ is bounded from above in $(0,1)\times(0,T_1)$. □

Proof of Main Theorem. By linear parabolic regularity theory [7] for u-equation we see that for $\sigma\in(0,T_0)$ the function $h^\sigma(r,t)=h(r,t+\sigma)$ belongs to $C^{2,1}([0,1]\times[0,T_\sigma])$ with $T_\sigma=T_0-2\sigma$, since we have assumed that $|\nabla u|\neq 0$. Moreover, $J(h)>0$ on $[0,1]\times[0,T_\sigma]$.

We take f in §3 so that f is a subsolution of (2.7)-(2.9) with initial value $h_0^\sigma=h^\sigma(\cdot,0)$. By Lemma 4.1 and by our assumption $0<h_0<\pi$ on (0,1) we see that $0<h<\pi$ on $(0,1)\times[0,T_0)$. Since $0\le f<\pi$ on $[0,1]\times[0,T_{\lambda_0})$ we see that

$$v=\theta h^\sigma+(1-\theta)f$$

satisfies $J(v)>0$ for all $r\in[0,1]$ (up to $r=1$), $t\in[0,\min(T_{\lambda_0},T_\sigma))$ (This is the only place we need the property that $0<h<\pi$ on (0,1).)

If $T_\sigma>T_{\lambda_0}$, then we apply Lemma 4.2 and conclude that

$$h^\sigma(r,t)\ge f(r,t) \quad \text{in} \quad [0,1]\times(0,T_{\lambda_0}).$$

Since $h(0,t)=f(0,t)=0$ so that

$$\frac{h^\sigma(r,t)-h^\sigma(0,t)}{r}\ge\frac{f(r,t)-f(0,t)}{r},$$

we observe that $h_r^\sigma(0,t)\ge f_r(0,t),\ t\in(0,T_{\lambda_0})$. By (3.3) we see that $f_r(0,t)\to\infty$ as $t\to T_{\lambda_0}$ which yields a contradiction: $\lim_{t\to T_{\lambda_0}} h_r^\sigma(0,t)=\infty$. This implies that $T_{\lambda_0}\ge T_\sigma$. Since σ is arbitrary and since $h\in C^{1,0}([0,1)\times[0,T_\sigma])\cap C^0([0,1]\times[0,T_\sigma])$, we send σ to zero by modifying λ_0 in an appropriate way to get $T_0<T_{\lambda_0}$ with λ_0 satisfying $f(r,0)\le h_0(r)$. Since $T_{\lambda_0}=\sqrt{2}\lambda_0/(1-\varepsilon)$ and ε is arbitrary, we conclude that $T_0<\sqrt{2}\lambda_0$. □

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