# ATTRACTORS OF ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GOVERNED BY TIME-DEPENDENT SUBDIFFERENTIALS

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**Abstract.** Let us consider a nonlinear evolution equation associated with time-dependent subdifferential in a separable Hilbert space. In this paper we treat an asymptotically periodic system which means that time-dependent terms converge to some time-periodic ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without uniqueness. In such a situation the corresponding dynamical systems are multivalued. In fact we discuss the stability of multivalued semiflows from the view-point of attractors. Namely, the main object of this paper is to construct a global attractor for the asymptotically periodic multivalued dynamical system, and to discuss the relationship to one for the limiting periodic system.

# 1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space $H$ of the form

$$v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \quad t > s \ (\geq 0), \tag{1.1}$$

where $v' = \frac{dv}{dt}$, $\partial\varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous (l.s.c.) convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued perturbation small relative to $\varphi^t$, and $f$ is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness, asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13], [14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Ôtani has already shown the existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was

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discussed by [28] from the view-point of attractors. For the time periodic case, assuming the periodicity conditions with same period $T_0$, $0 < T_0 < +\infty$, i.e.

$$\varphi^t = \varphi^{t+T_0}, \quad G(t,\cdot) = G(t+T_0,\cdot), \quad f(t) = f(t+T_0), \qquad \forall t \in R_+ := [0,\infty),$$

the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic stability was discussed in [29]. In fact, the author showed the existence and characterization of time-periodic global attractors for (1.1).

In this paper, for a given positive number $T_0 > 0$ let us treat the case when $\varphi^t$, $G(t,\cdot)$ and $f(t)$ are asymptotically $T_0$-periodic in time. Namely we assume that

$$\varphi^t - \varphi_p^t \longrightarrow 0, \quad G(t,\cdot) - G_p(t,\cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \tag{1.2}$$

in appropriate senses as $t \to +\infty$, where $\varphi_p^t = \varphi_p^{t+T_0}$, $G_p(t,\cdot) = G_p(t+T_0,\cdot)$ and $f_p(t) = f_p(t+T_0)$ for any $t \in R_+$. By the asymptotically $T_0$-periodic stability (1.2), we have the limiting $T_0$-periodic system for (1.1) of the form:

$$u'(t) + \partial\varphi_p^t(u(t)) + G_p(t,u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s \ (\geq 0). \tag{1.3}$$

In the case when $G(t,\cdot)$ and $G_p(t,\cdot)$ are single-valued, the asymptotically $T_0$-periodic problem has already been discussed in [11]. In order to guarantee the uniqueness of solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on $\varphi^t$, $\varphi_p^t$, $G(t,\cdot)$ and $G_p(t,\cdot)$. Then, they discussed the asymptotically $T_0$-periodic stability for (1.1) from the view-point of attractors (cf. [11]). The main object of this paper is to develop the result obtained in [11] in order to consider the large-time behaviour of solution for (1.1) without uniqueness. Namely, we would like to construct the attractor for the asymptotically $T_0$-periodic multivalued flows associated with (1.1). Moreover we shall discuss the relationship to the $T_0$-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In Section 3 we consider the limiting $T_0$-periodic problem (1.3) and recall the abstract results obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family $\{\varphi^t; t \geq 0\}$ which was constructed in [16]. And we present and prove the main results in this paper. In proving main results, we generalize the results obtained in [11] and [30]. In the final section we apply our abstract results to the parabolic variational inequality with asymptotically $T_0$-periodic double obstacles. Then we can discuss the asymptotic stability for the asymptotically $T_0$-periodic double obstacle problem without uniqueness of solutions.

**Notation.** Throughout this paper, let $H$ be a (real) separable Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)_H$. For a proper l.s.c. convex function $\varphi$ on $H$ we use the notation $D(\varphi)$, $\partial\varphi$ and $D(\partial\varphi)$ to indicate the effective domain, subdifferential and its domain of $\varphi$, respectively; for their precise definitions and basic properties see [4].

For two non-empty sets $A$ and $B$ in $H$, we define the so-called Hausdorff semi-distance

$$\mathrm{dist}_H(A,B) := \sup_{x\in A} \inf_{y\in B} |x-y|_H.$$

## 2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in $H$ of the form:

$$u'(t) + \partial\varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \in J, \tag{2.1}$$

where $J$ is an interval in $R_+$, $\partial\varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function $\varphi^t$ on $H$, $G(t,\cdot)$ is a multivalued operator from a subset $D(G(t,\cdot)) \subset H$ into $H$ for each $t \in R_+$ and $f$ is a given function in $L^2_{loc}(J; H)$.

We begin with the definition of solution for (2.1).

**Definition 2.1.** (i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if $u \in C(J; H) \cap W^{1,2}_{loc}((t_0, t_1]; H)$, $\varphi^{(\cdot)}(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{loc}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial\varphi^t(u(t)), \quad \text{a.e. } t \in J.$$

(ii) For any interval $J$ in $R_+$ and $f \in L^2_{loc}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if it is a solution of (2.1) on every compact subinterval of $J$ in the sense of (i).

(iii) Let $J$ be any interval in $R_+$ with initial time $s \in R_+$. For $f \in L^2_{loc}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on $J$ with given initial value $u_0 \in H$, if it is a solution of (2.1) on $J$ satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{loc}(R_+)$ and $W^{1,1}_{loc}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a_r'|_{L^2(t,t+1)} + \sup_{t \in R_+} |b_r'|_{L^1(t,t+1)} < +\infty \quad \text{for each } r \geq 0.$$

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function $\varphi^t$.

**Definition 2.2.** $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if $\varphi^t$ is a proper l.s.c. convex function on $H$ satisfying the following properties ($\Phi$1)-($\Phi$3):

**($\Phi$1)** For each $r > 0$, $s$, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

**($\Phi$2)** There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1|z|_H^2, \quad \forall t \in R_+, \ \forall z \in D(\varphi^t).$$

**($\Phi$3)** For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^t(z) \leq k\}$ is compact in $H$.

Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent multivalued perturbation $G(t,\cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$.

**Definition 2.3.** $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ if and only if $G(t,\cdot)$ is a multivalued operator from $D(G(t,\cdot)) \subset H$ into $H$ which fulfills the following conditions (G1)-(G5):

**(G1)** $D(\varphi^t) \subset D(G(t,\cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{loc}(J;H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on $J$ such that

$$g(t) \in G(t, v(t)) \text{ for a.e. } t \in J.$$

**(G2)** $G(t,z)$ is a convex subset of $H$ for any $z \in D(\varphi^t)$ and $t \in R_+$.

**(G3)** There are positive constants $C_2$, $C_3$ such that

$$|g|^2_H \leq C_2\varphi^t(z) + C_3, \quad \forall t \in R_+,\ \forall z \in D(\varphi^t), \forall g \in G(t,z).$$

**(G4)** (demi-closedness) If $z_n \in D(\varphi^{t_n})$, $g_n \in G(t_n, z_n)$, $\{t_n\} \subset R_+$, $\{\varphi^{t_n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t,z)$.

**(G5)** For each bounded subset $B$ of $H$, there exist positive constants $C_4(B)$ and $C_5(B)$ such that

$$\varphi^t(z) + (g, z-b)_H \geq C_4(B)|z|^2_H - C_5(B),$$
$$\forall t \in R_+,\ \forall g \in G(t,z), \forall z \in D(\varphi^t),\ \forall b \in B.$$

For given $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and a forcing term $f \in L^2_{loc}(R_+;H)$, we consider the following evolution equation

$$(\mathrm{E})_s \qquad u'(t) + \partial\varphi^t(u(t)) + G(t,u(t)) \ni f(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(\mathrm{E})_s$:

**(A)** [Existence of solution for $(\mathrm{E})_s$] (cf. [21, Theorem II, III])
The Cauchy problem for $(\mathrm{E})_s$ has at least one solution $u$ on $J = [s,+\infty)$ such that $(\cdot - s)^{\frac{1}{2}}u' \in L^2_{loc}(J;H)$, $(\cdot - s)\varphi^{(\cdot)}(u(\cdot)) \in L^\infty_{loc}(J)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s,+\infty)$, provided that $u_0 \in \overline{D(\varphi^s)}$. In particular, if $u_0 \in D(\varphi^s)$, then the solution $u$ satisfies that $u' \in L^2_{loc}(J;H)$ and $\varphi^{(\cdot)}(u(\cdot))$ is absolutely continuous on any compact interval in $J$.

**(B)** [Global boundedness of solutions for $(\mathrm{E})_s$] (cf. [25, Theorem 2.2])
Suppose that

$$S_f := \sup_{t \in R_+} |f|_{L^2(t,t+1;H)} < +\infty.$$

Then, the solution $u$ of the Cauchy problem for $(\mathrm{E})_s$ on $[s, +\infty)$ satisfies the following global estimate:

$$\sup_{t\geq s}|u(t)|_H^2+\sup_{t\geq s}\int_t^{t+1}\varphi^\tau(u(\tau))d\tau\leq N_1(1+S_f^2+|u_0|_H^2),$$

where $N_1$ is a positive constant independent of $f$, $s\in R_+$ and $u_0\in\overline{D(\varphi^s)}$. Moreover, for each $\delta>0$ and each bounded subset $B$ of $H$, there is a constant $N_2(\delta, B)>0$, depending only on $\delta>0$ and $B$, such that

$$\sup_{t\geq s+\delta}|u'|^2_{L^2(t,t+1;H)}+\sup_{t\geq s+\delta}\varphi^t(u(t))\leq N_2(\delta, B)$$

for the solution $u$ of the Cauchy problem for $(\mathrm{E})_s$ on $[s, +\infty)$ with $s\in R_+$ and $u_0\in\overline{D(\varphi^s)}\cap B$.

Next, let us remember a notion of convergence of convex functions.

**Definition 2.4.** (cf. [20]) Let $\psi$, $\psi_n$ $(n\in N)$ be proper l.s.c. and convex functions on $H$. Then we say that $\psi_n$ converges to $\psi$ on $H$ as $n\to+\infty$ in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

**(i)** for any subsequence $\{\psi_{n_k}\}\subset\{\psi_n\}$, if $z_k\to z$ weakly in $H$ as $k\to+\infty$, then

$$\liminf_{k\to+\infty}\psi_{n_k}(z_k)\geq\psi(z).$$

**(ii)** for any $z\in D(\psi)$, there is a sequence $\{z_n\}$ in $H$ such that

$$z_n\to z \text{ in } H \text{ as } n\to+\infty,\qquad \lim_{n\to+\infty}\psi_n(z_n)=\psi(z).$$

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

**(C)** Let $\{\varphi_n^t\}\in\Phi(\{a_r\},\{b_r\})$, $\{G_n(t,\cdot)\}\in\mathcal{G}(\{\varphi_n^t\})$ with common positive constants $C_1$, $C_2$, $C_3$, $C_4(B)$ and $C_5(B)$, $\{f_n\}\subset L^2(J;H)$, $J=[s,t_1]\subset R_+$ and $u_{0,n}\in\overline{D(\varphi_n^s)}$ for $n=1,2,\cdots$. Assume that

- **(i)** $\varphi_n^t$ converges to $\varphi^t$ on $H$ in the sense of Mosco [20] for each $t\in J$ (as $n\to+\infty$) and $\bigcup_{n=1}^{+\infty}\{z\in H;\ \varphi_n^t(z)\leq k\}$ is relatively compact in $H$ for every real $k>0$ and $t\in J$, where $\{\varphi^t\}\in\Phi(\{a_r\},\{b_r\})$ and $\varphi_n^t=\varphi^t$ if $n=+\infty$.
- **(ii)** if $z_n\in D(\varphi_n^{t_n})$, $g_n\in G_n(t_n,z_n)$, $\{t_n\}\subset R_+$, $\{\varphi_n^{t_n}(z_n)\}$ is bounded, $z_n\to z$ in $H$, $t_n\to t$ and $g_n\to g$ weakly in $H$ as $n\to+\infty$, then $g\in G(t,z)$, where $\{G(t,\cdot)\}\in\mathcal{G}(\{\varphi^t\})$.
- **(iii)** $f_n\to f$ weakly in $L^2(J;H)$ for some $f\in L^2(J;H)$ and $u_{0,n}\to u_0$ in $H$ for some $u_0\in\overline{D(\varphi^s)}$.

Denote by $u$ the solution of the Cauchy problem for $(\mathrm{E})_s$ on $J$ with $u(s) = u_0$ and by $u_n$ the solution of the Cauchy problem for $(\mathrm{E})_s$ with $\varphi^t$, $G$, $f$ replaced by $\varphi^t_n$, $G_n$, $f_n$, and with $u_n(s) = u_{0,n}$. Then $u_n$ converges to $u$ on $J$ in the sense that

$$u_n \to u \text{ in } C(J;H),\ (\cdot - s)^{\frac{1}{2}} u_n' \to (\cdot - s)^{\frac{1}{2}} u' \text{ weakly in } L^2(J;H),$$

$$\int_J \varphi^t_n(u_n(t))dt \to \int_J \varphi^t(u(t))dt \qquad \text{as } n \to +\infty.$$

# 3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a $T_0$-periodic system in $H$, of the form:

$$(\mathrm{P})_s \qquad u'(t) + \partial\varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s$$

for each $s \in R_+$, where $\varphi^t_p$, $G_p(t,\cdot)$ and $f_p(t)$ are $T_0$-periodic, namely periodic in time with the same period $T_0$, $0 < T_0 < +\infty$.

**Definition 3.1.** Let $T_0$ be a positive number. Then
(i) $\Phi_p(\{a_r\},\{b_r\};T_0)$ is the set of all $\{\varphi^t_p\} \in \Phi(\{a_r\},\{b_r\})$ satisfying $T_0$-periodicity condition:

$$\varphi^{t+T_0}_p(\cdot) = \varphi^t_p(\cdot) \quad \text{on } H, \quad \forall t \in R_+. \tag{3.1}$$

(ii) $\mathcal{G}_p(\{\varphi^t_p\};T_0)$ is the set of all $\{G_p(t,\cdot)\} \in \mathcal{G}(\{\varphi^t_p\})$ satisfying $T_0$-periodicity condition:

$$G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in } H, \quad \forall t \in R_+. \tag{3.2}$$

Throughout this section we assume that $\{\varphi^t_p\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$, $\{G_p(t,\cdot)\} \in \mathcal{G}_p(\{\varphi^t_p\};T_0)$ and $f_p \in L^2_{loc}(R_+;H)$ is $T_0$-periodic in time, namely

$$f_p(t + T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_+. \tag{3.3}$$

Here we note that $(\mathrm{P})_s$ can be considered as $(\mathrm{E})_s$ in Section 2. So, by the result (A) in Section 2, the Cauchy problem for $(\mathrm{P})_s$ has at least one solution $u$ on $[s, +\infty)$. Hence we can define the multivalued dynamical process associated with $(\mathrm{P})_s$ as follows:

**Definition 3.2.** For every $0 \le s \le t < +\infty$ we denote by $U(t,s)$ the mapping from $\overline{D(\varphi^s_p)}$ into $\overline{D(\varphi^t_p)}$ which assigns to each $u_0 \in \overline{D(\varphi^s_p)}$ the set

$$U(t,s)u_0 := \left\{ z \in H \;\middle|\; \begin{array}{c} \text{There is a solution } u \text{ of } (\mathrm{P})_s \text{ on } [s,+\infty) \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right\}. \tag{3.4}$$

Then we easily see the following properties of $\{U(t,s)\} := \{U(t,s); 0 \le s \le t < +\infty\}$:

**(U1)** $U(s,s) = I$ on $\overline{D(\varphi_p^s)}$ for any $s \in R_+$;

**(U2)** $U(t_2,s)z = U(t_2,t_1)U(t_1,s)z$ for any $0 \le s \le t_1 \le t_2 < +\infty$ and $z \in \overline{D(\varphi_p^s)}$;

**(U3)** $U(t+T_0, s+T_0)z = U(t,s)z$ for any $0 \le s \le t < +\infty$ and $z \in \overline{D(\varphi_p^s)}$, that is, $U$ is $T_0$-periodic.

**(U4)** $\{U(t,s)\}$ has the following demi-closedness:

- If $0 \le s_n \le t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in \overline{D(\varphi_p^{s_n})}$, $z \in \overline{D(\varphi_p^s)}$, $z_n \to z$ in $H$ and a element $w_n \in U(t_n,s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(t,s)z$

Next we define the discrete dynamical system in order to construct a global attractor for $(\mathrm{P})_s$.

**Definition 3.3.** Let $U(\cdot,\cdot)$ be the solution operator for $(\mathrm{P})_s$ defined by Definition 3.2. Then
(i) For each $\tau \in R_+$, we denote by $U_\tau$ the $T_0$-step mapping from $\overline{D(\varphi_p^\tau)}$ into $\overline{D(\varphi_p^{\tau+T_0})} = \overline{D(\varphi_p^\tau)}$, namely,

$$U_\tau := U(\tau + T_0, \tau).$$

(2) For any $k \in Z_+ := N \cup \{0\}$, we define

$$U_\tau^k := \underbrace{U_\tau \circ U_\tau \circ \cdots \circ U_\tau}_{k\text{-th iteration}}.$$

Clearly we have $U_\tau^k = U(\tau + kT_0, \tau)$ for any $\tau \in R_+$ and $k \in Z_+$.

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems $U_\tau$ associated with $(\mathrm{P})_s$.

**Theorem 3.1.** (cf. [29, Theorem 3.1]) *Assume that* $\{\varphi_p^t\} \in \Phi_p(\{a_r\},\{b_r\};T_0)$, $\{G_p(t,\cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\};T_0)$, $f_p \in L^2_{loc}(R_+;H)$ *satisfies the* $T_0$*-periodicity condition (3.3). Then, for each* $\tau \in R_+$*, there exists a subset* $\mathcal{A}_\tau$ *of* $D(\varphi_p^\tau)$ *such that*

**(i)** $\mathcal{A}_\tau$ *is non-empty and compact in* $H$*;*

**(ii)** *for each bounded set* $B$ *in* $H$ *and each number* $\epsilon > 0$ *there exists* $N_{B,\epsilon} \in N$ *such that*

$$dist_H(U_\tau^k z, \mathcal{A}_\tau) < \epsilon$$

*for all* $z \in \overline{D(\varphi_p^\tau)} \cap B$ *and all* $k \ge N_{B,\epsilon}$*;*

**(iii)** $U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ *for any* $k \in N$.

**Remark 3.1.** By [29, Lemma 3.1] we can get the compact absorbing set $B_{0,\tau}$ of $\overline{D(\varphi_p^\tau)}$ for $U_\tau$ such that for each bounded subset $B$ of $H$ there is a positive integer $n_B$ (independent of $\tau \in R_+$) satisfying

$$U_\tau^n \left(\overline{D(\varphi_p^\tau)} \cap B\right) \subset B_{0,\tau} \quad \text{for all } n \ge n_B.$$

Then we observe that the global attractor $\mathcal{A}_\tau$ is given by the $\omega$-limit set of the absorbing set $B_{0,\tau}$ for $U_\tau$, i.e.

$$\mathcal{A}_\tau = \bigcap_{n\in Z_+} \overline{\bigcup_{k\geq n} U_\tau^k B_{0,\tau}}.$$

The next theorem is concerned with a relationship between two global attractors $\mathcal{A}_s$ and $\mathcal{A}_\tau$. For detail proof, see [29].

**Theorem 3.2.** (cf. [29, Theorem 3.2]) *Suppose the same assumptions are made as in Theorem 3.1. Let $\mathcal{A}_s$ and $\mathcal{A}_\tau$ be the global attractors for $U_s$ and $U_\tau$, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have*

$$\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s,$$

*where $U(\tau, s)$ is the $T_0$-periodic process given in Definition 3.2.*

**Remark 3.2.** By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor $\mathcal{A}_\tau$ for $U_\tau$ is $T_0$-periodic in $\tau$. In fact, for each $\tau \in R_+$ choose $m_\tau \in Z_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $\mathcal{A}_\tau = \mathcal{A}_{\sigma_\tau}$.

The third known result is the existence of a global attractor for the $T_0$-periodic multivalued dynamical system $(\mathrm{P})_s$.

**Theorem 3.3.** (cf. [29, Theorem 3.3]) *Under the same assumptions as Theorem 3.1, put*

$$\mathcal{A} := \bigcup_{0\leq\tau\leq T_0} \mathcal{A}_\tau,$$

*where $\mathcal{A}_\tau$ is as obtained in Theorem 3.1. Then, $\mathcal{A}$ has the following properties:*

**(i)** *$\mathcal{A}$ is non-empty and compact in $H$;*

**(ii)** *for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists a finite time $T_{B,\epsilon} > 0$ such that*

$$dist_H(U(t+\tau, \tau)z, \mathcal{A}) < \epsilon$$

*for all $\tau \in R_+$, all $z \in \overline{D(\varphi_p^\tau)} \cap B$ and all $t \geq T_{B,\epsilon}$.*

**Remark 3.3.** In [29, Section 4] the characterization of the $T_0$-periodic global attractor was discussed. The author proved that for each time $\tau \in R_+$ the global attractor $\mathcal{A}_\tau$ for the discrete multivalued dynamical system $U_\tau$ coincides with the cross-section of the family of all global bounded complete trajectories for the $T_0$-periodic system $(\mathrm{P})_s$.

# 4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \begin{array}{c} \psi \text{ is proper, l.s.c. and convex on } H, \\ \exists z \in D(\psi) \text{ s.t. } |z|_H \leq M, \ \psi(z) \leq M \end{array} \right\}.$$

Then let us introduce the notion of a metric topology on $\Psi_M$ which was introduced in [16].

Given $\varphi,\ \psi \in \Psi_M$, we define $\rho(\varphi, \psi; \cdot):\ D(\varphi) \to R$ by putting

$$\rho(\varphi, \psi; z) = \inf\{\max(|y - z|_H, \psi(y) - \varphi(z)); y \in D(\psi)\}$$

for each $z \in D(\varphi)$, and for each $r \geq M$

$$\rho_r(\varphi, \psi) := \sup_{z \in L_\varphi(r)} \rho(\varphi, \psi; z),$$

where $L_\varphi(r) := \{z \in D(\varphi); |z|_H \leq r,\ \varphi(z) \leq r\}$. Moreover, for each $r \geq M$, we define the functional $\pi_r(\cdot, \cdot)$ on $\Psi_M \times \Psi_M$ by

$$\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \qquad \text{for } \varphi, \psi \in \Psi_M.$$

Then, according to [16, Proposition 3.1], we can define a complete metric topology on $\Psi_M$ so that the convergence $\psi_n \to \psi$ in $\Psi_M$ (as $n \to +\infty$) if and only if

$$\pi_r(\psi_n, \psi) \to 0 \quad \text{for every } r \geq M.$$

Now by using the above topology on $\Psi_M$, we consider an asymptotically $T_0$-periodic system as follows.

**Definition 4.1.** Assume $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M,\ \{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $f \in L^2_{loc}(R_+; H)$. Then the system

$$(\mathrm{AP})_s \qquad v'(t) + \partial\varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H,\ t > s\ (\geq 0)$$

is asymptotically $T_0$-periodic, if there are $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M,\ \{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^t_p\}; T_0)$ and a $T_0$-periodic function $f_p \in L^2_{loc}(R_+; H)$ such that

**(A1)** (Convergence of $\varphi^t - \varphi^t_p \to 0$ as $t \to +\infty$) For each $r \geq M$,

$$J^{(r)}_m := \sup_{\sigma \in [0, T_0]} \pi_r(\varphi^{mT_0 + \sigma}, \varphi^\sigma_p) \to 0 \quad \text{as } m \to +\infty;$$

**(A2)** (Convergence of $G(t, \cdot) - G_p(t, \cdot) \to 0$ as $t \to +\infty$) If $\{\tau_n\} \subset [0, T_0]$, $\{m_n\} \subset Z_+$, $m_n \to +\infty$, $z_n \in D(\varphi^{m_n T_0 + \tau_n})$, $g_n \in G(m_n T_0 + \tau_n, z_n)$, $\{\varphi^{m_n T_0 + \tau_n}(z_n)\}$ is bounded, $z_n \to z$ in $H$, $\tau_n \to \tau$ and $g_n \to g$ weakly in $H$ (as $n \to +\infty$), then

$$g \in G_p(\tau, z);$$

**(A3)** (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0, T_0; H)} \longrightarrow 0 \text{ as } m \to +\infty.$$

By Definition 4.1 we easily see that a limiting system for $(\mathrm{AP})_s$ is a $T_0$-periodic one of the form:

$$(\mathrm{P})_s \qquad u'(t) + \partial\varphi_p^t(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H,\ t > s\ (\geq 0).$$

Here we note that $(\mathrm{AP})_s$ is also considered as $(\mathrm{E})_s$. So, by the result (A) in Section 2, the Cauchy problem for $(\mathrm{AP})_s$ has at least one solution $v$ on $[s, +\infty)$. Hence we can define the multivalued dynamical system associated with $(\mathrm{AP})_s$ as follows:

**Definition 4.2.** For every $0 \leq s \leq t < +\infty$ we denote by $E(t,s)$ the mapping from $\overline{D(\varphi^s)}$ into $\overline{D(\varphi^t)}$ which assigns to each $v_0 \in \overline{D(\varphi^s)}$ the set

$$E(t,s)v_0 := \left\{ z \in H \;\middle|\; \begin{array}{c} \text{There is a solution } v \text{ of } (\mathrm{AP})_s \text{ on } [s,+\infty) \\ \text{such that} \\ v(s) = v_0 \text{ and } v(t) = z. \end{array} \right\}.$$

Then we easily see that $\{E(t,s)\} := \{E(t,s); 0 \leq s \leq t < +\infty\}$ has the following evolution properties:

**(E1)** $E(s,s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$;

**(E2)** $E(t_2,s)z = E(t_2,t_1)E(t_1,s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$;

**(E3)** $\{E(t,s)\}$ has the following demi-closedness:

- If $0 \leq s_n \leq t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in \overline{D(\varphi^{s_n})}$, $z \in \overline{D(\varphi^s)}$, $z_n \to z$ in $H$ and a element $w_n \in E(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in E(t,s)z$

We begin with the definition of a discrete $\omega$-limit set for $E(\cdot,\cdot)$.

**Definition 4.3.** (Discrete $\omega$-limit set for $E(\cdot,\cdot)$) Let $\tau \in R_+$ be fixed. Let $\mathcal{B}(H)$ be a family of bounded subsets of $H$. Then for each $B \in \mathcal{B}(H)$, the set

$$\omega_\tau(B) := \bigcap_{n \in Z_+} \overline{\bigcup_{k \geq n, m \in Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{D(\varphi^{mT_0+\tau})} \cap B)}$$

is called the discrete $\omega$-limit set of $B$ under $E(\cdot,\cdot)$.

**Remark 4.1.** By definition of the discrete $\omega$-limit set $\omega_\tau(B)$, it is easy to see that $x \in \omega_\tau(B)$ if and only if there exist sequences $\{k_n\} \subset Z_+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_nT_0+\tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that

$$x_n \longrightarrow x \text{ in } H \text{ as } n \to +\infty.$$

Now let us mention main theorems in this paper.

**Theorem 4.1.** (Discrete attractors of $(\mathrm{AP})_\tau$) *For each $\tau \in R_+$, let $\mathcal{A}_\tau$ be the global attractor of $T_0$-periodic dynamical systems $U_\tau$, which is obtained in Section 3. For $\{\varphi^t\} \in$*

$\Phi(\{a_r\},\{b_r\}) \cap \Psi_M$, $\{G(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ *and* $f \in L^2_{loc}(R_+;H)$, *we assume that the system* $(AP)_s$ *is asymptotically* $T_0$*-periodic. Here we put*

$$\mathcal{A}^*_\tau := \overline{\bigcup_{B \in \mathcal{B}(H)} \omega_\tau(B)}. \tag{4.1}$$

*Then, we have*

**(i)** $\mathcal{A}^*_\tau (\subset D(\varphi^\tau_p))$ *is non-empty and compact in* $H$*;*

**(ii)** *for each bounded set* $B \in \mathcal{B}(H)$ *and each number* $\epsilon > 0$ *there exists* $N_{B,\epsilon} \in N$ *such that*

$$dist_H(E(kT_0+\tau,\tau)z, \mathcal{A}^*_\tau) < \epsilon$$

*for all* $z \in \overline{D(\varphi^\tau)} \cap B$ *and all* $k \geq N_{B,\epsilon}$*;*

**(iii)** $\mathcal{A}^*_\tau \subset U^l_\tau \mathcal{A}^*_\tau \subset \mathcal{A}_\tau$ *for any* $l \in N$, *where* $U_\tau$ *is the discrete dynamical system for* $(P)_\tau$ *given in Definition 3.3.*

**Remark 4.2.** By the definition of the discrete $\omega$-limit set $\omega_\tau(B)$ and $\mathcal{A}^*_\tau$, we easily see that

$$\mathcal{A}^*_\tau = \mathcal{A}^*_{\tau+nT_0}, \quad \forall n \in N.$$

Hence $\mathcal{A}^*_\tau$ is $T_0$-periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$.

**Theorem 4.2.** *Suppose the same assumptions are made as in Theorem 4.1. Let* $\mathcal{A}^*_s$ *and* $\mathcal{A}^*_\tau$ *be discrete attractors for* $E(\cdot,s)$ *and* $E(\cdot,\tau)$ *with* $0 \leq s \leq \tau < +\infty$, *respectively. Then,*

$$\mathcal{A}^*_\tau \subset U(\tau,s)\mathcal{A}^*_s.$$

*where* $U(\tau,s)$ *is the* $T_0$*-periodic process for* $(P)_s$ *which is given in Definition 3.2.*

By Theorems 4.1-4.2, we can get the attractor for asymptotic $T_0$-periodic system $(\text{AP})_\tau$.

**Theorem 4.3.** (Global attractor for $(\text{AP})_\tau$) *Suppose the same assumptions are made as in Theorem 4.1. For any* $\tau \in R_+$, *let* $\mathcal{A}^*_\tau$ *be the discrete attractor for* $E(\cdot,\tau)$ *obtained in Theorem 4.1. Here we put*

$$\mathcal{A}^* := \bigcup_{\tau \in [0,T_0]} \mathcal{A}^*_\tau. \tag{4.2}$$

*Then, for any bounded set* $B \in \mathcal{B}(H)$,

$$\bigcap_{s \geq 0} \overline{\bigcup_{t \geq s, \tau \in R_+} E(t+\tau,\tau)(\overline{D(\varphi^\tau)} \cap B)} \subset \mathcal{A}^*. \tag{4.3}$$

By Theorem 4.3, the set $\mathcal{A}^*$ can be called the global attractor of $(\mathrm{AP})_\tau$.

Here we give some key lemmas.

**Lemma 4.1.** *If $\{s_n\} \subset R_+$, $\{\tau_n\} \subset R_+$, $s \in R_+$, $\tau \in R_+$, $s_n \to s$, $\tau_n \to \tau$, $\{m_n\} \subset Z_+$ with $m_n \to +\infty$, $z_n \in \overline{D(\varphi^{m_n T_0 + s_n})}$, $z \in \overline{D(\varphi_p^s)}$, $z_n \to z$ in $H$ and a element $w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(\tau + s, s) z$*

**Proof.** Since $\tau_n \to \tau$, without loss of generality we may assume that there exists a finite time $T > 0$ such that $\{\tau_n\} \subset [0, T]$ and $\tau \in [0, T]$. By $w_n \in E(m_n T_0 + \tau_n + s_n, m_n T_0 + s_n) z_n$, there is a solution $v_n$ of $(\mathrm{AP})_{m_n T_0 + s_n}$ on $[m_n T_0 + s_n, +\infty)$ such that

$$v_n(m_n T_0 + \tau_n + s_n) = w_n \text{ and } v_n(m_n T_0 + s_n) = z_n.$$

Now we put $u_n(t) := v_n(t + m_n T_0 + s_n)$, then we easily see that $u_n$ is the solution for

$$\begin{cases} u_n'(t) + \partial\varphi^{t + m_n T_0 + s_n}(u_n(t)) + G(t + m_n T_0 + s_n, u_n(t)) \ni f(t + m_n T_0 + s_n), \quad t > 0, \\ u_n(0) = z_n. \end{cases}$$

Let $\delta \in (0, 1)$ be fixed. Since $z_n \to z$ in $H$ as $n \to +\infty$, $\{z_n\}$ is bounded in $H$. Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant $M_\delta > 0$ (independent of $n$) satisfying

$$\sup_{t \geq \delta} |u_n(t)|_H^2 + \sup_{t \geq \delta} |u_n'|^2_{L^2(t, t+1; H)} + \sup_{t \geq \delta} \varphi^{t + m_n T_0 + s_n}(u_n(t)) \leq M_\delta. \tag{4.4}$$

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies

$$\varphi^{t + m_n T_0 + s_n} \longrightarrow \varphi_p^{t+s} \text{ in the sense of Mosco [20]} \tag{4.5}$$

for each $t \geq 0$ as $n \to +\infty$. Moreover by the same argument in [10, Lemma 3.1] we can prove that

$$\bigcup_{n=1}^{+\infty} \{z \in H;\ \varphi^{t + m_n T_0 + s_n}(z) \leq k\} \text{ is relatively compact in } H \tag{4.6}$$

for every real $k > 0$ and $t \geq 0$, where $\varphi^{t + m_n T_0 + s_n} = \varphi_p^{t+s}$ if $n = +\infty$. Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of $\{n\}$, if necessary) we see that there is a function $u_\delta$ such that

$$u_\delta'(t) + \partial\varphi_p^{t+s}(u_\delta(t)) + G_p(t + s, u_\delta(t)) \ni f_p(t + s), \quad t > \delta.$$

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution $u$ on $[0, +\infty)$ satisfying

$$\begin{cases} u'(t) + \partial\varphi_p^{t+s}(u(t)) + G_p(t + s, u(t)) \ni f_p(t + s), \quad t > 0, \\ u(0) = z \end{cases}$$

and

$$u_n \longrightarrow u \text{ in } C([0, T]; H) \text{ as } n \to +\infty. \tag{4.7}$$

Then, by (4.7) and $u_n(\tau_n) = w_n$ we have $u(\tau) = w$, which implies that $w \in U(\tau + s, s)z$. $\diamondsuit$

By (B) in Section 2, for each $B \in \mathcal{B}(H)$ we can choose constants $r_B > 0$ and $M_B > 0$ so that

$$|v|_H \le r_B \quad \text{and} \quad \varphi^{t+s}(v) \le M_B, \tag{4.8}$$

for any $s \in R_+$, $t \ge T_0$, $z \in \overline{D(\varphi^s)} \cap B$ and $v \in E(t+s,s)z$. Hence it follows from condition (A1) that for each $m \in Z_+$, $\tau \in [0, T_0]$, $n \in N$ and $z \in \overline{D(\varphi^{mT_0+\tau})} \cap B$ there is $\tilde{z} := \tilde{z}_{mT_0+\tau,z,nT_0} \in D(\varphi_p^\tau)$ such that

$$|\tilde{z} - v|_H \le J_{m+n}^{(r_B+M_B+M)},$$

$$\Big(\text{hence } |\tilde{z}|_H \le r_B + J_{m+n}^{(r_B+M_B+M)}\Big)$$

and

$$\varphi_p^\tau(\tilde{z}) - \varphi^{nT_0+mT_0+\tau}(v) \le J_{m+n}^{(r_B+M_B+M)},$$

$$\Big(\text{hence } \varphi_p^\tau(\tilde{z}) \le M_B + J_{m+n}^{(r_B+M_B+M)}\Big).$$

where $v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z$.

Since $J_k^{(r_B+M_B+M)} \longrightarrow 0$ as $k \to +\infty$, there is a number $N_0 \in N$ such that

$$J_k^{(r_B+M_B+M)} \le 1, \quad \forall k > N_0.$$

Now, put $J_0 := 1 + \sup_{1 \le k \le N_0} J_k^{(r_B+M_B+M)} < +\infty$. Then, we define the bounded set $\widetilde{B_\tau}$ by

$$\widetilde{B_\tau} := \{z \in H; |z|_H \le r_B + J_0\} \cap \overline{D(\varphi_p^\tau)}.$$

Let $B_{0,\tau}$ be the compact absorbing set for $U_\tau$ introduced by Remark 3.1. Then, we see that there exists a number $\widetilde{N} \in N$ so that

$$U_\tau^l \widetilde{B_\tau} \subset B_{0,\tau}, \quad \forall l \ge \widetilde{N}. \tag{4.9}$$

The next lemma is very important to prove Theorem 4.1 (iii).

**Lemma 4.2.** *Let $\tau \in R_+$ and $B_{0,\tau}$ be the compact absorbing set for $U_\tau$. Then we have*

$$\omega_\tau(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).$$

**Proof.** At first we assume $\tau \in [0, T_0]$.

For each $B \in \mathcal{B}(H)$, let $x$ be any element of $\omega_\tau(B)$. Then, it follows from Remark 4.1 that there exist sequences $\{k_n\} \subset Z_+$ with $k_n \to +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_nT_0+\tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{as } n \to +\infty. \tag{4.10}$$

Let $\widetilde{N}$ be the positive integer obtained in (4.9). Then by (E2) we have

$$x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \widetilde{N}T_0 + m_nT_0 + \tau)$$

$$\circ E(k_nT_0 - \widetilde{N}T_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \tag{4.11}$$
$$\text{for any } n \text{ with } k_n \geq \widetilde{N} + 1.$$

Hence, there exists an element $y_n \in E(k_nT_0 - \widetilde{N}T_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that

$$x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \widetilde{N}T_0 + m_nT_0 + \tau)y_n. \tag{4.12}$$

Since $\{z_n\} \subset B$, we see that

$$|y_n|_H \leq r_B \ \text{ and } \ \varphi^{k_nT_0 - \widetilde{N}T_0 + m_nT_0 + \tau}(y_n) \leq M_B \ \text{ for any } n \text{ with } k_n \geq \widetilde{N} + 1,$$

where $r_B$ and $M_B$ are same positive constants in (4.8).

From the convergence condition (A1) it follows that for $y_n \in E(k_nT_0 - \widetilde{N}T_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ there is $\tilde{z}_n \in D(\varphi_p^\tau)$ such that

$$|\tilde{z}_n - y_n|_H \leq J^{(r_B + M_B + M)}_{k_n - \widetilde{N} + m_n},$$
$$(\text{hence } |\tilde{z}_n|_H \leq r_B + J^{(r_B + M_B + M)}_{k_n - \widetilde{N} + m_n})$$

and

$$\varphi_p^\tau(\tilde{z}_n) \ \leq M_B + J^{(r_B + M_B + M)}_{k_n - \widetilde{N} + m_n}.$$

Since $\{\tilde{z}_n \in D(\varphi_p^\tau)\ ;\ n \in N \text{ with } k_n \geq \widetilde{N} + 1\}(\subset \widetilde{B}_\tau)$ is relatively compact in $H$, we may assume that

$$\tilde{z}_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{ as } n \to +\infty$$

for some $\tilde{z}_\infty \in H$. Then we easily see that $\tilde{z}_\infty \in \widetilde{B}_\tau$ and

$$y_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{ as } n \to +\infty. \tag{4.13}$$

By Lemma 4.1 and (4.10)-(4.13), we observe that

$$x \in U(\widetilde{N}T_0 + \tau, \tau)\tilde{z}_\infty,$$

which implies that

$$x \in U(\widetilde{N}T_0 + \tau, \tau)\widetilde{B}_\tau = U_\tau^{\widetilde{N}}\widetilde{B}_\tau \subset B_{0,\tau}.$$

Hence we have

$$\omega_\tau(B) \subset B_{0,\tau}.$$

For the general case of $\tau \in R_+$, choose positive numbers $i_\tau \in N$ and $\tau_0 \in [0, T_0]$ so that $\tau = \tau_0 + i_\tau T_0$. Then, we can show $\omega_\tau(B) \subset B_{0,\tau}$ by the same argument as above. $\diamondsuit$

**Proof of Theorem 4.1.** On account of Lemma 4.2 we can get $\mathcal{A}_\tau^* \subset B_{0,\tau}$. Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that $\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^*$ for any $l \in N$.

Let $x$ be any element of $\mathcal{A}_\tau^*$. By the definition of $\mathcal{A}_\tau^*$, there are sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{ as } n \to +\infty. \tag{4.14}$$

Then, for each $n$ it follows from Remark 4.1 that there exist sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{ as } j \to +\infty. \tag{4.15}$$

Let $l$ be any number in $N$, then we see that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)$$
$$\circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j}$$

for $j$ with $k_{n,j} \geq l + 1$. So, there exists an element $w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \tag{4.16}$$

By global estimates (B) in Section 2, $\{w_{n,j} \in H\ ;\ j \in N \text{ with } k_{n,j} \geq l+1\}$ is relatively compact in $H$ for each $n$. Therefore we may assume that the element $w_{n,j}$ converges to some element $\widetilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\widetilde{w}_{n,\infty} \in \omega_\tau(B_n)$. Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

$$x_n \in U(lT_0 + \tau, \tau)\widetilde{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_\tau(B_n),$$

hence, we have

$$x_n \in \bigcup_{n \geq 1} U_\tau^l \omega_\tau(B_n), \quad \forall n \geq 1. \tag{4.17}$$

Here, by the closedness of $U(\cdot,\cdot)$ we note that for each subset $X$ of $B_{0,\tau}$,

$$\overline{U_\tau^l X} \subset U_\tau^l \overline{X}, \quad \forall l \in N. \tag{4.18}$$

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

$$\begin{aligned} x &\in \overline{\bigcup_{n\geq 1} U_\tau^l \omega_\tau(B_n)} \\ &= \overline{U_\tau^l \bigcup_{n \geq 1} \omega_\tau(B_n)} \\ &\subset U_\tau^l \overline{\bigcup_{n\geq 1} \omega_\tau(B_n)} \\ &\subset U_\tau^l \mathcal{A}_\tau^*, \end{aligned}$$

which implies that $\mathcal{A}_\tau^*$ is semi-invariant under the $T_0$-periodic dynamical systems $U_\tau$, i.e.

$$\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^*, \quad \forall l \in N. \tag{4.19}$$

Next we shall prove that $U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for any $l \in N$. By (4.19), for each $l \in N$

$$U_\tau^l \mathcal{A}_\tau^* \subset U_\tau^l U_\tau^n \mathcal{A}_\tau^* = U_\tau^{l+n} \mathcal{A}_\tau^*, \quad \forall n \in N. \tag{4.20}$$

By $\mathcal{A}_\tau^* \subset B_{0,\tau}$, (4.20) and the attractive property of $\mathcal{A}_\tau$, we have

$$U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall l \in N.$$

Therefore we conclude that

$$\mathcal{A}_\tau^* \subset U_\tau^l \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall l \in N.$$

$\diamondsuit$

**Proof of Theorem 4.2.** Let $x$ be any element of $\mathcal{A}_\tau^*$. Then by the definition of $\mathcal{A}_\tau^*$, there exist sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{ as } n \to +\infty. \tag{4.21}$$

From Remark 4.1 it follows that for each $n$, there are sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{ as } j \to +\infty. \tag{4.22}$$

Note that for given $s, \tau \in R_+$ with $s \le \tau$ there is a positive number $l_s \in N$ satisfying

$$s \le \tau \le l_s T_0 + s.$$

By using the property (E2) we see that

$$\begin{aligned} v_{n,j} \in E(&k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s) \\ &\circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s) \\ &\circ E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for any $j \in Z_+$ with $k_{n,j} \ge l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \tag{4.23}$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)y_{n,j} \tag{4.24}$$

and

$$y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_sT_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \tag{4.25}$$

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}|_H \le C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_sT_0 + s,\ m_{n,j}T_0 + \tau)z_{n,j}. \tag{4.26}$$

Here we define the bounded set $B_{C_n}$ by

$$B_{C_n} := \{b \in H \ ; |b|_H \le C_n\}.$$

From (4.26) and the result (B) in Section 2 it follows that the set

$$\left\{ w_{n,j} \in H \ ; \begin{array}{c} w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_sT_0 + s)y_{n,j} \\ \text{for any } j \in Z_+ \text{ with } k_{n,j} \ge l_s + 2 \end{array} \right\}$$

is relatively compact in $H$. Hence, we may assume that the element $w_{n,j}$ converges to some element $\widetilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\widetilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset \overline{D(\varphi_p^s)}.$$

Moreover, by Lemma 4.1 and (4.22)-(4.23) we have

$$x_n \in U(\tau, s)\widetilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \ge 1,$$

hence, we see that

$$x_n \in \bigcup_{n\ge1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \ge 1. \tag{4.27}$$

Here, by the closedness of $U(\cdot, \cdot)$, we note that for each subset $X$ of $B_{0,s}$,

$$\overline{U(\tau, s)X} \subset U(\tau, s)\overline{X}. \tag{4.28}$$

On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that

$$\begin{aligned} x \ &\in \overline{\bigcup_{n\ge1} U(\tau, s)\omega_s(B_{C_n})} \\ &= \overline{U(\tau, s) \bigcup_{n\ge1} \omega_s(B_{C_n})} \\ &\subset U(\tau, s)\overline{\bigcup_{n\ge1} \omega_s(B_{C_n})} \\ &\subset U(\tau, s)\mathcal{A}_s^*, \end{aligned}$$

which implies that $\mathcal{A}_\tau^*$ is the subset of $U(\tau, s)\mathcal{A}_s^*$, namely

$$\mathcal{A}_\tau^* \subset U(\tau, s)\mathcal{A}_s^*.$$

$\diamondsuit$

**Proof of Theorem 4.3.** For any $B \in \mathcal{B}(H)$, let $z_0$ be any element of the $\omega$-limit set $\omega_E(B)$ which is define by

$$\omega_E(B) := \bigcap_{s\ge0} \overline{\bigcup_{t\ge s, \tau\in R_+} E(t + \tau, \tau)(\overline{D(\varphi^\tau)} \cap B)}.$$

Then we easily see that there exist sequences $\{t_n\} \subset R_+$ with $t_n \uparrow +\infty$, $\{\tau_n\} \subset R_+$, $\{y_n\} \subset B$ with $y_n \in \overline{D(\varphi^{\tau_n})}$ and $\{z_n\} \subset H$ with $z_n \in E(t_n + \tau_n, \tau_n)y_n$ such that

$$t_n := k_n T_0 + t'_n,\ k_n \in Z_+,\ k_n \nearrow +\infty,\ t'_n \in [T_0, 2T_0],\ t'_n \to t'_0,$$
$$\tau_n := l_n T_0 + \tau'_n,\ l_n \in Z_+,\ \tau'_n \in [0, T_0],\ \tau'_n \to \tau'_0$$

and

$$z_n \longrightarrow z_0 \quad \text{in } H \tag{4.29}$$

as $n \to +\infty$. Without loss of generality, we may assume that

$$\text{(a)} \quad t'_n + \tau'_n \nearrow t'_0 + \tau'_0 \quad \text{or} \quad \text{(b)} \quad t'_n + \tau'_n \searrow t'_0 + \tau'_0.$$

Now, assume that (a) holds. Then let us consider the multivalued semiflow

$$v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0,\ k_n T_0 + l_n T_0 + t'_n + \tau'_n) z_n. \tag{4.30}$$

Then, there is a solution $u_n$ on $[k_n T_0 + l_n T_0 + t'_n + \tau'_n, +\infty)$ for

$$\begin{cases} u'_n(t) + \partial\varphi^{t+k_nT_0+l_nT_0+t'_n+\tau'_n}(u_n(t)) + G(t + k_nT_0 + l_nT_0 + t'_n + \tau'_n, u_n(t)) \\ \qquad\qquad \ni f(t + k_nT_0 + l_nT_0 + t'_n + \tau'_n), \quad t > 0, \\ u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n. \end{cases}$$

Since $z_n \to z_0$ in $H$, $\{z_n\}$ is bounded in $H$. Therefore by the global estimate (B) in Section 2, we see that

$$\left\{ v_n \in H;\ \begin{array}{c} v_n \in E(1 + k_nT_0 + l_nT_0 + t'_0 + \tau'_0,\ k_nT_0 + l_nT_0 + t'_n + \tau'_n)z_n \\ \text{for any } n \in N \end{array} \right\}$$

is relatively compact in $H$. Hence we may assume that

$$v_n \longrightarrow v \text{ in } H \text{ for some } v \in H. \tag{4.31}$$

Now applying Lemma 4.1 with (4.29)-(4.31), we can get

$$v \in U(1 + t'_0 + \tau'_0,\ t'_0 + \tau'_0) z_0,$$

more precisely, (taking the subsequence of $\{n\}$ if necessary) we observe that

$$u_n \longrightarrow u \quad \text{in } C([0,2]; H) \quad \text{as } n \to +\infty, \tag{4.32}$$

where $u$ is the solution $[t'_0 + \tau'_0, +\infty)$ satisfying

$$\begin{cases} u'(t) + \partial\varphi_p^{t+t'_0+\tau'_0}(u(t)) + G_p(t + t'_0 + \tau'_0, u(t)) \ni f_p(t + t'_0 + \tau'_0), \quad t > 0, \\ u(0) = z_0 \quad \text{and} \quad u(1) = v. \end{cases}$$

By (4.32) we easily see that

$$u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \longrightarrow z_0 \quad \text{as } n \to +\infty. \tag{4.33}$$

Note that

$$
\begin{aligned}
& u_n(t_0' + \tau_0' - t_n' - \tau_n') \\
\in\ & E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ k_nT_0 + l_nT_0 + t_n' + \tau_n')z_n \\
=\ & E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ l_nT_0 + \tau_n')y_n \\
=\ & E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ l_nT_0 + t_0' + \tau_0')E(l_nT_0 + t_0' + \tau_0',\ l_nT_0 + \tau_n')y_n.
\end{aligned}
$$

So, we can take a element $x_n \in E(l_nT_0 + t_0' + \tau_0',\ l_nT_0 + \tau_n')y_n$ such that

$$u_n(t_0' + \tau_0' - t_n' - \tau_n') \in E(k_nT_0 + l_nT_0 + t_0' + \tau_0',\ l_nT_0 + t_0' + \tau_0')x_n. \tag{4.34}$$

By $\{y_n\} \subset B$ and the global estimate (B) in Section 2, we easily see that $\{x_n\}$ is bounded, i.e.

$$\{x_n\} \subset \widetilde{B} \text{ for some } \widetilde{B} \in \mathcal{B}(H). \tag{4.35}$$

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that

$$z_0 \in \omega_{t_0'+\tau_0'}(\widetilde{B}) \subset \mathcal{A}^*_{t_0'+\tau_0'} \subset \mathcal{A}^*.$$

Thus (4.3) holds.

In the case (b) when $t_n' + \tau_n' \searrow t_0' + \tau_0'$, we can prove (4.3) by the slight modification of the proof as above. $\diamondsuit$

Theorem 4.1 implies that the attracting set $\mathcal{A}^*_\tau$ for $(\mathrm{AP})_\tau$ is semi-invariant under $U_\tau$ associated with the limiting $T_0$-periodic system $(\mathrm{P})_s$, in general. Moreover, from Theorem 4.2 we observe that

$$\mathcal{A}^*_\tau \subset U(\tau, s)\mathcal{A}^*_s \quad \text{for any } 0 \le s \le \tau < +\infty.$$

In order to get the invariance of $\mathcal{A}^*_\tau$ under $U_\tau$ and $\mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s$, let us use a concept of a regular approximation, which was introduced in [17].

**Definition 4.4.** (Regular approximation) Let $s \in R_+$ be fixed. Let $z \in D(\varphi_p^s)$. Then, we say that $U(t+s, s)z$ is regularly approximated by $E(t + kT_0 + s,\ kT_0 + s)$ as $k \to +\infty$, if for each finite $T > 0$ there are sequences $\{k_n\} \subset Z_+$ with $k_n \to +\infty$ and $\{z_n\} \subset H$ with $z_n \in D(\varphi^{k_nT_0+s})$ and $z_n \to z$ in $H$ satisfying the following property: for any function $u \in W^{1,2}(0, T; H)$ satisfying $u(t) \in U(t + s, s)z$ for all $t \in [0, T]$ there is a sequence $\{u_n\} \subset W^{1,2}(0, T; H)$ such that $u_n(t) \in E(t + k_nT_0 + s,\ k_nT_0 + s)z_n$ for all $t \in [0, T]$ and $u_n \to u$ in $C([0, T]; H)$ as $n \to +\infty$.

Using the above concept, we can show that the invariance of $\mathcal{A}^*_\tau$ under $U_\tau$. Moreover we can get

$$\mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s.$$

**Theorem 4.4** Suppose all assumptions in Theorem 4.1. Let $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$ be discrete attractors for $E(\cdot, s)$ and $E(\cdot, \tau)$, with $0 \le s \le \tau < +\infty$, respectively. Assume that for any point $z$ of $\mathcal{A}_s^*$, $U(t+s,s)z$ is regularly approximated by $E(t+kT_0+s, kT_0+s)$ as $k \to +\infty$. Then we have

$$\mathcal{A}_\tau^* = U(\tau,s)\mathcal{A}_s^*.$$

**Proof.** By Theorem 4.2, we have only to show that

$$U(\tau,s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*.$$

To do so, let $x$ be any element of $U(\tau,s)\mathcal{A}_s^*$.

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each $n \in N$

$$\begin{aligned} & U_\tau^n U(\tau,s)\mathcal{A}_s^* \\ = \;& U(nT_0+\tau,\tau)U(\tau,s)\mathcal{A}_s^* \\ = \;& U(nT_0+\tau, nT_0+s)U(nT_0+s,s)\mathcal{A}_s^* \\ = \;& U(\tau,s)U_s^n\mathcal{A}_s^* \\ \supset \;& U(\tau,s)\mathcal{A}_s^*. \end{aligned} \tag{4.36}$$

Hence, there exists a element $y_n \in \mathcal{A}_s^*$ such that

$$x \in U_\tau^n U(\tau,s)y_n = U(nT_0+\tau-s+s,s)y_n.$$

By using our assumption as $t = nT_0+\tau-s$, we observe that for each $n$, there are sequences $\{k_{n,j}\} \subset Z_+$, $\{x_{n,j}\} \subset H$ and $\{y_{n,j}\} \subset H$ such that

$$k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi^{k_{n,j}T_0+s}), \quad y_{n,j} \to y_n \;\text{ in } H$$

and

$$x_{n,j} \in E(nT_0+\tau-s+k_{n,j}T_0+s,\; k_{n,j}T_0+s)y_{n,j}, \quad x_{n,j} \to x \quad \text{in } H \tag{4.37}$$

as $j \to +\infty$. Therefore, by the usual diagonal argument, we can find a subsequence $\{j_n\}$ of $\{j\}$ such that $\tilde{x}_n := x_{n,j_n}$, $\tilde{y}_n := y_{n,j_n}$ and $\tilde{k}_n := k_{n,j_n}$ satisfy

$$|\tilde{x}_n - x|_H < \frac{1}{n}, \quad \tilde{x}_n \in E(nT_0+\tau-s+\tilde{k}_nT_0+s,\; \tilde{k}_nT_0+s)\tilde{y}_n, \quad |\tilde{y}_n - y_n|_H < \frac{1}{n} \tag{4.38}$$

for every $n = 1, 2, ....$ Since $\{\tilde{y}_n\}$ is bounded in $H$, there is a bounded set $B \in \mathcal{B}(H)$ so that $\{\tilde{y}_n\} \subset B$.

By (E2), we see that

$$\begin{aligned} \tilde{x}_n &\in E(nT_0+\tau-s+\tilde{k}_nT_0+s, \tilde{k}_nT_0+s)\tilde{y}_n \\ &= E(nT_0+\tilde{k}_nT_0+\tau, T_0+\tilde{k}_nT_0+\tau)E(T_0+\tilde{k}_nT_0+\tau, \tilde{k}_nT_0+s)\tilde{y}_n, \end{aligned}$$

hence there is an element $\tilde{z}_n \in E(T_0+\tilde{k}_nT_0+\tau,\; \tilde{k}_nT_0+s)\tilde{y}_n$ such that

$$\tilde{x}_n \in E(nT_0+\tilde{k}_nT_0+\tau,\; T_0+\tilde{k}_nT_0+\tau)\tilde{z}_n. \tag{4.39}$$

Since $\{\tilde{y}_n\} \subset B$ and the global estimate (B) in Section 2, we see that $\{\tilde{z}_n\}$ is also bounded in $H$. Hence, there is a bounded set $\tilde{B} \in \mathcal{B}(H)$ so that $\{\tilde{z}_n\} \subset \tilde{B}$. The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that $x \in \omega_\tau(\tilde{B}) \subset \mathcal{A}_\tau^*$. Thus we have $U(\tau,s)\mathcal{A}_s^* \subset \mathcal{A}_\tau^*$. $\diamondsuit$

By the same argument in Theorem 4.4, we can get the following corollary:
**Corollary.** (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that $\mathcal{A}_s^*$ is invariant under the $T_0$-periodic dynamical system $U_s(:= U(T_0+s,s))$. Namely,

$$\mathcal{A}_s^* = U_s^l \mathcal{A}_s^* \quad \text{for any } l \in N.$$

(ii) Assume that for any point $z$ of $\mathcal{A}_\tau$, $U(t+\tau,\tau)z$ is regularly approximated by $E(t+kT_0+\tau, kT_0+\tau)$ as $k \to +\infty$. Then, we have $\mathcal{A}_\tau^* \supset \mathcal{A}_\tau(= U_\tau \mathcal{A}_\tau)$. Hence by Theorem 4.1 (iii) we conclude that

$$\mathcal{A}_\tau^* = \mathcal{A}_\tau.$$

**Remark 4.3.** If the solution operator $U(t,s)$ is singlevalued, namely the solution for the Cauchy problem of $(\mathrm{P})_s$ is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic $T_0$-periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

# 5 Application to obstacle problems for PDE's

Let $\Omega$ be a bounded domain in $R^N$ $(1 \le N < +\infty)$ with smooth boundary $\Gamma = \partial\Omega$, $q$ be a fixed number with $2 \le q < +\infty$ and $T_0$ be a fixed positive number. We use the notation

$$a_q(v,z) := \int_\Omega |\nabla v|^{q-2} \nabla v \cdot \nabla z dx, \quad \forall v,\ z \in W^{1,q}(\Omega)$$

and denote by $(\cdot,\cdot)$ the usual inner product in $L^2(\Omega)$.

For prescribed obstacle functions $\sigma_0 \le \sigma_1$ and each $t \in R_+$ we define the set

$$K(t) := \left\{ z \in W^{1,q}(\Omega);\ \sigma_0(t,\cdot) \le z \le \sigma_1(t,\cdot) \ \text{ a.e. on } \Omega \right\}.$$

Let $f$ be a function in $L^2_{loc}(R_+; L^2(\Omega))$ and $h$ be a non-negative function on $R_+ \times R$.

Then for given $\mathbf{b} \in L^\infty(\Omega)^N$ we consider an interior asymptotically $T_0$-periodic double obstacle problem $(\mathrm{OP})_s^{AP}$ $(s \in R_+)$ :

- Find functions $v \in C([s,+\infty); L^2(\Omega))$ and $\theta \in L^2_{loc}((s,+\infty); L^2(\Omega))$ such that

$$(\mathrm{OP})_s^{AP} \begin{cases} v \in L^q_{loc}((s,+\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{loc}((s,+\infty); L^2(\Omega)); \\ v(t) \in K(t) \text{ for a.e. } t \ge s; \\ 0 \le \theta(t,x) \le h(t, v(t,x)) \quad \text{a.e. on } (s,+\infty) \times \Omega; \\ (v'(t) + \theta(t) + \mathbf{b} \cdot \nabla v(t) - f(t),\ v(t) - z) + a_q(v(t),\ v(t) - z) \le 0 \\ \text{for any } z \in K(t) \text{ and a.e. } t \ge s. \end{cases}$$

The main object of this section is to consider the large-time behaviour of solution for $(\mathrm{OP})_s^{AP}$ assuming asymptotically $T_0$-periodicity conditions

$$\sigma_i(t) - \sigma_{i,p}(t) \longrightarrow 0 \ \ (i = 0, 1), \quad h(t, \cdot) - h_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0$$

as $t \to \infty$ in the sense specified below, where $\sigma_{i,p}(t)$, $h_p(t, \cdot)$, $f_p(t)$ are periodic in time with the same period $T_0$. By the above assumptions, the limiting system of $(\mathrm{OP})_s^{AP}$ is a $T_0$-periodic one $(\mathrm{OP})_s^{P}$ as follows:

- Find functions $u \in C([s, +\infty); L^2(\Omega))$ and $\theta \in L^2_{loc}((s, +\infty); L^2(\Omega))$ such that

$$(\mathrm{OP})_s^P \begin{cases} u \in L^q_{loc}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{loc}((s, +\infty); L^2(\Omega)); \\ u(t) \in K_p(t) \text{ for a.e. } t \geq s; \\ 0 \leq \theta(t, x) \leq h_p(t, u(t, x)) \quad \text{a.e. on } (s, +\infty) \times \Omega; \\ (u'(t) + \theta(t) + \mathbf{b} \cdot \nabla u(t) - f_p(t),\ u(t) - z) + a_q(u(t),\ u(t) - z) \leq 0 \\ \text{for any } z \in K_p(t) \text{ and a.e. } t \geq s, \end{cases}$$

  where $K_p(t) := \{ z \in W^{1,q}(\Omega);\ \sigma_{0,p}(t, \cdot) \leq z \leq \sigma_{1,p}(t, \cdot) \ \text{ a.e. on } \Omega \}$.

Now we suppose the following conditions:

- $\sigma_i$ and $\sigma_{i,p}$ are functions on $R_+ \times \Omega$ such that

$$\sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_i}{dt} \right|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty,$$

$$\sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d\sigma_{i,p}}{dt} \right|_{L^2(t,t+1;L^\infty(\Omega))} < +\infty$$

  and $\sigma_{i,p}$ is a $T_0$-periodic obstacle function, i.e.

$$\sigma_{i,p}(t + T_0, x) = \sigma_{i,p}(t, x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+$$

  for $i = 0, 1$. Moreover, there are positive constants $k_1 > 0$ and $k_2 > 0$ such that

$$\sigma_1 - \sigma_0 \geq k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \geq k_1 \quad \text{a.e. on } R_+ \times \Omega$$

  and

$$|\sigma_i|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_i|_{L^\infty(R_+ \times \Omega)} + |\sigma_{i,p}|_{L^\infty(R_+;W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+ \times \Omega)} \leq k_2$$

  for $i = 0, 1$.

- $h$ and $h_p$ are non-negative continuous functions on $R_+ \times R$. There is a positive constant $L$ such that

$$|h(t, z_1) - h(t, z_2)| \leq L|z_1 - z_2|$$
$$|h_p(t, z_1) - h_p(t, z_2)| \leq L|z_1 - z_2|$$

  for all $t \in R_+$, $z_i \in R$ and $i = 1, 2$. Moreover, $h_p$ is a $T_0$-periodic function, i.e. for any $z \in R$, $h_p(t + T_0, z) = h_p(t, z)$ for any $t \in R_+$.

- $f,\ f_p \in L^2_{loc}(R_+; L^2(\Omega))$, and $f_p$ is a $T_0$-periodic function, i.e.

$$f_p(t+T_0) = f_p(t) \quad \text{in } L^2(\Omega), \quad \forall t \in R_+.$$

Moreover, we suppose the following convergence conditions:

- (Convergence of $\sigma_i(t) - \sigma_{i,p}(t) \longrightarrow 0$ as $t \to +\infty$) Put

$$I_m := \sup_{t\in[0,T_0]} |\sigma_0(mT_0+t) - \sigma_{0,p}(t)|_{W^{1,q}(\Omega)} + \sup_{t\in[0,T_0]} |\sigma_1(mT_0+t) - \sigma_{1,p}(t)|_{W^{1,q}(\Omega)}$$

$$+ \sup_{t\in[0,T_0]} |\sigma_0(mT_0+t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)} + \sup_{t\in[0,T_0]} |\sigma_1(mT_0+t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)}$$

Then,

$$I_m \longrightarrow 0 \quad \text{as } m \to +\infty;$$

- (Convergence of $h(t,\cdot) - h_p(t,\cdot) \longrightarrow 0$ as $t \to +\infty$) For any $z \in R$,

$$\sup_{t\in[0,T_0]} |h(mT_0+t,z) - h_p(t,z)| \longrightarrow 0 \text{ as } m \to +\infty; \tag{5.1}$$

- (Convergence of $f(t) - f_p(t) \longrightarrow 0$ as $t \to +\infty$)

$$|f(mT_0+\cdot) - f_p|_{L^2(0,T_0;L^2(\Omega))} \longrightarrow 0 \quad \text{as } m \to +\infty. \tag{5.2}$$

Under the above assumptions, let us consider problems $(\mathrm{OP})_s^{AP}$ and $(\mathrm{OP})_s^{P}$.

In order to apply the abstract results in Sections 2-4, we choose $L^2(\Omega)$ as a real separable Hilbert space $H$. And we define a family $\{\varphi^t\}$ of proper l.s.c. convex functions $\varphi^t$ on $L^2(\Omega)$ by

$$\varphi^t(z) = \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega |\nabla z|^q dx & \text{if } z \in K(t), \\ +\infty & \text{if } z \in L^2(\Omega) \setminus K(t), \end{cases} \tag{5.3}$$

and define $\varphi_p^t$ by replacing $K(t)$ by $K_p(t)$ in (5.3).

Also, we define a multivalued operator $G(\cdot,\cdot)$ from $R_+ \times H^1(\Omega)$ into $L^2(\Omega)$ by

$$G(t,z) := \left\{ g \in L^2(\Omega); \begin{array}{ll} g = l + \mathbf{b}\cdot\nabla z & \text{in } L^2(\Omega) \\ 0 \le l(x) \le h(t,z(x)) & \text{a.e. on } \Omega \end{array} \right\} \tag{5.4}$$

for all $t \in R_+$ and $z \in H^1(\Omega)$. And we define $G_p(\cdot,\cdot)$ by replacing $h(t,\cdot)$ by $h_p(t,\cdot)$ in (5.4).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

**Lemma 5.1.** (cf. [27, Lemma 5.1]) *Put for any $r>0$ and $t \in R_+$*

$$a_r(t) = b_r(t) := k_3 \int_0^t \left\{ |\sigma'_{0,p}|_{L^\infty(\Omega)} + |\sigma'_{0,p}|_{W^{1,q}(\Omega)} + |\sigma'_{1,p}|_{L^\infty(\Omega)} + |\sigma'_{1,p}|_{W^{1,q}(\Omega)} \right\} d\tau$$

$$+k_3 \int_0^t \left\{ |\sigma_0'|_{L^\infty(\Omega)} + |\sigma_0'|_{W^{1,q}(\Omega)} + |\sigma_1'|_{L^\infty(\Omega)} + |\sigma_1'|_{W^{1,q}(\Omega)} \right\} d\tau,$$

*where $k_3$ is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ and $\{\varphi_p^t\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)$.*

*Moreover we have $\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})$ and $\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi_p^t\}; T_0)$.*

**Lemma 5.2.** *The convergence assumptions (A1)-(A3) hold.*

**Proof.** We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each $t \in R_+$ there are $m \in Z_+$ and $\tau \in [0, T_0]$ so that $t = mT_0 + \tau$.

For each $z_p \in D(\varphi_p^t) = K_p(t)$, we put

$$z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).$$

Then we easily see that $z \in D(\varphi^t) = K(t)$. Moreover, by the same argument in [27, Lemma 5.1], we see that

$$|z - z_p|_{L^2(\Omega)} \le k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^q(\Omega)} \le k_4 I_m (1 + |\nabla z_p|_{L^q(\Omega)}) \tag{5.5}$$

for some constant $k_4 > 0$. Hence we have

$$\varphi^t(z) - \varphi_p^t(z_p) \le k_5 I_m (1 + \varphi_p^t(z_p)), \tag{5.6}$$

for a sufficiently large $k_5 > 0$.

Conversely, let $z \in D(\varphi^t) = K(t)$ and we put

$$z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).$$

Then, we observe that $z_p \in D(\varphi_p^t) = K_p(t)$ and

$$|z_p - z|_{L^2(\Omega)} \le k_4 I_m \quad \text{and} \quad \varphi_p^t(z_p) - \varphi^t(z) \le k_5 I_m (1 + \varphi^t(z)). \tag{5.7}$$

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. $\diamondsuit$

Clearly, the obstacle problem $(\mathrm{OP})_s^{AP}$ can be reformulated as an evolution equation $(\mathrm{AP})_s$ involving the subdifferential of $\varphi^t$ given by (5.3) and the multivalued operator $G(t, \cdot)$ defined by (5.4). Also, the limiting $T_0$-periodic problem $(\mathrm{OP})_s^P$ can be reformulated as an evolution equation $(\mathrm{P})_s$. Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor $\mathcal{A}_s^*$ for $(\mathrm{OP})_s^{AP}$, a $T_0$-periodic attractor $\mathcal{A}_s$ for $(\mathrm{OP})_s^P$ and the relationships between $(\mathrm{OP})_s^{AP}$ and $(\mathrm{OP})_s^P$.

Additionally, we assume that $f(t) \equiv f_p(t)$ for any $t \in R_+$ and

$$\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \le h(t, z)$$

for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get $\mathcal{A}_s^* = \mathcal{A}_s$ by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for $\sigma_i(t,\cdot)$, $h(t,\cdot)$ and $f(t)$ in order to get

$$U(\tau, s)\mathcal{A}_s^* = \mathcal{A}_\tau^* \subset \mathcal{A}_\tau \text{ for any } 0 \leq s \leq \tau < +\infty. \tag{5.8}$$

It seems difficult to show (5.8), so it is the open problem.

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