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Stability of Discrete Ground State

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Abstract

We present new criteria for a self-adjoint operator to have a ground state. As an application, we consider models of “quantum particles” coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

Key words: Ground state; discrete ground state; generalized spin-boson model; Fock space; Dereziński-Gérard model.

1 INTRODUCTION

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and bounded from below. We say that T has a discrete ground state if the bottom of the spectrum of T is an isolated eigenvalue of T . In that case a non-zero vector

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in $\ker(T - E_0(T))$ is called a ground state of T . Let S be a symmetric operator on $\mathcal{H}$. Suppose that T has a discrete ground state and S is T -bounded. By the regular perturbation theory [8, XII] it is already known that $T + \lambda S$ has a discrete ground state for “sufficiently small” $\lambda \in \mathbb{R}$. Our aim is to present new criteria for $T + \lambda S$ to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model [2] with a self-interaction term of a Bose field, which we call the GSB + ϕ^2 model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in [2]. In [2] A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian A has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where A does not have necessarily compact resolvent [4, 3]. In this paper, using a theorem in Section 2, we prove the existence of a ground state of the GSB + ϕ^2 model in the case where A does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model [5]. The Dereziński-Gérard model introduced in [5] and J. Dereziński and C. Gérard prove an existence of a ground state for their model under some conditions including that A has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where A does not have compact resolvent. Our strategy to establish a ground state is the same as in Section 3.

2 BASIC RESULTS

Let $\mathcal{H}$ be a separable complex Hilbert space. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the scalar product on Hilbert space $\mathcal{H}$ and by $\|\cdot\|_{\mathcal{H}}$ the associated norm. Scalar product $\langle f, g \rangle_{\mathcal{H}}$ is linear in g and antilinear in f . We omit $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, respectively if there is no danger of confusion. For a linear operator T in Hilbert space, we denote by $D(T)$ and $\sigma(T)$ the domain and the spectrum of T respectively. If T is self-adjoint and bounded from below, then we define

$$E_0(T) := \inf \sigma(T), \quad \Sigma(T) := \inf \sigma_{\text{ess}}(T),$$

where $\sigma_{\text{ess}}(T)$ is the essential spectrum of T . If T has no essential spectrum, then we set $\Sigma(T) = \infty$. For a self-adjoint operator T , we denote the form domain of T by $Q(T)$. In this paper, an eigenvector of a self-adjoint operator T with eigenvalue $E_0(T)$ is called a ground state of T (if it exists). We say that T has a ground state if $\dim \ker(T - E_0(T)) > 0$.

The basic results are as follows:

Theorem 2.1. *Let H be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator V on $\mathcal{H}$ satisfying the following conditions (i)-(iii):*

- (i) $D(H) \subset D(V)$.
- (ii) V is bounded from below, and $\Sigma(V) > 0$.
- (iii) $H - E_0(H) \geq V$ on $D(H)$.

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, H has a ground state.

Proof. For all $u_1, \dots, u_{n-1} \in \mathcal{H}$, we have

$$\inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, H\Psi \rangle - E_0(H) \geq \inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, V\Psi \rangle,$$

where $\text{L.h.}[\dots]$ denotes the linear hull of the vectors in $[\dots]$. Since $D(H) \subset D(V)$, we have that

$$\inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, V\Psi \rangle \geq \inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(V)}} \langle \Psi, V\Psi \rangle.$$

Hence, for all $n \in \mathbb{N}$

$$\mu_n(H) - E_0(H) \geq \mu_n(V).$$

where

$$\mu_n(H) := \sup_{u_1, \dots, u_{n-1} \in \mathcal{H}} \inf_{\substack{\Psi \in \text{L.h.}[u_1, \dots, u_{n-1}]^\perp \\ \|\Psi\|=1, \Psi \in D(H)}} \langle \Psi, H\Psi \rangle.$$

By the min-max principle ([8, Theorem XIII.1]), $\lim_{n \rightarrow \infty} \mu_n(H) = \Sigma(H)$ and $\lim_{n \rightarrow \infty} \mu_n(V) = \Sigma(V)$. Therefore we obtain

$$\Sigma(H) - E_0(H) \geq \Sigma(V) > 0.$$

This means that H has purely discrete spectrum in $[E_0(H), E_0(H) + \Sigma(V))$. ■

Theorem 2.2. *Let H be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator V on $\mathcal{H}$ satisfying the following conditions (i)-(iii):*

- (i) $Q(H) \subset Q(V)$.
- (ii) V is bounded from below, and $\Sigma(V) > 0$.
- (iii) $H - E_0(H) \geq V$ on $Q(H)$.

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(V))$. In particular, H has a ground state.

Proof. Similar to the proof of Theorem 2.1. ■

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

Theorem 2.3. *Let A be a self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$, and let B be a symmetric operator on $D(A)$. Suppose that $A + B$ is self-adjoint on $D(A)$ and that there exist constants $a \in [0, 1)$ and $b \geq 0$ such that*

$$|\langle \psi, B\psi \rangle| \leq a\langle \psi, A\psi \rangle + b\|\phi\|^2, \quad \psi \in D(A).$$

Assume

$$\frac{b + E_0(A + B)}{1 - a} < \Sigma(A). \quad (1)$$

Then $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state.

Proof. By the assumption we have

$$A + B - E_0(A + B) \geq (1 - a)A - b - E_0(A + B)$$

on $D(A)$, and $(1 - a)\Sigma(A) - b - E_0(A + B) > 0$. Hence we can apply Theorem 2.1, to conclude that $A + B$ has purely discrete spectrum in $[E_0(A + B), (1 - a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state. ■

Remark. It is easily to see that $-b \leq E_0(A + B) \leq b$. Therefore condition (1) is satisfied if

$$\frac{2b}{1 - a} < \Sigma(A).$$

Theorem 2.4. *Let $\mathcal{H}, \mathcal{K}$ be complex separable Hilbert spaces. Let A and B be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that $E_0(A) = E_0(B) = 0$. We set*

$$T_0 := A \otimes I + I \otimes B.$$

Let Z be a symmetric sesquilinear form on $Q(T_0)$, and assume that there exist constants $a_1 \in [0, 1)$, $a_2 \in [0, 1)$ and $b \geq 0$ such that, for all $\Psi \in Q(T_0)$

$$|Z(\Psi, \Psi)| \leq a_1 \langle \Psi, A \otimes I \Psi \rangle_{\text{form}} + a_2 \langle \Psi, I \otimes B \Psi \rangle_{\text{form}} + b \|\Psi\|^2,$$

where $\langle \Psi, A \otimes I \Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I \Psi\|^2$. Therefore, by the KLMN theorem there exists a unique self-adjoint operator T on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and $T = T_0 + Z$ in the sense of sesquilinear form on $Q(T_0)$. We set

$$s := \min\{(1 - a_1)\Sigma(A), (1 - a_2)\Sigma(B)\}.$$

Assume

$$s > b + E_0(T). \quad (2)$$

Then, T has purely discrete spectrum in the interval $[E_0(T), s - b)$. In particular, T has a ground state.

Proof. Similar to the proof of Theorem 2.3. ■

Remark. It is easy to see that $-b \leq E_0(T) \leq b$. Therefore the condition (2) is satisfied if

$$s > 2b.$$

Remark. Theorem 2.4 is essentially same as [4 , Theorem B.1] But our proof is very simple.

3 Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{F}_b$ be the Boson Fock space over $L^2(\mathbb{R}^d)$:

$$\mathcal{F}_b := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^d) \right].$$

The Hilbert space of the quantum field model we consider is

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b.$$

Let $\omega : \mathbb{R}^d \rightarrow [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for all most everywhere (a.e.) $k \in \mathbb{R}^d$. We denote the multiplication operator by the function ω acting in $L^2(\mathbb{R}^d)$ by the same symbol ω . We set

$$H_b := d\Gamma_b(\omega)$$

the second quantization of ω (e.g. [7, Section X.7]). We denote by $a(f)$, $f \in L^2(\mathbb{R}^d)$, the smeared annihilation operators on $\mathcal{F}_b$. It is a densely defined closed linear operator on $\mathcal{F}_b(\mathbb{R}^d)$ (e.g. [7, Section X.7]). The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$, $g \in L^2(\mathbb{R}^d)$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in L^2(\mathbb{R}^d)$ on the dense subspace

$$\mathcal{F}_0 := \{\psi = (\psi^{(n)})_{n=0}^\infty \in \mathcal{F}_b \mid \text{there exists a number } n_0 \text{ such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X, Y] = XY - YX$. The symmetric operator

$$\phi(f) := \frac{1}{\sqrt{2}}[a(f)^* + a(f)],$$

called the Segal field operator, is essentially self-adjoint on $\mathcal{F}_0$ (e.g. [7, Section X.7]). We denote its closure by the same symbol. Let A be a positive self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then, the unperturbed Hamiltonian of the model is defined by

$$H_0 := A \otimes I + I \otimes H_b$$

with domain $D(H_0) = D(A \otimes I) \cap D(I \otimes H_b)$. For $g_j, f_j \in L^2(\mathbb{R}^d)$ $j = 1, \dots, J$, and $B_j (j = 1, \dots, J)$ a symmetric operator on $\mathcal{H}$, we define a symmetric operator

$$H_1 := \sum_{j=1}^J B_j \otimes \phi(g_j),$$

$$H_2 := \sum_{j=1}^J I \otimes \phi(f_j)^2.$$

The Hamiltonian of the model we consider is of the form

$$H(\lambda, \mu) := H_0 + \lambda H_1 + \mu H_2,$$

where $\lambda \in \mathbb{R}$ and $\mu \geq 0$ are coupling parameters.

For $H(\lambda, \mu)$ to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

- [H.1] $g_j \in D(\omega^{-1/2})$, $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$, $j = 1, \dots, J$.
[H.2] $D(A^{1/2}) \subset \cap_{j=1}^J D(B_j)$ and there exist constants $a_j \geq 0$, $b_j \geq 0$, $j = 1, \dots, J$, such that,

$$\|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in D(A^{1/2}).$$

$$[H.3] \quad |\lambda| \sum_{j=1}^J a_j \|g_j / \sqrt{\omega}\| < 1.$$

Proposition 3.1. *Assume [H.1], [H.2] and [H.3]. Then, $H(\lambda, \mu)$ is self-adjoint with $D(H(\lambda, \mu)) = D(H_0) \subset D(H_1) \cap D(H_2)$ and bounded from below. Moreover, $H(\lambda, \mu)$ is essentially self-adjoint on every core of H_0 .*

Remark. This proposition has no restriction of the coupling parameter $\mu \geq 0$.

* * *

To perform a finite volume approximation, we need an additional condition:

- [H.4] The function $\omega(k)$ ($k \in \mathbb{R}^d$) is continuous with

$$\lim_{|k| \rightarrow \infty} \omega(k) = \infty,$$

and there exist constants $\gamma > 0$, $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C |k - k'|^\gamma [1 + \omega(k) + \omega(k')], \quad k, k' \in \mathbb{R}^d.$$

Let

$$m := \inf_{k \in \mathbb{R}^d} \omega(k). \tag{3}$$

If A has compact resolvent, we can prove the extension of the previous theorem [2, Theorem 1.2]

Theorem 3.2. *Consider the case $m > 0$. Suppose that A has entire purely discrete spectrum. Assume Hypotheses [H.1]-[H.4]. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval $[E_0(H(\lambda, \mu)), E_0(H(\lambda, \mu)) + m)$. In particular, $H(\lambda, \mu)$ has a ground state.*

Remark. This theorem has no restriction of the coupling parameter $\mu \geq 0$.

Remark. In the case $m > 0$, the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1, \dots, J.$$

For a vector $v = (v_1, \dots, v_J) \in \mathbb{R}^J$ and $h = (h_1, \dots, h_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d)$, we define

$$M_v(h) = \sum_{j=1}^J v_j \|h_j\|.$$

We set

$$g = (g_1, \dots, g_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d), \quad f = (f_1, \dots, f_J) \in \bigoplus_{j=1}^J L^2(\mathbb{R}^d),$$

and

$$a = (a_1, \dots, a_J), \quad b = (b_1, \dots, b_J).$$

For $\theta, \epsilon, \epsilon'$, we introduce the following constants:

$$\begin{aligned} C_{\theta, \epsilon} &:= \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g), \\ D_{\theta, \epsilon'} &:= M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}), \\ E_{\epsilon, \epsilon'} &:= M_a(g)/8\epsilon + M_b(g/\sqrt{\omega})/2\epsilon' + M_b(g)/\sqrt{2}. \end{aligned}$$

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda, g} := \begin{cases} \left(\frac{|\lambda| M_a(g/\sqrt{\omega})}{2}, \frac{1}{|\lambda| M_a(g/\sqrt{\omega})} \right), & |\lambda| M_a(g/\sqrt{\omega}) \neq 0 \\ [0, \infty], & |\lambda| M_a(g/\sqrt{\omega}) = 0 \end{cases}$$

It is easy to see that $[1/2, 1] \subset I_{\lambda, g}$. Therefore, for all $\theta \in I_{\lambda, g}$,

$$\begin{aligned} 1 - \theta |\lambda| M_a(g/\sqrt{\omega}) &> 0, \\ 1 - \frac{|\lambda| M_a(g/\sqrt{\omega})}{2\theta} &> 0. \end{aligned}$$

We define for $\theta \in I_{\lambda,g}$,

$$\mathbf{S}_\theta := \{(\epsilon, \epsilon') | \epsilon, \epsilon' > 0, |\lambda|C_{\theta,\epsilon} < 1, |\lambda|D_{\theta,\epsilon'} < 1\}.$$

Next we set

$$\tau_{\theta,\epsilon,\epsilon'} := (1 - |\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'},$$

and

$$\mathbf{T} := \{(\theta, \epsilon, \epsilon') \in \mathbb{R}^3 | \theta \in I_{\lambda,g}, (\epsilon, \epsilon') \in \mathbf{S}_\theta, \tau_{\theta,\epsilon,\epsilon'} > E_0(H(\lambda, \mu))\}.$$

Theorem 3.3. *Consider the case $m > 0$. Suppose that $\sigma_{\text{ess}}(A) \neq \emptyset$. Assume Hypothesis [H.1]-[H.4], and $\mathbf{T} \neq \emptyset$. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval*

$$[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \sup_{(\theta,\epsilon,\epsilon') \in \mathbf{T}} \tau_{\theta,\epsilon,\epsilon'}\}]. \quad (4)$$

In particular, $H(\lambda, \mu)$ has a ground state.

Remark. $\mathbf{T} \neq \emptyset$ is necessary condition for A to have a discrete ground state. Conversely, if A has a discrete ground state, then $\mathbf{T} \neq \emptyset$ holds for sufficiently small λ, μ . Therefore the condition $\mathbf{T} \neq \emptyset$ is a restriction for the coupling constants λ, μ .

* * *

3.1 Proof of Proposition 3.1

In what follows, we write simply

$$H := H(\lambda, \mu).$$

For $\mathcal{D}$ a dense subspace of $L^2(\mathbb{R}^d)$, we define

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) := \text{L.h}[\{\Omega, a(h_1)^* \cdots a(h_n)^* \Omega | n \in \mathbb{N}, h_j \in \mathcal{D}, j = 1, \dots, n\}],$$

where $\Omega := (1, 0, 0, \dots)$ is the Fock vacuum in $\mathcal{F}_{\text{b}}$. We introduce a dense subspace in $\mathcal{F}$

$$\mathcal{D}_\omega := D(A) \hat{\otimes} \mathcal{F}_{\text{fin}}(D(\omega)),$$

where $\hat{\otimes}$ denotes algebraic tensor product. The subspace $\mathcal{D}_\omega$ is a core of H_0 .

Let

$$H_{\text{GSB}} := H_0 + \lambda H_1$$

be a GSB Hamiltonian. The Hamiltonian H and H_{GSB} has the following relation:

Proposition 3.4. *Let $D(A) \subset D(B_j)$, $j = 1, \dots, J$ and $f_j \in D(\omega^{1/2})$. Assume that H_{GSB} is bounded from below. Then, for all $\Psi \in D_\omega$,*

$$\|(H_{\text{GSB}} - E_0)\Psi\|^2 + \|\mu H_2 \Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \quad (5)$$

where $D = \mu \sum_{j=1}^J \|\omega^{1/2} f_j\|^2$ and

$$E_0 := \inf_{\substack{\Psi \in D(H_{\text{GSB}}) \\ \|\Psi\|=1}} \langle \Psi, H_{\text{GSB}} \Psi \rangle.$$

Proof. It is enough to show (5) the case $\lambda = \mu = 1$. First we consider the case where $f_j \in D(\omega)$. Inequality (5) is equivalent to

$$-2 \operatorname{Re} \langle (H_{\text{GSB}} - E_0)\Psi, H_2 \Psi \rangle \leq D\|\Psi\|^2. \quad (6)$$

By $H_{\text{GSB}} - E_0 \geq 0$, we have

$$\begin{aligned} \langle (H_{\text{GSB}} - E_0)\Psi, I \otimes \phi(f_j)^2 \Psi \rangle &= \langle [I \otimes \phi(f_j), (H_{\text{GSB}} - E_0)]\Psi, I \otimes \phi(f_j)\Psi \rangle \\ &\quad + \langle (H_{\text{GSB}} - E_0)I \otimes \phi(f_j)\Psi, I \otimes \phi(f_j)\Psi \rangle \\ &\geq \langle [I \otimes \phi(f_j), H_{\text{GSB}} - E_0]\Psi, I \otimes \phi(f_j)\Psi \rangle. \end{aligned}$$

Therefore we have

$$2 \operatorname{Re} \langle (H_{\text{GSB}} - E_0)\Psi, \phi(f_j)^2 \Psi \rangle \geq -\|\sqrt{\omega} f_j\|^2 \|\Psi\|^2.$$

This means inequality (6). Next, we set $f_j \in D(\sqrt{\omega})$. Then, there exists a sequence $\{f_{jn}\}_{n=0}^\infty \subset D(\omega)$ such that $f_{jn} \rightarrow f_j$, $\omega^{1/2} f_{jn} \rightarrow \omega^{1/2} f_j$ ($n \rightarrow \infty$). By limiting argument, (6) holds with $f_j \in D(\omega^{1/2})$. $\blacksquare$

Lemma 3.5. *Suppose that H_{GSB} is self-adjoint with $D(H_{\text{GSB}}) = D(H_0)$, essentially self-adjoint on $\mathcal{D}_\omega$, and bounded from below. Let $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$. Then H is self-adjoint with $D(H) = D(H_0)$ and essentially self-adjoint on any core of H_{GSB} with*

$$\|(H_{\text{GSB}} - E_0)\Psi\|^2 + \|\mu H_2 \Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \quad \Psi \in D(H_0).$$

Proof. It is well known that $D(H_b) \subset D(\phi(f_j)^2)$, and $\phi(f_j)^2$ is H_b -bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants $\eta \geq 0$, $\theta \geq 0$ such that

$$\left\| \sum_{j=1}^J \phi(f_j)^2 \psi \right\| \leq \eta \|H_b \psi\| + \theta \|\psi\|, \quad \psi \in D(H_b). \quad (7)$$

Since H_{GSB} is self-adjoint on $D(H_0)$, by the closed graph theorem, we have

$$\|H_0 \Psi\| \leq \lambda \|H_{\text{GSB}} \Psi\| + \nu \|\Psi\|, \quad \Psi \in D(H_0), \quad (8)$$

where λ and ν are non-negative constant independent of Ψ . Hence

$$\|H_2 \Psi\| \leq \eta \lambda \|H_{\text{GSB}} \Psi\| + (\eta \nu + \theta) \|\Psi\|, \quad \Psi \in D(H_0).$$

We fix a positive number μ_0 such that $\mu_0 < 1/(\mu \lambda)$. Then, by the Kato-Rellich theorem, $H(\lambda, \mu_0)$ is self-adjoint on $D(H_{\text{GSB}})$, bounded from below and essentially self-adjoint on any core of H_{GSB} . For a constant a ($0 < a < 1$), we set $\mu_n := (1 + a)^n \mu_0$. Since H_{GSB} is self-adjoint on $D(H_0)$, for each $j = 1, \dots, J$ we have $D(A) \subset D(B)$. Thus by Proposition 3.4, for all $\Psi \in \mathcal{D}_\omega$

$$\|(H_{\text{GSB}} - E_0) \Psi\|^2 + \|\mu_n H_2 \Psi\|^2 \leq \|(H(\lambda, \mu_n) - E_0) \Psi\|^2 + D \|\Psi\|^2.$$

If $H(\lambda, \mu_n)$ is self-adjoint on $D(H_{\text{GSB}})$, bounded from below and essentially self-adjoint on any core of H_{GSB} , then $H(\lambda, \mu_{n+1})$ has the same property. On the other hand, we have $\mu_n \rightarrow \infty$ ($n \rightarrow \infty$). Hence we conclude that H is self-adjoint with $D(H) = D(H_{\text{GSB}})$, bounded from below and essentially self-adjoint on any core of H_{GSB} . $\blacksquare$

Now, we assume conditions [H.1], [H.2] and [H.3].

Then H_{GSB} is self-adjoint on $D(H_0)$, bounded from below and essentially self-adjoint on any core of H_0 (see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows. $\blacksquare$

3.2 Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and $m > 0$.

For a parameter $V > 0$, we define the set of lattice points by

$$\Gamma_V := \frac{2\pi\mathbb{Z}^d}{V} := \left\{ k = (k_1, \dots, k_d) \mid k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \dots, d \right\}$$

and we denote by $l^2(\Gamma_V)$ the set of l^2 sequences over Γ_V . For each $k \in \Gamma_V$ we introduce

$$C(k, V) := \left[k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V} \right) \times \cdots \times \left[k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V} \right) \subset \mathbb{R}^d,$$

the cube centered about k . By the map

$$U : l^2(\Gamma_V) \ni \{h_l\}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l,V}(\cdot) \in L^2(\mathbb{R}^d),$$

we identify $l^2(\Gamma_V)$ with a subspace in $L^2(\mathbb{R}^d)$, where $\chi_{l,V}(\cdot)$ is the characteristic function of the cube $C(l, V) \subset \mathbb{R}^d$. It is easy to see that $l^2(\Gamma_V)$ is a closed subspace of $L^2(\mathbb{R}^d)$. Let

$$\mathcal{F}_{b,V} := \mathcal{F}_b(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n l^2(\Gamma_V) \right],$$

the boson Fock space over $l^2(\Gamma_V)$. We can identify $\mathcal{F}_{b,V}$ the closed subspace of $\mathcal{F}_b$ by the operator $\Gamma(U) := \bigoplus_{n=0}^{\infty} \bigotimes^n U$, where we define $\bigotimes^0 U = 0$. For each $k \in \mathbb{R}^d$, there exists a unique point $k_V \in \Gamma_V$ such that $k \in C(k_V, V)$. Let

$$\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d$$

be a lattice approximate function of $\omega(k)$ and let

$$H_{b,V} := d\Gamma(\omega_V)$$

be the second quantization of ω_V . We define a constant

$$C_V := Cd^\gamma \left(\frac{\pi}{V} \right) \left(\frac{1}{2m} + 1 \right),$$

where C and γ were defined in [H.4]. In what follows we assume that

$$C_V < 1.$$

This is satisfied for all sufficiently large V .

Lemma 3.6. ([2 , Lemma 3.1]). *We have*

$$D(H_{b,V}) = D(H_b),$$

and

$$\|(H_b - H_{b,V})\Psi\| = \frac{2C_V}{1 - C_V} \|H_b\Psi\|, \quad \Psi \in D(H_b).$$

First we consider the case where g_j 's and f_j 's are continuous, and finally, by limiting argument, we treat a general case. For a constant $K > 0$, we define $g_{j,K}$, $f_{j,K}$, and $g_{j,K,V}$, $f_{j,K,V}$ as follows:

$$g_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d) g_j(k), \quad g_{j,K,V}(k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1, \dots, d}} g_j(\ell) \chi_{\ell,V}(k),$$

$$f_{j,K}(k) := \chi_K(k_1) \cdots \chi_K(k_d) f_j(k), \quad f_{j,K,V}(k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1, \dots, d}} f_j(\ell) \chi_{\ell,V}(k),$$

where χ_K denotes the characteristic function of $[-K, K]$.

Lemma 3.7. *For all $j = 1, \dots, J$,*

$$\begin{aligned} \lim_{V \rightarrow \infty} \|g_{j,K,V} - g_{j,K}\| &= 0, & \lim_{V \rightarrow \infty} \|g_{j,K,V}/\sqrt{\omega_V} - g_{j,K}/\sqrt{\omega}\| &= 0, \\ \lim_{K \rightarrow \infty} \|g_{j,K} - g_j\| &= 0, & \lim_{K \rightarrow \infty} \|g_{j,K}/\sqrt{\omega} - g_j/\sqrt{\omega}\| &= 0, \\ \lim_{V \rightarrow \infty} \|f_{j,K,V} - f_{j,K}\| &= 0, & \lim_{V \rightarrow \infty} \|f_{j,K,V}/\sqrt{\omega_V} - f_{j,K}/\sqrt{\omega}\| &= 0, \\ \lim_{K \rightarrow \infty} \|f_{j,K} - f_j\| &= 0, & \lim_{K \rightarrow \infty} \|f_{j,K}/\sqrt{\omega} - f_j/\sqrt{\omega}\| &= 0, \\ \lim_{K \rightarrow \infty} \|\sqrt{\omega} f_{j,K} - \sqrt{\omega} f_j\| &= 0, & \lim_{V \rightarrow \infty} \|\sqrt{\omega_V} f_{j,K,V} - \sqrt{\omega} f_{j,K}\| &= 0. \end{aligned}$$

Proof. Similar to the proof of [2 , Lemma 3.10] ■

We introduce a new operator:

$$\begin{aligned} H_{0,V} &:= A \otimes I + I \otimes H_{b,V}, \\ H_{1,K} &:= \sum_{j=1}^J B_j \otimes \phi(g_{j,K}), \\ H_{1,K,V} &:= \sum_{j=1}^J B_j \otimes \phi(g_{j,K,V}), \end{aligned}$$

$$H_{2,K} := \sum_{j=1}^J I \otimes \phi(f_{j,K})^2,$$

$$H_{2,K,V} := \sum_{j=1}^J I \otimes \phi(f_{j,K,V})^2,$$

and define

$$H_K := H_0 + \lambda H_{1,K} + \mu H_{2,K},$$

$$H_{K,V} := H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}.$$

- Lemma 3.8.** (i) H_K is self-adjoint with $D(H_K) = D(H_0) \subset D(H_{1,K}) \cap D(H_{2,K})$, bounded from below, and essentially self-adjoint on any core of H_0 .
- (ii) For all large V , $H_{K,V}$ is self-adjoint with $D(H_{K,V}) = D(H_0) \subset D(H_{1,K,V}) \cap D(H_{2,K,V})$, bounded from below, and essentially self-adjoint on any core of $H_{0,V}$.

Proof. Similar to the proof of Proposition 3.1. ■

Lemma 3.9. For all $z \in \mathbb{C} \setminus \mathbb{R}$, and $K > 0$,

$$\lim_{K \rightarrow \infty} \|(H_K - z)^{-1} - (H - z)^{-1}\| = 0,$$

$$\lim_{V \rightarrow \infty} \|(H_{K,V} - z)^{-1} - (H_K - z)^{-1}\| = 0.$$

Proof. Similar to the proof of [2, Lemma 3.5] ■

The following fact is well known:

Lemma 3.10. The operator $H_{b,V}$ is reduced by $\mathcal{F}_{b,V}$ and $H_{b,V} \upharpoonright \mathcal{F}_{b,V}$ equal to the second quantization of $\omega_V \upharpoonright l^2(\Gamma_V)$ on $\mathcal{F}_{b,V}$.

Lemma 3.11. $H_{K,V}$ is reduced by $\mathcal{F}_V$.

Proof. Similar to the proof of [2, Lemma 3.7] ■

Lemma 3.12. We have

$$H_{K,V} \upharpoonright \mathcal{F}_V^\perp \geq E_0(H_{K,V}) + m.$$

Proof. Similar to the proof of [2 , Lemma 3.10] ■

Lemma 3.13. *Let T_n and T be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that $T_n \rightarrow T$ in norm resolvent sense as $n \rightarrow \infty$ and T_n has purely discrete spectrum in the interval $[E_0(T_n), E_0(T_n) + c_n)$ with some constant c_n . If $c := \limsup_{n \rightarrow \infty} c_n > 0$, then T has purely discrete spectrum in $[E_0(T), E_0(T) + c)$.*

Proof. There exists a sequence $\{c_{n_j}\}_{j=1}^\infty \subset \{c_n\}_{n=1}^\infty$ so that $c_{n_j} \rightarrow c$ ($j \rightarrow \infty$). So, for all $\epsilon > 0$ and for sufficiently large j , the spectrum of T_{n_j} in $[E_0(T_{n_j}), E_0(T_{n_j}) + c - \epsilon)$ is discrete. Therefore, applying [2 , Lemma 3.12], we find that the spectrum of T in $[E_0(T), E_0(T) + c - \epsilon)$ is discrete. Since $\epsilon > 0$ is arbitrary, we get the conclusion. ■

Now, if A has compact resolvent, by a method similar to the proof of [2 , Theorem 1.2] we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known [2 , (2.12)]

$$|\langle \Psi, H_1 \Psi \rangle| \leq C_{\theta, \epsilon} \langle \Psi, A \otimes I \Psi \rangle + D_{\theta, \epsilon} \langle \Psi, I \otimes H_b \Psi \rangle + E_{\epsilon, \epsilon'} \|\Psi\|^2,$$

where $\Psi \in D(H_0)$ is arbitrary. Thus we have,

$$H \geq (1 - |\lambda|C_{\theta, \epsilon})A \otimes I + (1 - |\lambda|D_{\theta, \epsilon'})I \otimes H_b + \mu H_2 - |\lambda|E_{\epsilon, \epsilon'}.$$

Let $I_{\lambda, g}(K)$, $C_{\theta, \epsilon}(K)$, $D_{\theta, \epsilon}(K)$ and $E_{\epsilon, \epsilon'}(K)$ are $I_{\lambda, g}$, $C_{\theta, \epsilon}$, $D_{\theta, \epsilon}$, $E_{\epsilon, \epsilon'}$ with g_j , f_j replaced by $g_{j, K}$, $f_{j, K}$ respectively, and let $I_{\lambda, g}(K, V)$, $C_{\theta, \epsilon}(K, V)$, $D_{\theta, \epsilon}(K, V)$ and $E_{\epsilon, \epsilon'}(K, V)$ are $I_{\lambda, g}$, $C_{\theta, \epsilon}$, $D_{\theta, \epsilon}$, $E_{\epsilon, \epsilon'}$ with g_j , f_j and ω replaced by $g_{j, K, V}$, $f_{j, K, V}$ and ω_V respectively. Then we have

Lemma 3.14. *The following operator inequalities hold:*

$$\begin{aligned} H_K &\geq (1 - |\lambda|C_{\theta, \epsilon}(K))A \otimes I + (1 - |\lambda|D_{\theta, \epsilon'}(K))I \otimes H_b \\ &\quad + \mu H_{2, K} - |\lambda|E_{\epsilon, \epsilon'}(K) \quad \text{on} \quad D(H_0), \\ H_{K, V} &\geq (1 - |\lambda|C_{\theta, \epsilon}(K, V))A \otimes I + (1 - |\lambda|D_{\theta, \epsilon'}(K, K))I \otimes H_{b, V} \\ &\quad + \mu H_{2, K, V} - |\lambda|E_{\epsilon, \epsilon'}(K, V) \quad \text{on} \quad D(H_0). \end{aligned}$$

Proof. Similar to the calculation of [2 , (2.12)] ■

By Lemma 3.7, we have

$$\lim_{V \rightarrow \infty} C_{\theta, \epsilon}(K, V) = C_{\theta, \epsilon}(K), \quad \lim_{K \rightarrow \infty} C_{\theta, \epsilon}(K) = C_{\theta, \epsilon}, \quad (9)$$

$$\lim_{V \rightarrow \infty} D_{\theta, \epsilon'}(K, V) = D_{\theta, \epsilon'}(K), \quad \lim_{K \rightarrow \infty} D_{\theta, \epsilon'}(K) = D_{\theta, \epsilon'}, \quad (10)$$

$$\lim_{V \rightarrow \infty} E_{\epsilon, \epsilon'}(K, V) = E_{\epsilon, \epsilon'}(K), \quad \lim_{K \rightarrow \infty} E_{\epsilon, \epsilon'}(K) = E_{\epsilon, \epsilon'}. \quad (11)$$

Let $(\theta, \epsilon, \epsilon') \in \mathbb{T}$, namely

$$\tau_{\theta, \epsilon, \epsilon'} = (1 - |\lambda|C_{\theta, \epsilon})\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'} > E_0(H).$$

Formulas (9)-(11) and Lemma 3.9 imply that for all large V there exists a constant $K_0 > 0$ such that for all $K > K_0$,

$$(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V) > E_0(H_{K, V}), \quad (12)$$

$$|\lambda|C_{\theta, \epsilon}(K, V) < 1, \quad |\lambda|D_{\theta, \epsilon'}(K, V) < 1. \quad (13)$$

By Lemma 3.11, $H_{K, V}$ is reduced by $\mathcal{F}_V$. Therefore, $H_{K, V}$ satisfies the following inequality:

$$\begin{aligned} H_{K, V} \lceil \mathcal{F}_V &\geq (1 - |\lambda|C_{\theta, \epsilon}(K, V))A \otimes I \lceil \mathcal{F}_V \\ &\quad + (1 - |\lambda|D_{\theta, \epsilon'}(K, V))I \otimes H_{b, V} \lceil \mathcal{F}_V \\ &\quad - |\lambda|E_{\epsilon, \epsilon'}(K, V). \end{aligned} \quad (14)$$

Since $H_{b, V} \lceil \mathcal{F}_V$ has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to

$$(1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K, V).$$

By Lemma 3.12, we have $E_0(H_{K, V} \lceil \mathcal{F}_V) = E_0(H_{K, V})$. Thus, applying Theorem 2.1 with $H_{K, V} \lceil \mathcal{F}_V$, we have that $H_{K, V} \lceil \mathcal{F}_V$ has purely discrete spectrum in $[E_0(H_{K, V}), (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma_A - E_{\epsilon, \epsilon'}(K, V)]$. Since this fact and Lemma 3.12, $H_{K, V}$ has purely discrete spectrum in

$$[E_0(H_{K, V}), \min\{E_0(H_{K, V}) + m, (1 - |\lambda|C_{\theta, \epsilon}(K, V))\Sigma_A - E_{\epsilon, \epsilon'}(K, V)\}].$$

By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large $K > 0$, H_K has purely discrete spectrum in $[E_0(H_K), \min\{E_0(H_K) + m, (1 - |\lambda|C_{\theta, \epsilon}(K))\Sigma(A) - |\lambda|E_{\epsilon, \epsilon'}(K)\}]$. Similarly, H has purely discrete spectrum in $[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \tau_{\theta, \epsilon, \epsilon'}\}]$. Since $(\theta, \epsilon, \epsilon') \in \mathbb{T}$ is arbitrary, H has purely discrete spectrum in (4). Finally, we have to consider the case where g_j 's and f_j 's are not necessarily continuous. But, that argument were already discussed in [4]. So we skip that argument. ■

4 Ground State of the Dereziński-Gérard Model

We consider a model discussed by J. Dereziński and C. Gérard [5]. We take the Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^N).$$

The Hilbert space for the Dereziński-Gérard (DG) model is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_b(L^2(\mathbb{R}^d)).$$

We identify $\mathcal{F}$ as

$$\bigoplus_{n=0}^{\infty} \left[\mathcal{H} \otimes \bigotimes_s^n L^2(\mathbb{R}^d) \right].$$

Hence, if we denote that $\Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$, each $\Psi^{(n)}$ belongs to $\mathcal{H} \otimes [\otimes_s^n L^2(\mathbb{R}^d)]$. We denote by $\mathcal{B}(\mathcal{K}, \mathcal{J})$ the set of bounded linear operators from $\mathcal{K}$ to $\mathcal{J}$. For $v \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$, we define an operator $\tilde{a}^*(v)$ by

$$\begin{aligned} (\tilde{a}^*(v)\Psi)^{(0)} &:= 0, \\ (\tilde{a}^*(v)\Psi)^{(n)} &:= \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\otimes_s^{n-1} L^2(\mathbb{R}^d)})\Psi^{(n-1)}, \quad (n \geq 1), \\ \Psi \in D(\tilde{a}^*(v)) &:= \left\{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \left| \sum_{n=0}^{\infty} \|(\tilde{a}^*(v)\Psi)^{(n)}\|^2 < \infty \right. \right\}. \end{aligned}$$

We set

$$\begin{aligned} \mathcal{D}_0 &:= \{ \Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \mid \text{there exists a constant } n_0 \in \mathbb{N}, \\ &\quad \text{such that, for all } n \geq n_0, \Psi^{(n)} = 0 \}. \end{aligned}$$

Throughout this section, we write simply $I_n := I_{\otimes_s^n L^2(\mathbb{R}^d)}$. It is easy to see that:

Proposition 4.1. *$\tilde{a}^*(v)$ is a closed linear operator and $\mathcal{D}_0$ is a core of $\tilde{a}^*(v)$.*

So we set

$$\tilde{a}(v) := (\tilde{a}^*(v))^*$$

the adjoint operator of $\tilde{a}^*(v)$.

Proposition 4.2. *The operator $\tilde{a}(v)$ has the following properties:*

$$D(\tilde{a}(v)) = \left\{ \Psi = (\Psi^{(n)})_{n=0}^\infty \left| \sum_{n=0}^\infty (n+1) \|(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n) \Psi^{(n+1)}\|^2 < \infty \right. \right\} \quad (15)$$

$$(\tilde{a}(v)\Psi)^{(n)} = \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n) \Psi^{(n+1)}, \quad \Psi \in D(\tilde{a}(v)), \quad (16)$$

and $\mathcal{D}_0$ is a core of $\tilde{a}(v)$.

Proof. For $\Phi \in \mathcal{F}$, $\Psi \in D(\tilde{a}^*(v))$,

$$\begin{aligned} \langle \Phi, \tilde{a}^*(v)\Psi \rangle &= \sum_{n=1}^\infty \langle \Phi^{(n)}, \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{n-1}) \Psi^{(n-1)} \rangle \\ &= \sum_{n=0}^\infty \sqrt{n+1} \langle v^* \otimes I_n \Phi^{(n+1)}, \Psi^{(n)} \rangle \\ &= \sum_{n=0}^\infty \langle \sqrt{n+1}(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n) \Phi^{(n+1)}, \Psi^{(n)} \rangle. \end{aligned}$$

This implies (15) and (16). It is easy to prove that $\mathcal{D}_0$ is a core of $\tilde{a}(v)$. $\blacksquare$

An analogue of the Segal field operator is defined by

$$\tilde{\phi}(v) := \frac{1}{\sqrt{2}}(\tilde{a}(v) + \tilde{a}^*(v)).$$

Let A be a non-negative self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then the Hamiltonian of the DG model is defined by

$$H_{\text{DG}} := A \otimes I + I \otimes H_{\text{b}} + \tilde{\phi}(v).$$

We call it the *Derezinski-Gérard Hamiltonian*. Here H_{b} is the second quantization of ω introduced in Section 3. Let

$$H_0 := A \otimes I + I \otimes H_{\text{b}}.$$

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function $v(x, k) \in \mathbb{C}$, $(x \in \mathbb{R}^N, k \in \mathbb{R}^d)$, such that

$$(vf)(x, k) = v(x, k)f(x), \quad f \in L^2(\mathbb{R}^d).$$

We need also the following assumption:

[DG.2]

$$\operatorname{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \frac{v(x, k)}{\sqrt{\omega(k)}} \right|^2 dk < \infty.$$

Proposition 4.3. *Assume [DG.1] and [DG.2]. Then H_{DG} is self-adjoint with $D(H_{\text{DG}}) = D(H_0)$, and essentially self-adjoint on any core of H_0 .*

For a finite volume approximation, we introduce the following hypotheses:

[DG.3] There exists a nonnegative function $\tilde{v} \in L^2(\mathbb{R}^d)$ and function $\tilde{o} : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\begin{aligned} \operatorname{ess.sup}_{x \in \mathbb{R}^n} |v(x, k) - v(x, \ell)| &\leq \tilde{v}(k) \tilde{o}(|k - \ell|), \quad \text{a.e. } k, \ell \in \mathbb{R}^d \\ \lim_{t \downarrow 0} \tilde{o}(t) &= 0. \end{aligned}$$

[DG.4]

$$\operatorname{ess.sup}_{x \in \mathbb{R}^n} \int_{([-K, K]^d)^c} |v(x, k)|^2 dk = o(K^0).$$

where

$$([-K, K]^d)^c := \mathbb{R}^d \setminus (I \times \cdots \times I), \quad I := [-K, K]$$

and, $o(t^0)$ satisfies $\lim_{t \rightarrow 0} o(t^0) = 0$.

Let m be defined by (3). Let

$$D := \frac{1}{2} \inf_{0 < \epsilon' < \frac{\|v\|}{\|v/\sqrt{\omega}\|^2}} \left(\epsilon' + \frac{1}{\epsilon'} \right). \quad (17)$$

Here, $v/\sqrt{\omega}$ is a multiplication operator by the function $v(x, k)/\sqrt{\omega(k)}$ from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d)$. In the case $m > 0$, we can establish the existence of a ground state of H_{DG} :

Theorem 4.4. *Let $m > 0$. Suppose that [DG.1]-[DG.4] and [H.4] hold, and suppose*

$$\Sigma(A) - \|v\|D - E_0(H_{\text{DG}}) > 0.$$

Then, H_{DG} has purely discrete spectrum in

$$[E_0(H_{\text{DG}}), \min\{E_0(H_{\text{DG}}) + m, \Sigma(A) - \|v\|D\}).$$

In particular H_{DG} has a ground state.

Remark. In the case where A has compact resolvent, this theorem has been proved in [5]. A new aspect here is in that A does not necessarily have compact resolvent. Also our method is different from that in [5].

4.1 Proof of Proposition 4.3

Lemma 4.5. Let $M(x) = (\int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$, $x \in \mathbb{R}^N$ and $M : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ be a multiplication operator by the function $M(x)$. Then

$$\|vf\|^2 = \|Mf\|^2, \quad f \in L^2(\mathbb{R}^N).$$

In particular, $\|v\| = \|M\| = (\text{ess. sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} |v(x, k)|^2 dk)^{1/2}$ hold.

Proof. By the Fubini's theorem, we have

$$\|vf\|^2 = \int_{\mathbb{R}^d} dk \int_{\mathbb{R}^N} dx |v(x, k)|^2 |f(x)|^2 = \int_{\mathbb{R}^N} dx \left(|f(x)|^2 \int_{\mathbb{R}^d} |v(x, k)|^2 dk \right) dx.$$

This means the result. ■

The adjoint v^* has the following form:

Lemma 4.6. For all $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$,

$$(v^*g)(x) = \int_{\mathbb{R}^d} v(x, k)^* g(x, k) dk, \quad \text{a.e. } x \in \mathbb{R}^d. \quad (18)$$

Proof. For all $f \in \mathcal{H}$, we have

$$\begin{aligned} \langle g, vf \rangle &= \int dx \int dk g(x, k)^* v(x, k) f(x) \\ &= \int dx \left(\int g(x, k)^* v(x, k) dk \right) f(x). \end{aligned}$$

Since f is arbitrary, this proves (18). ■

Lemma 4.7. $\tilde{a}(v)$ is

$$\begin{aligned} D(\tilde{a}(v)) &= \left\{ \Psi \in \mathcal{F} \left| \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{N+dn}} dx dk_1 \cdots dk_n \right. \right. \\ &\quad \left. \left| \int_{\mathbb{R}^d} dk v(k, x)^* \Psi^{(n+1)}(x, k, k_1, \dots, k_n) \right|^2 < \infty \right\} \\ &\quad (\tilde{a}(v)\Psi)^{(n)}(x, k_1, \dots, k_n) \\ &= \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \dots, k_n), \quad \text{a.e.} \quad (\Psi \in D(\tilde{a}(v))) \end{aligned}$$

Proof. Using Lemma 4.6, we have

$$(v^* \otimes I_n) \Psi^{(n+1)}(x, k_1, \dots, k_n) = \int_{\mathbb{R}^d} v^*(x, k) \Psi^{(n+1)}(x, k, k_1, \dots, k_n) dk. \quad (19)$$

This is invariant for all permutations of $k_1, \dots, k_n$. Therefore, using Proposition 4.2, we get

$$(\tilde{a}(v)\Psi)^{(n)}(x, k_1, \dots, k_n) = \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \dots, k_n) dk.$$

■

Lemma 4.8. Suppose that [DG.1] and [DG.2] hold. Then, $D(\tilde{a}(v)) \supset D(I \otimes H_b^{1/2})$ and

$$\|\tilde{a}(v)\Phi\| \leq \|v/\sqrt{\omega}\| \|I \otimes H_b^{1/2}\Phi\|, \quad \Phi \in D(I \otimes H_b^{1/2}).$$

Proof. By (19), we have for all $\Phi \in D(\tilde{a}(v))$

$$\begin{aligned} \|(\tilde{a}(v)\Phi)^{(n)}\|^2 &= (n+1) \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_n \left| \int_{\mathbb{R}^d} \sqrt{\omega(k)} \right. \\ &\quad \left. \times \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \dots, k_n) dk \right|^2. \end{aligned}$$

Using the Schwarz inequality, one has

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \sqrt{\omega(k)} \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \dots, k_n) dk \right|^2 \\ &\leq \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 dk \cdot \int_{\mathbb{R}^d} \omega(k) |\Phi^{(n+1)}(x, k, k_1, \dots, k_n)|^2 dk. \end{aligned}$$

Hence, for every $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_b^{1/2})$, we have

$$\begin{aligned}
& \|(\tilde{a}(v)\Phi)^{(n)}\|^2 \\
& \leq \left(\operatorname{ess.\sup}_x \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 dk \right) (n+1) \times \\
& \quad \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_n dk \omega(k) |\Phi^{(n+1)}(x, k, k_1, \dots, k_n)|^2 \\
& = \left(\operatorname{ess.\sup}_x \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 dk \right) \times \\
& \quad \int_{\mathbb{R}^{dn+N}} dx dk_1 \cdots dk_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x, k_1, \dots, k_{n+1})|^2 \\
& = \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H_b^{1/2} \Phi)^{(n+1)} \right\|^2.
\end{aligned}$$

Therefore

$$\|\tilde{a}(v)\Phi\| \leq \left\| \frac{v}{\sqrt{\omega}} \right\| \left\| (I \otimes H_b^{1/2} \Phi) \right\|^2.$$

Since, $\mathcal{D}_0 \cap D(I \otimes H_b^{1/2})$ is a core of $I \otimes H_b^{1/2}$, one can extend this inequality to all $\Phi \in D(I \otimes H_b^{1/2})$, and $D(I \otimes H_b^{1/2}) \subset D(\tilde{a}(v))$ holds. $\blacksquare$

Lemma 4.9. *On $\mathcal{D}_0$, $\tilde{a}(v)$ and $\tilde{a}^*(v)$ satisfy the following commutation relation:*

$$[\tilde{a}(v), \tilde{a}(v)^*] = \int_{\mathbb{R}^d} |v(\cdot, k)|^2 dk.$$

where the right hand side is a multiplication operator by the function $: x \mapsto \int_{\mathbb{R}^d} |v(x, k)|^2 dk$.

Proof. Let $\Phi \in \mathcal{D}_0$. By the definition of $\tilde{a}^*(v)$, and using Proposition 4.2, we get

$$\begin{aligned}
([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)} &= (\tilde{a}(v)\tilde{a}(v)^*\Phi)^{(n)} - (\tilde{a}(v)^*\tilde{a}(v)\Phi)^{(n)} \\
&= \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n)(\tilde{a}(v)^*\Phi)^{(n+1)} \\
&\quad - \sqrt{n}(I \otimes S_n)(v \otimes I_{n-1})(\tilde{a}(v)\Phi)^{(n-1)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
& ([\tilde{a}^*(v), \tilde{a}(v)]\Phi)^{(n)}(x, k_1, \dots, k_n) \\
&= (n+1) \int_{\mathbb{R}^d} v(x, k)^* (I \otimes S_{n+1}(v \otimes I_{n-1})\Phi^{(n)})(x, k, k_1, \dots, k_n) dk \\
&\quad - n \frac{1}{n} \sum_{j=1}^n v(x, k_j) (v^* \otimes I_{n-1}\Phi^{(n)})(x, k_1, \dots, \widehat{k}_j, \dots, k_n) \\
&= \int_{\mathbb{R}^d} dk v(x, k)^* \left(v(x, k)\Phi^{(n)}(x, k_1, \dots, k_n) \right. \\
&\quad \left. + \sum_{j=1}^n v(x, k_j)\Phi^{(n)}(x, k, k_1, \dots, \widehat{k}_j, \dots, k_n) \right) \\
&\quad - \sum_{j=1}^n v(x, k_j) \int_{\mathbb{R}^d} dk v(x, k)^* \Phi^{(n)}(x, k, k_1, \dots, \widehat{k}_j, \dots, k_n) \\
&= \left(\int_{\mathbb{R}^d} |v(x, k)|^2 \right) \Phi(x, k_1, \dots, k_n).
\end{aligned}$$

Here ' $\wedge$ ' indicates the omission of the object wearing the hat. ■

Lemma 4.10. *Assume, [DG.1] and [DG.2]. Then $D(I \otimes H_b^{1/2}) \subset D(\tilde{a}^*(v))$ and for all $\Phi \in D(I \otimes H_b^{1/2})$,*

$$\|\tilde{a}^*(v)\Phi\|^2 \leq \|v/\sqrt{\omega}\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \|v\|^2 \|\Phi\|^2. \quad (20)$$

Proof. For all $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_b^{1/2})$, we have

$$\begin{aligned}
\|\tilde{a}^*(v)\Phi\|^2 &= \langle \Phi, \tilde{a}(v)\tilde{a}^*(v)\Phi \rangle = \langle \Phi, \tilde{a}^*(v)\tilde{a}(v)\Phi \rangle + \left\langle \left(\int_{\mathbb{R}^d} |v(\cdot, k)|^2 \right) \Phi, \Phi \right\rangle \\
&\leq \|\tilde{a}(v)\Phi\|^2 + \|v\|^2 \|\Phi\|^2.
\end{aligned}$$

Thus we can apply Lemma 4.8 to obtain the result. ■

Now we can prove Proposition 4.3:

Proof of Proposition 4.3. By Lemma 4.8 and 4.10, the operator $\tilde{\phi}(v)$ is $I \otimes H_b^{1/2}$ -bounded. Hence $\tilde{\phi}(v)$ is infinitesimally small with respect to $I \otimes H_b$. Namely, for all $\epsilon > 0$, there exists a constant $c_\epsilon > 0$, such that,

$$\|\tilde{\phi}(v)\Phi\| \leq \epsilon \|I \otimes H_b \Phi\| + c_\epsilon \|\Phi\|, \quad \Phi \in D(I \otimes H_b).$$

Since $A \geq 0$, we have

$$\|\tilde{\phi}(v)\Phi\| \leq \epsilon\|H_0\Phi\| + c\|\Phi\|, \quad \Phi \in D(H_0).$$

Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3. $\blacksquare$

4.2 Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds. Let $\mathcal{F}_{b,V}$, ω_V , $H_{b,V}$, $H_{0,V}$, $\mathcal{F}_V$, Γ_V , $\chi_{\ell,V}(k)$ be an object already defined in Section 3, respectively. Suppose that χ_K is a characteristic function of $[-K, K]$.

For a parameter $K > 0$, we define $v_K \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by

$$(v_K f)(x, k) := \chi_{[-K, K]}(k) v(x, k) f(x).$$

and $v_{K,V} \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by

$$(v_{K,V} f)(x, k) := \sum_{\substack{\ell \in \Gamma_V, |\ell_i| < K \\ i=1, \dots, d}} \chi_{\ell,V}(k) v(x, \ell) f(x).$$

Lemma 4.11. *The following hold:*

$$\|v_K - v_{K,V}\| \rightarrow 0 \ (V \rightarrow \infty), \quad \|v_K - v\| \rightarrow 0 \ (K \rightarrow \infty). \quad (21)$$

$$\left\| \frac{v_K}{\sqrt{\omega}} - \frac{v_{K,V}}{\sqrt{\omega_V}} \right\| \rightarrow 0 \ (V \rightarrow \infty), \quad \left\| \frac{v}{\sqrt{\omega}} - \frac{v_K}{\sqrt{\omega}} \right\| \rightarrow 0 \ (K \rightarrow \infty). \quad (22)$$

Proof. By [DG.3] and [DG.4], we have

$$\begin{aligned} \|v_K - v_{K,V}\|^2 &= \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \chi_K(k) v(x, k) - \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(x, \ell) \chi_{\ell,V}(k) \right|^2 dk \\ &= \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |v(x, k) - v(x, \ell)|^2 dk \\ &\leq \text{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |\tilde{v}(k)|^2 \tilde{o}(|k - \ell|)^2 dk \\ &\leq \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |\tilde{v}(k)|^2 \tilde{o}(|k - \ell|)^2 dk. \end{aligned}$$

It follows from the property of $\tilde{o}$ that for every $\epsilon > 0$, there exists a constant $V_0 > 0$ such that, for all $V > V_0$,

$$\chi_{\ell,V}(k)\tilde{o}(|k - \ell|)^2 \leq \epsilon\chi_{\ell,V}(k).$$

Therefore,

$$\|v_K - v_{K,V}\|^2 \leq \epsilon \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell,V}(k) |\tilde{v}(k)|^2 dk = \epsilon \|\tilde{v}\|_{L^2(\mathbb{R}^d)}^2.$$

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4] :

$$\begin{aligned} \|v_K - v\|^2 &= \text{ess.sup}_x \int_{\mathbb{R}^d} |\chi_K(k) - 1|^2 |v(x, k)|^2 dk \\ &= \text{ess.sup}_x \int_{([-K, K]^d)^c} |v(x, k)|^2 dk = o(K^0) \rightarrow 0 \text{ (} K \rightarrow \infty \text{)}. \end{aligned}$$

Using [H.4], one can easily check (22) . ■

We introduce two operators:

$$\begin{aligned} H_{\text{DG}}(K) &:= A \otimes I + I \otimes H_b + \tilde{\phi}(v_K), \\ H_{\text{DG}}(K, V) &:= A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}). \end{aligned}$$

Lemma 4.12. (i) $H_{\text{DG}}(K)$ is self-adjoint with $D(H_{\text{DG}}(K)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of H_0 .
(ii) For sufficiently large $V > 0$, $H_{\text{DG}}(K, V)$ is self-adjoint with domain $D(H_{\text{DG}}(K, V)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core of H_0 .

Proof. Similar to the proof of Proposition 4.3. ■

Lemma 4.13. For all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned} \lim_{V \rightarrow \infty} \|(H_{\text{DG}}(K, V) - z)^{-1} - (H_{\text{DG}}(K) - z)^{-1}\| &= 0, \\ \lim_{K \rightarrow \infty} \|(H_{\text{DG}}(K) - z)^{-1} - (H_{\text{DG}} - z)^{-1}\| &= 0. \end{aligned}$$

Proof. Similar to the proof of [2, Lemma 3.5] ■

Lemma 4.14. *The operator $H_{\text{DG}}(K, V)$ is reduced by $\mathcal{F}_V$.*

Proof. We identify $v(x, \ell)$ with multiplication operator by $v(\cdot, \ell)$. By abuse of symbols, we denote $\chi_{\ell, V}(\cdot)$ by $\chi_{\ell, V}(k)$. Then

$$\begin{aligned} (\tilde{a}^*(v(x, \ell)\chi_{\ell, V}(k))\Phi)^{(n)} &= \sqrt{n}(I \otimes S_n)(v(x, \ell)\chi_{\ell, V}(k) \otimes I)\Phi^{(n-1)} \\ &= \sqrt{n}v(x, \ell)S_n(\chi_{\ell, V} \otimes \Phi^{(n-1)}) \\ &= \chi(x, \ell)\sqrt{n}S_n(\chi_{\ell, V} \otimes \Phi^{(n-1)}). \end{aligned}$$

Hence, we have

$$\tilde{a}^*(v(x, \ell)\chi_{\ell, V}(k))\Phi = v(x, \ell) \otimes a^*(\chi_{\ell, V})\Phi.$$

Therefore, we get

$$\tilde{a}^*(v_{K, V}) = \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(\cdot, \ell) \otimes a^*(\chi_{\ell, V}). \quad (23)$$

Hence, its adjoint is

$$\tilde{a}(v_{K, V}) = \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(\cdot, \ell)^* \otimes a(\chi_{\ell, V}). \quad (24)$$

This means that the operator $H_{\text{DG}}(K, V)$ is a special case of the GSB Hamiltonian(see [2]). Hence, by [2 , Lemma 3.7] $H_{\text{DG}}(K, V)$ is reduced by $\mathcal{F}_V$. ■

Lemma 4.15. $H_{\text{DG}}(K, V) \lceil \mathcal{F}_V^\perp \geq E_0(H_{\text{DG}}(K, V)) + m$

Proof. Similar to the proof of [2 , Lemma 3.10] ■

Lemma 4.16. *For all $\Phi \in D(I \otimes H_{\text{b}}^{1/2})$, and for all $\epsilon' > 0$,*

$$|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{\epsilon'}{\|v\|} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_{\text{b}}^{1/2}\|^2 + \frac{\|v\|}{2} \left(\epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2.$$

Proof. For all $\Phi \in D(I \otimes H_b^{1/2})$, $\epsilon' > 0$,

$$\begin{aligned} |\langle \Phi, \tilde{\phi}(v)\Phi \rangle| &\leq \frac{1}{\sqrt{2}} \left(\epsilon \|\tilde{a}(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 + \epsilon \|\tilde{a}^*(v)\Phi\|^2 + \frac{1}{4\epsilon} \|\Phi\|^2 \right) \\ &\leq \frac{1}{\sqrt{2}} \left(2\epsilon \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \epsilon \|v\|^2 \|\Phi\|^2 + \frac{1}{2\epsilon} \|\Phi\|^2 \right) \\ &= \sqrt{2}\epsilon \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \frac{\|v\|}{2} \left(\sqrt{2}\epsilon \|v\| + \frac{1}{\sqrt{2}\epsilon \|v\|} \right) \|\Phi\|^2, \end{aligned}$$

where we have used Lemma 4.8 and 4.10. Let $\sqrt{2}\epsilon\|v\| =: \epsilon'$. Then, for all $\epsilon' > 0$, we have

$$|\langle \Phi, \tilde{\phi}(v)\Phi \rangle| \leq \frac{\epsilon'}{\|v\|} \left\| \frac{v}{\sqrt{\omega}} \right\|^2 \|I \otimes H_b^{1/2}\Phi\|^2 + \frac{\|v\|}{2} \left(\epsilon' + \frac{1}{\epsilon'} \right) \|\Phi\|^2.$$

■

Proof of Theorem 4.4. From (23) and (24), $H_{\text{DG}}(K, V)$ is equal to the special case of the GSB model. Therefore, $H_{\text{DG}}(K, V)[\mathcal{F}_V$ has the same form with $H_{\text{DG}}(K, V)$. Using Lemma 4.16 we have on $D(H_0) \cap \mathcal{F}_V$

$$\begin{aligned} &H_{\text{DG}}(K, V) \\ &= A \otimes I + I \otimes H_{b,V} + \tilde{\phi}(v_{K,V}) \\ &\geq A \otimes I + I \otimes H_{b,V} - \frac{\epsilon'}{\|v_{K,V}\|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 I \otimes H_{b,V} - \frac{\|v_{K,V}\|}{2} \left(\epsilon' + \frac{1}{\epsilon'} \right) \\ &= A \otimes I + \left(1 - \frac{\epsilon'}{\|v_{K,V}\|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 \right) I \otimes H_{b,V} - \frac{\|v_{K,V}\|}{2} \left(\epsilon' + \frac{1}{\epsilon'} \right), \quad (25) \end{aligned}$$

where $\epsilon' > 0$ is an arbitrary constant. By Lemma 3.10, $H_{b,V}[\mathcal{F}_{b,V}$ has compact resolvent. Thus, for $\epsilon' > 0$ satisfying

$$1 - \frac{\epsilon'}{\|v_{K,V}\|} \left\| \frac{v_{K,V}}{\sqrt{\omega_V}} \right\|^2 > 0, \quad (26)$$

the bottom of the essential spectrum of (25) is equal to

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} \left(\epsilon' + \frac{1}{\epsilon'} \right).$$

Let, D_K and $D_{K,V}$ be D with v replaced by v_K , $v_{K,V}$, respectively. It is easy to see that

$$\lim_{K \rightarrow \infty} D_K = D, \quad \lim_{V \rightarrow \infty} D_{K,V} = D_K.$$

By Lemma 4.13, one has

$$\lim_{K \rightarrow \infty} E_0(H_{\text{DG}}(K)) = E_0(H_{\text{DG}}), \quad \lim_{V \rightarrow \infty} E_0(H_{\text{DG}}(K, V)) = E_0(\text{DG}(K)).$$

From the assumption of Theorem 4.4, for all $K > 0$, there exists a constant V_0 such that for $V > V_0$,

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} D_{K,V} - E_0(H_{\text{DG}}(K, V)) > 0.$$

By the definition of $D_{K,V}$, for all $K > 0$ and $V > V_0$, and for all ϵ' which satisfies (26), we have

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} \left(\epsilon' + \frac{1}{\epsilon'} \right) > E_0(H_{\text{DG}}(K, V)).$$

Therefore, by Theorem 2.1, we have that $H_{\text{DG}}(K, V)[\mathcal{F}_V]$ has purely discrete spectrum in

$$[E_0(H_{\text{DG}}(K, V)), \Sigma(A) - \|v_{K,V}\| D_{K,V}).$$

This fact and Lemma 4.15 mean that $H_{\text{DG}}(K, V)$ has purely discrete spectrum in

$$[E_0(H_{\text{DG}}(K, V)), \min\{E_0(H_{\text{DG}}(K, V)) + m, \Sigma(A) - \|v_{K,V}\| D_{K,V}\}).$$

Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that H_{DG} has purely discrete spectrum in the interval

$$[E_0(H_{\text{DG}}), \min\{E_0(H_{\text{DG}}) + m, \Sigma(A) - \|v\| D\})$$

■

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