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DOUBLY NONLINEAR EVOLUTION EQUATION ASSOCIATED WITH ELLIPTIC-PARABOLIC FREE BOUNDARY PROBLEMS

NORIAKI YAMAZAKI

Department of Mathematical Science, Common Subject Division
Muroran Institute of Technology
27-1 Mizumoto-chō, Muroran, 050-8585 Japan

Abstract. We study an abstract doubly nonlinear evolution equations associated with elliptic-parabolic free boundary problems. In this paper we show the existence and uniqueness of solution for the doubly nonlinear evolution equation. Moreover we apply our abstract results to an elliptic-parabolic free boundary problem.

1. INTRODUCTION

We study an abstract doubly nonlinear evolution equation in a real Hilbert space H of the form

$$(1.1) \quad (Bu)'(t) + \partial\varphi^t(Bu(t); u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T),$$

where B is a monotone operator in H , $(Bu)'(t) := \frac{d}{dt}Bu(t)$ and f is a given H -valued function. For each $t \in [0, T]$, a function $\varphi^t(\cdot; \cdot) : H \times H \rightarrow \mathbf{R} \cup \{\infty\}$ is given such that for all $w \in H$, $\varphi^t(w; \cdot) : H \rightarrow \mathbf{R} \cup \{\infty\}$ is a proper, l.s.c. (lower semi-continuous) and convex function, and $\partial\varphi^t(w; \cdot)$ is its subdifferential operator.

For a proper, l.s.c. and convex function $\psi^t(\cdot) : H \rightarrow \mathbf{R} \cup \{\infty\}$, many mathematicians studied the doubly nonlinear evolution equation of the form

$$(1.2) \quad (Bu)'(t) + \partial\psi^t(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T).$$

For instance, Kenmochi [9] showed the existence-uniqueness, stability and convergence of solutions to (1.2) in the case when B is bi-Lipschitz.

For the Lipschitz operator B , Kenmochi-Pawłow [12, 13] have already established the results on existence-uniqueness and asymptotic behavior of solutions to (1.2). Kenmochi-Kubo [10] proved the existence of periodic solutions to (1.2), when ψ^t and $f(t)$ are periodic functions in t with same period. The author [16] considered the almost periodic problem to (1.2), when ψ^t and $f(t)$ are almost periodic in t .

The main object of this paper is to establish an abstract result on existence-uniqueness of solution to (1.1). Since the function $\varphi^t(Bu; u)$ is not convex in u , we can not apply to the results of Kenmochi-Pawłow [12]. So we shall refine on the abstract theory of Kenmochi-Pawłow [12] in this paper. Using the idea of Kenmochi-Kubo [11] and Kubo-Yamazaki [14], we shall show the existence of solution to (1.1). In fact, for the given

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function $w : [0, T] \rightarrow H$, let us consider the problem

$$(1.3) \quad (Bu)'(t) + \partial\varphi^t(w(t); u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T).$$

Assuming some appropriate conditions on the t - and w -dependence of the function $\varphi^t(w; z)$, we can apply the result of Kenmochi-Pawłowski [12]. Then we see that the equation (1.3) has a solution u for each w , and that the mapping $w \mapsto Bu$ has some compactness property. Hence, using a fixed point argument, we can get the existence of solution to (1.1).

In Section 2 we present our main results on existence and uniqueness of solution to (1.1), and then the uniqueness is proved. In Section 3 we prove the main existence result. In Section 4 we apply our abstract results to an elliptic-parabolic free boundary problem.

Notation. Throughout this paper, let H be a real Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot, \cdot)$. For a proper l.s.c. convex function ψ on H we use the notation $D(\psi)$, $\partial\psi$ and $D(\partial\psi)$ to indicate the effective domain, subdifferential and its domain of $\partial\psi$, respectively. For their precise definitions and basic properties, see a monograph by Brézis [5].

2. ASSUMPTIONS AND MAIN RESULT

We consider a Cauchy problem $CP(u_0)$ for (1.1) of the following form:

$$CP(u_0) \begin{cases} (Bu)'(t) + \partial\varphi^t(Bu(t); u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0, T), \\ Bu(0) = Bu_0, \end{cases}$$

where T is a given positive number, $u_0 \in H$, $f \in L^2(0, T; H)$, B is a monotone operator and a function $\varphi^t(Bu(t); u(t))$ is introduced in section 1.

Definition 2.1. Given $u_0 \in H$ and $f \in L^2(0, T; H)$, the function $u : [0, T] \rightarrow H$ will be called a solution to $CP(u_0)$, if $Bu \in W^{1,2}(0, T; H)$, $u \in L^2(0, T; H)$, $Bu(0) = Bu_0$, $u(t) \in D(\partial\varphi^t(Bu(t); \cdot))$ and $f(t) - (Bu)'(t) \in \partial\varphi^t(Bu(t); u(t))$ for a.e. $t \in [0, T]$, namely

$$(f(t) - (Bu)'(t), y - u(t)) \leq \varphi^t(Bu(t); y) - \varphi^t(Bu(t); u(t))$$

for any $y \in H$, a.e. $t \in [0, T]$.

Now we assume that the single valued operator B from $D(B) (\subset H)$ into H satisfies the following five conditions:

(B1) There is a proper l.s.c. convex function j_B on H such that its subdifferential ∂j_B coincides with B ;

(B2) There is a positive constant $C_1 > 0$ such that

$$C_1 |Bz_1 - Bz_2|_H^2 \leq (Bz_1 - Bz_2, z_1 - z_2), \quad \forall z_1, z_2 \in H;$$

(B3) $Bz \in D(\varphi^t(0; \cdot))$ for any $t \in [0, T]$ and $z \in D(\varphi^t(0; \cdot))$;

(B4) There are positive constants $C_2 > 0$ and $C_3 > 0$ such that

$$\varphi^t(0; Bz) \leq C_2 \varphi^t(0; z) + C_3, \quad \forall z \in H, \forall t \in [0, T];$$

(B5) B is bounded in H , namely, there is a positive constant $C_B > 0$ such that $|Bz|_H \leq C_B$ for any $z \in H$.

Definition 2.2. Given a positive number T and a function $\alpha \in W^{1,2}(0, T)$, we denote by $\{\varphi^t\} \in \Phi(\{\alpha\})$ the set of all time-dependent functions $\varphi^t(\cdot, \cdot)$ from $H \times H$ into $\mathbf{R} \cup \{\infty\}$ satisfying the following six conditions:

(Φ1) For each $w \in H$ and $t \in [0, T]$, $\varphi^t(w; \cdot) : H \rightarrow \mathbf{R} \cup \{\infty\}$ is a proper l.s.c. convex function;

(Φ2) There exists a positive constant $C_4 > 0$ such that

$$\varphi^t(w; z) \geq C_4 |z|_H^2, \quad \forall w \in H, \quad \forall t \in [0, T], \quad \forall z \in D(\varphi^t(w; \cdot));$$

(Φ3) For each $t \in [0, T]$, $w \in H$ and $r > 0$, the level set $\{z \in H; \varphi^t(w; z) \leq r\}$ is compact in H ;

(Φ4) $D(\varphi^t(w; \cdot))$ is independent of $w \in H$ for any $t \in [0, T]$;

(Φ5) For any $s, t \in [0, T]$ with $s \leq t$, $w \in D(\varphi^s(0; \cdot))$ with $|w|_H \leq C_B$ and $z \in D(\varphi^s(w; \cdot))$, there exists an element $\tilde{z} \in D(\varphi^t(w; \cdot))$ such that

$$|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z)^{\frac{1}{2}}\right),$$

$$\varphi^t(w; \tilde{z}) - \varphi^s(w; z) \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z) + \varphi^s(0; w)^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}}\right);$$

(Φ6) There is a positive constants $C_5 > 0$ such that

$$|\varphi^t(\tilde{w}; z) - \varphi^t(w; z)| \leq C_5 |\tilde{w} - w|_H \varphi^t(0; z)^{\frac{1}{2}}$$

$$\forall t \in [0, T], w \in H \text{ with } |w|_H \leq C_B, \tilde{w} \in H \text{ with } |\tilde{w}|_H \leq C_B \text{ and } z \in D(\varphi^t(0; \cdot)).$$

Let us begin with the uniqueness of solution to $CP(u_0)$. To do so, we shall introduce a subclass of $\Phi(\{\alpha\})$.

Definition 2.3. Let γ be a non-negative continuous and convex function on H such that $\gamma(z) + \gamma(-z) = 0$ if and only if $z = 0$. Then $\{\varphi^t\} \in \Phi_{B, \gamma}(\{\alpha\})$ if and only if $\{\varphi^t\} \in \Phi(\{\alpha\})$ satisfies the γ -accretiveness $(\star)$ for φ^t and B as follows:

$(\star)$ For any $w_i \in H$, $z_i \in D(\partial\varphi^t(w_i; \cdot))$ and $z_i^* \in \partial\varphi^t(w_i; z_i)$ ($i = 1, 2$), there is an element $w_0 \in \partial\gamma(Bz_1 - Bz_2)$ such that $(z_1^* - z_2^*, w_0) \geq 0$, where $\partial\gamma$ is the subdifferential of γ in H .

Now let us mention our abstract uniqueness result in this paper.

Theorem 2.4. Let T be any positive number. Assume $\{\varphi^t\} \in \Phi_{B, \gamma}(\{\alpha\})$, $f \in L^2(0, T; H)$ and B satisfies the conditions (B1)-(B5).

(i) Let u and v be solutions to $CP(u_0)$ and $CP(v_0)$, respectively. Then, we have

$$\gamma(Bu(t) - Bv(t)) \leq \gamma(Bu(s) - Bv(s)) \quad \text{for any } 0 \leq s \leq t \leq T.$$

(ii) For each $u_0 \in H$, the function Bu is uniquely determined, where u is the solution to $CP(u_0)$.

(iii) Furthermore, we assume that for each $w \in H$ $\varphi^t(w; \cdot)$ is strictly convex on $D(\varphi^t(w; \cdot))$. Then the solution u to $CP(u_0)$ is unique.

Proof. (i) Let u and v be solutions to $CP(u_0)$ and $CP(v_0)$, respectively. By the γ -accretiveness of φ^t and B , for a.e. $\tau \in [0, T]$ there exists $z^*(\tau) \in \partial\gamma(Bu(\tau) - Bv(\tau))$ such that

$$\begin{aligned} 0 &\leq \left([f(\tau) - \frac{d}{d\tau} Bu(\tau)] - [f(\tau) - \frac{d}{d\tau} Bv(\tau)], z^*(\tau) \right) \\ (2.1) \quad &= \left(-\frac{d}{d\tau} Bu(\tau) + \frac{d}{d\tau} Bv(\tau), z^*(\tau) \right) = -\frac{d}{d\tau} \gamma(Bu(\tau) - Bv(\tau)). \end{aligned}$$

Hence, integrating (2.1) over (s, t) , we get

$$(2.2) \quad \gamma(Bu(t) - Bv(t)) \leq \gamma(Bu(s) - Bv(s)) \quad \text{for any } 0 \leq s \leq t \leq T.$$

Thus the assertion (i) holds.

Here if $u_0 = v_0$, then (2.2) implies that $\gamma(Bu(t) - Bv(t)) = 0$ for all $t \in [0, T]$.

Similarly we can get $\gamma(Bv(t) - Bu(t)) = 0$ for all $t \in [0, T]$. Hence we have $Bu = Bv$. Thus the assertion (ii) has been shown.

From the assertion of (ii) it follows that

$$(2.3) \quad \partial\varphi^t(Bu(t); u(t)) \cap \partial\varphi^t(Bv(t); v(t)) \neq \emptyset, \quad \text{a.e. } t \in [0, T].$$

Furthermore, if $\varphi^t(w; \cdot)$ is strictly convex on $D(\varphi^t(w; \cdot))$ for any $w \in H$, then $\partial\varphi^t(w; \cdot)$ is strictly monotone for a.e. $t \in [0, T]$. Hence from (2.3), the assertion (ii) and the strictly monotonicity of $\partial\varphi^t(Bu(t); \cdot)$ we have $u(t) = v(t)$ for a.e. $t \in [0, T]$. Thus, the assertion (iii) has been proved. $\square$

Next main result is concerned with the existence of solutions to $CP(u_0)$.

Theorem 2.5. *Let T be any positive number. Assume $\{\varphi^t\} \in \Phi_{B,\gamma}(\{\alpha\})$, $f \in W^{1,1}(0, T; H)$ and B satisfies the conditions (B1)-(B5). Then, for each $u_0 \in D(\varphi^0(0; \cdot))$ there exists at least one solution u of $CP(u_0)$ on $[0, T]$.*

3. PROOF OF THEOREM 2.5

In this section we shall show Theorem 2.5 by the fixed point argument and a regularization method. Let us begin with the existence of local solution to $CP(u_0)$.

3.1. Local solution. Using the fixed point argument we shall prove the existence of local solution to $CP(u_0)$. To do so, for given positive numbers $T > 0$ and $M > 0$, let us consider a Banach space

$$E_M(T) \equiv \left\{ w \in W^{1,2}(0, T; H) ; \begin{array}{l} \sup_{t \in [0, T]} \varphi^t(0; w(t)) \leq M, \quad |w'|_{L^2(0, T; H)}^2 \leq M, \\ w(0) = Bu_0, \quad \sup_{t \in [0, T]} |w(t)|_H \leq C_B. \end{array} \right\}.$$

Now, for each $w \in E_M(T)$ let us consider a following Cauchy problem $CP(w; u_0)$:

$$CP(w; u_0) \left\{ \begin{array}{l} (Bu)'(t) + \partial\varphi^t(w(t); u(t)) \ni f(t) \quad \text{in } H \text{ a.e. } t \in (0, T), \\ Bu(0) = Bu_0. \end{array} \right.$$

Lemma 3.1. *For each $w \in E_M(T)$ we put $\psi_w^t(z) := \varphi^t(w(t); z)$ for $z \in H$. Then, there is a positive constant $N_1 > 0$ independent of w satisfying the following : for any $s, t \in [0, T]$ with $s \leq t$ and $z \in D(\psi_w^s)$, there exists $\tilde{z} \in D(\psi_w^t)$ such that*

$$|\tilde{z} - z|_H \leq N_1 |\alpha(t) - \alpha(s)| \left(1 + \psi_w^s(z)\right)^{\frac{1}{2}}$$

and

$$(3.1) \quad \begin{aligned} \psi_w^t(\tilde{z}) - \psi_w^s(z) &\leq N_1 \left\{ |\alpha(t) - \alpha(s)| (1 + \psi_w^s(z)) + |w(t) - w(s)|_H (1 + \psi_w^s(z))^{\frac{1}{2}} \right. \\ &\quad \left. + |\alpha(t) - \alpha(s)| \varphi^s(0; w(s))^{\frac{1}{2}} (1 + \psi_w^s(z))^{\frac{1}{2}} \right\}. \end{aligned}$$

Proof. Taking $w = w(s)$ in $(\Phi 5)$, then for any $s, t \in [0, T]$ with $s \leq t$ and $z \in D(\varphi^s(w(s); \cdot))$ there exists $\tilde{z} \in D(\varphi^t(w(s); \cdot))$ such that

$$(3.2) \quad |\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z)^{\frac{1}{2}}\right),$$

$$(3.3) \quad \begin{aligned} & \varphi^t(w(s); \tilde{z}) - \varphi^s(w(s); z) \\ & \leq |\alpha(t) - \alpha(s)| \left(1 + \varphi^s(0; z) + \varphi^s(0; w(s))^{\frac{1}{2}} \varphi^s(0; z)^{\frac{1}{2}}\right). \end{aligned}$$

It follows from $(\Phi 4)$ that

$$(3.4) \quad z \in D(\varphi^s(w(s); \cdot)) = D(\psi_w^s), \quad \tilde{z} \in D(\varphi^t(w(s); \cdot)) = D(\psi_w^t).$$

Note that by $(\Phi 6)$ and $w \in E_M(T)$ we have

$$(3.5) \quad \varphi^s(0; z) \leq 2\varphi^s(w(s); z) + C_5^2 |w(s)|_H^2 \leq 2\psi_w^s(z) + C_5^2 C_B^2.$$

Then, by (3.2) and (3.5) there is a positive number $N_2 > 0$ independent of w satisfying

$$(3.6) \quad \begin{aligned} |\tilde{z} - z|_H & \leq |\alpha(t) - \alpha(s)| \left(1 + \sqrt{2}\psi_w^s(z)^{\frac{1}{2}} + C_5 C_B\right) \\ & \leq N_2 |\alpha(t) - \alpha(s)| \left(1 + \psi_w^s(z)^{\frac{1}{2}}\right). \end{aligned}$$

Moreover, we observe that by (3.3), (3.5), $(\Phi 6)$ there is a positive number $N_3 > 0$ independent of w satisfying the following :

$$(3.7) \quad \begin{aligned} & \psi_w^t(\tilde{z}) - \psi_w^s(z) = \varphi^t(w(t); \tilde{z}) - \varphi^t(w(s); \tilde{z}) + \varphi^t(w(s); \tilde{z}) - \varphi^s(w(s); z) \\ & \leq N_3 \left\{ |w(t) - w(s)|_H \psi_w^t(\tilde{z})^{\frac{1}{2}} + |w(t) - w(s)|_H \right. \\ & \quad \left. + |\alpha(t) - \alpha(s)| (1 + \psi_w^s(z)) + |\alpha(t) - \alpha(s)| \varphi^s(0; w(s))^{\frac{1}{2}} (1 + \psi_w^s(z))^{\frac{1}{2}} \right\}. \end{aligned}$$

From $\alpha \in W^{1,2}(0, T)$, $w \in E_M(T)$ and (3.7) it follows that

$$(3.8) \quad \psi_w^t(\tilde{z}) \leq N_4 \left(1 + \psi_w^s(z) + |\alpha(t) - \alpha(s)|^2 \varphi^s(0; w(s))\right)$$

for some constant $N_4 > 0$. Therefore, using (3.8) in the right hand side of (3.7), and by (3.4)-(3.6), we get this Lemma 3.1 for some constant $N_1 > 0$ independent of w . $\square$

Proposition 3.2. *For each $w \in E_M(T)$, there exists a solution u to $CP(w; u_0)$ such that the function Bu is uniquely determined.*

Proof. We note that $CP(w; u_0)$ can be regarded as the Cauchy problem for the doubly nonlinear evolution equation of the form:

$$\begin{cases} (Bu)'(t) + \partial \psi_w^t(u(t)) \ni f(t) & \text{in } H \text{ a.e. } t \in (0, T), \\ Bu(0) = Bu_0. \end{cases}$$

By Lemma 3.1 we get the time-dependence of ψ_w^t . Therefore it follows from the assumptions $(\Phi 1)$ -($\Phi 3$), $(B1)$ -($B2$) that we can apply the abstract theory established by Kenmochi-Pawłow [12]. Thus we can get the existence of solution u for $CP(w; u_0)$. For detail proof, see [12, Theorem 1.1].

Moreover by the same argument of Kenmochi-Pawłow [12, Theorem 1.2] or Theorem 2.4 (ii), we can get the uniqueness of the function Bu . $\square$

By Proposition 3.2, we can define a mapping $Q : E_M(T) \longrightarrow L^2(0, T; H)$ by $Qw = Bu$ for each $w \in E_M(T)$, where u is a solution for $CP(w; u_0)$.

Lemma 3.3. *There are positive constants T_0 and M_0 such that Q is a self-mapping on $E_{M_0}(T_0)$, i.e., $Qw(=Bu) \in E_{M_0}(T_0)$ for any $w \in E_{M_0}(T_0)$.*

Proof. We consider the approximate problem $\text{CP}(w; u_0)_{\varepsilon, \lambda}$ ($0 < \varepsilon, \lambda \leq 1$) of $\text{CP}(w; u_0)$:

$$\text{CP}(w; u_0)_{\varepsilon, \lambda} \begin{cases} (B_\varepsilon u_{\varepsilon, \lambda})'(t) + \partial \varphi_\lambda^t(w(t); u_{\varepsilon, \lambda}(t)) = f(t), & 0 < t < T, \\ B_\varepsilon u_{\varepsilon, \lambda}(0) = B_{0, \varepsilon} (:= B_\varepsilon u_0). \end{cases}$$

Here we put $B_\varepsilon := B + \varepsilon I$ and $\varphi_\lambda^t(w(t); \cdot)$ is the Moreau-Yosida approximation of $\varphi^t(w(t); \cdot)$ defined by $\varphi_\lambda^t(w(t); z) := \inf_{y \in H} \left\{ \frac{1}{2\lambda} |z - y|_H^2 + \varphi^t(w(t); y) \right\}$ for $z \in H$.

By the same argument in [12, Lemma 2.2] we see that there is a positive constant $C'_4 > 0$ independent of t, w, z and $0 < \lambda \leq 1$ so that

$$(3.9) \quad \varphi_\lambda^t(w; z) \geq C'_4 |z|_H^2.$$

Moreover, we observe that the problem $\text{CP}(w; u_0)_{\varepsilon, \lambda}$ has a unique solution $u_{\varepsilon, \lambda}$ which converges to the solution of $\text{CP}(w; u_0)$ in some sense. For detail proof, see [12]. By the slight modification of [12, Lemma 2.3], we see that $\Psi_{\varepsilon, \lambda}(t) := \varphi_\lambda^t(w(t); u_{\varepsilon, \lambda}(t))$ is of bounded variation on $[0, T]$ and satisfies the following inequality :

$$(3.10) \quad \begin{aligned} & \Psi_{\varepsilon, \lambda}(t) - \Psi_{\varepsilon, \lambda}(s) + \int_s^t ((B_\varepsilon u_{\varepsilon, \lambda})'(\tau) - f(\tau), u'_{\varepsilon, \lambda}(\tau)) d\tau \\ & \leq N_1 \int_s^t \left[|\alpha'(\tau)| |(B_\varepsilon u_{\varepsilon, \lambda})'(\tau) - f(\tau)| \{1 + \Psi_{\varepsilon, \lambda}(\tau)^{\frac{1}{2}}\} + |\alpha'(\tau)| (1 + \Psi_{\varepsilon, \lambda}(\tau)) \right. \\ & \quad \left. + (|w'(\tau)|_H + |\alpha'(\tau)| \varphi^t(0; w(\tau))^{\frac{1}{2}}) (1 + \Psi_{\varepsilon, \lambda}(\tau))^{\frac{1}{2}} \right] d\tau \end{aligned}$$

for $0 \leq s \leq t \leq T$. Note that the following inequalities hold (cf. [12, Section 3]) :

$$(3.11) \quad ((B_\varepsilon u_{\varepsilon, \lambda})'(\tau), u'_{\varepsilon, \lambda}(\tau)) \geq \frac{C_1}{1 + \varepsilon C_1} |(B_\varepsilon u_{\varepsilon, \lambda})'(\tau)|_H^2,$$

$$(3.12) \quad \begin{aligned} & \int_s^t (f(\tau), u'_{\varepsilon, \lambda}(\tau)) d\tau = - \int_s^t (f'(\tau), u_{\varepsilon, \lambda}(\tau)) d\tau + (f(t), u_{\varepsilon, \lambda}(t)) - (f(s), u_{\varepsilon, \lambda}(s)) \\ & \leq \int_s^t \{ |f'(\tau)|_H^2 + N_5 \Psi_{\varepsilon, \lambda}(\tau) \} d\tau + (f(t), u_{\varepsilon, \lambda}(t)) - (f(s), u_{\varepsilon, \lambda}(s)), \end{aligned}$$

$$(3.13) \quad \begin{aligned} & |\alpha'(\tau)| |(B_\varepsilon u_{\varepsilon, \lambda})'(\tau) - f(\tau)|_H \{1 + \Psi_{\varepsilon, \lambda}(\tau)^{\frac{1}{2}}\} \\ & \leq 2\delta |(B_\varepsilon u_{\varepsilon, \lambda})'(\tau)|_H^2 + 2\delta |f(\tau)|_H^2 + \delta^{-1} |\alpha'(\tau)|^2 (1 + \Psi_{\varepsilon, \lambda}(\tau)), \end{aligned}$$

where the positive constant $N_5 > 0$ in (3.12) depends on C'_4 and the constant $\delta > 0$ will be defined below. Using (3.10)-(3.13), we have

$$(3.14) \quad \begin{aligned} & X_{\varepsilon, \lambda}(t) - X_{\varepsilon, \lambda}(s) + N_6 \int_s^t |(B_\varepsilon u_{\varepsilon, \lambda})'(\tau)|_H^2 d\tau \\ & \leq N_7 \int_s^t \left\{ G(\tau) (1 + \Psi_{\varepsilon, \lambda}(\tau)) + W(\tau) (1 + \Psi_{\varepsilon, \lambda}(\tau))^{\frac{1}{2}} \right\} d\tau \\ & \quad \text{for } 0 \leq s \leq t \leq T \text{ and } 0 < \varepsilon, \lambda \leq 1, \end{aligned}$$

where we put $X_{\varepsilon, \lambda}(t) := \Psi_{\varepsilon, \lambda}(t) - (f(t), u_{\varepsilon, \lambda}(t))$, $N_6 := \frac{C_1}{1 + \varepsilon C_1} - 2\delta N_1$,

$$G(t) := |f(t)|_H^2 + |f'(t)|_H^2 + |\alpha'(t)|^2 + 1, \quad W(t) := |w'(t)|_H + |\alpha'(t)| \varphi^t(0; w(t))^{\frac{1}{2}}.$$

Now we take $\delta > 0$ so that $N_6 > 0$. Then, note that $N_7 > 0$ is dependent only on C_1, N_1, N_5 . Here by (3.9) we get

$$(3.15) \quad X_{\varepsilon, \lambda}(t) = \Psi_{\varepsilon, \lambda}(t) - (f(t), u_{\varepsilon, \lambda}(t)) \geq N_8 \Psi_{\varepsilon, \lambda}(t) - \delta_1^{-1} |f(t)|_H^2,$$

where $\delta_1 > 0$ is the constant so that $N_8 > 0$. Using (3.15) in (3.14), applying Gronwall's inequality and letting $\varepsilon \rightarrow 0, \lambda \rightarrow 0$, we obtain

$$(3.16) \quad \begin{aligned} & \sup_{0 \leq t \leq T} \varphi^t(w(t); u(t)) + \int_0^T |(Bu)'(t)|_H^2 dt \\ & \leq N_{10} e^{N_9(|G|_{L^1(0,T)} + |W|_{L^1(0,T)})} \{1 + N_9(|G|_{L^1(0,T)} + |W|_{L^1(0,T)})\}, \end{aligned}$$

where N_9 and N_{10} are positive constants dependent on the given data.

Now we show that Q is the self-mapping on $E_{M_0}(T_0)$ for some chosen $T_0 > 0$ and $M_0 > 0$. Taking account of (3.5) and (B4), for any $w \in E_M(T)$ we have

$$(3.17) \quad \varphi^t(0; Bu(t)) \leq C_2 \varphi^t(0; u(t)) + C_3 \leq 2C_2 \varphi^t(w(t); u(t)) + C_2 C_5^2 C_B^2 + C_3.$$

Here we put the constant $M_0 > 0$ so that

$$(1 + 2C_2)N_{10}e^{2N_9}(1 + 2N_9) + C_2 C_5^2 C_B^2 + C_3 \leq M_0,$$

and then take $T_0 > 0$ such that

$$|G|_{L^1(0,T_0)} \leq 1, \quad |W|_{L^1(0,T)} \leq T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} + T_0^{\frac{1}{2}} M_0^{\frac{1}{2}} |\alpha'|_{L^2(0,T_0)} \leq 1.$$

Then from (3.16) and (3.17) it follows that $Qw(= Bu) \in E_{M_0}(T_0)$ for any $w \in E_{M_0}(T_0)$, hence, Q is the self-mapping on $E_{M_0}(T_0)$. $\square$

Lemma 3.4. *Let $M_0 > 0$ and $T_0 > 0$ be constants obtained in Lemma 3.3. Let $\{w_n\} \subset E_{M_0}(T_0)$, $w \in E_{M_0}(T_0)$ and u_n be the solution of $CP(w_n; u_0)$. Suppose $w_n \rightarrow w$ in $C([0, T_0]; H)$ as $n \rightarrow +\infty$. Then, there is a solution u of $CP(w; u_0)$ on $[0, T_0]$ such that $Bu \in E_{M_0}(T_0)$ and $Bu_n \rightarrow Bu$ in $C([0, T_0]; H)$ as $n \rightarrow +\infty$.*

Proof. Since $\{w_n\} \subset E_{M_0}(T_0)$, Lemma 3.3 and (3.17), we have

$$(3.18) \quad \sup_{t \in [0, T_0]} \varphi^t(0; Bu_n(t)) \leq M_0, \quad |(Bu_n)'|_{L^2(0, T_0; H)}^2 \leq M_0, \quad \forall n = 1, 2, \dots,$$

$$(3.19) \quad \sup_{t \in [0, T_0]} \varphi^t(0; u_n(t)) \leq C_2^{-1} M_0, \quad \forall n = 1, 2, \dots.$$

Here we note that the function Bu_n is uniquely determined (cf. Theorem 2.4 (ii)).

By $(\Phi 2)$, $(\Phi 3)$, (3.18), (3.19) there are a subsequence $\{n_k\}$ of $\{n\}$, a countable dense subset J_D of $[0, T_0]$ and functions $\tilde{u} \in W^{1,2}(0, T_0; H)$, $u \in L^\infty(0, T_0; H)$ such that

$$(3.20) \quad Bu_{n_k}(t) \rightharpoonup \tilde{u}(t) \quad \text{weakly in } H \quad \text{for all } t \in [0, T_0],$$

$$(3.21) \quad (Bu_{n_k})' \rightharpoonup (\tilde{u})' \quad \text{weakly in } L^2(0, T_0; H),$$

$$(3.22) \quad u_{n_k} \rightharpoonup u \quad \text{weakly-}^* \text{ in } L^\infty(0, T_0; H),$$

$$(3.23) \quad u_{n_k}(t) \rightarrow u(t) \quad \text{strongly in } H, \quad \text{for } t \in J_D$$

as $k \rightarrow +\infty$.

Since $C_1 |Bu_{n_k}(t) - Bu(t)|_H \leq |u_{n_k}(t) - u(t)|_H$, we observe that $Bu_{n_k}(t) \rightarrow Bu(t)$ strongly in H as $k \rightarrow +\infty$ for all $t \in J_D$. On account of (3.20)-(3.21), we see that $Bu(t) = \tilde{u}(t)$ for all $t \in J_D$. Therefore by (3.18)-(3.23) and the uniqueness of the function Bu_n , we observe that

$Bu \in E_{M_0}(T_0)$ and $Bu_n \longrightarrow Bu$ strongly in $C([0, T_0]; H)$ and weakly in $W^{1,2}(0, T_0; H)$ as $n \rightarrow +\infty$.

Now, let us show that u is a solution of $\text{CP}(w; u_0)$ on $[0, T_0]$. To do so, we define $\Phi(w; z) = \int_0^{T_0} \varphi^t(w(t); z(t)) dt$. Then by the assumption $(\Phi 6)$ we see that

$$(3.24) \quad \Phi(w_n; z) \longrightarrow \Phi(w; z) \text{ as } n \rightarrow +\infty$$

for any $z \in L^2(0, T_0; H)$ with $\varphi^{(\cdot)}(0; z(\cdot)) \in L^1(0, T_0)$. From (3.19), $(\Phi 1)$, $(\Phi 2)$, $(\Phi 6)$ and the Fatou's lemma, it follows that

$$(3.25) \quad \begin{aligned} \liminf_{k \rightarrow +\infty} \Phi(w_{n_k}; u_{n_k}) &= \liminf_{k \rightarrow +\infty} \{\Phi(w_{n_k}; u_{n_k}) - \Phi(w; u_{n_k}) + \Phi(w; u_{n_k})\} \\ &\geq \liminf_{k \rightarrow +\infty} \Phi(w; u_{n_k}) \geq \Phi(w; u). \end{aligned}$$

Moreover, let j_B^* be a conjugate function of j_B on H . Clearly, j_B^* is a proper l.s.c. convex function on H such that $\partial j_B^* = B^{-1}$. Then, we have

$$(3.26) \quad \begin{aligned} \liminf_{k \rightarrow +\infty} \int_0^{T_0} ((Bu_{n_k})'(t), u_{n_k}(t)) dt &= \liminf_{k \rightarrow +\infty} \{j_B^*(Bu_{n_k}(T_0)) - j_B^*(Bu_0)\} \\ &\geq j_B^*(Bu(T_0)) - j_B^*(Bu_0) = \int_0^{T_0} ((Bu)'(t), u(t)) dt. \end{aligned}$$

Now, let z be any function in $L^2(0, T_0; H)$ with $\varphi^{(\cdot)}(0; z(\cdot)) \in L^1(0, T_0)$. Since u_{n_k} is the solution of $\text{CP}(w_{n_k}; u_0)$, then the following inequality holds:

$$(3.27) \quad \int_0^{T_0} (f(t) - (Bu_{n_k})'(t), z(t) - u_{n_k}(t)) dt \leq \Phi(w_{n_k}; z) - \Phi(w_{n_k}; u_{n_k}).$$

Taking account of (3.21), (3.24)-(3.26) and letting $k \rightarrow +\infty$ in (3.27), we get

$$\int_0^{T_0} (f(t) - (Bu)'(t), z(t) - u(t)) dt \leq \Phi(w; z) - \Phi(w; u),$$

which implies that $f(t) - (Bu)'(t) \in \partial \varphi^t(w(t); u(t))$ for a.e. $t \in [0, T_0]$ (cf. [3, Proposition 3.3]). Thus u is the solution of $\text{CP}(w; u_0)$. $\square$

Proof. [Proof of Theorem 2.5; Local existence] By Lemma 3.3, we can define a self-mapping $Q : E_{M_0}(T_0) \longrightarrow E_{M_0}(T_0)$ by $Qw = Bu$ for each $w \in E_{M_0}(T_0)$, where u is a solution of $\text{CP}(w; u_0)$. Clearly, $E_{M_0}(T_0)$ is compact in $C([0, T_0]; H)$.

Moreover, it follows from Lemma 3.4 that Q is continuous with respect to the topology of $C([0, T_0]; H)$. Therefore, the Schauder's fixed point theorem implies that the self-mapping Q has a fixed point Bu in $E_{M_0}(T_0)$, i.e. $QB u = Bu$. Clearly u is the solution of $\text{CP}(u_0)$, thus we can get the local existence of solution u of $\text{CP}(u_0)$. $\square$

3.2. Global solution. Now let us begin with the inequality (3.14). Applying Schwarz inequality to the term $W(\tau)(1 + \Psi_{\varepsilon, \lambda}(\tau))^{1/2}$ and using (3.15), we get

$$(3.28) \quad \begin{aligned} &X_{\varepsilon, \lambda}(t) - X_{\varepsilon, \lambda}(s) + N_6 \int_s^t |(B_\varepsilon u_{\varepsilon, \lambda})'(\tau)|_H^2 d\tau \\ &\leq N_{11} \int_s^t G(\tau)(1 + X_{\varepsilon, \lambda}(\tau)) d\tau + \frac{N_6}{2} \int_s^t (|w'(\tau)|_H^2 + \varphi^\tau(0; w(\tau))) d\tau \end{aligned}$$

for $0 \leq s \leq t \leq T$ and $0 < \varepsilon, \lambda \leq 1$, where $N_{11} > 0$ is dependent on N_6, N_7, N_8, δ_1 and $|f|_{L^\infty(0, T; H)}$.

Applying Gronwall's inequality to (3.28), and letting $\varepsilon, \lambda \rightarrow 0$, we have

$$(3.29) \quad \begin{aligned} & \varphi^t(w(t); u(t)) \\ & + \int_0^t e^{N_{11} \int_\tau^t G(s) ds} \{N_6 |(Bu)'(\tau)|_H^2 - \frac{N_6}{2} \{|w'(\tau)|_H^2 + \varphi^\tau(0; w(\tau))\}\} d\tau \leq N_{12} \end{aligned}$$

for $0 \leq t \leq T$, where $N_{12} > 0$ is dependent on the given data.

Note that the inequality (3.29) holds for any $w \in E_{M_0}(T_0)$. Then by the result in Section 3.1, we can take $w = Bu \in E_{M_0}(T_0)$, where u is the solution of $\text{CP}(u_0)$ on $[0, T_0]$. Hence, using (3.17) and (3.29), we have

$$(3.30) \quad \begin{aligned} & \varphi^t(Bu(t); u(t)) + \int_0^t e^{N_{11} \int_\tau^t G(s) ds} \frac{N_6}{2} |(Bu)'(\tau)|_H^2 d\tau \\ & \leq N_{13} \left(1 + \int_0^t \varphi^\tau(Bu(\tau); u(\tau)) d\tau \right) \quad \text{for } 0 \leq t \leq T_0, \end{aligned}$$

where N_{13} depends on $C_2, C_3, C_5, C_B, N_6, N_{11}, N_{12}, |G|_{L^1(0,T)}$.

Applying Gronwall's inequality to (3.30), we get

$$(3.31) \quad \sup_{t \in [0, T_0]} \varphi^t(Bu(t); u(t)) + \int_0^{T_0} |(Bu)'(t)|_H^2 dt \leq N_{14},$$

where N_{14} depends only on the given data and is independent of $T_0 (\leq T)$.

Now let us prove the global existence of solution to $\text{CP}(u_0)$.

Proof. [**Proof of Theorem 2.5; Global existence**] Assume that

$$T^* := \sup\{T_0; \text{CP}(u_0) \text{ has a solution on } [0, T_0]\} < +\infty.$$

By the local existence results in Section 3.1, we note $T^* > 0$. By the definition of T^* , there is a function $u : [0, T^*) \rightarrow H$ such that for any $T_0 (< T^*)$ u is the solution of $\text{CP}(u_0)$ on $[0, T_0]$. By (B5) and (3.31) we have

$$Bu \in W^{1,2}(0, T^*; H), \quad \varphi^{(\cdot)}(Bu(\cdot); u(\cdot)) \in L^\infty(0, T^*).$$

Hence we observe that the limit $B^* := \lim_{t \uparrow T^*} Bu(t)$ strongly in H exists. By assumptions (B2), ($\Phi 1$), ($\Phi 3$), ($\Phi 5$), ($\Phi 6$) we see that

$$B^* = Bu^* \text{ for some } u^* \in D(\varphi^{T^*}(0; \cdot)).$$

Now, taking B^* as the initial value at $t = T^*$, we can get the solution u beyond the time interval $[0, T^*]$. Thus $T^* = +\infty$, so we have the global existence of solution. $\square$

4. APPLICATION

In this section we consider an elliptic-parabolic free boundary problem with Signorini-Dirichlet-Neumann mixed boundary condition :

$$(4.1) \quad b(u)_t - \nabla \cdot a(x, b(u), \nabla u) = f(t, x) \quad \text{in } (0, T) \times \Omega,$$

$$(4.2) \quad u \leq p(t), \quad \nu \cdot a(x, b(u), \nabla u) \leq 0, \quad (u - p(t))\nu \cdot a(x, b(u), \nabla u) = 0 \quad \text{on } \Gamma_S,$$

$$(4.3) \quad u = p(t) \quad \text{on } \Gamma_D,$$

$$(4.4) \quad \nu \cdot a(x, b(u), \nabla u) = 0 \quad \text{on } \Gamma_N,$$

$$(4.5) \quad b(u(0, \cdot)) = b_0 \quad \text{in } \Omega.$$

Here ν is an outward normal vector on the boundary, and p is a given function. Ω is a bounded domain in $\mathbf{R}^N$ ($1 \leq N < +\infty$) with smooth boundary Γ consisting of three disjoint parts Γ_ν , $\nu = S, D, N$, namely, $\Gamma = \Gamma_S \cup \Gamma_D \cup \Gamma_N$.

The problem (4.1)-(4.5) is a model of partially saturated porous media, in which Γ_S, Γ_D and Γ_N refer to the parts of the boundary in contact with the atmosphere, reservoirs and impervious layer, respectively. The function p is the pressure in the reservoirs on Γ_D , and to the atmospheric pressure on Γ_S .

Many mathematicians have already studied the various models of porous media. For instance, see [1, 2, 4, 8, 11, 13, 14, 15].

The aim of this section is to consider the problem (4.1)-(4.5) as a application of the abstract evolution equation $\text{CP}(u_0)$. To do so, we suppose that

- (A1) $a(x, s, p) = \partial_p A(x, s, p)$ for some potential function $A(x, s, p)$. There exist constants $\mu_1 > 0$, $\mu_2 = \mu_2(a) > 0$ and $\mu_3 = \mu_3(a) > 0$ such that

$$[a(x, s, p) - a(x, s, \hat{p})] \cdot (p - \hat{p}) \geq \mu_1 |p - \hat{p}|^2,$$

$$|a(x, s, p)|^2 + |A(x, s, p)| + |\partial_s A(x, s, p)|^2 \leq \mu_2(1 + |s|^2 + |p|^2),$$

$$|a(x, s, p) - a(x, \hat{s}, p)| \leq \mu_3(1 + |p|)|s - \hat{s}|$$

for all $x \in \Omega$, $s, \hat{s} \in \mathbf{R}$, $p, \hat{p} \in \mathbf{R}^N$.

- (A2) $b : \mathbf{R} \rightarrow \mathbf{R}$ is bounded, nondecreasing and Lipschitz continuous.

Here for each $t \in [0, T]$ let us define the convex set $K(t)$ by

$$K(t) := \{z \in H^1(\Omega); z \leq p(t) \text{ on } \Gamma_S \text{ and } z = p(t) \text{ on } \Gamma_D\}.$$

As a direct application of Theorems 2.4 and 2.5, we have:

Proposition 4.1. *Assume (A1) and (A2). Then, for each $f \in W^{1,1}(0, T; L^2(\Omega))$, $p \in W^{1,2}(0, T; H^1(\Omega))$ and $b_0 = b(u_0)$ for some $u_0 \in K(0)$, the problem (4.1)-(4.5) has a solution u on $[0, T]$.*

Proof. To apply Theorems 2.4 and 2.5 to the problem (4.1)-(4.5), we choose $L^2(\Omega)$ as a real Hilbert space H , and define a function $\varphi^t : L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbf{R} \cup \{\infty\}$ by

$$\varphi^t(w; z) := \begin{cases} \int_{\Omega} A(x, w(x), \nabla z(x)) dx, & \text{if } z \in K(t), \\ +\infty, & \text{otherwise.} \end{cases}$$

Let us define an operator $B : L^2(\Omega) \rightarrow L^2(\Omega)$ by $Bz := b(z)$ in $L^2(\Omega)$. Also we define a function γ by $\gamma(z) := \int_{\Omega} z^+(x) dx$ for $z \in L^2(\Omega)$, where $z^+ := \max\{z, 0\}$.

Now we put for any $t \in [0, T]$

$$\alpha(t) := N_{15} \int_0^t |p'(\tau)|_{H^1(\Omega)} d\tau,$$

where N_{15} is a (sufficient large) positive constant. Then, we easily see that $\{\varphi^t\} \in \Phi_{B, \gamma}(\{\alpha\})$ and the operator B satisfies the assumptions (B1)-(B5). Clearly, the problem (4.1)-(4.5) can be reformulated in the abstract evolution equation $\text{CP}(u_0)$. Thus, applying Theorems 2.4 and 2.5, we see that the problem (4.1)-(4.5) has a solution u on $[0, T]$ such that the function $b(u)$ is uniquely determined. $\square$

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E-mail address: noriaki@mmm.muroran-it.ac.jp