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The convergence proof of the no-response test for localizing an inclusion.

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Abstract

In this paper, we use the no-response test idea, introduced in ([L-P], [P1]) for the inverse obstacle problem, to identify the interface of the discontinuity of the coefficient γ of the equation $\nabla \cdot \gamma(x) \nabla + c(x)$ with piecewise regular γ and *bounded* function $c(x)$. We use infinitely many Cauchy data as measurement and give a reconstructive method to localize the interface. We will base this multiwave version of the no-response test on two different proofs. The first one contains a pointwise estimate as used by the singular sources method. The second one is built on an energy (or an integral) estimate which is the basis of the probe method. As a conclusion of this, the no response can be seen as a unified framework for the probe and the singular sources method. As a further contribution, we provide a formula to reconstruct the values of the jump of $\gamma(x)$, $x \in \partial D$ at the boundary.

1 Introduction and statement of the result

1.1 Introduction

The inverse boundary value problem for identifying an inclusion inside a conductive medium from infinitely many measurements was initiated in [Isa]. Isakov proved uniqueness for identifying the inclusion D . Later, in [I], a method for identifying the inclusion was proposed. This *probe method* has been generalized to deal with general scalar equations with mixed boundary conditions and with source term [D-N] and for anisotropic elastic systems [I-N-T]. In a recent paper [A-D], a stability result concerning this problem of localization of the interface of discontinuity is given.

For the inverse obstacle problem, in [L-P] and [P1] the *no response test* is proposed to localize an obstacle from finitely or infinitely many measurements and in [P2] we find the description of the *singular sources method* for shape reconstruction. The

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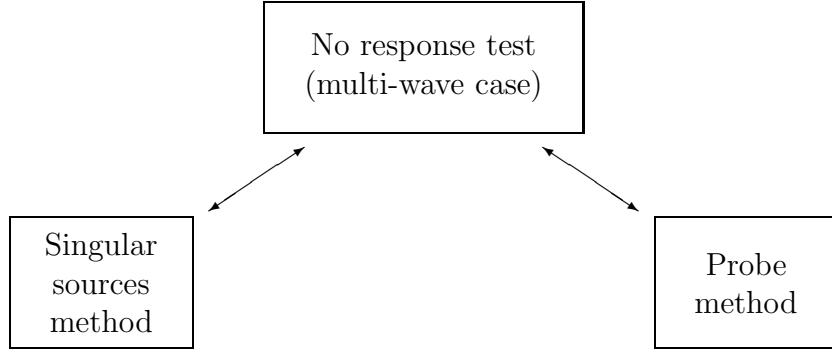


Figure 1: *The no response test is a method with a general minimization formulation which unifies the singular sources method (SSM) (pointwise estimate of fundamental solution) and the probe method (PM) (energy estimate of fundamental solution). Convergence of the SSM or PM is used to prove the convergence of the multi-wave no response test. However, the no response test also admits a one-wave version which tests for analytic continuability. In the one-wave case the SSM or PM do not work. Here, the no response test may be based on the point source method or the enclosure method.*

purpose of this paper is to use the idea of the no-response test to reconstruct the inclusion from infinitely many measurements and to clarify its relation to the probe and the singular sources method.

We show that the functional of the no-response has two different versions of lower estimates. One is of energy type. It is exactly the one of the probe method as it is given in [I]. This implies that in any case where the probe method converges then the no-response test also converges. The other version is of pointwise behavior. Its behavior is exactly the one of the singular sources method, see [P2]. We will use this second version to give another convergence proof of the no-response test. The relation of the methods is visualized in Figure 1 and explained in the section 3.5

As a further contribution, we derive a formula to reconstruct the values of $A(x)$, $x \in \partial D$. Similar formula has been given in [I-N] using the probe method. In their formula, one needs to compute the integrals of the gradient of the fundamental solution on D . We will explain more about our formula after the statement of the main result in the next section.

The idea to justify the blowup in a pointwise sense of this indicator function is to transform this behavior to the one of the Green's function of the equation $\nabla \cdot \gamma(x)\nabla + c(x)$. Then we prove that this Green's function is locally (near any point $a \in \partial D$) equivalent, in the L^∞ norm sense, to the fundamental solution of $\nabla \cdot (1 + A(a)\chi^-)\nabla$ where χ^- is the characteristic function of the negative half-space. The explicit form of this last fundamental solution gives the result. The proof of this equivalence is given by freezing and flattening the coefficient $\gamma(x)$ near the point a . To justify these two steps, we combined some estimates of the corresponding Green's

functions given in [A-D], $L^p, 1 < p < \infty$, and L^∞ estimates of solutions for scalar divergence form elliptic equations with discontinuous coefficients, see [S] and [L-V] respectively. The term $c(x)$ is added to include the case $c(x) = k^2$ for the singular sources method, see section 2.

The paper is organized as follow. In the following subsection we formulate the problem and describe the no-response test for this problem. In section 2, we recall the probe method and the singular sources method in details and state the result. In the section 3, we describe the proof of the result and the relations of these three methods. In section 4, we give the details of the proof.

1.2 Statement of the results.

Let Ω be a bounded domain in $\mathbb{R}^n, n = 2, 3$. We assume that Ω contains a bounded domain D with its boundary ∂D . We suppose that ∂D has the $C^{1,1}$ regularity. We consider a function γ of the form

$$\gamma(x) := 1 + \chi_D A(x),$$

where χ_D is the characteristic function of D and $A(x)$ is a $C^1(\overline{D})$ function satisfying $A(x) > 0$ in $\overline{D}$. We denote by

$$L_\gamma := \nabla \cdot \gamma \nabla \quad \text{and} \quad M_\gamma := L_\gamma + c(x),$$

where $c(x)$ is a bounded measurable function.

Let Φ be the fundamental solution of M_1 and Φ' be the one of L_1 where M_1 and L_1 are M_γ and L_γ when $\gamma(x) = 1, x \in \Omega$, extended by 1 to $\mathbb{R}^n \setminus \Omega$ and $c(x)$ extended by zero to $\mathbb{R}^n \setminus \Omega$.

Further, consider $f \in H^{\frac{1}{2}}(\partial\Omega)$ and let u^f be the $H^1(\Omega)$ solution of

$$\begin{cases} M_\gamma u^f = 0 & \text{in } \Omega, \\ u^f = f & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

This problem is well posed by assuming that *zero* is not an eigenvalue for the related operator. By taking all the functions $f \in H^{\frac{1}{2}}(\partial\Omega)$, we define the Dirichlet to Neumann map

$$\Lambda : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega), \quad f \mapsto \Lambda(f) := \frac{\partial u^f}{\partial \nu} \Big|_{\partial\Omega},$$

where ν is the exterior normal of $\partial\Omega$.

Definition 1.1 (Inverse Problem.) *Let the function $c(x)$ and the Dirichlet to Neumann map Λ be known. Our task is:*

- 1) *Reconstruct the interface ∂D of discontinuity of the coefficient $\gamma(x)$.*
- 2) *Recover the values of $A(x), x \in \partial D$.*

Now, we explain the idea of the no-response test introduced in [L-P, P1] for the inverse obstacle problem and show how to adapt it to our problem.

The no-response test. By (1.1) and Green's formula, we write

$$\begin{aligned} u^f(x) &= \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \\ &\quad + \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \end{aligned} \quad (1.2)$$

for $x \in \Omega \setminus \overline{D}$. Letting $x \rightarrow \partial\Omega$ in (1.2) and using Green's formula proven in the Appendix, we obtain

$$\begin{aligned} u^f(x) &= \frac{1}{2} u^f(x) + \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) + \\ &\quad + \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y) \end{aligned} \quad (1.3)$$

for $x \in \partial\Omega$. From our Cauchy data on ∂D , we know the function

$$J^f(x) := \frac{1}{2} u^f(x) - \int_{\partial\Omega} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in \partial\Omega.$$

By (1.3) we have

$$J^f(x) = \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right\} ds(y), \quad x \in \partial\Omega. \quad (1.4)$$

For $\varphi \in L^2(\partial\Omega)$ we define the single layer potential $v[\varphi](y)$ by

$$v[\varphi](y) = \int_{\partial\Omega} \Phi(x, y) \varphi(x) ds(x) \quad y \in \Omega.$$

Multiplying (1.4) by φ , integrating over $\partial\Omega$ and exchanging the order of integration, we obtain

$$\begin{aligned} &\int_{\partial\Omega} J^f(x) \varphi(x) ds(x) \\ &= \int_{\partial\Omega} \varphi(x) \left\{ \int_{\partial D} \left(\frac{\partial u^f}{\partial \nu}(y) \Phi(x, y) - u^f(y) \frac{\partial \Phi(x, y)}{\partial \nu(y)} \right) ds(y) \right\} ds(x) \\ &= \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) \int_{\partial\Omega} \varphi(x) \Phi(x, y) ds(x) - u^f(y) \int_{\partial\Omega} \varphi(x) \frac{\partial \Phi(x, y)}{\partial \nu(y)} ds(x) \right\} ds(y). \end{aligned} \quad (1.5)$$

Hence

$$\int_{\partial\Omega} J^f(x) \varphi(x) ds(x) = \int_{\partial D} \left\{ \frac{\partial u^f}{\partial \nu}(y) v[\varphi] - u^f(y) \frac{\partial v[\varphi]}{\partial \nu(y)} \right\} ds(y). \quad (1.6)$$

Let now B be a domain inside Ω . We define the functional

$$I_{\epsilon_1, \epsilon_2}(B) := \sup_{(f, \varphi) \in \mathbf{M}_{\epsilon_1, \epsilon_2}(B)} \left| \int_{\partial\Omega} J^f(x) \varphi(x) ds(x) \right| \quad (1.7)$$

where

$$\begin{aligned} \mathbf{M}_{\epsilon_1, \epsilon_2}(B) := & \left\{ (f, \varphi) \in H^{\frac{1}{2}}(\partial\Omega) \times L^2(\partial\Omega) : \|v[\varphi]\|_{H^1(\bar{B})} \leq \epsilon_1 \right. \\ & \left. \text{and } \|f - v[\varphi]\|_{L^2(\partial\Omega)} \leq \epsilon_2 \right\}. \end{aligned} \quad (1.8)$$

Our main *indicator function* is defined by

$$I(B) := \lim_{\epsilon_1, \epsilon_2 \rightarrow 0^+} I_{\epsilon_1, \epsilon_2}(B). \quad (1.9)$$

Please note that it is defined on a set of domains, not in the underlying 'physical' space. Now, using the data given by the Dirichlet to Neumann map we may calculate the functional (1.6) or the indicator function $I(B)$ defined in (1.7), respectively. In section 2, we give the proof of the following theorem which gives a reconstructive way how to localize ∂D and how to reconstruct the values of $A(x)$, $x \in \partial D$.

Theorem 1.1 *1) We have the following characterization of D from the Dirichlet to Neumann map:*

$$D = \bigcap_{B \in \mathbf{B}} B,$$

where $\mathbf{B} := \{B \subset \Omega : I(B) = 0\}$.

2) Knowing ∂D , then for every $x \in \partial D$, we reconstruct a sequence $(f_n^p, \varphi_n^p) \in H^{\frac{1}{2}}(\partial\Omega) \times L^2(\partial\Omega)$ such that the following formula is valid:

$$\frac{A(x)}{A(x) + 2} = \lim_{p, n \rightarrow \infty} (4\pi) |z_p - z_p^*| \int_{\partial\Omega} J^{f_n^p}(x) \varphi_n^p(x) ds(x), \quad (1.10)$$

where z_p is any sequence of points in $\Omega \setminus \bar{D}$ tending to x as p tends to ∞ and z_p^* is the point symmetric to z_p with respect to the plane tangent to ∂D at the point x .

Based on our Lemma 3.1, the functions ϕ_n^p , and hence f_n^p , can be reconstructed using the Tikhonov regularization scheme. Please also note that from (1.10) we don't need to know D everywhere to reconstruct $A(x)$. It is enough to know D near the point x . In fact it is enough to know the point x and the plane tangent to ∂D at the point x .

Remark 1.1 *The conditions on ∂D and $A(x)$ can be weakened by considering ∂D having the $C^{1, \alpha}$ regularity, $A(x) \in C^{0, \alpha}(\bar{D})$ where $0 < \alpha \leq 1$ and $A(x) \neq 0$ near ∂D . We take the limit cases to simplify the exposition.*

2 The probe and singular sources methods

Now we recall the probe and the singular sources methods.

The probe method. The functional of the probe method is defined by

$$\int_{\partial\Omega} (\Lambda - \Lambda_0) f(x) \cdot f(x) \, ds(x),$$

where Λ_0 is the Dirichlet-Neumann map when $\gamma = 1$ in Ω .

Let now $z_p \in \Omega \setminus \overline{D}$ such that z_p tends to $z \in \Omega$ when p tends to ∞ . We set $E(z_p)$ any regular domain such that $z_p \in \Omega \setminus \overline{E(z_p)}$ and $D \subset\subset E(z_p) \subset \Omega$. Using the Runge approximation, we can find a sequence of functions, v_n^p , such that $\|v_n^p - \Phi(\cdot, z_p)\|_{H^1(E(z_p))}$ tends to zero when n tends to ∞ .

We take now $f_{n,p} := v_n^p|_{\partial\Omega}$ and evaluate $\int_{\partial\Omega} (\Lambda - \Lambda_0) f_n^p(x) \cdot f_n^p(x) ds(x)$, then, see [I], for every p fixed we obtain

$$\lim_{n \rightarrow \infty} \int_{\partial\Omega} (\Lambda - \Lambda_0) f_n^p(x) \cdot f_n^p(x) ds(x) = \int_D A(x) (\nabla w_p + \nabla \Phi)(x) \cdot \nabla \Phi(x) dx \quad (2.1)$$

where w_p is the H^1 -solution of

$$\begin{cases} M_\gamma w_p = -\nabla \cdot \chi_D A(x) \nabla \Phi(\cdot, z_p) & \text{in } \Omega, \\ w_p = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.2)$$

The characterization of z to be in ∂D is given by the testing

$$\lim_{p, n \rightarrow \infty} \int_{\partial\Omega} (\Lambda - \Lambda_0) f_n^p(x) \cdot f_n^p(x) ds(x) = \infty.$$

The singular sources method. For this method we take $c(x) = k^2 > 0$, constant. One can find a sequence of densities $g_n^p(\xi)$ such that $v_n^p := \int_S e^{ikx \cdot \xi} g_n^p(\xi) d\xi$ tends to $\Phi(\cdot, z_p)$ in $E(z_p)$ with the H^1 -norm, see [C-K] or [P2].

We define u_n^p as the solution of

$$\begin{cases} M_\gamma u_n^p = 0 & \text{in } \Omega, \\ u_n^p = v_n^p & \text{on } \partial\Omega. \end{cases}$$

Then $w_n^p := u_n^p - v_n^p$ satisfies:

$$\begin{cases} M_\gamma w_n^p = -\nabla \cdot \chi_D A(x) \nabla \Phi(\cdot, z_p) & \text{in } \Omega, \\ w_n^p = u_n^p - v_n^p & \text{on } \partial\Omega. \end{cases}$$

Tending n to ∞ , we deduce that w_n^p tends to w^p in $H^1(\Omega)$ which is the solution of (2.2). From the data $(u_n^p, \frac{\partial u_n^p}{\partial \nu})|_{\partial\Omega}$, we compute via the point source method the values $u_n^p(z_p)$, then we compute

$$\lim_{n \rightarrow \infty} (u_n^p(z_p) - v_n^p(z_p)) = w^p(z_p). \quad (2.3)$$

The characterization of z to be in ∂D is given by the testing $\lim_{p \rightarrow \infty} w^p(z_p) = \infty$.

3 Description of the proof of Theorem 1.1 and a relation between the three methods

3.1 Description of the proof

We give the proof for the case $n = 3$. The case $n = 2$ can be treated similarly with the appropriate changes for the behavior of the related Green's functions. We denote by $\mathbb{N}$ the set of positive integers. We start by proving the first part of Theorem 1.1.

3.1.1 Case one

Let D be such that $D \subset B$. Let also $\varphi \in L^2(\partial\Omega)$ be such that $\|v[\varphi]\|_{H^1(B)} < \epsilon_1$ and $f \in H^{\frac{1}{2}}(\partial\Omega)$ be such that $\|f - v[\varphi]\|_{L^2(\partial\Omega)} \leq \epsilon_2$. Then the function $w := u^f - v[\varphi] \in L^2(\Omega)$ satisfies

$$\begin{cases} M_\gamma w = -\nabla \cdot \chi_D \nabla(v[\varphi]) & \text{in } \Omega, \\ w = f - v[\varphi] & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

We decompose this function into $w := \tilde{w} + \tilde{\tilde{w}}$ where $\tilde{w}$ satisfies (3.1) with zero boundary condition and $\tilde{\tilde{w}}$ the solution of (3.1) with homogeneous equation in Ω . Hence we have $\|\tilde{w}\|_{H^1(\Omega)} \leq c\epsilon_1$ and $\|\tilde{\tilde{w}}\|_{H^1(F)} \leq c\epsilon_2$ for every $F \subset\subset \Omega$. Then also $\|w\|_{H^1(F)} \leq c\epsilon_2$ for every $F \subset\subset \Omega$. Taking $F = B$, we deduce that $\|u^f\|_{H^1(B)} \leq c(\epsilon_1 + \epsilon_2)$. Hence $I_{\epsilon_1, \epsilon_2}(B) \leq c(\epsilon_1 + \epsilon_2)^2$. This means that if we have $D \subset B$, then:

$$I_B = 0.$$

3.1.2 Case two

We suppose that $\partial B \cap D \neq \{\}$. We take a point a in $\partial D \setminus \overline{B}$ and a sequence $z_p \in \Omega \setminus (\overline{D} \cup \overline{B})$ such that z_p tends to a . We denote by $E(z_p)$ an open domain containing D and B such that $z_p \in \Omega \setminus E(z_p)$. We consider the sequence of functions $\Phi(\cdot, z_p)$. We have the following lemma whose proof will be given in section 4.2.

Lemma 3.1 *For every $p \in \mathbb{N}$, we can find a sequence of functions $\varphi_n^p(x) \in L^2(\partial\Omega)$ such that $\|v[\varphi_n^p] - \beta\Phi(\cdot, z_p)\|_{H^1(E(z_p))}$ tends to zero when n tends to ∞ , where β is a constant independent on x .*

Now for every $p \in \mathbb{N}$ fixed, we have:

$$\int_{\partial\Omega} J^f(x) \phi_n^p(x) dx = \int_{\partial D} \left(\frac{\partial u^f}{\partial \nu}(y) v[\varphi_n^p] - u^f(y) \frac{\partial v[\phi_n^p]}{\partial \nu(y)} \right) ds(y).$$

For every $p \in \mathbb{N}$, let $(f_n^p)_{n,p} \subset H^{\frac{1}{2}}(\partial\Omega)$ be such that $\|f_n^p - v[\varphi_n^p]\|_{L^2(\partial\Omega)}$ tends to zero when n tends to ∞ . We define $u^{f_n^p}$ the $H^1(\Omega)$ -solution of the problem:

$$\begin{cases} M_\gamma u^{f_n^p} = 0 & \text{in } \Omega, \\ u^{f_n^p} = f_n^p & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

Proposition 3.1 *We have:*

$$\lim_{p,n \rightarrow \infty} \int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) dx = \infty$$

By Proposition 3.1 and case one, we easily deduce the first part of Theorem 1.1. There are two ways to prove Proposition 3.1. In the following subsection we explain them.

3.2 Two representations for the blowup

We set $w_n^p := u^{f_n^p} - v[\varphi_n^p]$. Hence w_n^p satisfies:

$$\begin{cases} M_\gamma w_n^p = -\nabla \cdot \chi_D A(x) \nabla v[\varphi_n^p] & \text{in } \Omega, \\ w_n^p = f_n^p - v[\phi_n^p] & \text{on } \partial\Omega. \end{cases} \quad (3.3)$$

We write $w_n^p = \tilde{w}_n^p + \tilde{\tilde{w}}_n^p$ where $\tilde{w}_n^p$ satisfies

$$\begin{cases} M_\gamma \tilde{w}_n^p = -\nabla \cdot \chi_D A(x) \nabla v[\phi_n^p] & \text{in } \Omega, \\ \tilde{w}_n^p = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

and $\tilde{\tilde{w}}_n^p$ is a solution of

$$\begin{cases} M_\gamma \tilde{\tilde{w}}_n^p = 0 & \text{in } \Omega, \\ \tilde{\tilde{w}}_n^p = f_n^p - v[\phi_n^p] & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

For $p \in \mathbb{N}$ fixed, we obtain that $v[\phi_n^p]$ tends to $\beta\Phi(\cdot, z_p)$ on any subset of $E(z_p)$ in the H^1 norm (see Lemma 3.1). Hence, the right hand side of (3.4) tends to $-\beta\nabla \cdot \chi_D A \nabla \Phi(\cdot, z_p)$ in $H^{-1}(\Omega)$. From Lax-Milgram lemma we deduce that $\tilde{w}_n^p$ is bounded in $H^1(\Omega)$ and tends weakly to some $w \in H^1(\Omega)$ which satisfies in the distribution sense $M_\gamma w = -\beta\nabla \cdot \chi_D A \nabla \Phi(\cdot, z_p)$. Similarly $\tilde{\tilde{w}}_n^p$ is bounded in $H^{\frac{1}{2}}(\partial\Omega)$ and, hence, converges strongly to w in $L^2(\partial\Omega)$. Thus, $w = 0$ on $\partial\Omega$.

Consider the problem (3.5). Since $\|f_n^p - v[\varphi_n^p]\|_{L^2(\partial\Omega)}$ tends to zero as n tends to ∞ , by interior estimates we deduce that $\tilde{\tilde{w}}_n^p$ tends to zero in $H^1(B)$ For all $B \subset\subset \Omega$. Finally, we deduce that w_n^p tends to $w(\cdot, z_p) \in H^1(\Omega)$ in $H^1(B)$ for every $B \subset\subset \Omega$, where w satisfies:

$$\begin{cases} M_\gamma w = -\beta\nabla \cdot \chi_D A(x) \nabla \Phi(\cdot, z_p) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.6)$$

Since $D \subset E(z_p)$, the previous argument and Lemma 3.1 imply that:

$$\int_{\partial\Omega} J^{f_n^p}(x) \varphi_n^p(x) ds(x) = \int_{\partial D} \left\{ v[\varphi_n^p] \left[\frac{\partial u^{f_n^p}}{\partial \nu} - \frac{\partial v[\varphi_n^p]}{\partial \nu} \right] + [v[\varphi_n^p] - u^{f_n^p}] \frac{\partial v[\varphi_n^p]}{\partial \nu} \right\} ds(x)$$

tends to

$$\beta \int_{\partial D} \left\{ \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} - w \frac{\partial \Phi}{\partial \nu}(\cdot, z_p) \right\} ds(x).$$

I. Integration in $\Omega \setminus \overline{D}$: The pointwise version of the no-response test. Using the Green's representation formula applied in $\Omega \setminus \overline{D}$, we deduce that $\int_{\partial\Omega} J^{f_n^p}(x) \varphi_n^p(x) ds(x)$ tends to:

$$\beta w(z_p, z_p) - \beta \int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x). \quad (3.7)$$

II. Integration in D : The integral version of the no-response test. We write:

$$\begin{aligned} & \int_{\partial D} \left\{ \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} - w \frac{\partial \Phi}{\partial \nu}(\cdot, z_p) \right\} ds(x) \\ &= \int_{\partial D} \left\{ \Phi(\cdot, z_p) \frac{\partial(w + \Phi)}{\partial \nu} - (w + \Phi) \frac{\partial \Phi}{\partial \nu}(\cdot, z_p) \right\} ds(x). \end{aligned} \quad (3.8)$$

We remark that $w + \Phi$ satisfies $\nabla \cdot \gamma \nabla(w + \Phi) = 0$ in D and recalling that $\Delta \Phi = 0$ in D we deduce that:

$$\begin{aligned} & \int_{\partial D} \left\{ \Phi(\cdot, z_p) \frac{\partial(w + \Phi)}{\partial \nu} - (w + \Phi) \frac{\partial \Phi}{\partial \nu}(\cdot, z_p) \right\} ds(x) = \\ &= - \int_{\partial D} \left\{ \Phi(\cdot, z_p) (1 + A(x)) \frac{\partial(w + \Phi)}{\partial \nu^+} - (w + \Phi) \frac{\partial \Phi}{\partial \nu^+}(\cdot, z_p) \right\} ds(x) \\ &= \int_D A(x) \nabla \Phi \cdot \nabla(\Phi + w) dx \end{aligned} \quad (3.9)$$

where ν^+ is the unit normal oriented into $\Omega \setminus D$. Hence, $\int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) ds(x)$ tends to

$$\beta \int_D A(x) \nabla \Phi \cdot \nabla(\Phi + w) dx. \quad (3.10)$$

3.3 Proof of Proposition 3.1

Now for every $\epsilon > 0$ fixed, we choose:

$$\beta := \beta(z_p, \epsilon) = \frac{\epsilon}{4} \left[\max \left(\int_B |\Phi(\cdot, z_p)|^2 dx, \int_B |\nabla_x \Phi(\cdot, z_p)|^2 dx \right) \right]^{-1}.$$

With this choice, we have $\|\beta \Phi(\cdot, z_p)\|_{H^1(B)} \leq \frac{\epsilon}{2}$. Since $\|v[\varphi_n^p] - \beta \Phi(\cdot, z_p)\|_{H^1(E(z_p))}$ tends to zero as n tends to ∞ , for n large enough we obtain $\|v[\varphi_n^p]\|_{H^1(\overline{B})} \leq \epsilon$ and $\|v[\varphi_n^p] - f_n^p\|_{L^2(\partial\Omega)} \leq \epsilon$.

As a conclusion we have a sequence of functions ϕ_n^p such that for every fixed $p \in \mathbb{N}$ there is $N(p, \epsilon) \in \mathbb{N}$ such that for all $n \geq N(p, \epsilon)$ we have

$$\|v[\varphi_n^p]\|_{H^1(B)} \leq \epsilon \quad \text{and} \quad \|f_n^p - v[\varphi_n^p]\|_{L^2(\partial\Omega)} \leq \epsilon.$$

This sequence has the property: for p fixed, $\int_{\partial\Omega} J^{f_n^p}(x) \varphi_n^p(x) ds(x)$ tends to

$$\beta w(z_p, z_p) - \beta \int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x).$$

The function w is called the reflected solution to the system M_γ . In the next section, we justify the following result:

Proposition 3.2 1) The sequence $w(z_p, z_p)$ tends to ∞ when z_p tends to a .
 2) The sequence $\int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x)$ is bounded with respect to p .

We remark that $\beta(z_p, \epsilon)$ is bounded with respect to z_p since $(z_p)_{p \in \mathbb{N}} \subset \subset \Omega \setminus \overline{B}$. This implies that

$$\lim_{p, n \rightarrow \infty} \int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) ds(x) = \infty.$$

Hence $I_\epsilon(B) = \infty$. We proved the theorem. $\square$

On the other hand we can show the blowup of $\int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) dx$ by using (3.10). This the way of the probe method, see [I].

3.4 Reconstruction of the values of $A(a)$, $a \in \partial D$.

In this part we show how to recover the values of $A(a)$, $a \in \partial D$ from the Dirichlet to Neumann map. Let $x \in \partial D$. From the proof of the reconstruction of ∂D , we reconstructed a sequence f_n^p such that

$$\int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) ds(x)$$

tends to

$$\beta(z_p, \epsilon) w(z_p, z_p) - \beta(z_p, \epsilon) \int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x),$$

when n tends to ∞ . The constant $\beta(z_p, \epsilon)$ has been introduced just for normalization, we take it here equal to 1. Then, we have

$$w(z_p, z_p) = \lim_{n \rightarrow \infty} \int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) ds(x) + \int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x).$$

Now from Proposition 4.1, we deduce that

$$\frac{A(a)}{A(a) + 2} = \lim_{p \rightarrow \infty} \frac{|z_p - z_p^*| w(z_p, z_p)}{(4\pi)^{-1}}.$$

Knowing that $\int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x)$ is bounded, we deduce the following formula to compute the values of $A(a)$, $a \in \partial D$

$$\frac{A(a)}{A(a) + 2} = \lim_{p, n \rightarrow \infty} (4\pi) |z_p - z_p^*| \int_{\partial\Omega} J^{f_n^p}(x) \phi_n^p(x) ds(x), \quad (3.11)$$

where z_p is in $\Omega_{a, \theta}$ and z_p tends to a as p tends to ∞ . $\square$

3.5 Some comments on the relation between the three methods

The limit (2.1) is the one related to the probe method (3.10). The behavior of the pointwise estimate of the no-response test (2.3) is exactly the one of the singular sources method (3.7). The convergence of the no-response test is a consequence of the convergence of either the probe method or the singular sources method.

Since it is known that the integral (3.10) diverges as z_p tends to ∂D , see [I], then $\int_{\partial\Omega} J_n^{f_p}(x) \phi_n^p(x) ds(x)$ diverges also. We will show that also the one given in the pointwise sense is diverging as z_p tends to ∂D . This will give a formula to reconstruct the values of $A(x)$, $x \in \partial D$.

To prove the convergence of these methods, the energy version is easier, see [I], since the pointwise estimates are more difficult to establish than the energy ones. Regarding the stability, the pointwise version is more suitable, see [P2].

4 Behavior of the reflected solution and proof of Proposition 3.2.

Our first goal is the justification of the first point of Theorem 3.2. We recall that the reflected solution satisfies:

$$\begin{cases} M_\gamma w = -\beta \nabla \cdot \chi_D A \nabla \Phi(\cdot, z_p) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

Let us consider the sequence $w(z_p, z_p)$. From (4.1) we see that the distribution $G := \frac{1}{\beta} w + \Phi$ satisfies

$$\begin{cases} M_\gamma G = -\delta(x - z_p) & \text{in } \Omega, \\ G = \Phi(\cdot, z_p) & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

The purpose of the following lemmas is to localize the dominant part of G in the pointwise sense. The proofs will be given in the section 3.1.

We set $\tilde{G}$ the Green's function of M_γ on Ω with homogeneous boundary Dirichlet condition.

Lemma 4.1 *For every $B \subset\subset \Omega$, the function $(G - \tilde{G})(x, z)$ is bounded for $x \in \Omega$ and $z \in B$.*

Let us define the Green's function G' of L_γ on Ω with homogenous Dirichlet boundary condition, i.e:

$$\begin{cases} L_\gamma G' = -\delta(x - z) & \text{in } \Omega, \\ G' = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.3)$$

Lemma 4.2 *The function $(\tilde{G} - G')(x, z)$ is bounded for $(x, z) \in \Omega^2$.*

Let now $a \in \partial D$. We set $L_{\gamma(a)}$ as the expression L_γ with γ replaced by $\gamma(a) := 1 + \chi_D A(a)$. We denote by G'_0 the Green's function of $L_{\gamma(a)}$ on Ω with homogenous Dirichlet boundary condition. We denote also by $\Omega_{a,\theta}$ the positive cone of center a with axis $\nu(a)$ and angle $\theta \in [0, \frac{\pi}{2})$ where $\nu(a)$ is the normal to ∂D on the point a oriented outside D . The positivity of this cone is to be understood in terms of the direction of $\nu(a)$.

Lemma 4.3 *Let $\epsilon > 0$, $\theta \in [0, \frac{\pi}{2})$ and $B \subset\subset \Omega$ such that $D \subset B$ be fixed. There exists a constant $c(\epsilon, B, \theta) > 0$ such that*

$$|(G' - G'_0)(x, y)| \leq c(\epsilon, B, \theta) [d(y, \partial D)]^{-\frac{\epsilon}{3-\epsilon}}$$

for $x \in \Omega$ and $y \in \Omega_{a,\theta} \cap B$ where $0 < c(\epsilon, B, \theta)$ is a constant.

Let $\Phi'_{\gamma(a)}$ be the fundamental solution of $\nabla \cdot (1 + A(a)\chi^-) \nabla \cdot$, where χ^- is the characteristic function of the negative half-space of $\mathbb{R}^3$ given by $\mathbb{R}^3_- := \{x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$. Let T be the transformation of coordinates which transforms the half-space given by the points which are below the plane tangent to ∂D on a to the half-space $\mathbb{R}^3_-$.

Lemma 4.4 *Let $\theta \in [0, \frac{\pi}{2})$ be fixed. For $\epsilon > 0$ and $\delta > 0$ small enough, there exists a constant $c(\delta, \theta)$ such that*

$$|(G'_0 - \Phi'_{\gamma(a)} \circ T)(x, y)| \leq c(\delta, \theta) [d(y, \partial D)]^{-\epsilon}$$

for $x \in \Omega$ and $y \in \Omega_{a,\theta} \cap B(a, \delta)$ where $0 < c(\delta, \theta)$ is a constant.

Now, we have the following formula, see [A-I-P]:

$$\Phi'_{\gamma(a)}(x, y) - \frac{(4\pi)^{-1}}{|x - y|} = \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|x - y^*|},$$

with $(x, y) \in \mathbb{R}^n_+$, where y^* is the point symmetric to y with respect to the plane $\{y \in \mathbb{R}^3 : y = (y_1, y_2, 0)\}$.

Proposition 4.1 *Let $a \in \partial D$ and $\theta \in [0, \frac{\pi}{2})$ be fixed. There exists $\delta > 0$ small enough such that the function*

$$|y - y^*| \left[\frac{1}{\beta} w(y, y^*) - \frac{A(a)}{A(a) + 2} \frac{(4\pi)^{-1}}{|y - y^*|} \right]$$

is bounded for y in $\Omega_{a,\theta} \cap B(a, \delta)$.

As a consequence of Proposition 4.1, for every $\theta \in [0, \frac{\pi}{2})$ we have:

$$w(z_p, z_p) \rightarrow \infty \quad \text{as } z_p \rightarrow a.$$

where $z_p \in \Omega_{a,\theta}$.

Proof of Proposition 4.1. We write

$$G = (G - \tilde{G}) + (\tilde{G} - G') + (G' - G'_0) + G'_0$$

and we recall that $\frac{1}{\beta}w = G - \Phi$, then from the previous lemmas we obtain

$$|\frac{1}{\beta}w(x, y) - (G'_0 - \Phi)| \leq c \left[d(y, D) \right]^{\frac{-\epsilon}{3-\epsilon}}$$

Now, we write

$$G'_0 - \Phi = G'_0 - \Phi'_{\gamma(a)} \circ T + [\Phi'_{\gamma(a)} \circ T - \frac{(4\pi)^{-1}}{|x-y|} + \frac{(4\pi)^{-1}}{|x-y|} - \Phi].$$

By Lemma 4.4 the term $G'_0 - \Phi'_{\gamma(a)} \circ T$ is bounded by $c \left[d(y, D) \right]^{-\epsilon}$ and arguing as for Lemma 4.2, $\frac{(4\pi)^{-1}}{|x-z|} - \Phi$ is also bounded. Recall also that T is an isometry since it is given by a combination of a translation and a rotation, then using the identity

$$\Phi'_{\gamma(a)} \circ T(x, y) - \frac{(4\pi)^{-1}}{|x-y|} = \frac{A(a)}{A(a)+2} \frac{(4\pi)^{-1}}{|x-y^*|}$$

we deduce that there exists a constant c such that

$$\left| G'_0 - \Phi - \frac{A(a)}{A(a)+2} \frac{(4\pi)^{-1}}{|x-y^*|} \right| \leq c \left[d(y, \partial D) \right]^{-\epsilon}$$

for every x and y in $\Omega_{a,\theta} \cap B(a, \delta)$. Hence we have the estimate:

$$\begin{aligned} |y - y^*| \left[\frac{1}{\beta}w(y, y) - \frac{A(a)}{A(a)+2} \frac{(4\pi)^{-1}}{|y-y^*|} \right] &\leq c \left[d(y, \partial D) \right]^{-\frac{\epsilon}{3-\epsilon}} |y - y^*| \\ &\leq c \left[d(y, \partial D) \right]^{1-\frac{\epsilon}{3-\epsilon}} + c \left[d(y, \partial D) \right]^{-\frac{\epsilon}{3-\epsilon}} \left[d(y^*, \partial D) \right]. \end{aligned}$$

Choosing $\delta > 0$ small enough such that $\Omega_{a,\theta} \cap B(a, \delta) \cap \partial D = \{a\}$, then there exists a constant $c(\delta) > 0$ such that $d(y, \partial D) \geq c(\delta)d(y, a)$.

Taking $\epsilon > 0$ satisfying $-\frac{\epsilon}{3-\epsilon} + 1 > 0$, we deduce the result.

□

4.1 Proofs of the lemmas.

The proof of Lemma 3.1 can be adapted from the one given in [C-K] or [P1] for the denseness of the range of the Herglotz wave operator. Here, we provide a concise version of the proof.

Proof of Lemma 3.1. Let $\tilde{E}(z_p)$ be any domain such that $E(z_p) \subset \subset \tilde{E}(z_p) \subset \Omega$ and $z_p \in \Omega \setminus \tilde{E}(z_p)$ and let $H: L^2(\partial\Omega) \rightarrow L^2(\partial\tilde{E}(z_p))$ defined by

$$H(\varphi)(x) := \int_{\partial\Omega} \Phi(x, y) \varphi(y) ds(y).$$

We want to prove that $\Phi(x, z_p) \in \overline{R(H)}^{L^2(\partial\tilde{E}(z_p))}$, where $R(H)$ is the range of H .

We know that the adjoint of H is given by the operator, $H^*: L^2(\partial\tilde{E}(z_p)) \rightarrow L^2(\partial\Omega)$ with $H^*(\varphi)(y) := \int_{\partial\tilde{E}(z_p)} \overline{\Phi(x, y)} \varphi(x) ds(x)$. We write

$$L^2(\partial\tilde{E}(z_p)) = \overline{R(H)} \oplus N(H^*),$$

where

$$N(H^*) := \{\varphi \in L^2(\partial\tilde{E}(z_p)) : H^*(\varphi) = 0\}.$$

To prove our lemma, it is enough to prove that $\Phi(x, z_p) \in N(H^*)^\perp$. Let $\varphi \in N(H^*)$, then $\int_{\tilde{E}(z_p)} \overline{\Phi(x, y)} \varphi(x) ds(x) = 0$, for $y \in \partial\Omega$. Passing to the conjugate, for

$$v(y) := \int_{\tilde{E}(z_p)} \Phi(x, y) \overline{\varphi(x)} ds(x)$$

we derive $v(y) = 0$ for $y \in \partial\Omega$. Solving the exterior problem

$$\begin{cases} L_1 v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v \text{ satisfies the Sommerfeld radiation conditions at infinity.} \end{cases} \quad (4.4)$$

we deduce that $v(y) = 0$, $\forall y \in \mathbb{R}^n \setminus \Omega$. By unique continuation, we have $v(y) = 0$ $\forall y \in \mathbb{R}^n \setminus \tilde{E}(z_p)$. Since $z_p \in \Omega \setminus \tilde{E}(z_p)$, then $v(z_p) = 0$, hence $\int_{\partial\tilde{E}(z_p)} \Phi(x, z_p) \overline{\varphi(x)} ds(x) = 0$.

This proves that $\Phi(x, z_p) \in \overline{R(H)}^{L^2(\partial\tilde{E}(z_p))}$. It means that for every p fixed, there exists a sequence $\varphi_n^p \in L^2(\partial\Omega)$ such that $H(\varphi_n^p)$ tends to $\Phi(x, z_p)$ in $L^2(\partial\tilde{E}(z_p))$, as n tends to ∞ . Since both of $H(\varphi_n^p)$ and $\Phi(x, z_p)$ satisfy the same elliptic equation in $\tilde{E}(z_p)$, then using interior estimates, we deduce Lemma 3.1. $\square$

Proof of Lemma 4.1. We have

$$M_\gamma(G - \tilde{G}) = 0 \quad \text{in } \Omega, G - \tilde{G} = \Phi(\cdot, y) \quad \text{on } \partial\Omega. \quad (4.5)$$

Since for $y \in B \subset \subset \Omega$, $\Phi(\cdot, y)$ is very smooth and bounded in $C^2(\partial\Omega)$, then ([L-V], corollary 1.3) gives the result.

□

Proof of Lemma 4.2. We have

$$L_\gamma(\tilde{G} - G') = c(x)\tilde{G} \quad \text{in } \Omega, \quad \tilde{G} - G' = 0 \quad \text{in } \partial\Omega. \quad (4.6)$$

Using the Green's representation in Ω , we write:

$$(\tilde{G} - G')(x, y) = - \int_{\Omega} c(z)\tilde{G}(z, y)G'(z, x)dz. \quad (4.7)$$

We know that G' is bounded in $L^2(\Omega)$ (see [D-N]). Solving the problem

$$\begin{cases} M_\gamma(\tilde{G} - G') = -c(x)G' & \text{in } \Omega, \\ \tilde{G} - G' = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.8)$$

we deduce that $\tilde{G} - G'$ is uniformly bounded in $H^1(\Omega)$, hence G is also bounded in $L^2(\Omega)$. From (4.7), we deduce that $(\tilde{G} - G')$ is bounded in $(x, y) \in \Omega^2$. This ends the proof. □

Proof of Lemma 4.3. Let $B \subset\subset \Omega$ be a fixed subdomain such that $D \subset B$. We set

$$R(x, y) := G'(x, y) - G'_0(x, y).$$

Then, for every $y \in \Omega$ fixed, $R(x, y)$ satisfies

$$\begin{cases} L_\gamma(R(x, y)) = -\nabla \cdot (\chi_D(\gamma - \gamma(a))\nabla G'_0) & \text{in } \Omega, \\ R(x, y) = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.9)$$

Since γ is of class C^1 , there exists $c > 0$ such that $|\gamma(x) - \gamma(a)| \leq c|x - a|$ for $x \in D$.

Let $\tilde{G}'_0(x, y)$ be the fundamental function of $\nabla \cdot \gamma(a)\nabla \tilde{G}'_0 = -\delta(x - y)$, $(x, y) \in (\mathbb{R}^3)^2$. It is proved in [A-D] that

$$|\nabla_x \tilde{G}'_0(x, y)| \leq c|x - y|^{-2}. \quad (4.10)$$

$(x, y) \in (\mathbb{R}^3)^2$ with $c > 0$ is a constant. In Ω , we have $\nabla \cdot \gamma(a)\nabla(\tilde{G}'_0 - G'_0) = 0$. Using [L-V] and knowing that both of $\tilde{G}'_0$ and G'_0 are bounded, in $L^2(\Omega)$, see [D-N], we deduce that $(\tilde{G}'_0 - G'_0)(x, y)$ is bounded for $(x, y) \in B$. From (4.10), we deduce that:

$$|\nabla_x G'_0(x, y)| \leq c(B)|x - y|^{-2}. \quad (4.11)$$

$(x, y) \in B^2$ with $c(B) > 0$ is a constant depending only on B . For $\theta \in [0, \frac{\pi}{2})$ fixed, there exists a constant $c(\theta) > 0$ such that for $x \in D$ and $y \in \Omega_{a, \theta}$, we have

$$|x - a| \leq c(\theta)|x - y|. \quad (4.12)$$

Hence, using (4.11), we have:

$$\left| \chi_D(\gamma(x) - \gamma(a)) \nabla_x G'_0(x, y) \right| \leq c(B, \theta) |x - y|^{-1} \quad (4.13)$$

for every x in Ω and y in $\Omega_{a, \theta} \cap B$. For every $\epsilon > 0$, we have the estimate

$$\begin{aligned} & \int_{\Omega} |\chi_D(\gamma(z) - \gamma(a)) \nabla_x G'_0(x, y)|^{3-\epsilon} dx \\ & \leq c(B, \theta) \int_D |x - y|^{-3+\epsilon} dx \\ & \leq c'(B, \theta). \end{aligned} \quad (4.14)$$

We return now to (4.9). Using the L^p regularity of this problem, see [S], we deduce that

$$\|\nabla R(\cdot, y)\|_{L^{3-\epsilon}(D)}^{3-\epsilon} \leq c(B, \theta).$$

From (4.9), we write

$$R(x, y) = \int_D (\gamma(z) - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z G'(x, z) dz, \quad (x, y) \in \Omega.$$

We rewrite it as

$$\begin{aligned} R(x, y) &= \int_D (\gamma - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z G'_0(x, z) dz \\ &\quad + \int_D (\gamma - \gamma(a)) \nabla_z G'_0(x, z) \cdot \nabla_z R(y, z) dz. \end{aligned} \quad (4.15)$$

Hence, the first part satisfies:

$$\begin{aligned} & \left| \int_D (\gamma(z) - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z G'_0(x, z) dz \right| \\ & \leq c \int_D |z - a| |y - z|^{-2} |x - z|^{-2} dz. \end{aligned} \quad (4.16)$$

For $x \in \overline{D}$ and $y \in \Omega_{a, \theta}$, arguing as in ([A-D], page 11 inequality (4.12)), this last integral is bounded by $\tilde{c} |\ln(|x - y|)|$ where the authors took y on the normal $\nu(a)$. Their proof still justified also for $y \in \Omega_{a, \theta}$ since the critical point is the inequality (4.12). Using the inequalities $|x - y| \leq d(x, \partial D) + d(y, \partial D)$ and $|\ln(|x - y|)| \leq c|x - y|^{-t}$ locally for every $t > 0$, we deduce that:

$$\left| \int_D (\gamma(z) - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z G'_0(x, z) dz \right| \leq c(d(x, \partial D) + d(y, \partial D))^{-t} \quad (4.17)$$

Let us now consider the term $\int_D (\gamma(z) - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z R(x, z) dz$. Since $\nabla_x R(x, y)$

is bounded in $(L^{3-\epsilon}(\Omega))^3$, then by the Holder inequality we have:

$$\begin{aligned}
& \left| \int_D (\gamma(z) - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z R(x, z) dz \right| \\
& \leq cc(B) \int_D |z - a| |y - z|^{-2} |\nabla_z R(x, z)| dz \\
& \leq cc(B) \left[\int_D [|z - a| |y - z|^{-2}]^{p'} dz \right]^{\frac{1}{p'}} \left[\int_D |\nabla_z R(x, z)|^{3-\epsilon} dz \right]^{\frac{1}{3-\epsilon}}, \quad (4.18)
\end{aligned}$$

where $p' := p'(\epsilon) = \frac{3-\epsilon}{2-\epsilon} > \frac{3}{2}$. Using

$$\int_D [|z - a| |y - z|^{-2}]^{p'} dz \leq c' \int_D |y - z|^{-2p'} dz \leq c'' \frac{2-\epsilon}{\epsilon} [d(y, \partial D)]^{-\frac{\epsilon}{2-\epsilon}},$$

we estimate

$$\left| \int_D (\gamma(z) - \gamma(a)) \nabla_z G'_0(y, z) \cdot \nabla_z R(x, z) dz \right| \leq c(B, \theta, \epsilon) [d(y, \partial D)]^{-\frac{\epsilon}{3-\epsilon}},$$

where $c(B, \theta, \epsilon)$ depends on B, θ and ϵ .

This means that for $x \in \bar{D}$ and $y \in \Omega_{a, \theta} \cap B$, we have the estimate $|R(x, y)| \leq c(B, \theta, \epsilon) [d(y, \partial D)]^{-\frac{\epsilon}{3-\epsilon}}$. Now, on $(\Omega \setminus D)$, $R(x, y)$ satisfies $\Delta_x R(x, y) = 0$ with the estimate

$$|R(x, y)| \leq c(B, \theta, \epsilon) [d(y, \partial D)]^{-\frac{\epsilon}{3-\epsilon}}$$

for $x \in \partial D$ and $R(x, y) = 0$ for $x \in \partial\Omega$. Then we have the result. $\square$

Proof of Lemma 4.4. To prove Lemma 4.4, it is enough to prove that:

$$|(G'_0 \circ T^{-1} - \Phi'_{\gamma(a)})(x, y)| \leq c(\delta, \theta) (d(y, \partial T(D)))^{-t}$$

for $x \in T(\Omega)$ and $y \in T(\Omega_{a, \theta})$. The set $T(\Omega_{a, \theta})$ is the intersection of $T(\Omega)$ and the cone with vertex at the origin $O = (0, 0, 0)$, the axis in the direction $\nu(O) = (0, 0, 1)$ and angle θ .

Arguing as in ([A-D], Proposition 3.2), we get the following estimate:

$$\left| (G'_0 \circ T^{-1} - \Phi'_{\gamma(a)})(x, y) \right| \leq c(r) |\ln(|x - y|)| \quad (4.19)$$

for $x \in \overline{\mathbb{R}^3} \cap B(0, \frac{r}{2})$ and $y = t\nu(O)$ with t small enough such that $y \in B(0, \frac{r}{2})$ with $r > 0$ depending on ∂D via its parametrization. We deduce that $|(G'_0 \circ T^{-1} - \Phi'_{\gamma(a)})(x, y)|$ is bounded by $c(r) |\ln(|x - y|)|$ for $x \in \overline{T(D)} \cap B(0, \frac{r}{2})$ and $y = t\nu(O)$ with t small enough such that $y \in B(0, \frac{r}{2})$.

In [A-D], the proof depends on the inequality $|x| \leq |x - y|$, which is true for these points, and the fact that the change of variable they used fix the points of the form

$y = t\nu(0)$ i.e. $\Psi(y) = y$, where Ψ is the change of variables introduced to flatten $\partial T(D)$ near the point O .

Following their proof, we find that the same result is true by taking

$$x \in \overline{\mathbb{R}_+^3} \cap T(\Omega) \cap B(0, r) \quad \text{and} \quad y \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2}),$$

i.e the inequality (4.12) of [A-D] is valid for those points. This is due to the inequalities $|x| \leq c(\theta)|x - y|$ and $|\Psi(x)| \leq c'(\theta)|\Psi(x) - \Psi(y)|$, where $c(\theta)$ and $c'(\theta)$ depend only on θ , which are satisfied for these points. The second inequality is a consequence of the first and the form of Ψ .

Hence as for the proof of Lemma 4.3, we have:

$$|(G'_0 \circ T^{-1} - \Phi'_{\gamma(a)})(x, y)| \leq c(r, \theta)(d(x, \partial T(D)) + d(y, \partial T(D)))^{-t}$$

for every $t > 0$, $x \in \overline{\mathbb{R}_+^3} \cap T(\Omega) \cap B(0, r)$ and $y \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$.

Now, we show that $|(G'_0 \circ T^{-1} - \Phi'_{\gamma(a)})(x, y)| \leq c(r, \theta)[d(y, \partial T(D))]^{-t}$ for $x \in \mathbb{R}_+^3 \cap T(\Omega) \cap B(0, r)$ $y \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$. To do so we observe that on $\mathbb{R}_+^3 \cap T(\Omega) \cap B(0, r)$, we have $\tilde{G}(x, y) := G'_0 \circ T^{-1}(x, y) - \Phi'_{\gamma(a)}(x, y)$ satisfies $\Delta \tilde{G} = 0$ with uniformly bounded boundary conditions. The uniform boundedness of the boundary conditions is justified by:

- 1) the fact that $|\tilde{G}(x, y)| \leq c(r, \theta)[d(y, \partial T(D))]^{-t}$ for x on the boundary of $\mathbb{R}_+^3 \cap T(\Omega) \cap B(0, r)$ and $y \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{2})$ and
- 2) both of $G'_0 \circ T^{-1}(x, y)$ and $\Phi'_{\gamma(a)}(x, y)$ are bounded for x in $\partial B(0, r)$ and $y \in B(0, \frac{r}{2})$.

The second point is justified since for $x \in T(\Omega) \setminus B(0, \frac{2}{3}r)$ and $y \in B(0, \frac{r}{2})$, both of $G'_0 \circ T^{-1}(x, y)$ and $\Phi'_{\gamma(a)}(x, y)$ satisfy divergence form elliptic equation with discontinuous coefficient and homogenous second member. Since $G'_0 \circ T^{-1}(x, y)$ and $\Phi'_{\gamma(a)}(x, y)$ are bounded in $L^2(T(\Omega))$, then [L-V] implies 2).

We deduce that $|\tilde{G}(x, y)| \leq c(r, \theta)[d(y, \partial T(D))]^{-t}$ for $x \in \mathbb{R}_+^3 \cap T(\Omega) \cap B(0, \frac{r}{2})$ and $y \in T(\Omega_{a,\theta}) \cap B(0, \frac{r}{3})$. Hence, taking all together, we showed that $\tilde{G}(x, y)$ is bounded by $c(r, \theta)[d(y, \partial T(D))]^{-t}$ for $x \in T(\Omega)$ and $y \in B(0, \frac{r}{2})$. To finish the proof we take $\delta = \frac{r}{2}$. This ends the proof. $\square$

Proof of the second point of Theorem 3.2. To show that the term

$$\int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu}(\cdot, z_p) ds(x)$$

is bounded with respect to p , it is enough to remark that the sequence z_p is near ∂D hence $\Phi(\cdot, z_p)$ is very regular and bounded in $H^{\frac{1}{2}}(\partial\Omega)$.

Next, we investigate the term $\frac{\partial w}{\partial \nu}(\cdot, z_p)$. Near $\partial\Omega$ the function w solves an elliptic equation with zero second member. From Lemma 4.2, we have $(G - G')(\cdot, z)$ is

bounded in $L^2(\Omega)$ with respect to $z \in \Omega$. Since $G'(\cdot, z)$ is bounded in $L^2(\Omega)$ with respect to $z \in \Omega$, hence also $G(\cdot, z)$ has the same property. Now, by interior estimates we deduce that $w(\cdot, z_p)$ is bounded in $H^1(B)$ for every $B \subset\subset (\Omega \setminus \bar{D})$. We take now B as any corona surrounding D and denote by ∂B the exterior part of the boundary of B . Then solving the problem $M_\gamma w = 0$ in Ω_B , where Ω_B is the domain limited by $\partial\Omega$ and ∂B , with Dirichlet condition on $\partial\Omega \cup \partial B$, we deduce that $\frac{\partial w}{\partial \nu}(\cdot, z_p)$ is bounded in $H^{-\frac{1}{2}}(\partial\Omega)$. This implies that $\int_{\partial\Omega} \Phi(\cdot, z_p) \frac{\partial w}{\partial \nu} ds(x)$ is bounded with respect to z_p . $\square$

5 Appendix: Justification of the Green and jumps formulas for the equation $-\Delta + c(x)$ where $c(x)$ is a bounded function.

In this appendix, we justify the Green and jump formulas for the equation $-\Delta + c(x)$ where $c(x)$ is a bounded function. In the case where $c(x)$ is continuous these results are known, see [Isa1]. We assume that $c(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$. The following argument is true in $\mathbb{R}^n$, $n = 2, 3$. We give the details for $n = 3$. Let us consider the distribution $\Psi(x, y) := \Phi(x, y) - \frac{(4\pi)^{-1}}{|x-y|}$. Then $\Psi(\cdot, y)$ satisfies

$$\begin{cases} (-\Delta_x + c(x))\Psi = -c(x)\frac{(4\pi)^{-1}}{|x-y|} & \text{in } \tilde{\Omega}, \\ \Psi(x, y) = \Phi(x, y) - \frac{(4\pi)^{-1}}{|x-y|} & \text{on } \partial\tilde{\Omega}, \end{cases} \quad (5.1)$$

where $\Omega \subset\subset \tilde{\Omega}$. Since the right hand side of (5.1) is continuous with respect to $y \in \Omega$ with values in $L^2(\tilde{\Omega})$, we deduce that Ψ is continuous with respect to $y \in \bar{\Omega}$ with values in $H^2(\Omega)$ by interior estimates.

1) To prove the Green's formula, it is enough to prove that

$$\lim_{r \rightarrow 0} \int_{\partial B(y, r)} u(x) \frac{\partial \Phi}{\partial \nu}(x, y) - \frac{\partial u}{\partial \nu}(x) \Phi(x, y) ds(x) = u(y).$$

We write

$$\begin{aligned} & \int_{\partial B(y, r)} u(x) \frac{\partial \Phi}{\partial \nu}(x, y) - \frac{\partial u}{\partial \nu}(x) \Phi(x, y) ds(x) \\ &= \int_{\partial B(y, r)} u(x) \frac{\partial \Phi'}{\partial \nu}(x, y) - \frac{\partial u}{\partial \nu}(x) \Phi'(x, y) ds(x) \\ & \quad + \int_{\partial B(y, r)} u(x) \frac{\partial \Psi}{\partial \nu}(x, y) - \frac{\partial u}{\partial \nu}(x) \Psi(x, y) ds(x), \end{aligned} \quad (5.2)$$

where we used the notation $\Phi'(x, y) = \frac{(4\pi)^{-1}}{|x-y|}$. Now, since u satisfies

$$(-\Delta + c(x))u(x) = 0 \quad \text{in } \Omega,$$

then $u \in H_{loc}^2(\Omega)$, hence $u(x)$ is continuous in Ω since $\Omega \subset \mathbb{R}^3$. Since u is continuous and $\frac{\partial \Phi'}{\partial \nu} = \frac{\nu(x)}{4\pi r^2}$, where $r = |x - y|$, then using the mean value theorem we deduce that the first term is tending to $u(y)$ as r tends to zero. Using the Cauchy-Schwartz inequality, the other terms tends to zero since $|\Phi'(x, y)| = \frac{(4\pi)^{-1}}{r}$ and $\Psi(x, y)$ is bounded, with respect to x , in $H^2(\Omega)$ for every $y \in \Omega$. This ends the proof of the point 1).

For the case where the dimension $n > 3$ we need the continuity of the coefficient $c(x)$, in which case the solution of $(-\Delta + c(x))u(x) = 0$ in Ω is continuous.

2) To justify the jump formula, we use the decomposition $\Phi(x, y) = \Psi(x, y) + \Phi'(x, y)$. Then we have:

$$\int_{\partial\Omega} \frac{\partial \Phi(x, y)}{\partial x} \phi(x) ds(x) = \int_{\partial\Omega} \frac{\partial \Phi'(x, y)}{\partial x} \phi(x) ds(x) + \int_{\partial\Omega} \frac{\partial \Psi(x, y)}{\partial x} \phi(x) ds(x)$$

where $y \in \Omega$. Now, letting y tend to $\partial\Omega$, using the jump formula for the Green's function $\Phi'(x, y)$ and the continuity of $\Psi(\cdot, y)$ with respect to $y \in \overline{\Omega}$ with values in $H^2(\Omega)$, we deduce the desired jump formula for the Green's function $\Phi(x, y)$.

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