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Open string mirror maps from Picard-Fuchs equations on relative cohomology

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Abstract

A definition of open string period integrals for noncompact Calabi-Yau manifolds is given. It is shown that the open string Picard-Fuchs operators, originally derived through physical considerations, follow from these period integrals. Also, we find that the natural extension to the compact case does not yield the expected results.

1 Introduction.

For some time now, mirror symmetry and its mathematical implications have provided unexpected connections between seemingly unrelated families of Calabi-Yau manifolds. In particular, closed string mirror symmetry, under which the complexified Kähler moduli space $\mathcal{X}$ of one Calabi-Yau family is locally isomorphic to the complex moduli space $\mathcal{Y}$ of another family, has proven to be a powerful tool of enumerative geometry [5],[18].

Closer to the present, there is the developing story of local mirror symmetry [9],[4],[10]. Here, the relevant space $\mathcal{X}$ may be described as the variation

of complexified Kähler moduli of canonical bundle K_S of a Fano surface S. The associated Gromov-Witten invariants have the (conjectural) interpretation as the number of holomorphic curves lying in the surface S. When compared with the compact case, the local periods take on a simplified form.

Finally, we come to the topic of interest, which is open string mirror symmetry [1],[14],[16]. Mathematically, this represents an extension of the moduli spaces considered in closed string mirror symmetry. We are now interested, on one side of the duality, in the moduli space of a pair (X, L), where we look at the complexified Kähler moduli space of X together with deformations of a Lagrangian submanifold $L \subset X$. Throughout, we will denote this moduli space as $(\mathcal{X}, \mathcal{L})$. On the other side, we consider the moduli space of (Y, C) with C a holomorphic curve in Y, in the sense of the complex structure moduli space of Y and normal deformations of C in X; this will be referred to as $(\mathcal{Y}, \mathcal{C})$. Then $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{Y}, \mathcal{C})$ are supposed to be locally isomorphic, and the enumerative significance of this duality is given the interpretation of counting the number of holomorphic maps $f : D \to X$ such that $f|_{\partial D}(\partial D) \subset L$, where $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

In this paper, we will give a definition of period integrals defining the moduli space $(\mathcal{Y}, \mathcal{C})$ in the noncompact case, and show that this is consistent with physicists' open string PF operators [14]. Moreover, we attempt to extend this construction to include compact considerations, and show that this does not seem to be the right approach. Physically, this is due to the D-brane charge on C, which must be dealt with in a compact theory.

The organization of this paper is as follows. In section 2, we will review the noncompact pairs of spaces (X, L) and (Y, C) that are used in local open string mirror symmetry. Section 3 contains a definition of "open string period integrals" on $(\mathcal{Y}, \mathcal{C})$, which are then shown to reproduce the known Picard-Fuchs operators. Section 4 shows the failure of extending this idea to the compact case, and finally, in section 5 we give an example in detail.

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2 Review of the spaces

2.1 Construction of $(\mathcal{X}, \mathcal{L})$.

Although the primary focus of the present work is to define the moduli space $(\mathcal{Y}, \mathcal{C})$, it is instructive to first carefully review the mirror $(\mathcal{X}, \mathcal{L})$; moreover, the holomorphic discs that are "counted" by open string mirror symmetry actually lie in (X, L).

Following [2], we can define the spaces X as

$$X_r = \{(z_0, z_1, z_2, z_3) \in (\mathbb{C}^4 - Z) | \sum_{i=0}^{3} l_i |z_i|^2 = r\}/S^1, \tag{1}$$

where the S^1 action is

$$S^1 : z_i \longrightarrow e^{\sqrt{-1} l_i \theta} z_i, \quad i = 0, \ldots, 3,$$

and $r \in \mathbb{R}^+$. Also, Z is the singular set implied by the relation $\sum_{i=0}^{3} l_i |z_i|^2 = r$. For example, for the local $\mathbb{P}^2$ case, we have $l = (-3, 1, 1, 1)$, and this means that $Z = \{z_1 = z_2 = z_3 = 0\}$ here. Finally, we must impose the condition $\sum_{i=0}^{3} l_i = 0$, which is equivalent to $c_1(X_r) = 0$ (that is, so X_r satisfies the definition of a local Calabi-Yau manifold).

Notice also that r is the single Kähler parameter on this space. To see this, recall that in such a toric construction of X, the vector l corresponds to a 2 cycle $C_l \hookrightarrow X$. Then there is an element of cohomology ω_l dual to C_l, and we find that

$$\int_{C_l} \omega_l = r$$

when appropriately scaled. Hence, if t is the coordinate on the complexified Kähler moduli space of X, we may identify $Re(t) = r$.

In the interests of clarity, throughout the paper, we will consider spaces with only one Kähler modulus. The inclusion of additional parameters is in fact trivial [14].

Next, we should construct Lagrangian submanifolds L of X. These have been worked out previously [1], and are given by

$$L_{r,c} = X_r \cap \{\sum_i k_i^1 |z_i|^2 = c, \ \sum_i k_i^2 |z_i|^2 = 0, \ \sum_i arg(z_i) = 0\}.$$

The convention used is that $k^j = (k_0^j, \ldots, k_3^j)$. Again, we must insist that $\sum_{i=0}^3 k_i^j = 0$ for $j = 1, 2$, although this now has the interpretation of making L a Lagrangian submanifold of X.

Then, according to local open string mirror symmetry, there should be a family of spaces $(\mathcal{Y}, \mathcal{C})$, such that $Y \in \mathcal{Y}$ is Calabi-Yau and C is a holomorphic curve in Y, such that $(\mathcal{Y}, \mathcal{C}) \cong (\mathcal{X}, \mathcal{L})$. This isomorphism is expected to count genus 0 "open string Gromov-Witten invariants" on (X, L); these do not yet have a good mathematical description [13], [6], but their supposed interpretation is a count of holomorphic discs in X with boundary on L.

2.2 The mirror family (Y, C).

From the mirror constructions of e.g. [9],[1], we can immediately give a description of a mirror family Y for X, in the usual (closed string) sense of mirror symmetry. This is

$$Y_z = \{uv + \sum_{i=0}^{3} y_i = 0,\ \prod_{i=0}^{3} y_i^{l_i} = z,\ y_0 = 1\}, \tag{2}$$

where u, v are $\mathbb{C}$ variables and $y_i \in \mathbb{C}^*$. Also, the condition $y_0 = 1$ is of course arbitrary, and in practice we vary which y_i should be set to 1 based on convenience in studying the problem at hand.

One way to see that this is the right definition of a mirror Y to X is to look at the local period integrals for Y. So, first use the constraints $\prod_{i=0}^3 y_i^{l_i} = z, y_0 = 1$ on the equation $uv + \sum_{i=0}^3 y_i = 0$, and rewrite

$$Y_z = \{uv + f(z) = 0\},$$

where f is a function of e.g. y_1, y_2. Then we can define the period integrals on Y to be [10]

$$\Pi_\Gamma(z) = \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{uv + f(z)} \tag{3}$$

Here we have $\Gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2 - Y, \mathbb{Z})$. These period integrals will reproduce the enumerative information relevant to counting holomorphic curves in X.

The reason for this is the same as in the usual formulation of Batyrev mirror symmetry [3], because of the use of l to define the polynomial for Y.

Next, we would like to see how mirror symmetry acts on the Lagrangian submanifolds L of X. In a sense, the mirror family $\mathcal{Y}$ (parameterized by z) of X was simply defined by exponentiation of the defining vector l. In the same way, we have but to exponentiate k^1, k^2 to get the mirror curve C to L. This yields

$$C_{z,w} = Y_z \cap \{\prod_{i=0}^{3} y_i^{k_i^1} = w,\ \prod_{i=0}^{3} y_i^{k_i^2} = 1\}. \tag{4}$$

In practice, the equation $\prod_{i=0}^{3} y_i^{k_i^1} = w$ will always take the form $y_i y_j^{-1} = w$, after accounting for the imposition of the constraint $y_0 = 1$.

Then, the moduli space $(\mathcal{Y}, \mathcal{C})$ is supposed to be locally isomorphic to $(\mathcal{X}, \mathcal{L})$. The next section will explore a way in which to make this correspondence precise.

3 Period integrals for $(\mathcal{Y}, \mathcal{C})$.

3.1 Periods and Picard-Fuchs operators for $\mathcal{Y}$.

From the above, we have integrals

$$\Pi_\Gamma(z) = \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{uv + f(z)} \tag{5}$$

which determine the complex structure of Y. Rather than directly integrating the Π_Γ, it is often easter to determine a set of Picard-Fuchs operators which annihilate these, and then write down a basis of solutions. Here, this process will be interpreted in a slightly different way from the conventional, which will make the constructions of the following section natural.

First, looking back at (2), note that this can be written as

$$Y_z = \{uv + 1 + y_1 + y_2 + (zy_1^{l_1}y_2^{l_2})^{1/l_3} = 0\};$$

From here, define

$$Y_{a_0,\ldots,a_3} = \{uv + a_0 + a_1y_1 + a_2y_2 + (a_3y_1^{l_1}y_2^{l_2})^{1/l_3} = uv + f(a,y) = 0\}.$$

This means that the $\mathbb{C}^*$ variables $(a_0, \ldots, a_3)$ are considered to be homogeneous coordinates for the complex structure moduli space $\mathcal{Y}$. Then the period integrals

$$\Pi_\Gamma(a_0, \ldots, a_3) = \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{uv + a_0 + a_1y_1 + a_2y_2 + (a_3y_1^{l_1}y_2^{l_2})^{1/l_3}} \tag{6}$$

are annihilated by the GKZ differential operators, given by

$$\mathcal{L} = \prod_{l_i>0} \partial_{a_i}^{l_i} - \prod_{l_i<0} \partial_{a_i}^{-l_i} \tag{7}$$

and another operator. However, the addition of the second GKZ operator is equivalent to the statement that $\Pi_\Gamma(a_0, \ldots, a_3)$ is actually a function of only one complex parameter, which we can identify as

$$z = \frac{\prod_{l_i>0} {a_i}^{l_i}}{\prod_{l_i<0} {a_i}^{-l_i}}. \tag{8}$$

Then, as is well known, we can reduce $\mathcal{L}$ via (8) to obtain a Picard-Fuchs differential operator

$$\mathcal{D} = \prod_{l_k>0} \prod_{n=0}^{l_k-1} (l_k\theta - n) - z \prod_{l_k<0} \prod_{n=0}^{-l_k-1} (l_k\theta - n). \tag{9}$$

As usual, $\theta = z\frac{d}{dz}$. Then the solution set of $\mathcal{D}f = 0$ is supposed to yield the periods of $\mathcal{Y}$.

3.2 Extension to $(\mathcal{Y}, \mathcal{C})$.

It was proposed in [16] that the moduli space $(\mathcal{Y}, \mathcal{C})$ should be thought of as a variation of mixed Hodge structure of relative cohomology $H^3(Y, C; \mathbb{C})$. Taking this as a starting point, note that there is an isomorphism $H^3(Y, C) \cong H_c^3(Y - C)$,where the subscript c denotes forms with compact support. Now, from the defining equations (5), since Y is given as a hypersurface, we are led to consider derivatives of the differential form

$$\frac{dudvdy_1dy_2/(y_1y_2)}{uv + f(z)}$$

to determine the Picard-Fuchs operator, and hence the period integrals.

In the same spirit, from the equation of the curve C,

$$C_{z,w} = Y_z \cap \{\prod_{i=0}^{3} y_i^{k_i^1} = w,\ \prod_{i=0}^{3} y_i^{k_i^2} = 1\},$$

set

$$g(w) = \prod_{i=0}^{3} y_i^{k_i^1} - w, \quad h = \prod_{i=0}^{3} y_i^{k_i^2} - 1.$$

Then the natural differential form to look at, which lies in $H_c^3(Y - C)$, is

$$\frac{dudvdy_1dy_2/(y_1y_2)}{(uv + f(z))g(w)h}. \tag{10}$$

This leads us to make the

Definition. *The relative period integrals on the moduli space* $(\mathcal{Y}, \mathcal{C})$ *are given as*

$$\Pi_\Gamma(z, w) = \int_\Gamma \frac{dudvdy_1dy_2/(y_1y_2)}{(uv + f(z))g(w)h} \tag{11}$$

for $\Gamma \in H_4(\mathbb{C}^2 \times (\mathbb{C}^*)^2 - Y, Y - C, \mathbb{Z})$.

An important feature of these integrals is that, as g is independent of z, the usual Picard-Fuchs operator (9) still annihilates $\Pi_\Gamma(z, w)$. Thus, the period integrals for $\mathcal{Y}$ are a subset of those in the definition.

Next, it will be shown that this definition indeed reproduces the extended Picard-Fuchs system associated to the pair $(\mathcal{Y}, \mathcal{C})$ [14]. To this end, it is useful, as in the previous section, to first consider a different set of integrals which are annihilated by an extended GKZ system. Afterwards, the resulting GKZ operators will be reduced via canonically defined coordinates, and the result will be a relative Picard-Fuchs system consistent with that of the literature.

Proposition 1 *The period integrals defined in (11) reproduce the known extended Picard-Fuchs differential operators definind the moduli space* $(\mathcal{Y}, \mathcal{C})$.

Proof. It is useful to write the period integrals $\Pi_\Gamma(z,w)$ in a slightly different form, as was done for the periods on Y in the previous section. For simplicity, take the defining equation of g to be given as $g(w) = y_1 - w$. Then, set $g(b) = b_0 + b_1 y_1$, and consider the integrals

$$\Pi_\Gamma(a_0,\ldots,a_3,b_0,b_1) = \int_\Gamma \frac{dudvdy_1dy_2/y_1y_2}{(uv+f(a,y))g(b)h} =$$

$$\int_\Gamma \frac{dudvdy_1dy_2/y_1y_2}{(uv+a_0+a_1y_1+a_2y_2+(a_3y_1^{l_1}y_2^{l_2})^{1/l_3})(b_0+b_1y_1)h}. \tag{12}$$

This is, of course, inspired by the process used above when considering (6) in place of (5). The interpretation is that $(a_0,\ldots,a_3,b_0,b_1) \in (\mathbb{C}^*)^6$ are homogeneous coordinates for the moduli space $(\mathcal{Y},\mathcal{C})$.

From the explicit form (12), it is easy to see that $\Pi_\Gamma(a,b)$ is still annihilated by $\mathcal{L}$ (7), and one immediately has that

$$\mathcal{L}' = \partial_{a_0}\partial_{b_1} - \partial_{b_0}\partial_{a_1} \tag{13}$$

also sends $\Pi_\Gamma(a,b)$ to 0. So, take $\mathcal{L} = \mathcal{L}_z$, $\mathcal{L}' = \mathcal{L}_w$. There are then two canonically defined coordinates, based on these operators:

$$z = \frac{\prod_{l_i>0} {a_i}^{l_i}}{\prod_{l_i<0} {a_i}^{-l_i}}, \quad w = \frac{a_0b_1}{a_1b_0}. \tag{14}$$

The moduli space defined by (12) is larger than that of (6). This extended moduli space can be realized as a pair of vectors:

$$l_z = (l_0,\ldots,l_3,0,0), \quad l_w = (1,-1,0,0,-1,1). \tag{15}$$

These new vectors, in turn, are equivalent to the GKZ operators $\{\mathcal{L},\mathcal{L}'\}$. Hence, we may take $\{l_z,l_w\}$ as toric data for the moduli space of the pair $(\mathcal{Y},\mathcal{C})$. □

3.3 Extended Picard-Fuchs operators and solutions.

The vectors (15) were found previously through physical considerations [14]. Thus, at this point the equivalence of the definition (11) to the physical calculations of [14],[16] has already been demonstrated. For completeness, we will present the extended Picard-Fuchs operators and solutions that follow from (15).

Let $z = z_1$, $w = z_2$, and change all other indices accordingly, so e.g. $l_z = l^1$. Writing the operators $\{\mathcal{L}_1, \mathcal{L}_2\}$ in the variables (14), we find

$$\mathcal{D}_k = \prod_{l_i^k>0} \prod_{n=0}^{l_i^k-1} (\sum_{k=1}^{2} l_i^k \theta_k - n) - z_k \prod_{l_i^k<0} \prod_{n=0}^{-l_i^k-1} (\sum_{k=1}^{2} l_i^k \theta_k - n). \tag{16}$$

The notation used here is $l^k = (l_0^k, \ldots, l_3^k)$. This is nothing other than the noncompact Picard-Fuchs system associated to the vectors $\{l^1, l^2\}$, so we can use the results of [4] to immediately write down a generating function for the solutions:

$$\omega_0(z, \rho) = \sum_{n \geq 0} c(n, \rho) z_1^{n_1+\rho_1} z_2^{n_2+\rho_2}, \tag{17}$$

where

$$c(n, \rho)^{-1} = \prod_i \Gamma(1 + \sum_k l_i^k (n_k + \rho_k)).$$

Then $\omega_0(z, 0)$ is constant, and there are two logarithmic solutions $t_i(z) = \partial_{\rho_i} \omega_0(z, \rho)|_{\rho=0}$ providing the mirror map between $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{Y}, \mathcal{C})$. According to [16], the function that is relevant for counting discs on $(\mathcal{X}, \mathcal{L})$ is

$$W(z_1, z_2) = \sum_{n \geq 0} (\partial_{\rho_2}^2 c(n, \rho)|_{\rho=0}) z_1^{n_1} z_2^{n_2}$$

This is obtained by ignoring the logarithmic terms of $\partial_{\rho_2}^2 \omega_0(z, \rho)|_{\rho=0}$.

Numerous examples have already been worked out [16] as evidence that, upon insertion of the inverse mirror map into W, the conjectural open Gromov-Witten invariants can be extracted, after accounting for the multiple cover formula [13]. Next, we would like to see whether the above considerations can be extended to the compact setting.

4 An attempt at the compact case.

From the mirror symmetry constructions of [1], it is evident that we cannot distinguish the compact and noncompact cases at the level of GKZ operators. That is to say, the GKZ system associated to the mirror family of the quintic (i.e., a zero section of $\mathcal{O}(5) \to \mathbb{P}^4$) is identical to that derived from the mirror family of the (complex dimension 5) noncompact Calabi-Yau space $\mathcal{O}(-5) \to \mathbb{P}^4$. This statement continues to hold true in the open string setting.

This means that the distinction of compact and noncompact geometry is made through the transition from GKZ to Picard-Fuchs operators. The reason for this is easy to explain: in the noncompact case, the differential form

$$\frac{dudvdy_1dy_2/(y_1y_2)}{uv + a_0 + a_1y_1 + a_2y_2 + (a_3y_1^{l_1}y_2^{l_2})^{1/l_3}}$$

is invariant when we pass to the quotient defined by the variable (14). However, in the compact case, we are working with a differential form

$$\Omega = \frac{\prod_i dy_i/y_i}{a_0 + \sum_{i>0} a_i^{l_i}},$$

which has an additional $\mathbb{C}^*$ action on the denominator. Hence, we must modify the differential form as well as the GKZ operator, to maintain invariance under this new $\mathbb{C}^*$ action:

$$\Omega \to a_0\Omega, \quad \mathcal{L} \to \mathcal{L}a_0^{-1}.$$

Naturally, the Picard-Fuchs operator arising from $\mathcal{L}a_0^{-1}$ will be different from that in (9).

From these considerations, as well as the mirror construction in [1], it is evident that the form

$$\Omega_1 = \frac{\prod_i dy_i/y_i}{(a_0 + \sum_{i>0} a_i^{l_i})(b_0 + b_1y_1)h}$$

will be the natural object of interest in the context of compact open string mirror symmetry. This now takes on the form from the complete intersection

case [11], and it follows that invariance under the quotient given by (14) requires the modification

$$\Omega_1 \to \Omega_1 a_0 b_0, \quad \mathcal{L}_i \to \mathcal{L}_i a_0^{-1} b_0^{-1}.$$

Then we have new Picard-Fuchs operators

$$\mathcal{D}_k = \prod_{l_i^k>0} \prod_{n=0}^{l_i^k-1} (\sum_{k=1}^{2} l_i^k \theta_k - n) - z_k \prod_{l_i^k<0} \prod_{n=0}^{-l_i^k} (\sum_{k=1}^{2} l_i^k \theta_k - n)$$

and their associated solution set, which is given in full detail in [11]. The point now, though, is that the associated double logarithmic solution of $\{\mathcal{D}_1, \mathcal{D}_2\}$ will yield nontrivial numbers, after the insertion of the inverse mirror map. However, as was pointed out in [1], a holomorphic family of curves in a compact Calabi-Yau, such as that given by $\{b_0 + b_1 y_1 = 0\}$ in $\{a_0 + \sum_{i>0} a_i^{l_i} = 0\}$, should have no double log solution. This is because such a family will have a vanishing disc number in the mirror. Thus, we cannot naively apply the techniques of [16] to compact situations.

5 An example.

Finally, it will be shown how the application of the above ideas reproduces the results of both [6] and [14], in the case of $K_{\mathbb{P}^2} = \mathcal{O}(-3) \to \mathbb{P}^2$.

First, we take $K_{\mathbb{P}^2} =$

$$X_r = \{(z_1, \ldots, z_4) \in \mathbb{C} - Z : |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 = r\}/S^1,$$

where $Z = \{z_1 = z_2 = z_3 = 0\}$, and the S^1 action is determined by the vector $l^1 = (1, 1, 1, -3)$. Also, consider the Lagrangian submanifolds

$$L_{r,c} = X_r \cap \{|z_1|^2 - |z_4|^2 = c, |z_2|^2 - |z_4|^2 = 0, \sum_i arg(z_i) = 0\},$$

$$L'_{r,-c} = X_r \cap \{|z_1|^2 - |z_4|^2 = -c, |z_2|^2 - |z_1|^2 = 0, \sum_i arg(z_i) = 0\}.$$

Here, $r, c \in \mathbb{R}^+$.

Then, as in previous sections, the moduli space $(\mathcal{Y}, \mathcal{C})$ can be specified by the toric data

$$l^1 = (1, 1, 1, -3, 0, 0), \quad l^2 = (1, 0, 0, -1, -1, 1),$$

while that of $(\mathcal{Y}, \mathcal{C}')$ is given as $\{l^1, -l^2\}$.

The vectors $\{l^1, l^2\}$ immediately give GKZ operators

$$\mathcal{L}_1 = \partial_{a_1}\partial_{a_2}\partial_{a_3} - \partial_{a_0}^3, \quad \mathcal{L}_2 = \partial_{a_1}\partial_{b_0} - \partial_{a_0}\partial_{b_1},$$

and these are the annihilators of the modified period vectors

$$\Pi_\Gamma(a, b) = \int_\Gamma \frac{dudv\frac{dy_1}{y_1}\frac{dy_2}{y_2}}{(uv + a_0 + a_1y_1 + a_2y_2 + a_3y_1^{-1}y_2^{-1})(b_0 + b_1y_1)(y_2 - 1)}.$$

These period vectors are, of course, arising from the mirror geometry to $(\mathcal{X}, \mathcal{L})$. This may be described by a pair $(Y, C) \in (\mathcal{Y}, \mathcal{C})$, where $\mathcal{Y}$ is the family of hypersurfaces

$$Y_{z_1} = \{(u, v, y_1, y_2) \in \mathbb{C}^2 \times (\mathbb{C}^*)^2 : uv + 1 + y_1 + y_2 + z_1y_1^{-1}y_2^{-1} = 0\}$$

together with the curves

$$C_{z_1,z_2} = Y_{z_1} \cap \{1 - z_2y_1 = 1 - y_2 = 0\}.$$

Then, the GKZ operators $\{\mathcal{L}_1, \mathcal{L}_2\}$ produce Picard-Fuchs operators

$$\mathcal{D}_1 = (\theta_1)^2(\theta_1 + \theta_2) + z_1(3\theta_1 + \theta_2)(3\theta_1 + \theta_2 + 1)(3\theta_1 + \theta_2 + 2),$$

$$\mathcal{D}_2 = (\theta_1 + \theta_2)\theta_2 - z_2(3\theta_1 + \theta_2)\theta_2.$$

The double logarithmic solution of the system $\{\mathcal{D}_1, \mathcal{D}_2\}$ is, modulo logarithmic terms,

$$W_{K_{\mathbb{P}^2}} = \sum_{n_1 \geq 0, n_2 \geq 1} \frac{(-1)^{n_1}(3n_1 + n_2 - 1)!}{n_2(n_1 + n_2)!(n_1!)^2} z_1^{n_1} z_2^{n_2}.$$

This is exactly the superpotential of [6], which was computed through explicit localization calculations on $(\mathcal{X}, \mathcal{L})$.

To reproduce the results of [14], use instead the moduli space of $(\mathcal{Y}, \mathcal{C}')$, which is defined by the vectors $\{l^1, -l^2\}$. Then there are new Picard-Fuchs operators $\{\mathcal{D}'_1, \mathcal{D}'_2\}$ which have a double log solution (again suppressing log terms)

$$W'_{K_{\mathbb{P}^2}} = \sum_{n_1 \geq 0, n_2 > n_1} \frac{(-1)^{n_1}(n_2 - n_1 - 1)!}{n_2(n_2 - 3n_1)!(n_1!)^2} z_1^{n_1} z_2^{n_2}.$$

This is the superpotential of [14].

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