Invariant Subspaces And Hankel Type Operators On A Bergman Space

by

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*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education of Japan.

2000 Mathematics Subject Classification : Primary 47 B 35, 47 A 15

Key words and phrases : Bergman space, invariant subspace, Hankel type operator

Abstract. Let $L^2 = L^2(D, rdrd\theta/\pi)$ be the Lebesgue space on the open unit disc D and let $L_a^2 = L^2 \cap Hol(D)$ be a Bergman space on D. In this paper, we are interested in a closed subspace $\mathscr{M}$ of L^2 which is invariant under the multiplication by the coordinate function z, and a Hankel type operator from L_a^2 to $\mathscr{M}^\perp$. In particular, we study an invariant subspace $\mathscr{M}$ such that there does not exist a finite rank Hankel type operator except a zero operator.

§1. Introduction

Let D be the open unit disc in $\mathbb{C}$ and $Hol(D)$ be the set of all holomorphic functions on D. Let $d\mu = rdrd\theta/\pi$ and $L^2 = L^2(D, d\mu)$ the Lebesgue space. The Bergman space L_a^2 on D is defined by $L_a^2 = L^2 \cap Hol(D)$. Then L_a^2 is the closed subspace of L^2. When $\mathscr{M}$ is a closed subspace of L^2 and $z\mathscr{M} \subseteq \mathscr{M}$, $\mathscr{M}$ is called an invariant subspace. For φ in $L^\infty = L^\infty(D, d\mu)$, a Hankel type operator is defined by

$$H_\varphi^{\mathscr{M}} f = \left(I - P^{\mathscr{M}}\right)(\varphi f) \qquad \left(f \in L_a^2\right)$$

where $P^{\mathscr{M}}$ is the orthogonal projection from L^2 onto $\mathscr{M}$. When $\mathscr{M} = L_a^2$, $H_\varphi^{\mathscr{M}}$ is called a big Hankel operator and when $\mathscr{M} = \left(\overline{zL_a^2}\right)^\perp$, $H_\varphi^{\mathscr{M}}$ is called a small Hankel operator. When $L_a^2 \subseteq \mathscr{M} \subseteq \left(\overline{zL_a^2}\right)^\perp$, $H_\varphi^{\mathscr{M}}$ is called an intermediate Hankel operator.

It is easy to see that there does not exist a finite rank big Hankel operator except a zero one (see [5], [6]). On the other hand, there exist a lot of finite rank nonzero small Hankel operators (see [5]). In fact, it is easy to see the results. E. Strouse [7] described completely all finite rank intermediate Hankel operators for some invariant subspace. In the previous paper [5], we began to study finite rank intermediate Hankel operators for arbitrary invariant subspace. In $[5, Theorem\, 3.2]$, we gave three necessary and sufficient conditions for $\mathscr{M}$ such that does not exist a finite rank intermediate Hankel operator except a zero one. In this paper, without the hypothesis on an invariant subspace $\mathscr{M}$, we give a new necessary and sufficient condition for $\mathscr{M}$ which have a finite rank Hankel type operator except a zero one.

For an invariant subspace $\mathscr{M}$ in L^2, $ker\, H_\varphi^{\mathscr{M}}$ denotes the kernel of $H_\varphi^{\mathscr{M}}$ and then $ker\, H_\varphi^{\mathscr{M}} = \{f \in L_a^2;\, \varphi f \in \mathscr{M}\}$. Hence $ker\, H_\varphi^{\mathscr{M}}$ is also an invariant subspace in L_a^2. Thus each invariant subspace $\mathscr{M}$ in L^2 is related to an invariant subspace in L_a^2 by a Hankel type operator. In this paper, the following property of invariant subspaces in L^2 is important.

Definition. *Let $\mathscr{M}$ be an invariant subspace of L^2. $\mathscr{M}$ is called weakly divisible if whenever $f \in \mathscr{M}$ and $\mid f(z) \mid \leq \gamma \mid z - a \mid$ for some $a \in D$ and some $\gamma \geq 0$ then $f(z) = (z-a)g(z)$ and g is a function in $\mathscr{M}$.*

In Section 2, we generalize a theorem of S. Axler and P. Bourdon [1]. Then they will be used in the latter sections. In Section 3, we show that there does not exist a finite rank Hankel type operator $H_\varphi^{\mathscr{M}}$ except a zero one if and only if $\mathscr{M}$ is weakly divisible. In Section 4, we give several examples of weakly divisible invariant subspaces.

In this paper $[S]_*$ denotes the weak* closed linear span of a subset S in L^∞ and $[S]_2$ denotes the closed linear span of a subset S in L^2.

§2. An invariant subspace and the index

In this section, for a given invatiant subspace $\mathscr{M}$ we are interested in two invariant subspaces $\mathscr{M}'$ and $\mathscr{M}''$ such that $\mathscr{M}' \subseteq \mathscr{M} \subseteq \mathscr{M}''$, $dim \mathscr{M} \ominus \mathscr{M}' < \infty$ and $dim \mathscr{M}'' \ominus \mathscr{M} < \infty$. Under some conditions on $\mathscr{M}, \mathscr{M}' and \mathscr{M}''$, we describe $\mathscr{M}'$ and $\mathscr{M}''$ using $\mathscr{M}$. Corollary 3 will be used in Sections 3 and 4. (1) of Corollary 3 is known in [1].

When $\mathscr{M}$ is an invariant subspace of L^2, for $a \in \mathbb{C}$ put $ind_a \mathscr{M} = dim\{\mathscr{M} \ominus (z-a)\mathscr{M}\}$. $ind_a \mathscr{M}$ is called index of $\mathscr{M}$ at a. It is known (cf. [1]) that for each $n\,(0 \leq n \leq \infty)$ and for any $a\,(\in D)$ there exists an invariant subspace $\mathscr{M}$ with $ind_a \mathscr{M} = n$.

Theorem 1. *Let $\mathscr{M}, \mathscr{M}_1$ and $\mathscr{M}_2$ be invariant subspaces of L^2 and $\mathscr{M}_1 \subseteq \mathscr{M}_2$.*
(1) *$ind_a \mathscr{M} = 0$ for any $a \notin D$.*
(2) *If $dim \mathscr{M}_2 \ominus \mathscr{M}_1 < \infty$ then there exists a polynomial b such that $b\mathscr{M}_2 \subseteq \mathscr{M}_1$, $Z(b) \subset D$ and the degree of $b \leq dim \mathscr{M}_2 \ominus \mathscr{M}_1$ and $\sum(ind_a \mathscr{M}_2\,; a \in Z(b)) \geq dim \mathscr{M}_2 \ominus \mathscr{M}_1$.*

Proof. (1) If $|\,a\,| > 1$ then $(z-a)^{-1} \in H^\infty$ and $\mathscr{M} = (z-a)\mathscr{M}$. Hence $ind_a \mathscr{M} = 0$. If $|\,a\,| = 1$ then $(z-a)\mathscr{M} = (z-a)\{z-a(1+\varepsilon)\}^{-1}\mathscr{M}$. For any $f \in \mathscr{M}$, it is easy to see that

$$\int_D \left| \frac{z-a}{z-a(1+\varepsilon)} f - f \right|^2 d\mu \longrightarrow 0 \qquad (\varepsilon \to 0)$$

by a Lebesgue's convergence theorem. This implies that $(z-a)\mathscr{M}$ is dense in $\mathscr{M}$ and so $ind_a \mathscr{M} = 0$ for $|\,a\,| = 1$. (2) Put $\mathscr{N} = \mathscr{M}_2 \ominus \mathscr{M}_1$ and $\mathscr{S}_z = PM_z|\mathscr{N}$ where M_z is a multiplication operator on L^2 by the coordinate function z and P is the orthgonal projection from L^2 to $\mathscr{N}$. If $n = dim \mathscr{N} < \infty$, then there exists a polynomial b of degree n such that $\mathscr{S}_b = b(\mathscr{S}_z) = 0$ and so $b\mathscr{M}_2 \subseteq \mathscr{M}_1$. By (1), we may assume that $Z(b) \subset D$. We will prove that $\sum(ind_a \mathscr{M}_2\,; a \in Z(b)) \geq n$. We can write that $b = a_0 \prod_{j=1}^{n}(z-a_j)$ and so $Z(b) = \{a_1, a_2, \ldots, a_n\}$ where $a_0 \in \mathbb{C}$. If $\sum(ind_a \mathscr{M}_2\,; a \in Z(b)) \leq n-1$ then we may assumue $ind_{a_1} \mathscr{M}_2 = 0$. Since $[\,(z-a_1)\mathscr{M}_2]_2 = \mathscr{M}_2$, $\prod_{j=2}^{n}(z-a_j)\mathscr{M}_2 \subseteq \mathscr{M}_1 \subset \mathscr{M}_2$. Then it is easy to see that $dim \mathscr{M}_2 \ominus [\prod_{j=2}^{n}(z-a_j)\mathscr{M}_2]_2 \leq n-1$ because $ind_{a_j} \mathscr{M}_2 \leq 1$ for $2 \leq j \leq n$. This contradicts that $dim \mathscr{M}_2 \ominus \mathscr{M}_1 = n$. $\square$

Corollary 1. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be invariant subspaces of L^2 and $\mathscr{M}_1 \subseteq \mathscr{M}_2$. If $dim \mathscr{M}_2 \ominus \mathscr{M}_1 = 1$ then $(z-a)\mathscr{M}_2 \subseteq \mathscr{M}_1 \subsetneq \mathscr{M}_2$ for some $a \in D$ and $ind_a \mathscr{M}_2 \geq 1$. If $ind_a \mathscr{M}_1 = 1$ or $ind_a \mathscr{M}_2 = 1$ then $\mathscr{M}_1 = [(z-a)\mathscr{M}_2]_2$.*

Proof. By Theorem 1, $(z-a)\mathscr{M}_2 \subseteq \mathscr{M}_1$ for some $a \in D$ and so $ind_a \mathscr{M}_2 \geq 1$. Since $(z-a)\mathscr{M}_1 \subseteq (z-a)\mathscr{M}_2 \subseteq \mathscr{M}_1 \subsetneq \mathscr{M}_2, \mathscr{M}_1 = [(z-a)\mathscr{M}_2]_2$ if $ind_a \mathscr{M}_1 = 1$ or $ind_a \mathscr{M}_2 = 1$. $\square$

Corollary 2. *Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be invariant subspaces such that $\mathscr{M}_1 \subsetneq \mathscr{M}_2$ and $dim\,\mathscr{M}_2 \ominus \mathscr{M}_1 = n < \infty$. Suppose that $(z-a)\mathscr{M}_j$ is closed for any a in D when $j = 1,2$. If $ind_a\mathscr{M}_1 = 1$ for any a in D or $ind_a\mathscr{M}_2 = 1$ for any a in D then $\mathscr{M}_1 = b\mathscr{M}_2$ and $\mathscr{M}_2 = \langle f_1/b, \dots, f_n/b\rangle \oplus \mathscr{M}_1$ where $b = \prod_{j=1}^{n}(z-a_j), \{a_j\} \subset D$ and $\{f_j\} \subset \mathscr{M}_1$.*

Proof. By Theorem 1 there exists a polynomial b such that $b\mathscr{M}_2 \subseteq \mathscr{M}_1$ and $Z(b) \subset D$ and the degree of $b \leq n$. Hence $b = \prod_{j=1}^{\ell}(z-a_j)$ and $\{a_j\} \subset D$ and $\ell \leq n$. When $ind_a\mathscr{M}_2 = 1$ for any a in D, $dim\,\mathscr{M}_2 \ominus b\mathscr{M}_2 = \ell$ because $(z-a_j)\mathscr{M}_2$ is closed for $1 \leq j \leq \ell$ and so $\ell = n$. Hence $\mathscr{M}_1 = b\mathscr{M}_2$. When $ind_a\mathscr{M}_1 = 1$ for any a in D, $dim\mathscr{M}_1 \ominus b\mathscr{M}_1 = \ell$ by the same reason. Since $b\mathscr{M}_1 \subseteq b\mathscr{M}_2 \subseteq \mathscr{M}_1$ and $dim\,b\mathscr{M}_2 \ominus b\mathscr{M}_1 = n$, $\ell = n$ and so $\mathscr{M}_1 = b\mathscr{M}_2$. Put $\mathscr{M}_2 = \langle \varphi_1, \dots, \varphi_n\rangle \oplus \mathscr{M}_1$ where $\{\varphi_j\}$ are orthogonal to $\mathscr{M}_1$. What was just proved above, $b\mathscr{M}_2 = \mathscr{M}_1$ and so $b\mathscr{M}_2 = \langle b\varphi_1, \dots, b\varphi_n\rangle \oplus b\mathscr{M}_1 = \mathscr{M}_1$. Put $f_j = b\varphi_j$ for $j = 1, \dots, n$ then $\{f_j\}$ are in $\mathscr{M}_1$ and $\mathscr{M}_2 = \langle f_1/b, \dots, f_n/b\rangle \oplus \mathscr{M}_1$. $\square$

Corollary 3. *Let $\mathscr{M}$ be an invariant subspace of L^2.*
(1) *If $dim\,L_a^2 \ominus \mathscr{M} = n < \infty$ and $n \neq 0$ then $\mathscr{M} = bL_a^2$ where $b = \prod_{j=1}^{n}(z-a_j)$ and $\{a_j\} \subset D$.*
(2) *If $dim\,\mathscr{M} \ominus L_a^2 = n < \infty$ then $\mathscr{M} = L_a^2$.*

Proof. It is known that $ind_aL_a^2 = 1$ and $(z-a)L_a^2$ is closed for each $a \in D$. Hence we can apply Corollary 2 $\mathscr{M}_1 = L_a^2$ or $\mathscr{M}_2 = L_a^2$. If $\mathscr{M}_1 = \mathscr{M}$ and $\mathscr{M}_2 = L_a^2$ then (1) follows. If $\mathscr{M}_1 = L_a^2$ and $\mathscr{M}_2 = \mathscr{M}$ then $\mathscr{M} = \langle f_1/b, \dots, f_n/b\rangle \oplus L_a^2$ where $b = \prod_{j=1}^{n}(z-a_j)$, $\{a_j\} \subset D$ and $\{f_j\} \subset L_a^2$. For each $1 \leq \ell \leq n$, $f_\ell/b \in L^2$ and so $f_\ell(a_j) = 0$ for $1 \leq j \leq n$. Then f_ℓ/b belongs to L_a^2 and so $f_\ell/b = 0$ for each ℓ. Thus $\mathscr{M} = L_a^2$ and so (2) follows. $\square$

§3. Finite rank Hankel type operators

In this section, we study the relation between finite rank Hankel type operators and invariant subspaces.

Theorem 2. *Let $\mathscr{M}$ be an invariant subspace of L^2. Then there does not exist a finite rank Hankel type operator $H_\varphi^{\mathscr{M}}$ except a zero one if and only if $\mathscr{M}$ is weakly divisible.*

Proof. Suppose $\mathscr{M}$ is weakly divisible. If $H_\varphi^{\mathscr{M}}$ is of finite rank then $ker\,H_\varphi^{\mathscr{M}}$ is an invariant subspace in L_a^2 and $dim\,L_a^2/ker\,H_\varphi^{\mathscr{M}} < \infty$. By (1) of Corollary 3, $ker\,H_\varphi^{\mathscr{M}} = bL_a^2$ for some polynomial b with $Z(b) \subset D$ and so $b\varphi$ belongs to $\mathscr{M}$. Put $f = b\varphi$ then $\mid f(z) \mid \leq \gamma \mid$

$b(z) \mid \ (z \in D)$ where $\gamma = \|\varphi\|_\infty$. Suppose $b(z) = a_0 \prod_{j=1}^{n}(z - a_j)$ where $\{a_j\} \subset D$. For any ℓ with $1 \leq \ell \leq n$,

$$\left| \frac{f(z)}{z - a_\ell} \right| \leq \gamma \mid a_0 \mid \prod_{j \neq \ell} \mid z - a_j \mid \quad (z \in D)$$

and $f(z)/(z - a_\ell)$ belongs to $\mathscr{M}$ because $a_\ell \in D$ and $\mathscr{M}$ is weakly divisible. Thus $\varphi(z) = f(z)/b(z)$ belongs to $\mathscr{M}$. Hence $H_\varphi^{\mathscr{M}} = 0$.

Conversely if $\mathscr{M}$ is not weakly divisible then there exists a function f in $\mathscr{M}$ and a point a in D such that $\mid f(z) \mid \leq \gamma \mid z - a \mid \ (z \in D)$ and $f(z)/(z - a)$ does not belong to $\mathscr{M}$. Put $\varphi = f(z)/(z - a)$ then $\varphi \in L^\infty$ and $H_\varphi^{\mathscr{M}}$ is not zero because $\varphi \notin \mathscr{M}$. On the other hand, $(z - a)\varphi \in \mathscr{M}$ and so the kernel of $H_\varphi^{\mathscr{M}}$ contains $(z - a)L_a^2$. This implies that $H_\varphi^{\mathscr{M}}$ is of rank one because $L_a^2/(z - a)L_a^2 = \mathbb{C}$. $\square$

Proposition 1. *If there exists a symbol φ such that $r\left(H_\varphi^{\mathscr{M}}\right) = n \geq 1$ then there exists a symbol φ_j such that $r\left(H_{\varphi_j}^{\mathscr{M}}\right) = j$ for any j with $0 \leq j \leq n - 1$.*

Proof. Suppose $1 \leq n = r\left(H_\varphi^{\mathscr{M}}\right) < \infty$. Then $ker H_\varphi^{\mathscr{M}} =$ the kernel of $H_\varphi^{\mathscr{M}}$ is an invariant subspace of L_a^2 and $L_a^2/ker H_\varphi^{\mathscr{M}}$ is of finite dimension n. By Corollary 3, $ker H_\varphi^{\mathscr{M}} = bL_a^2$ where $b = \prod_{\ell=1}^{n}(z - a_\ell)$ and $(a_\ell) \subset D$. Hence $b\varphi$ belongs to $\mathscr{M}$. Put $\varphi_j = \varphi \prod_{\ell=j+1}^{n}(z - a_\ell)$ for $1 \leq j \leq n - 1$ then $\varphi_j \notin \mathscr{M}$ for $1 \leq j \leq n - 1$ and $\varphi_0 = b\varphi$. Since $ker H_{\varphi_j}^{\mathscr{M}} = b_j L_a^2$ for $1 \leq j \leq n - 1$ where $b_j = \prod_{\ell=1}^{j}(z - a_\ell)$, $H_{\varphi_j}^{\mathscr{M}}$ is of finite rank j for $0 \leq j \leq n - 1$. $\square$

Corollary 4. *The following* (1) *and* (2) *are equivalent for an invariant subspace* $\mathscr{M}$.

(1) *If* $r\left(H_\varphi^{\mathscr{M}}\right) < \infty$ *then* $r\left(H_\varphi^{\mathscr{M}}\right) = 0$.

(2) *If* $r\left(H_\varphi^{\mathscr{M}}\right) \leq 1$ *then* $r\left(H_\varphi^{\mathscr{M}}\right) = 0$.

Proof. $(1) \Rightarrow (2)$ is clear. $(2) \Rightarrow (1)$. If (1) is not true then there exists a symbol φ with $r\left(H_\varphi^{\mathscr{M}}\right) = n \geq 2$. By Proposition 1 there exists a symbol φ_1 such that $r\left(H_{\varphi_1}^{\mathscr{M}}\right) = 1$. This contradicts (2). $\square$

§4. Weakly divisible invariant subspaces

For a function f in L_a^2, put $Z(f) = \{a \in D;\, f(a) = 0\}$ and $Z(G) = \cap\{Z(f);\, f \in G\}$ for a subset G in L_a^2. For $1 \le p \le \infty$, if E is a open set in D, H_E^p denotes the set of all functions in L^p that are analytic on E. In Corollary 5, a weakly divisible invariant subspace $\mathscr{M}$ is described completely when $\mathscr{M}$ is in L_a^2. There exists a nonzero invariant subspace $\mathscr{M}$ in L_a^2 such that $\mathscr{M} \cap L^\infty = \langle 0 \rangle$. For it is known (see [4]) that there exists a nonzero function f in L_a^2 such that $Z(f)$ does not satisfy the Blaschke condition.

Theorem 3. *Let $\mathscr{M}$ be an invariant subspace of L^2.*
(1) *If $\mathscr{M} \cap L^\infty \subseteq H^\infty$ and $Z(\mathscr{M} \cap L^\infty) = \emptyset$ then $\mathscr{M}$ is weakly divisible.*
(2) *If $\mathscr{M} \cap L^\infty = H_E^\infty$ for some open set E, then $\mathscr{M}$ is weakly divisible.*
(3) *If $\mathscr{M} \cap L^\infty = \langle 0 \rangle$ then $\mathscr{M}$ is weakly divisible.*

Proof. (1) If $\{f_n\}$ is a sequence in $\mathscr{M} \cap L^\infty$which converges pointwise boundedly to f, then $f \in \mathscr{M}$. By the Krein-Schmulian criterion (see[3, IV 2.1]), $\mathscr{M} \cap L^\infty$ is weak-star closed. Hence, by a well known theorem of Beurling [2] $\mathscr{M} \cap L^\infty = qH^\infty$ for some inner function q. Hence if $f \in \mathscr{M}$ and $|\, f(z) \,| \le \gamma \,|\, z - a \,|$ $(z \in D)$ for some $a \in D$ then $f = qh$ for some $h \in H^\infty$. Since $Z(\mathscr{M} \cap L^\infty) = \emptyset$, $|\, q(z) \,| > 0\, (z \in D)$ and so $h(a) = 0$. Hence $f(z)/(z-a) = q(z) \times (h(z)/(z-a)) \in qH^\infty$. Thus $f(z)/(z-a)$ belongs to $\mathscr{M}$. (2) If $f \in H_E^\infty$ and $|\, f(z) \,| \le \gamma \,|\, z - a \,|$ $(z \in D)$ for some $a \in D$ then $f(z)/(z-a) \in L^\infty$ and $f(z)/(z-a)$ is analytic on E. Hence $f(z)/(z-a)$ belongs to H_E^∞ and so $\mathscr{M}$ is weakly divisible. (3) is clear. $\square$

Corollary 5. *Let $\mathscr{M}$ be an invariant subspace of L_a^2. Then $\mathscr{M}$ is weakly divisible if and only if $\mathscr{M} \cap L^\infty = \langle 0 \rangle$ or $Z(\mathscr{M} \cap L^\infty) = \emptyset$.*

Proof. The part of ‘ if ’ is a result of (1) and (3) of Theorem 3. Conversely suppose that $\mathscr{M}$ is weakly divisible. If $\mathscr{M} \cap L^\infty \neq \langle 0 \rangle$ then by a theorem of Beurling there exists an inner function q with $\mathscr{M} \cap L^\infty = qH^\infty$. If $q(a) = 0$ for some $a \in D$ then there exists a finite positive constant γ such that $|\, q(z) \,| \le \gamma \,|\, z - a \,|$ $(z \in D)$ and $q/(z-a) \notin \mathscr{M}$. This contradicts the weakly divisibility of $\mathscr{M}$ and so $Z(q) = Z(\mathscr{M} \cap L^\infty) = \emptyset$. $\square$

Corollary 6. *Let $\mathscr{M}$ be an invariant subspace of L^2.*
(1) *If $\mathscr{M} \subsetneq L_a^2$ and $\dim L_a^2/\mathscr{M} < \infty$ then $\mathscr{M}$ is not weakly divisible.*
(2) *If $\mathscr{M} \supseteq L_a^2$ and $\dim \mathscr{M}/L_a^2 < \infty$ then $\mathscr{M}$ is weakly divisible.*

Proof. (1) If $\mathscr{M} \subsetneq L_a^2$ and $\dim L_a^2/\mathscr{M} = \ell < \infty$ then by (1) of Corollary 3 $\mathscr{M} = bL_a^2$ where $b = \prod_{j=1}^{\ell}(z - a_j)$ and $a_j \in D\, (1 \le j \le \ell)$. Hence $Z(\mathscr{M} \cap L^\infty) = Z(b) \neq \emptyset$ and so by Corollary 5 $\mathscr{M}$ is not weakly divisible. (2) By (2) of Corollary 3 $\mathscr{M} = L_a^2$ and so $\mathscr{M} \cap L^\infty = H^\infty$.

Hence (1) of Theorem 3 implies that $\mathscr{M}$ is weakly divisible. □

Corollary 7. *If $\mathscr{M} = H^2_E$ for some open set E in D then $\mathscr{M}$ is weakly divisible.*
Proof. It is a result of (2) of Theorem 3. □

Proposition 2. *Suppose that $\mathscr{M}_j$ is a weakly divisible invariant subspace of L^2 for $j = 1, 2, \ldots$ and $\mathscr{M}_j \times \mathscr{M}_\ell = \{fg;\, f \in \mathscr{M}_j \, and \, g \in \mathscr{M}_\ell\} = \langle 0 \rangle$ if $j \neq \ell$. If $\mathscr{M} = \sum_{j=1}^{\infty} \oplus \mathscr{M}_j$ then $\mathscr{M}$ is a weakly divisible invariant subspace.*
Proof. If $f \in \mathscr{M}$ then $f = \sum_{j=1}^{\infty} f_j$ and $\mid f(z) \mid = \sum_{j=1}^{\infty} \mid f_j(z) \mid$ $(z \in D)$ by hypothesis. This implies that $\mathscr{M}$ is weakly divisible. □

Corollary 8. *Let $1 \leq \ell \leq \infty$. Suppose D_j is an open set in D with $\mu(\partial D_j) = 0$ for $1 \leq j \leq \ell$, $D_i \cap D_j = \emptyset$ $(i \neq j)$ and $D = \cup_{j=1}^{\ell} D_j$. Then $\mathscr{M} = \sum_{j=1}^{\ell} \oplus L^2_a(D_j)$ is weakly divisible.*
Proof. This is a result of Corollary 7 and Proposition 2. □

Proposition 3. *If $\mathscr{M}$ is a weakly divisible invariant subspace of L^2 and φ is a unimodular function in L^∞ then $\varphi\mathscr{M}$ is a weakly divisible invariant subspace.*
Proof. From the definition of weakly divisibility, the proposition follows trivially. □

Corollary 9. *If φ is a unimodular function in L^∞ then φL^2_a is weakly divisible.*

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