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Author(s)	Nakamura, Gen; Wang, Shengzhang; Wang, Yanbo
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NUMERICAL DIFFERENTIATION FOR THE SECOND ORDER DERIVATIVE OF FUNCTIONS WITH SEVERAL VARIABLES

G. NAKAMURA, S. Z. WANG, AND Y. B. WANG†

Abstract. We propose a regularized optimization problem for computing numerical differentiation for the second order derivative for functions with two variables from the noisy values of the function at scattered points, and give the proof of the existence and uniqueness of the solution of this problem. The reconstruction scheme is also given during the proof, which is based on biharmonic Green function. The convergence estimate of the regularized solution to the exact solution for the regularized optimization problem as the regularized parameter and discrepancy of noisy data tending to zero is provided under a simple choice of regularization parameter. In the end we give the numerical examples and analyze the computational results.

1. Introduction

Numerical differentiation is a problem to determine the derivatives from the given noisy values of the function at scattered points. It arises from many scientific researches and applications. The differentiation of noisy data is an ill-posed problem, which means, the small errors in the measurement of the function may lead to large errors in its computed derivatives ([7], [9], [13]). There have been many methods developed ([8], [9], [12]) for treating the numerical differentiation problem. One group of methods uses Tikhonov regularization for solving the ill-posed problem([5], [7], [11]).

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A simple but very useful solution to the problem for one variable case based on Tikhonov regularization method has been developed in ([15], [9]). This method was used to find discontinuous solutions of Abel integral equations [3] and edge detection of image [10]. The results showed that this method was quite efficient. The higher order numerical differentiation by this method was given in [14]. And for the two variables case, a scheme for computing the first order derivative was given in [16] and the numerical example also showed this method was efficient. But in many applications, It is quite necessary to compute the higher order derivatives, for example, in the plate bending problem, the bending moments are obtained from the second derivatives of the primal solution [2], so in this paper we will give the solution for two variables case. For the cases of the number of variables more than two, the same argument given in this paper still works.

The paper is organized as follows: in section 2, we discribe the problem in detail and prove the existence and uniqueness of the solution; in section 3 we prove the convergence of our method by error estimate; the numerical examples are given in section 4 and we conclude this method in terms of the results of the numerical examples in section 5; we give the algorithm for computing the Green function in Appendix.

2. Problem and some results

Suppose that $\Omega \subset \mathbb{R}^2$ is a simply connected bounded domain and $\varrho = \varrho(x)$ is a function defined in Ω. Let N be a natural number and $\{x_i\}_{i=1}^N$ be a group of points in Ω. We assume that Ω is divided into N parts $\{\Omega_i\}_{i=1}^N$, and there is only one point of $\{x_i\}_{i=1}^N$ in each part. For simplicity we also assume that the volumes of all Ω_i are same. We denote d_i as the diameter of Ω_i and let $d = \max\{d_i\}$.

We will discuss the following problem:

Suppose that we know the approximate value $\tilde{\varrho}_i$ of $\varrho(x)$ at point x_i, i.e.

$$|\tilde{\varrho}_i - \varrho(x_i)| \leq \delta, \qquad i = 1, 2, \cdots, N, \tag{2.1}$$

where $\delta > 0$ is a given constant called the error level.

We want to find a function $f_*(x)$ approximates function $\varrho(x)$ such that $\|f_* - \varrho\|_{H^2(\Omega)}$ is small and

$$\lim_{h\to 0,\delta\to 0} \|f_* - \varrho\|_{H^2(\Omega)} = 0.$$

Assuming that there are two functions $\phi(x) \in H^{7/2}(\partial\Omega)$ and $\varphi(x) \in H^{3/2}(\partial\Omega)$ satisfying $\|\phi(x)-\varrho(x)\|_{H^{7/2}(\partial\Omega)} \leq \delta$ and $\|\varphi(x)-\triangle\varrho(x)\|_{H^{3/2}(\partial\Omega)} \leq \delta$, we treat this problem as the following optimization problem by using Tikhonov regularization method.

Problem 2.1. *Define a cost functional* $\Phi(f)$*:*

$$\Phi(f) = \frac{1}{N}\sum_{j=1}^{N}(f(x_j) - \tilde{\varrho_j})^2 + \alpha\|\triangle^2 f\|^2_{L^2(\Omega)}, \quad f \in H$$

where $H = \{f|f \in H^4(\Omega), f|_{\partial\Omega} = \phi, \triangle f|_{\partial\Omega} = \varphi\}$*, and* $\alpha > 0$ *is a regularization parameter.*

The problem is then to find $f_* \in H$ *such that* $\Phi(f_*) \leq \Phi(f)$*for every* $f \in H$*.*

Then we will prove the existence and uniqueness of the minimizer of Problem 2.1.

Theorem 2.2. *Suppose that* $f_* \in H$ *is the solution of the following variational problem:*

$$\int_\Omega \triangle^2 f \triangle^2 h dx = -\frac{1}{\alpha N}\sum_{j=1}^{N}(f(x_j) - \tilde{\varrho_j})h(x_j) \tag{2.2}$$

for all $h \in \hat{H} = \{h|h \in H^4(\Omega), h|_{\partial\Omega} = \triangle h|_{\partial\Omega} = 0\}$*. Then* f_* *is the minimizer of Problem 2.1. Moreover, the minimizer of Problem 2.1 is unique.*

Remark 2.3. *We will prove the existence of a solution of* (2.2) *later in Theorem 2.4.*

Proof of Theorem 2.2: For any $f \in H$, let $h = f - f_*$, then $h\,|_{\partial\Omega} = 0$ and $\triangle h\,|_{\partial\Omega} = 0$. It is easy to have the following equations:

$$\begin{aligned} \Phi(f) - \Phi(f_*) &= \frac{1}{N}\sum_{j=1}^{N}(f(x_j) - f_*(x_j))(f(x_j) + f_*(x_j) - 2\tilde{\varrho}_j) \\ &\quad +\alpha\int_\Omega \left[(\triangle^2 f)^2 - (\triangle^2 f_*)^2\right]dx \qquad (2.3) \\ &= I_1 + \alpha I_2 \end{aligned}$$

and

$$\begin{aligned} I_1 &= \frac{1}{N}\sum_{j=1}^{N} h(x_j)(2f_*(x_j) - 2\tilde{\varrho}_j + h(x_j)) \\ &= \frac{1}{N}\sum_{j=1}^{N} 2(f_*(x_j) - \tilde{\varrho}_j)h(x_j) + h^2(x_j)\,. \end{aligned}$$

By the definition of f_*, we have

$$\begin{aligned} I_2 &= \int_\Omega [(\triangle^2 f)^2 - (\triangle^2 f_*)^2]dx = \|\triangle^2 h\|^2_{L^2(\Omega)} + 2\int_\Omega \triangle^2 h \cdot \triangle^2 f_* dx \\ &= \|\triangle^2 h\|^2_{L^2(\Omega)} - \frac{2}{\alpha N}\sum_{j=1}^{N}(f_*(x_j) - \tilde{\varrho}_j)h(x_j)\,. \end{aligned}$$

Substituting the equations I_1 and I_2 into (2.3) gives

$$\Phi(f) - \Phi(f_*) = \frac{1}{N}\sum_{j=1}^{N} h^2(x_j) + \alpha\|\triangle^2 f - \triangle^2 f_*\|^2_{L^2(\Omega)} \geq 0\,.$$

Thus, f_* is a minimizer of Problem 2.1.

If there is another $f^* \in H$ minimizing Problem 2.1, denote $g = f^* - f_*$, then function g satisfies: $\int_\Omega (\triangle^2 g)^2 dx = 0$ and $g|_{\partial\Omega} = 0, \triangle g|_{\partial\Omega} = 0$. Hence, $g(x) \equiv 0$ for $x \in \Omega$. So $f^* = f_*$. Therefore, the uniqueness of the minimizer of Problem 2.1 has been proven. □

To completely solve numerical differentiation problem, it is necessary to provide a scheme for constructing f_*. For that, by an a priori argument using Green function of bi-harmonic operator, we construct

f_*. A theorem will also be given to prove that the constructed f_* is the solution of (2.2).

Let's recall the definition of a bi-harmonic Green function before the construction. Function $G(x, y)$ with fixed $y \in \Omega$ is called a bi-harmonic Green function if it satisfies the following equations:

$$\triangle_x^2 G(x, y) = \delta(x - y) \quad \text{in } \Omega$$

and

$$G|_{\partial\Omega} = 0, \triangle_x G|_{\partial\Omega} = 0.$$

We can obtain $G(x, y)$ by solving

$$\begin{aligned} &\triangle_x F(x, y) = \delta(x - y) \quad \text{in } \Omega \\ &F(x, y)|_{\partial\Omega} = 0 \end{aligned}$$

and

$$\begin{aligned} &\triangle_x G(x, y) = F(x, y) \quad \text{in } \Omega \\ &G(x, y)|_{\partial\Omega} = 0\,. \end{aligned}$$

We denote $\triangle_1$ as the Laplacian operator for the first argument, and $\triangle_2$ as the Laplacian operator for the second argument. Since $G(x, y) = G(y, x)$ and $F(x, y) = F(y, x)$ for $x, y \in \Omega$, then we will have

$$\triangle_2 G(y, x) = \triangle_1 G(x, y) = F(x, y) = F(y, x) = \triangle_1 G(y, x)\,. \tag{2.4}$$

Now we will propose a scheme to obtain the solution of the Eq. (2.2). Taking $h = G(x, y)$ in (2.2) and using the definition of Green function, we obtain

$$\begin{aligned} -\frac{1}{N}\sum_{j=1}^{N}(f_*(x_j) - \tilde{\varrho_j})G(x_j, y) &= \int_\Omega \alpha\triangle^2 f_*(x) \cdot \triangle_x^2 G(x, y)dx \\ &= \alpha\triangle^2 f_*(y). \end{aligned}$$

Multiply two sides of the above equation with $G(x,y)$ and integrate it on Ω, we obtain by integrating by parts

$$\begin{aligned}
& -\frac{1}{N}\sum_{j=1}^{N}(f_*(x_j)-\tilde{\varrho}_j)\int_\Omega G(x_j,x)G(x,y)dx \\
= \;& \alpha\int_\Omega \triangle^2 f_*(x)\cdot G(x,y)dx \\
= \;& \alpha\int_{\partial\Omega}(\frac{\partial}{\partial\nu}\triangle f_*(x)\cdot G(x,y)-\frac{\partial}{\partial\nu}G(x,y)\cdot\triangle f_*(x))ds(x) \\
& +\alpha\int_\Omega \triangle f_*(x)\cdot\triangle_x G(x,y)dx \\
= \;& -\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}G(x,y)\cdot\varphi(x)ds(x)-\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_x G(x,y)\cdot\phi(x)ds(x) \\
& +\alpha f_*(y)\,,
\end{aligned}$$

where ν is the unit normal of $\partial\Omega$ directed outside Ω. Rewrite the above equation in the form:

$$\begin{aligned}
& \alpha f_*(x)+\frac{1}{N}\sum_{j=1}^{N}(f_*(x_j)-\tilde{\varrho}_j)\int_\Omega G(x_j,y)G(y,x)dy \\
= \;& \alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}G(y,x)\cdot\varphi(y)ds(y)+\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_y G(y,x)\cdot\phi(y)ds(y)\,.
\end{aligned}\tag{2.5}$$

By defining

$$a_j(x)=\int_\Omega G(x_j,y)G(y,x)dy,\tag{2.6}$$

$$b(x)=\int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_y G(y,x)\cdot\phi(y)ds(y)+\int_{\partial\Omega}\frac{\partial}{\partial\nu}G(y,x)\cdot\varphi(y)ds(y),\tag{2.7}$$

and

$$c_j=-\frac{1}{\alpha N}(f_*(x_j)-\tilde{\varrho}_j)\tag{2.8}$$

then (2.5) becomes

$$f_*(x)=\sum_{j=1}^{N}c_j a_j(x)+b(x)\,.\tag{2.9}$$

Now the problem of constructing f_* reduces to computing the coefficients c_j from $\tilde{y_j}$ $\varphi(x)$ and $\phi(x)$. From (2.8) and (2.9) we obtain

$$c_j = -\frac{1}{\alpha N}(f_*(x_j) - \tilde{\varrho_j}) = -\frac{1}{\alpha N}(\sum_{k=1}^{N} a_k(x_j)c_k + b(x_j) - \tilde{\varrho_j})\,. \tag{2.10}$$

Let

$$\mathbf{A} = \begin{pmatrix} \alpha N + a_1(x_1) & a_2(x_1) & a_3(x_1) & \cdots & a_N(x_1) \\ a_1(x_2) & \alpha N + a_2(x_2) & a_3(x_2) & \cdots & a_N(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1(x_N) & a_2(x_N) & a_3(x_N) & \cdots & \alpha N + a_N(x_N) \end{pmatrix}$$

and

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_N \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \tilde{y_1} - b(x_1) \\ \tilde{y_2} - b(x_2) \\ \cdots \\ \tilde{y_N} - b(x_N)\,. \end{pmatrix},$$

Then (2.10) becomes the linear equations $\mathbf{Ac} = \mathbf{b}$. Solving this equations, we will obtain coefficients c_j, which finishes the construction of f_*.

Theorem 2.4. *Suppose function $f_* = \sum_{j=1}^{N} c_j a_j(x) + b(x)$ where $a_j(x)$ and $b(x)$ are defined in* (2.5) *and* (2.6)*, $\{c_j\}_{j=1}^{N}$ is the solution of linear system* (2.10)*, then f_* is the solution of* (2.2)*.*

Proof. For every $x \in \partial\Omega$, from the definition of Green function, we know that $G(x, y) = G(y, x) = 0$ for $y \in \Omega$. So

$$a_j(x) = \int_\Omega G(x_j, y) \cdot G(y, x)dy = 0.$$

Assume that $\hat{\phi} \in H^2(\Omega)$ is an extension of ϕ to Ω and $\hat{\varphi} \in H^2(\Omega)$ is an extension of φ over Ω, then integrating by parts yields

$$b(x) = \phi(x) \quad (x \in \partial\Omega).$$

Thus we have $f_*(x)|_{\partial\Omega} = \phi(x)$.

We also have

$$\triangle a_j(x) = \int_\Omega G(x_j, y)\triangle_x G(x, y)dy = 0 \qquad (x \in \partial\Omega)$$

Since

$$\begin{aligned} b(x) &= \int_\Omega \hat{\phi}(y)\triangle_y^2 G(y,x)dy - \int_\Omega \triangle_y G(y,x)\triangle\hat{\phi}(y)dy \\ &\quad + \int_\Omega \hat{\varphi}(y)\triangle_y G(y,x)dy - \int_\Omega G(y,x)\triangle\hat{\varphi}(y)dy \quad (x\in\partial\Omega), \end{aligned}$$

then we will have using the definition of $F(x,y)$ and (2.4),

$$\begin{aligned} \triangle b(x) &= \triangle\hat{\phi}(x) - \int_\Omega \triangle_x(\triangle_y G(y,x))\triangle\hat{\phi}(y)dy \\ &\quad + \int_\Omega \hat{\varphi}(y)\triangle_x(\triangle_y G(y,x))dy - \int_\Omega \triangle_x G(x,y)\triangle\hat{\varphi}(y)dy \\ &= \triangle\hat{\phi}(x) - \int_\Omega \triangle_x^2 G(x,y)\triangle\hat{\phi}(y)dy + \int_\Omega \hat{\varphi}(y)\triangle_x^2 G(x,y))dy \\ &= \varphi(x) \qquad (x\in\partial\Omega). \end{aligned}$$

Thus we have $\triangle f_*(x)|_{\partial\Omega} = \varphi(x)$.

Moreover, from the definition of $a_j(x)$ and $b(x)$, we know that for every $x\in\Omega$

$$\triangle^2 a_j(x) = \int_\Omega G(x_j,y)\triangle_x^2 G(x,y)dy = G(x_j,x) \tag{2.11}$$

and

$$\triangle^2 b(x) = \triangle\hat{\varphi}(x) - \int_\Omega \triangle_x^2 G(x,y)\triangle\hat{\varphi}(y)dy = \triangle\hat{\varphi}(x) - \triangle\hat{\varphi}(x) = 0. \tag{2.12}$$

Since $G(x_j,x)\in L^2(\Omega)$, so $\triangle^2 f_*(x)\in L^2(\Omega)$. By the priori estimate of Poisson equation with Dirichlet boundary condition (see Lemma 3.2 in the next section), we know $f_*\in H^4(\Omega)$. Furthermore $f_*\in H$.

For any $h \in \hat{H}$, we have

$$\begin{aligned}
&\int_\Omega \triangle^2 f_* \triangle^2 h dx = \int_\Omega \sum_{j=1}^N c_j G(x_j, x) \triangle^2 h(x) dx \\
= &\sum_{j=1}^N c_j \left(\int_\Omega \triangle G(x_j, x) \triangle h(x) dx - \int_{\partial\Omega} \triangle h(x) \frac{\partial G(x_j, x)}{\partial \nu} ds(x) \right. \\
&\left. + \int_{\partial\Omega} G(x_j, x) \frac{\partial \triangle h(x)}{\partial \nu} ds(x) \right) \\
= &\sum_{j=1}^N c_j \left(\int_\Omega \triangle^2 G(x_j, x) h(x) dx - \int_{\partial\Omega} h(x) \frac{\partial \triangle G(x_j, x)}{\partial \nu} ds(x) \right. \\
&\left. + \int_{\partial\Omega} \triangle G(x_j, x) \frac{\partial h(x)}{\partial \nu} ds(x) \right) \\
= &\sum_{j=1}^N c_j h(x_j) = -\frac{1}{\alpha N} \sum_{j=1}^N (f_*(x_j) - \tilde{\varrho}_j) h(x_j).
\end{aligned}$$

So f_* is the solution of (2.2). This completes the proof. □

The solution of the linear equations exists and is unique since if we assume $\tilde{y}_i = 0, i = 1, \cdots, N$, and $\phi(x) = \varphi(x) = 0$, then we know that there is only one minimizer of Problem 2.1 , which is $f_*(x) \equiv 0, x \in \Omega$. It is obvious $\mathbf{c} = 0$ is a solution of $\mathbf{Ac} = 0$ and if there is another $\hat{\mathbf{c}}$ satisfying $\mathbf{A}\hat{\mathbf{c}} = \mathbf{0}$, then we will have a function $\hat{f} \neq 0$ which is also a minimizer of Problem 2.1. This is a contradiction so the homogenous linear equations only has a trivial solution. Thus the solution of the linear equations exists and is unique.

3. Error estimate

In this section we will prove a convergence estimate for our proposed solution under a priori choice of the regularization parameter. The proof uses the following two Lemmas.

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^n$ having the strong local Lipschitz property, $u \in W^{1,p}(\Omega)$, and suppose that $n < p \leq \infty$, then*

$$|u(x) - u(y)| \leq K |x - y|^{1 - \frac{n}{p}} \|u\|_{1,p,\Omega}$$

where K is independent of u.

This lemma can be obtained from Lemma 5.17 in page 108 of [1].

Lemma 3.2. *Let Ω be a $C^{1,1}$ domain in R^n, and let the operator $Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$ be strictly elliptic in Ω with coefficients $a^{ij} \in C^0(\overline{\Omega}), b^i, c \in L^\infty$, with $i, j = 1, \cdots, n$ and $c \leq 0$. Then there exists a constant C (independent of u) such that*

$$\|u\|_{2,p,\Omega} \leq C\|Lu\|_{p,\Omega}$$

for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), 1 < p < \infty$.

This lemma can be obtained from Lemma 9.17 in page 242 of [6].

According to the result of [4], we choose the regularization parameter $\alpha = \delta^2$. Such choice has been proven to be quite effective (see [15]). We give the error estimate in the following theorem:

Theorem 3.3. *Suppose Ω satisfies the conditions in Lemma 3.1 and Lemma 3.2, and f_* is the minimizer of Problem 2.1 and $y \in H^4(\Omega)$. Let $e = f_* - y$ and choose $\alpha = \delta^2$, then we have the following error estimate*

$$\|\triangle e\|_{L^2} \leq L_1\, d^{\frac{1}{2}} + L_2\delta^{\frac{1}{2}}, \qquad \|e\|_{L^2} \leq L_3 d + L_4\delta$$
$$\|\nabla e\|_{L^2} \leq L_5 d^{\frac{3}{4}} + L_6\delta^{\frac{1}{2}}, \qquad \|\nabla\triangle e\|_{L^2} \leq L_7 d^{\frac{1}{4}} + L_8\delta^{\frac{1}{4}}$$

where L_i are constants which depend on Ω, $\|\phi\|_{H^{7/2}(\partial\Omega)}$, $\|\varphi\|_{H^{3/2}(\partial\Omega)}$ and $\|\triangle^2 y\|_{L^2}$.

Proof. First since $\delta^2\|\triangle^2 f_*\|_{L^2}^2 \leq \Phi(f_*) \leq \Phi(y) \leq \delta^2 + \delta^2\|\triangle^2 y\|_{L^2}^2$, it is easy to see that $\|\triangle^2 e\|_{L^2} \leq 1 + 2\|\triangle^2 y\|_{L^2}$. Also, from the well-posedness of the boundary value problem :

$$\begin{cases} \triangle^2 e = g \quad \text{in} \quad \Omega \\ e = k, \quad \triangle e = \ell \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

with given $g \in L^2(\Omega)$, $k \in H^{7/2}(\partial\Omega)$, $\ell \in H^{3/2}(\partial\Omega)$ and the continuity of the trace operator, there are constants C_1 and C_2 such that $\|\frac{\partial}{\partial\nu}e\|_{L^2(\partial\Omega)} \leq C_1$ and $\|\frac{\partial}{\partial\nu}\triangle e\|_{L^2(\partial\Omega)} \leq C_2$. Hereafter, C_i's are general constants which may depend on Ω, $\|\phi\|_{H^{7/2}(\partial\Omega)}$, $\|\varphi\|_{H^{3/2}(\partial\Omega)}$ and $\|\triangle^2 y\|_{L^2}$.

So, by $\|e\|_{L^2(\partial\Omega)} \le \delta$, $\|\triangle e\|_{L^2(\partial\Omega)} \le \delta$,

$$\begin{aligned}\|\triangle e\|_{L^2}^2 &= \int_\Omega |\triangle e|^2 dx \\ &= \int_{\partial\Omega} \triangle e \cdot \frac{\partial}{\partial\nu} e dS - \int_{\partial\Omega} e \cdot \frac{\partial}{\partial\nu} \triangle e dS + \int_\Omega \triangle^2 e \cdot e dx \\ &\le \|e\|_{L^2} \cdot \|\triangle^2 e\|_{L^2} + C_1\delta + C_2\delta.\end{aligned}$$

Here, note that the general constants C_1, C_2 can be different in each estimate. We rewrite $\|e\|_{L^2}$ as

$$\begin{aligned}\|e\|_{L^2}^2 &= \int_\Omega e^2(x)dx = \sum_{i=1}^N \int_{\Omega_i} e^2(x)dx \\ &= \sum_{i=1}^N \int_{\Omega_i} e(x)(e(x)-e(x_i))dx + \sum_{i=1}^N \int_{\Omega_i} e(x_i)(e(x)-e(x_i))dx \\ &\quad + \sum_{i=1}^N \int_{\Omega_i} e^2(x_i)dx \\ &= I_3 + I_4 + I_5.\end{aligned}$$

Now we estimate I_3, I_4, and I_5.

$$\begin{aligned}I_3 &= \sum_{i=1}^N \int_{\Omega_i} e(x)(e(x)-e(x_i))dx \le \sum_{i=1}^N \int_{\Omega_i} |e(x)||(e(x)-e(x_i))|dx \\ &\le \sum_{i=1}^N \int_{\Omega_i} C_1|x-x_i|^{1-\frac{n}{p}}\|e\|_{1,p}|e(x)|dx \le d^{1-\frac{n}{p}}C_1\|e\|_{1,p}\int_\Omega |e(x)|dx \\ &\le d^{1-\frac{n}{p}}C_1\|e\|_{1,p}\|e\|_{L^2}(vol(\Omega))^{\frac{1}{2}}\end{aligned}$$

where $vol(\Omega)$ is the volume of Ω. The second inequality is obtained from Lemma 3.1 with $n = 2$. We may set $p = 4$, then

$$I_3 \le d^{\frac{1}{2}}C_1(vol(\Omega))^{\frac{1}{2}}\|e\|_{1,4}\|e\|_{L^2}.$$

From the imbedding theorem of Soblev spaces we know that $W^{2,2}(\Omega) \to W^{1,4}(\Omega)$ for Ω having cone property, which means, there is a constant C_1 independent of e satisfying $\|e\|_{1,4} \le C_1\|e\|_{2,2}$. By the well-posedness of (3.1)

$$\|e\|_{2,2} \le C_1\|\triangle e\|_{L^2} + C_2\delta.$$

Hence, we have $I_3 \le C_1 d^{\frac{1}{2}} \|e\|_{L^2}(\|\triangle e\|_{L^2} + \delta)$.

By the same way, we have

$$\begin{aligned} I_4 &= \sum_{i=1}^{N} \int_{\Omega_i} e(x_i)(e(x) - e(x_i))dx \le \sum_{i=1}^{N} \int_{\Omega_i} |e(x_i)||(e(x) - e(x_i))|dx \\ &\le d^{\frac{1}{2}} C_1 \|e\|_{1,4} \sum_{i=1}^{N} (\int_{\Omega_i} |e(x_i)|dx) = d^{\frac{1}{2}} C_1 \|e\|_{1,4} \frac{vol(\Omega)}{N} \sum_{i=1}^{N} |e(x_i)|. \end{aligned}$$

And since $\Phi(f_*) \le \Phi(y)$, we have

$$\frac{1}{N} \sum_{i=1}^{N} (f_*(x_i) - \tilde{\varrho}_i)^2 \le \delta^2 (1 + \|\triangle^2 \varrho\|^2).$$

So

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} |e(x_i)| &\le \frac{1}{N} \sum_{i=1}^{N} (|f_*(x_i) - \tilde{\varrho}_i| + |\tilde{\varrho}_i - \varrho(x_i)|) \\ &\le \sqrt{\frac{1}{N} \sum_{i=1}^{N} |f_*(x_i) - \tilde{\varrho}_i|^2} + \delta \\ &\le \delta(\sqrt{1 + \|\triangle^2 \varrho\|^2} + 1). \end{aligned}$$

Hence, we have $I_4 \le C_1 d^{\frac{1}{2}} \delta(\|\triangle e\| + \delta)$.

The estimation of I_5 is simple

$$\begin{aligned} I_5 &= \sum_{i=1}^{N} \int_{\Omega_i} e^2(x_i)dx = \sum_{i=1}^{N} e^2(x_i) \int_{\Omega_i} dx \le \frac{1}{N} vol(\Omega) \cdot \sum_{i=1}^{N} e^2(x_i) \\ &\le \frac{2}{N} vol(\Omega) \cdot \sum_{i=1}^{N} ((f_*(x_i) - \tilde{\varrho}_i)^2 + (\tilde{\varrho}_i - \varrho(x_i))^2) \\ &\le 2vol(\Omega)\delta^2(2 + \|\triangle^2 \varrho\|^2) = C_1 \delta^2. \end{aligned}$$

From all above, we can conclude that

$$\|e\|_{L^2}^2 \le C_1 d^{\frac{1}{2}} \|e\|_{L^2}(\|\triangle e\|_{L^2} + \delta) + C_2 d^{\frac{1}{2}} \delta \|\triangle e\|_{L^2} + C_3 \delta^2.$$

Then we will have

$$\begin{aligned} \|e\|_{L^2} &\le C_1 d^{\frac{1}{2}} (\|\triangle e\|_{L^2} + \delta) + C_2 d^{\frac{1}{4}} \delta^{\frac{1}{2}} \|\triangle e\|_{L^2} + C_3 \delta \\ &\le C_1 d^{\frac{1}{2}} \|\triangle e\|_{L^2} + C_2 \delta. \end{aligned}$$

So it comes out the results of the theorem:

$$\|\triangle e\|_{L^2} \leq C_1 d^{\frac{1}{2}} + C_2 \delta^{\frac{1}{2}}$$

and

$$\|e\|_{L^2} \leq C_1 d + C_2 \delta$$

Also, since

$$\begin{aligned} \|\nabla e\|_{L^2}^2 &= \int_\Omega \nabla e \cdot \nabla e dx \\ &= -\int_\Omega \triangle e \cdot e dx + \int_{\partial\Omega} e \cdot \frac{\partial}{\partial \nu} e dS \\ &\leq \|e\|_{L^2} \cdot \|\triangle e\|_{L^2} + C_1 \delta. \end{aligned}$$

$$\begin{aligned} \|\nabla \triangle e\|_{L^2}^2 &= \int_\Omega \nabla \triangle e \cdot \nabla \triangle e dx \\ &= -\int_\Omega \triangle^2 e \cdot \triangle e dx + \int_{\partial\Omega} \triangle e \cdot \frac{\partial}{\partial \nu} \triangle e dS \\ &\leq \|\triangle e\|_{L^2} \cdot \|\triangle^2 e\|_{L^2} + C_1 \delta. \end{aligned}$$

then we will have

$$\|\nabla e\|_{L^2} \leq C_1 d^{\frac{3}{4}} + C_2 \delta^{\frac{1}{2}}$$

$$\|\nabla \triangle e\|_{L^2} \leq C_1 d^{\frac{1}{4}} + C_2 \delta^{\frac{1}{4}}.$$

This completes the proof. □

Remark 3.4. *In this paper, for the simplicity, we assume that the volumes of all Ω_i are same. In the real application, this condition may be not easy to be satisfied. But if we denote $V1 = \max_i\{vol(\Omega_i)\}$ and $V2 = \min_i\{vol(\Omega_i)\}$ and let $\frac{V1}{V2}$ is bounded with some constant, then we still have the same error estimate.*

Remark 3.5. *In Theorem 3.3, we used Lemma 3.1 to estimate I_3 in which we choose the parameter p to be 4. Actually we can choose any p satisfying $2 \leq p < \infty$. And we can still use the imbedding theorem of Soblev spaces $W^{2,2}(\Omega) \to W^{1,p}(\Omega)$. The result will be*

$$\|\triangle e\|_{L^2} \leq L_{1p} \cdot d^{1-\frac{2}{p}} + L_{2p}\delta^{\frac{1}{2}}, \qquad \|e\|_{L^2} \leq L_{3p} d^{2-\frac{4}{p}} + L_{4p}\delta$$

$$\|\nabla e\|_{L^2} \leq L_{5p} d^{\frac{3}{2}-\frac{3}{p}} + L_{6p}\delta^{\frac{1}{2}}, \qquad \|\nabla \triangle e\|_{L^2} \leq L_{7p} d^{\frac{1}{2}-\frac{1}{p}} + L_{8p}\delta^{\frac{1}{4}}$$

where L_{ip} are constants depending on $\|\phi\|_{H^{\frac{7}{2}}(\partial\Omega)}$, $\|\varphi\|_{H^{\frac{3}{2}}(\partial\Omega)}$ and Ω, $\|\triangle^2 y\|_{L^2}$ and p. So when we choose a larger p we will get a better convergence rate.

4. Numerical examples

We provide numerical examples in this section.

For constructing f_* and $\triangle f_*$, we compute the Green function G by Fourier series and denote the number of terms in Fourier series by K. We give the detailed algorithm of the construction in Appendix.

We compute $\triangle f_*$ from $\tilde{\varrho}_i$ $(1 \le i \le N)$ satisfying (2.1) with

$$\varrho(x,y) = (2-(2x-1)^2)^2 \sin(ay)$$

where a is a constant. We set $\delta = 0.001$, divide Ω into N parts $\{\Omega_i\}_1^N$.

We evaluate the computed relative error by the formulae

$$e_1 = \frac{\left(\sum_{j=1}^N (f_*(x_j) - \varrho(x_j))^2\right)^{1/2}}{\left(\sum_{i=1}^N (\varrho(x_j))^2\right)^{1/2}},$$

$$e_2 = \frac{\left(\sum_{j=1}^N (\triangle f_*(x_j) - \triangle\varrho(x_j))^2\right)^{1/2}}{\left(\sum_{i=1}^N (\triangle\varrho(x_j))^2\right)^{1/2}}.$$

First, we set $a = 4$ in function $\varrho(x,y)$, and choose $N = 40^2$ and $K = 100$, then Fig. 1 and Fig. 2 show our numerical results. In Fig. 1, the left figure shows exact value of $\varrho(x,y)$, the middle one shows the constructed function $f_*(x,y)$, and the right one shows difference of $f_*(x,y)$ and $\varrho(x,y)$ which is $f_* - \varrho$. In Fig. 2, the left figure shows exact value of $\triangle\varrho(x,y)$, the middle one shows constructed function $\triangle f_*(x,y)$, and the right one shows the difference $\triangle f_* - \triangle\varrho$.

Fig. 3 and Fig. 4 show our numerical results when $a = 2$. Fig. 3 is about constructed f_* and Fig. 4 is about $\triangle f_*$, and each figure corresponds to the same item as in Fig. 1 and in Fig. 2.

Fig. 5 and Fig. 6 show our numerical results when $a = 0.5$. Fig. 3 is about constructed f_* and Fig. 4 is about $\triangle f_*$, and each figure corresponds to the same item as in Fig. 1 and in Fig. 2.

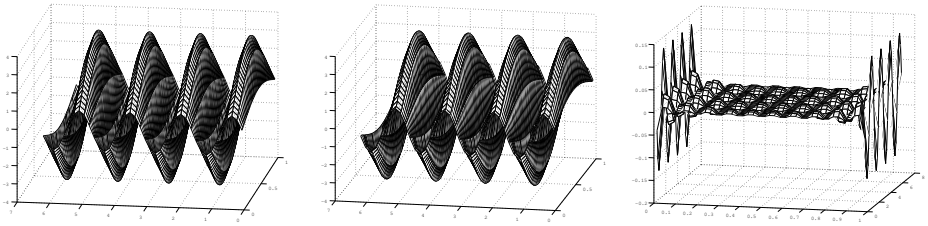

FIGURE 1. The left figure shows exact value of $\varrho(x, y)$ when $a = 4$, the middle figure shows constructed function $f_*(x, y)$, and the right figure shows the difference $f_* - \varrho$

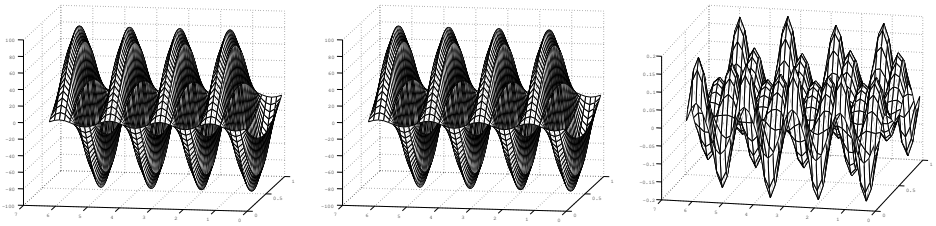

FIGURE 2. The left figure shows exact value of $\Delta\varrho(x, y)$ when $a = 4$, the middle figure shows constructed function $\Delta f_*(x, y)$, and the right figure shows the difference $\triangle f_* - \triangle\varrho$

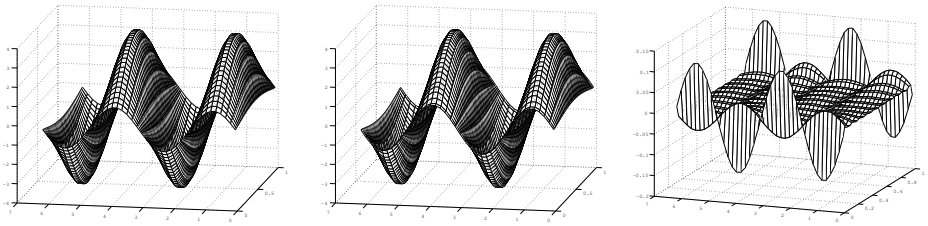

FIGURE 3. The left figure shows exact value of $\varrho(x, y)$ when $a = 2$, the middle figure shows constructed function $f_*(x, y)$, and the right figure shows the difference $f_* - \varrho$

In all the right figures, we can find the errors near the boundary of the domain Ω are much larger than the errors inside. Therefore, we define Ω' which equals Ω minus a band near the boundary with width ε and compute the relative errors e_1 and e_2 only in Ω'. We always choose $\varepsilon = 0.1$. Table 1 gives the value of the relative errors.

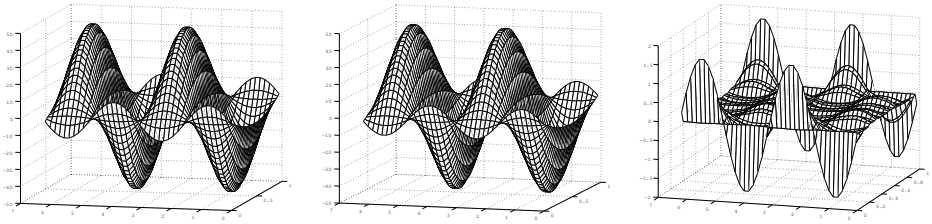

FIGURE 4. The left figure shows exact value of $\Delta\varrho(x,y)$ when $a=2$, the middle figure shows constructed function $\Delta f_*(x,y)$, and the right figure shows the difference $\triangle f_* - \triangle\varrho$

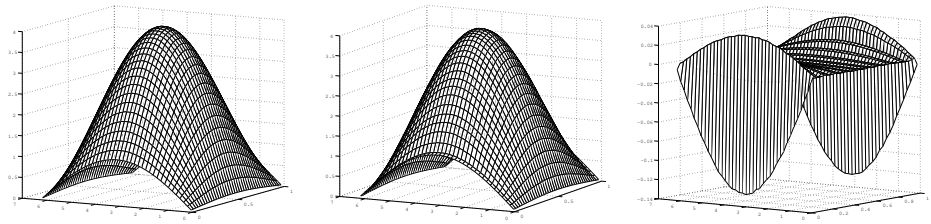

FIGURE 5. The left figure shows exact value of $\varrho(x,y)$ when $a=0.5$, the middle figure shows constructed function $f_*(x,y)$, and the right figure shows the difference $f_* - \varrho$

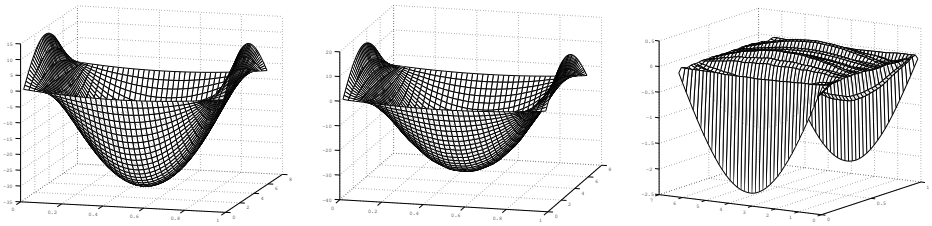

FIGURE 6. The left figure shows exact value of $\Delta\varrho(x,y)$ when $a=0.5$, the middle figure shows constructed function $\Delta f_*(x,y)$, and the right figure shows the difference $\triangle f_* - \triangle\varrho$

We also investigate the errors with K and N being changed.

We choose K equal 80, 100, 120 respectively and compute the relative errors. Table 2 gives the results.

TABLE 1. Relative errors(%) of constructed f_* and $\triangle f_*$ for different a in functions $\varrho(x,y)$ and $\triangle\varrho(x,y)$ ($N = 40^2, K = 100$), respectively

a	4	2	0.5
e_1	0.2794	0.2279	0.2312
e_2	0.1114	0.6175	1.3245

TABLE 2. Relative errors of f_* and $\triangle f_*$ with different K's(%)(fix $N = 40^2$)

		$K = 80$	$K = 100$	$K = 120$
a=4	e_1	0.2908	0.2794	0.1815
	e_2	1.3034	0.1114	0.1114
a=2	e_1	0.2144	0.2279	0.0846
	e_2	3.1704	0.6175	0.5595
a=0.5	e_1	0.2122	0.2312	0.0927
	e_2	4.9742	1.3245	1.2260

TABLE 3. Relative errors of f_* and $\triangle f_*$ with different N's(%)(fix $K = 100$)

		$N = 25^2$	$N = 30^2$	$N = 35^2$	$N = 40^2$	$N = 45^2$
a=4	e_1	0.4580	0.3865	0.3782	0.2774	0.3127
	e_2	0.5997	0.1250	0.1200	0.1114	0.1136
a=2	e_1	0.2257	0.2496	0.2949	0.2279	0.2717
	e_2	1.1406	0.8090	0.7031	0.6175	0.4953
a=0.5	e_1	0.2320	0.2538	0.2970	0.2312	0.2723
	e_2	1.2633	1.7458	1.5387	1.3245	1.0857

We choose $N = M^2$ and increase M from 25 to 45 with increment equal 5 in each time. Table 3 gives the computed results of relative errors.

5. Discussion and conclusion

In the numerical examples, we can see when we choose $K = 100$ and $N = 40^2$, the constructed f_* and $\triangle f_*$ are very close to the original

functions $\varrho(x,y)$ and $\triangle\varrho(x,y)$ with different a's in Fig. 1 - Fig 6. From Table 1, we can see that for f_* the relative errors e_1's are less than 0.3%, and for $\triangle f_*$ the relative error e_2 increases from 0.1114% to 1.3245%.

In Table 2, we can see that when increasing K from 80 to 120, both e_1 and e_2 decrease. So increasing the number of terms in Fourier series can improve the numerical results of our method.

In Table 3, we can see that when incresing N from 25^2 to 45^2, e_1's have small oscillation and the values are less than 0.5% for three cases. but e_2's change are a little bit complicated. For $a = 4$ case, e_2 decreases from 0.5997% to 0.1136%. For $a = 2$ case, e_2 decreases from 1.1406% to 0.4953%. For $a = 0.5$ case, e_2 decreases from 1.2633 to 1.0857. So by increasing N, we can improve the numerical results very much for $a = 4$, but the improvement is a little for $a = 0.5$. This fact tells us that increasing N for the domain decomposition $\{\Omega\}_{i=1}^N$ can also improve the numerical results of our numerical differentiation for the second derivative of functions with two variables, but the error depends on the property of the original function. As shown in Appendix B, we use sine series to compute the Green function, but computing in this way is not very accurate if the original function is not an odd function. Also for non-odd functions, we can observe that by using sine series, the result is better for rapidly oscillating functions than slowly oscillating functions. Due to the memory limitation problem of our computer, we could not take a larger N for the domain decomposition and a larger K which bounds (k_1, k_2) such that $1 \le k_1, k_2 \le K$ for the Fourier series to make our numerical results precise enough. We expect that if we can take N, K larger, we can make our results more precise.

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6. Appendix A: Proof of $G(x,y) = G(y,x)$

Here we will give the proof that the solution $G(x,y)$ with fixed $y \in \Omega$ of

$$\begin{aligned} &\triangle_x^2 G(x,y) = \delta(x-y) \quad \text{in } \Omega \\ &G(x,y)|_{\partial\Omega} = \triangle_x G(x,y)|_{\partial\Omega} = 0 \end{aligned}$$

satisfies $G(x,y) = G(y,x)$, for any $x, y \in \Omega$.

Proof: Suppose x, y are two points in Ω. We define $B_\delta(x) = z||z-x| < \delta, z \in \Omega$, $B_\delta(y) = \{z||z-y| < \delta, z \in \Omega\}$, and $\Omega_\delta = \Omega \setminus \overline{(B_\delta(x) \cup B_\delta(y))}$, $\Gamma_\delta(x) = \partial B_\delta(x)$, $\Gamma_\delta(y) = \partial B_\delta(y)$. According to Green formula, we know that for any $u, v \in H^4$

$$\begin{aligned} \int_{\Omega_\delta} \triangle_x^2 u \cdot v \, dx \;=\; & \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_x} \triangle_x u \cdot v \, ds - \int_{\partial\Omega_\delta} \triangle_x u \cdot \frac{\partial}{\partial \nu_x} v \, ds \\ & + \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_x} u \cdot \triangle_x v \, ds \\ & - \int_{\partial\Omega_\delta} u \cdot \frac{\partial}{\partial \nu_x} \triangle_x v \, ds + \int_{\Omega_\delta} u \cdot \triangle_x^2 v \, dx \end{aligned}$$

So

$$\begin{aligned} &\int_{\Omega_\delta} \triangle_z^2 G(z,x) \cdot G(z,y) \, dz = \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_z} \triangle_z G(z,x) \cdot G(z,y) \, ds \\ &- \int_{\partial\Omega_\delta} \triangle_z G(z,x) \cdot \frac{\partial}{\partial \nu_z} G(z,y) ds + \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_z} G(z,x) \cdot \triangle_z G(z,y) ds \\ &- \int_{\partial\Omega_\delta} G(z,x) \cdot \frac{\partial}{\partial \nu_z} \triangle_z G(z,y) ds + \int_{\Omega_\delta} G(z,x) \cdot \triangle_z^2 G(z,y) dz \\ &= I_1(\delta) - I_2(\delta) + I_3(\delta) - I_4(\delta) + \int_{\Omega_\delta} G(z,x) \cdot \triangle_z^2 G(z,y) dz \end{aligned}$$

Since $y, x \notin \Omega_\delta$, so for any $z \in \Omega_\delta$ with $z \neq x, z \neq y$, $\triangle_z^2 G(z,y) = 0$, $\triangle_z^2 G(z,x) = 0$.

Hence we have

$$I_1(\delta) - I_2(\delta) + I_3(\delta) - I_4(\delta) = 0$$

Next we will prove that $\lim_{\delta\to 0} I_2(\delta) = 0, \lim_{\delta\to 0} I_3(\delta) = 0$, and $\lim_{\delta\to 0} I_1(\delta) = G(x,y), \lim_{\delta\to 0} I_4(\delta) = G(y,x)$.

Let $F(z,x) = \triangle_z G(z,x)$, then

$$\begin{aligned} &\triangle_z F(z,x) = \delta(z-x) && \text{in } \Omega \\ &F(z,x)|_{\partial\Omega} = 0 \end{aligned}$$

So $F(z,x) \in C^\infty$ for $z \in \Omega \setminus \{x\}$, and $F(z,x) \sim \frac{1}{2\pi} \ln|z-x|, (z \to x)$. Since $G(z,x) = G(z,y) = \triangle_z G(z,x) = \triangle_z G(z,y) = 0 (z \in \partial\Omega)$

$$I_2(\delta) \;=\; \int_{\Gamma_\delta(x)} + \int_{\Gamma_\delta(y)}.$$

Here

$$\int_{\Gamma_\delta(x)} \sim \frac{1}{2\pi} \int_{\Gamma_\delta(x)} \ln|z-x| \frac{\partial}{\partial \nu_z} G(z,y) dz = \frac{1}{2\pi} \int_0^{2\pi} (\delta \ln \delta) \frac{\partial}{\partial \nu} G(\delta, \theta)$$

and$\frac{\partial}{\partial \nu_z} G(z,y)$ is bounded, so $\int_{\Gamma_\delta(x)} \to 0$ when $\delta \to 0$.

As for $\int_{\Gamma_\delta(y)}$,

$$\begin{aligned} \int_{\Gamma_\delta(y)} F(z,x) \frac{\partial}{\partial \nu_z} G(z,y) ds \;&=\; F(y,x) \int_{\Gamma_\delta(y)} \frac{\partial}{\partial \nu_z} G(z,y) ds \\ &\quad + \int_{\Gamma_\delta(y)} (F(z,x) - F(y,x)) \frac{\partial}{\partial \nu_z} G(z,y) ds. \end{aligned}$$

Here,

$$\int_{\Gamma_\delta(y)} \frac{\partial}{\partial \nu_z} G(z,y) = \int_{B_\delta(y)} F(z,y) \sim \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\delta r \ln r dr \to 0$$

Since $\triangle_z G(z,y) = F(z,y) \sim \frac{1}{2\pi} \ln|z-y| \in L^2$ near y, by the interior regularity for the Poison equation, we have $G(z,y) \in H^2$ near y and hence $\frac{\partial}{\partial \nu_z} G(z,y) \in H^{\frac{1}{2}}$ near y. So

$$\int_{\Gamma_\delta(y)} (F(z,x) - F(y,x) \frac{\partial}{\partial \nu_z} G(z,y) \to 0, \quad as \quad \delta \to 0$$

thus we have

$$I_2(\delta) \to 0, \quad as \quad \delta \to 0$$

By using the same way , we will have

$$I_3(\delta) \to 0, \quad as \quad \delta \to 0$$

Also,

$$I_1(\delta) = \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_z} F(z,x)G(z,y)ds = (\int_{\Gamma_\delta(x)} + \int_{\Gamma_\delta(y)}) \frac{\partial}{\partial \nu_z} F(z,x)G(z,y)ds$$

We know that $F(z,x) \in C^\infty$ near y, and $G(z,y) \in H^2$ which means $G(z,y) \in C^{1-\epsilon}$. So we have

$$\int_{\Gamma_\delta(y)} \frac{\partial}{\partial \nu_z} F(z,x)G(z,y)ds \to 0(\delta \to 0).$$

As for $\int)_{\Gamma_\delta(x)}$,

$$\begin{aligned}\int_{\Gamma_\delta(x)} \frac{\partial}{\partial \nu_z} F(z,x)G(z,y) &= G(x,y) \int_{\Gamma_\delta(x)} \frac{\partial}{\partial \nu_z} F(z,x) \\ &\quad + \int_{\Gamma_\delta(x)} (G(z,y) - G(x,y)) \frac{\partial}{\partial \nu_z} F(z,x).\end{aligned}$$

Since $F(z,x) \sim \frac{1}{2\pi} \ln|z-x|$, $\frac{\partial}{\partial \nu_z} F(z,x) \sim \frac{1}{2\pi|z-x|}$, $G(z,y) - G(x,y) = O(|z-x|)$ for z near x,

$$\int_{\Gamma_\delta(x)} (G(z,y) - G(x,y)) \frac{\partial}{\partial \nu_z} F(z,x)ds \to 0(\delta \to 0)$$

and

$$\int_{\Gamma_\delta(x)} \frac{\partial}{\partial \nu_z} F(z,x)ds \sim \frac{1}{2\pi} \int_0^{2*\pi} \delta\delta^{-1} d\theta = 1(\delta \to 0).$$

Thus $I_1(\delta) \to G(x,y)$ as $\delta \to 0$.

By the same way, we can prove that $I_4(\delta) \to G(y,x)$, as $\delta \to 0$. This completes the proof.

7. Appendix B: Algorithm of computing $G(x,y)$ and $\triangle f_*(x)$

Assume $\Omega = (0,L)\times(0,2\pi)$ and fix $y\in\Omega$. The problem of solving

$$\begin{aligned}&\triangle_x^2 G(x,y) = \delta(x-y) \quad \text{in } \Omega\\ &G(x,y)|_{\partial\Omega} = \triangle_x G(x,y)|_{\partial\Omega} = 0\end{aligned}$$

can be changed into solving

$$\begin{aligned}&\triangle_x F(x,y) = \delta(x-y) \quad \text{in } \Omega\\ &F(x,y)|_{\partial\Omega} = 0\end{aligned}$$

and

$$\begin{aligned}&\triangle_x G(x,y) = F(x-y) \quad \text{in } \Omega\\ &G(x,y)|_{\partial\Omega} = 0.\end{aligned}$$

From above we know that $F(x,y) = \triangle_x G(x,y)$. Define

$$u_k(x) = \sin\frac{k_1\pi x_1}{L}\sin\frac{k_2 x_2}{2},$$

where $x=(x_1,x_2), k=(k_1,k_2)$. Then, by a direct computation, $F(x,y)$ and $G(x,y)$ are given by

$$F(x,y) = \sum_k p_k(y)u(x) = \sum_k p_k(y)\sin\frac{k_1\pi x_1}{L}\sin\frac{k_2 x_2}{2}$$

$$G(x,y) = \sum_k q_k(y)u_k(x),$$

where

$$p_k(x) = \frac{-u_k(x)}{(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})\frac{\pi L}{2}}$$

$$q_k(y) = \frac{-p_k(y)}{(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})} = \frac{u_k(y)}{(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})^2\frac{\pi L}{2}}.$$

So the basis functions can be computed as following

$$\begin{aligned} a_j(x) &= \int_\Omega G(x_j,y)G(x,y)dy \\ &= \int_\Omega \sum_k q_k(x)u_k(y)\sum_k q_k(x_j)u_k(y)dy \\ &= \sum_k q_k(x)q_k(x_j)\int_\Omega u_k^2(y)dy \\ &= \sum_k q_k(x)q_k(x_j)\frac{\pi L}{2} \end{aligned}$$

and

$$\begin{aligned} b(x) &= \int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_y G(y,x)\cdot\phi(y)dy + \int_{\partial\Omega}\frac{\partial}{\partial\nu}G(y,x)\cdot\varphi(y)dy \\ &= -\sum_k q_k(x)(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})\int_{\partial\Omega}\frac{\partial}{\partial\nu}u_k(y)\cdot\phi(y)dy \\ &\quad +\sum_k q_k(x)\int_{\partial\Omega}\frac{\partial}{\partial\nu}u_k(y)\cdot\varphi(y)dy \\ &= I(x)+J(x). \end{aligned}$$

We divide $\partial\Omega$ into four parts: $\Gamma_1:(0,L)\times 0;\Gamma_2: L\times(0,2\pi);\Gamma_3:(L,0)\times 2\pi;\Gamma_4:(2\pi,0)\times 0$, and we denote the integral of I,J on each part as $I_1,I_2,I_3,I_4,\ J_1,J_2,J_3,J_4$. Then $b(x)=I_1+I_2+I_3+I_4+J_1+J_2+J_3+J_4=\sum_k(I_{1k}+I_{2k}+I_{3k}+I_{4k}+J_{1k}+J_{2k}+J_{3k}+J_{4k})$ and each $I_{\ell k}$ and $J_{\ell k}$ are given as follows:

$$I_{1k} = -p_k(x)\frac{k_2}{2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\phi(y_1,0)dy_1,$$

$$J_{1k} = -q_k(x)\frac{k_2}{2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\varphi(y_1,0)dy_1,$$

$$I_{2k} = p_k(x)\frac{k_1\pi}{L}(-1)^{k_1}\int_0^{2\pi}\sin\frac{k_2}{2}y_2\cdot\phi(L,y_2)dy_2,$$

$$J_{2k} = q_k(x)\frac{k_1\pi}{L}(-1)^{k_1}\int_0^{2\pi}\sin\frac{k_2}{2}y_2\cdot\varphi(L,y_2)dy_2,$$

$$I_{3k} = p_k(x)\frac{k_2}{2}(-1)^{k_2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\phi(y_1,2\pi)dy_1,$$

$$J_{3k} = q_k(x)\frac{k_2}{2}(-1)^{k_2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\varphi(y_1,2\pi)dy_1,$$

$$I_{4k} = -p_k(x)\frac{k_1\pi}{L}\int_0^{2\pi} \sin\frac{k_2}{2}y_2\cdot\phi(0,y_2)dy_2,$$

$$J_{4k} = -q_k(x)\frac{k_1\pi}{L}\int_0^{2\pi} \sin\frac{k_2}{2}y_2\cdot\varphi(0,y_2)dy_2.$$

Now we give the algorithm to compute $\triangle f_*(x)$.

We immediately have

$$\begin{aligned}
\triangle p_k(x) &= \frac{-\triangle u_k(x)}{(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})\frac{\pi L}{2}} = \frac{u_k(x)}{\frac{\pi L}{2}} = -p_k(x)(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2}),\\
\triangle q_k(x) &= \frac{\triangle u_k(x)}{(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})^2\frac{\pi L}{2}} = \frac{-u_k(x)}{(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2})\frac{\pi L}{2}} = p_k(x)\\
&= -q_k(x)(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2}).
\end{aligned}$$

Hence,

$$\triangle_x a_j(x) = \sum_k \triangle q_k(x)q_k(x_j)\frac{\pi L}{2} = \sum_k p_k(x)q_k(x_j)\frac{\pi L}{2}.$$

Since for $x\in\Omega$

$$\begin{aligned}
\triangle_x I(x) &= \int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_x\triangle_y G(y,x)\cdot\phi(y)dy\\
&= \int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_x^2 G(y,x)\cdot\phi(y)dy = 0,
\end{aligned}$$

$$\begin{aligned}
\triangle_x b(x) &= \triangle_x I(x)+\triangle_x J(x)\\
&= -\sum_k (J_{1k}+J_{2k}+J_{3k}+J_{4k})(\frac{k_1^2\pi^2}{L^2}+\frac{k_2^2}{2^2}).
\end{aligned}$$

Therefore

$$\triangle f_*(x) = \sum_{j=1}^N c_j\triangle_x a_j(x)+\triangle_x b(x)$$

can be easily calculated.

Department of Mathematics, Department of mathematics, Graduate School of Science, Hokkaido University, Sapporo, 060-0810 JAPAN
E-mail address: gnaka@math.sci.hokudai.ac.jp

Department of Mathematics, Department of mathematics, Graduate School of Science, Hokkaido University, Sapporo, 060-0810 JAPAN
E-mail address: wang@math.sci.hokudai.ac.jp

Department of Mathematics, Fudan University, Shanghai, 200433 P.R.CHINA
E-mail address: ybwang@fudan.edu.cn