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ON A DECAY PROPERTY OF SOLUTIONS TO THE HARAUX-WEISSLER EQUATION

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Dedicated to Professor Takahiko Nakazi on the occasion of his sixtieth birthday

ABSTRACT. We give a sufficient condition that non-radial H^1-solutions to the Haraux-Weissler equation should belong to the weighted Sobolev space $H^1_\rho(\mathbb{R}^n)$, where ρ is the weight function $\exp(|x|^2/4)$. Our result provides, in some sense, a connection between the solutions obtained by ODE method and those by variational approach in the space $H^1_\rho(\mathbb{R}^n)$.

1. INTRODUCTION

In this paper we study an asymptotic behavior at infinity of solutions to the Haraux-Weissler equation

$$-\Delta u - \frac{1}{2}x\cdot\nabla u - \frac{1}{p-1}u = |u|^{p-1}u, \tag{1.1}$$

where u is a complex-valued function on $\mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n$, $p>1$, and $n\geq 1$.

The purpose in this paper is to present weighted L^2 and H^1 estimates of solutions to (1.1) with $p > 1+4/n$ under a smallness assumption on solutions at infinity. We will give a sufficient condition that non-positive and non-radial solutions in $H^1(\mathbb{R}^n)$ of (1.1) belong to the weighted Sobolev space $H^1_\rho(\mathbb{R}^n)$

$$H^1_\rho(\mathbb{R}^n) := \left\{ v:\mathbb{R}^n\to\mathbb{C}\ ;\ \int_{\mathbb{R}^n}(|v|^2+|\nabla v|^2)\rho(x)dx<\infty \right\}, \tag{1.2}$$

where $\rho(x)=\exp(|x|^2/4)$.

Equation (1.1) was introduced by Haraux and Weissler [6] in the study of (forward) self-similar solutions of the semilinear heat equation

$$\partial_t w = \Delta w + w^p. \tag{1.3}$$

Equation (1.3) has a special scaling invariance in the sense that w is a solution if and only if w_λ, defined by

$$w_\lambda(x,t) = \lambda^{2/(p-1)}w(\lambda x,\lambda^2 t), \tag{1.4}$$

is a solution for some (equivalently, all) $\lambda>0$. A solution w is said to be self-similar if $w_\lambda = w$ for all $\lambda>0$. We see that w is a self-similar solution to (1.3) if and only if w has

the form

$$w(x,t) = t^{-1/(p-1)}u(x/\sqrt{t}),$$

where u satisfies (1.1).

We summarize basic results on solutions to (1.1). First, let us consider the radially symmetric case, where we are interested in the structure of C^2-solutions of the following ordinary differential equation

$$\begin{cases} u'' + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right)u' + \dfrac{1}{p-1}u + |u|^{p-1}u = 0, \\ u'(0) = 0, \qquad u(0) = \alpha, \end{cases} \tag{1.5}$$

where $r = |x| > 0$. The problem (1.5) has been analyzed extensively in [6, 10, 12, 13]. We recall that for every $\alpha \in \mathbb{R}$, there exists a unique solution $u \in C^2([0,\infty))$ of the problem (1.5) (see [6]). We denote by $u(r;\alpha)$ the unique solution of (1.5). In [6], it is shown that

$$L(\alpha) := \lim_{r\to\infty} r^{2/(p-1)}u(r;\alpha)$$

exists and is finite for every $\alpha \in \mathbb{R}$ and $p > 1$. Moreover, Peletier, Terman and Weissler [10] showed the following result (I) that is valid for solutions with sign changes.

(I)

(i) If $L(\alpha) = 0$, there exists a constant $A \neq 0$ such that

$$u(r;\alpha) = Ae^{-r^2/4}r^{2/(p-1)-n}\{1 + O(r^{-2})\} \quad \text{as} \quad r \to \infty. \tag{1.6}$$

(ii) If $L(\alpha) \neq 0$, then

$$u(r;\alpha) = L(\alpha)r^{-2/(p-1)}\{1 + O(r^{-2})\} \quad \text{as} \quad r \to \infty. \tag{1.7}$$

If the solution $u(r;\alpha)$ stays positive, structure of solutions was already established (see [2, 6, 10, 12, 14]). For the sake of simplicity, we mention the results only for the case $n \geq 3$.

(II)

(i) If $1 < p \leq 1 + 2/n$, then $u(r;\alpha)$ has a zero in $(0,\infty)$ for every $\alpha > 0$.

(ii) If $1 + 2/n < p < (n+2)/(n-2)$, then there exists a unique $\alpha_* > 0$ such that $u(r;\alpha_*)$ is positive on $[0,\infty)$ and $L(\alpha_*) = 0$. For every $\alpha \in (\alpha_*,\infty)$, $u(r;\alpha)$ has a zero in $(0,\infty)$. For every $\alpha \in (0,\alpha_*)$, $u(r;\alpha)$ is positive on $[0,\infty)$ and $L(\alpha) > 0$.

(iii) If $p \geq (n+2)/(n-2)$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and $L(\alpha) > 0$ for every $\alpha > 0$.

Here, we see that if $u \in C^2(\mathbb{R}^n)$ is a radial solution to (1.1) with $L(\alpha) = 0$, then $u \in L^2_\rho(\mathbb{R}^n) = \rho^{-1/2}L^2(\mathbb{R}^n)$ by (1.6) which partially corresponds to the case (II)-(ii). On the other hand, in the case $L(\alpha) \neq 0$, we see that $u \notin L^2_\rho(\mathbb{R}^n)$ and that $u \in L^2(\mathbb{R}^n)$ if and only if $p < 1 + 4/n$.

Next, let us consider the non-radial case, where we are interested in the following problem formulated by Naito and Suzuki: For any $a \in C(S^{n-1}; \mathbb{R}) \setminus \{0\}$, find a solution of (1.1) with

$$\lim_{r\to\infty} r^{2/(p-1)} u(r\omega) = a(\omega), \quad \omega \in S^{n-1}. \tag{1.8}$$

In [9] (see also Naito [8]), the existence of positive solutions of (1.1) with (1.8) was proved under the assumptions that $p > 1 + 2/n$, $0 \le a \le L$, where $L = \lim_{r\to\infty} r^{2/(p-1)} U(r)$ and $U(r)$ is a positive solution of (1.5) with $U'(0) = 0$ and $\lim_{r\to\infty} r^{2/(p-1)} U(r) > 0$.

Also, there is another approach to the existence of solutions to (1.1) on the basis of an equivalent equation

$$\nabla \cdot (\rho \nabla u) + \rho \left(\frac{1}{p-1} u + |u|^{p-1} u \right) = 0$$

and the associated functional

$$\frac{1}{2} \int \left(|\nabla u|^2 - \frac{1}{p-1} |u|^2 \right) \rho dx - \frac{1}{p+1} \int |u|^{p+1} \rho dx$$

defined on $H^1_\rho(\mathbb{R}^n)$. Weissler [13] and Escobedo and Kavian [3] showed that there exist infinitely many solutions to the problem

$$\begin{cases} -\Delta u - \dfrac{1}{2} x \cdot \nabla u - \dfrac{1}{p-1} u = |u|^{p-1} u, & x \in \mathbb{R}^n, \\ u \in H^1_\rho(\mathbb{R}^n) \end{cases} \tag{1.9}$$

if $p > 1$ and $p < (n+2)/(n-2)$ for $n \ge 3$ by variational methods using the compact embedding $H^1_\rho(\mathbb{R}^n) \subset L^2_\rho(\mathbb{R}^n)$. Especially the existence of positive solutions to (1.9) is proved for any p with $p > 1 + 2/n$ and $p < (n+2)/(n-2)$ if $n \ge 3$. Escobedo and Kavian also proved in [3] that solutions of (1.9) are of the class $C^2(\mathbb{R}^n)$ with asymptotic behavior $u(x) = O(\exp(-|x|^2/8))$ as $|x| \to \infty$. This decay is sufficient to apply the result by Naito and Suzuki [9] which says that if $u \in C^2(\mathbb{R}^n)$ is a positive solution of (1.1) satifying

$$u(x) = o(|x|^{-2/(p-1)}) \quad \text{as} \quad |x| \to \infty, \tag{1.10}$$

then u must be radially symmetric about the origin. Therefore we may conclude the uniqueness from the results by Yanagida [14] and Dohmen and Hirose [2].

Remark 1.1. It is known that (1.1) admits a positive solution only if $p > 1 + 2/n$ (see [4, 7, 11, 12, 13]).

Furthermore, it is shown in [9, Lemma 2.1] that if $u \in C^2(\mathbb{R}^n)$ is a positive solution to (1.1) with $n \ge 2$ and $p > 1 + 2/n$ satisfying (1.10), then, for every $m > 0$, $u(x) = o(|x|^{-m})$ as $|x| \to \infty$. However, it is not enough to ensure that $u \in H^1_\rho(\mathbb{R}^n)$. As we have seen above, there remains a gap between ODE approach and variational method. The ODE approach to (1.1)

assumes radial symmetry of solutions, while the variational approach assumes exponential decay of solutions.

The purpose in this paper is to examine the exponential decay of solutions in terms of $H^1_\rho(\mathbb{R}^n)$ in a general setting at the level of $H^1(\mathbb{R}^n)$. In view of (1.6) and (1.7), a reasonable borderline at the level of $L^2(\mathbb{R}^n)$ seems $p = 1 + 4/n$ at least radial and real solutions, since (1.7) implies the existence of L^2-solutions with algebraic decay if $p < 1 + 4/n$, while (1.6) implies that L^2-solutions decaying faster than an algebraic rate have the exponential decay.

The main result in this paper is the following.

Theorem 1. *Let $p > 1 + 4/n$ and let $p < (n+2)/(n-2)$ if $n \geq 3$. Let $u \in H^1(\mathbb{R}^n)$ be a solution to (1.1) satisfying*

$$\sup_{|x|\geq R} |u(x)| \leq \left(\frac{n}{4} - \frac{1}{p-1}\right)^{1/(p-1)} \tag{1.11}$$

for some $R > 0$. Then,

$$\rho(|\nabla u|^2 + |xu|^2 + |u|^2) \in L^1(\mathbb{R}^n).$$

Remark 1.2. H^1-solutions to (1.1) are understood to be functions $u \in H^1(\mathbb{R}^n)$ satisfying

$$\int \nabla u \cdot \nabla v - \frac{1}{2}\int \nabla u \cdot xv - \frac{1}{p-1}\int uv = \int |u|^{p-1}uv$$

for all $v \in H^1(\mathbb{R}^n)$ with $xv \in L^2(\mathbb{R}^n)$.

Remark 1.3. Theorem 1 ensures that H^1-solutions with a smallness assumption (1.11) belong to $H^1_\rho(\mathbb{R}^n)$ and satisfy the additional condition that $xu \in L^2_\rho(\mathbb{R}^n)$ as well.

Remark 1.4. We do not require the solutions to be positive and radially symmetric. We remark that it is easy to see that there is no H^1-solutions if $n \geq 3$ and $p \geq (n+2)/(n-2)$ by using Pohozaev identity (see [3]).

Remark 1.5. Assumption (1.11) holds for instance in the case where u vanishes at infinity, which holds when $n = 1$ or u is radial.

We prove Theorem 1 in Section 2 simply by repeating the integration by parts. We employ a similar method in [5] to prove a sharp exponential decay for solutions to a semilinear elliptic equation arising in the study of standing waves for nonlinear Schrödinger equations. We also refer to the proof by Cazenave [1] which has a relation with the method of proof in [5].

2. Proof of Theorem 1

For $m \geq 1$ and $l \geq 1$, we define ρ_m and ζ_l by

$$\rho_m(x) = \exp\left(\frac{m|x|^2}{4} / \left(m + \frac{|x|^2}{4}\right)\right), \quad \zeta_l(x) = \left(1 + \frac{1}{l}|x|^2\right)^{-1}.$$

Note that $\rho_m \leq \min(e^m, \rho)$ and $\zeta_l \leq \min(1, l/|x|^2)$. Taking the real part of the scalar product of (1.1) with $\rho_m \zeta_l u$, we obtain

$$\mathrm{Re} \int \nabla u \cdot \nabla(\rho_m \zeta_l \bar{u}) - \frac{1}{2}\mathrm{Re} \int \rho_m \zeta_l x\bar{u} \cdot \nabla u - \frac{1}{p-1}\int \rho_m \zeta_l |u|^2 = \int \rho_m \zeta_l |u|^{p+1}. \tag{2.1}$$

The first term on the LHS of (2.1) is equal to

$$\int \rho_m \zeta_l |\nabla u|^2 + \mathrm{Re} \int \frac{m^2}{2\left(m + |x|^2/4\right)^2} \rho_m \zeta_l \bar{u} x \cdot \nabla u - \frac{2}{l}\mathrm{Re} \int \rho_m \zeta_l^2 \bar{u} x \cdot \nabla u,$$

where the second term is estimated as

$$\begin{aligned}
\left|\mathrm{Re} \int \frac{m^2}{2\left(m + |x|^2/4\right)^2} \rho_m \zeta_l \bar{u} x \cdot \nabla u\right| &\leq \int \frac{m}{2} \cdot \frac{1}{m + |x|^2/4} \cdot \frac{m}{m + |x|^2/4} \rho_m \zeta_l |xu||\nabla u| \\
&\leq \int \frac{m}{2} \cdot \frac{1}{m + |x|^2/4} \rho_m \zeta_l |xu||\nabla u| \\
&\leq (1-\varepsilon)\int \rho_m \zeta_l |\nabla u|^2 \\
&\qquad + \frac{1}{1-\varepsilon}\int \frac{m^2}{16\left(m + |x|^2/4\right)^2} \rho_m \zeta_l^2 |xu|^2 \\
&\leq (1-\varepsilon)\int \rho_m \zeta_l |\nabla u|^2 \\
&\qquad + \frac{1}{1-\varepsilon}\int \frac{m^2}{16\left(m + |x|^2/4\right)^2} \rho_m \zeta_l |xu|^2
\end{aligned}$$

for $0 < \varepsilon < 1$ and the third term is estimated from below as

$$\begin{aligned}
-\frac{2}{l}\mathrm{Re} \int \rho_m \zeta_l^2 \bar{u} x \cdot \nabla u &\geq -\frac{2}{l}\int \rho_m \zeta_l^2 |xu||\nabla u| \\
&\geq -\frac{2}{l}\int \rho_m \zeta_l^2 |\nabla u|^2 - \frac{1}{2l}\int \rho_m \zeta_l^2 |xu|^2 \\
&\geq -\frac{2}{l}\int \rho_m \zeta_l |\nabla u|^2 - \frac{1}{2l}\int \rho_m \zeta_l^2 |xu|^2.
\end{aligned}$$

On the other hand, by integration by parts, the second term on the LHS of (2.1) is equal to

$$\begin{aligned}
-\frac{1}{2}\mathrm{Re} \int \rho_m \zeta_l x\bar{u} \cdot \nabla u &= \frac{n}{4}\int \rho_m \zeta_l |u|^2 + \frac{1}{4}\int (x \cdot \nabla \rho_m)\zeta_l |u|^2 + \frac{1}{4}\int (x \cdot \nabla \zeta_l)\rho_m |u|^2 \\
&= \frac{n}{4}\int \rho_m \zeta_l |u|^2 + \int \frac{m^2}{8(m + |x|^2/4)^2} \rho_m \zeta_l |xu|^2 - \frac{1}{2l}\int \rho_m \zeta_l^2 |xu|^2.
\end{aligned}$$

Combining these estimates with (2.1), we obtain

$$\begin{aligned}\left(\varepsilon-\frac{2}{l}\right)\int\rho_m\zeta_l|\nabla u|^2 &+ \int\frac{m^2}{8(m+|x|^2/4)^2}\cdot\frac{1-2\varepsilon}{2(1-\varepsilon)}\rho_m\zeta_l|xu|^2\\ &- \frac{1}{l}\int\rho_m\zeta_l^2|xu|^2+\left(\frac{n}{4}-\frac{1}{p-1}\right)\int\rho_m\zeta_l|u|^2\\ &\le\int\rho_m\zeta_l|u|^{p+1},\end{aligned}\tag{2.2}$$

where

$$\int\frac{m^2}{8(m+|x|^2/4)^2}\cdot\frac{1-2\varepsilon}{2(1-\varepsilon)}\rho_m\zeta_l|xu|^2\ge 0$$

by taking $\varepsilon<1/2$.

We now take $R>0$ as in the assumption (1.11) to estimate the RHS of (2.2) as

$$\int\rho_m\zeta_l|u|^{p+1}\le\rho(R)\int|u|^{p+1}+\left(\frac{n}{4}-\frac{1}{p-1}\right)\int\rho_m\zeta_l|u|^2,$$

where we have used the inequality $\rho_m\le\rho$. Therefore, (2.2) implies

$$\begin{aligned}\left(\varepsilon-\frac{2}{l}\right)\int\rho_m\zeta_l|\nabla u|^2+\int\frac{m^2}{8(m+|x|^2/4)^2}\cdot\frac{1-2\varepsilon}{2(1-\varepsilon)}\rho_m\zeta_l|xu|^2 &- \frac{1}{l}\int\rho_m\zeta_l^2|xu|^2\\ &\le\int\rho(R)|u|^{p+1}.\end{aligned}$$

Since $l^{-1}\zeta_l|x|^2\le 1$, by the Lebesgue dominated convergence theorem, we take the limit $l\to\infty$ of the last inequality to have

$$\varepsilon\int\rho_m|\nabla u|^2+\int\frac{m^2}{8(m+|x|^2/4)^2}\cdot\frac{1-2\varepsilon}{2(1-\varepsilon)}\rho_m|xu|^2\le\int\rho(R)|u|^{p+1}.$$

Moreover, it follows from the monotone convergence theorem that

$$\rho|\nabla u|^2+\rho|xu|^2\in L^1(\mathbb{R}^n).\tag{2.3}$$

Also, noting that

$$\begin{aligned}-\frac{n}{2}\int\rho|u|^2 &= \mathrm{Re}\int\overline{x\rho^{1/2}u}\cdot\nabla(\rho^{1/2}u)\\ &= \mathrm{Re}\int\overline{x\rho^{1/2}u}\cdot\rho^{1/2}\nabla u+\frac{1}{4}\int\rho|xu|^2,\end{aligned}$$

we obtain

$$\frac{n}{2}\int\rho|u|^2\le\|\rho^{1/2}xu\|_2\|\rho^{1/2}\nabla u\|_2,$$

namely,

$$\rho|u|^2\in L^1(\mathbb{R}^n).$$

□

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