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High-dimensional heteroclinic and homoclinic manifolds in odd point-vortex ring on sphere with pole vortices

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Abstract

We consider the motion of the N -vortex points that are equally spaced along a line of latitude on sphere with fixed pole vortices, called “ N -ring”. In particular, we focus on the evolution of the odd unstable N -ring. Since the eigenvalues that determine the stability of the odd N -ring are double, each of the unstable and stable manifolds corresponding to them is two-dimensional manifold. Accordingly, it is generally difficult to describe the global structure of the manifolds. In this article, based on the linear stability analysis, we propose a projection method to show the structure of the iso-surfaces of the Hamiltonian, in which the orbit of the vortex points exist. Then, applying the projection method to the motion of the 3-ring and 5-ring, we discuss the existence of the high-dimensional homoclinic and heteroclinic connections in the phase space, which characterize the evolution of the unstable N -ring.

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1 Introduction

In the mathematical study of fluid motions on Earth, we often assume that the incompressible and inviscid flow is confined to the surface of the sphere. Since the vorticity is conserved along the path of a fluid particle like two-dimensional flows, it is sufficient to consider the coherent local regions where the vorticity exists at the initial moment. The following element we need to introduce is the effect of rotation of the sphere. However, because of the Coriolis force due to the rotation, not the vorticity but the potential vorticity becomes the conserved quantity[1]. Accordingly, it is insufficient to consider only the coherent initial vortex structure, since the vorticity generates and disappears everywhere in the sphere during the evolution. Thus, in this article, instead of dealing with the Coriolis force directly, we incorporate two vortex points fixed at the poles of the sphere as an effect of rotation, and then investigate the evolution of coherent vortex structures. This is a simple model for the fluid motion on the sphere with the rotating effect, to which the analytic techniques for the two-dimensional flows are available.

One of the coherent vortex structures is a vortex sheet, which is a discontinuous surface of the velocity field. Numerical study of the vortex sheet on the sphere with

the pole vortices[20] indicates that it evolves into a structure with many rolling-up spirals whose centers are arranged equally along a line of latitude and that the number of the rolling-up spirals depends on the strengths of the pole vortices. In order to describe further long time evolution of the vortex sheet, by concentrating all the circulation contained in each of the spirals in its center point, we consider the motion of the vortex points that are equally spaced along the line of latitude. This configuration is called “ N -ring” configuration. Generally speaking, since each of the spirals has different size, we need to consider the motion of N -ring with various strengths. However, for the sake of simplicity, the strengths of all the vortex points are assumed to be identical, say Γ , in the present paper.

Let (Θ_m, Ψ_m) denote the position of the m th vortex point in the spherical coordinates. The strengths of the north and the south pole vortices are denoted by Γ_1 and Γ_2 respectively. Then the equations of the N -vortex points on the sphere with the pole vortices are given by

$$\dot{\Theta}_m = -\frac{\Gamma}{4\pi} \sum_{j \neq m}^N \frac{\sin \Theta_j \sin(\Psi_m - \Psi_j)}{1 - \cos \gamma_{mj}}, \quad (1)$$

$$\begin{aligned} \sin \Theta_m \dot{\Psi}_m &= -\frac{\Gamma}{4\pi} \sum_{j \neq m}^N \frac{\cos \Theta_m \sin \Theta_j \cos(\Psi_m - \Psi_j) - \sin \Theta_m \cos \Theta_j}{1 - \cos \gamma_{mj}} \\ &\quad + \frac{\Gamma_1}{4\pi} \frac{\sin \Theta_m}{1 - \cos \Theta_m} - \frac{\Gamma_2}{4\pi} \frac{\sin \Theta_m}{1 + \cos \Theta_m}, \quad m = 1, 2, \dots, N, \end{aligned} \quad (2)$$

in which γ_{mj} represents the central angle between the m th and the j th vortex points, and

$$\cos \gamma_{mj} = \cos \Theta_m \cos \Theta_j + \sin \Theta_m \sin \Theta_j \cos(\Psi_m - \Psi_j).$$

The equations define a Hamiltonian dynamical system in the phase space $\mathbb{P}_N \equiv [0, \pi]^N \times (\mathbb{R}/2\pi\mathbb{Z})^N \subset \mathbb{R}^{2N}$ [10, 16], whose Hamiltonian is given by

$$\begin{aligned} H &= -\frac{\Gamma^2}{8\pi} \sum_{m=1}^N \sum_{j \neq m}^N \log(1 - \cos \gamma_{mj}) \\ &\quad - \frac{\Gamma_1 \Gamma}{4\pi} \sum_{m=1}^N \log(1 - \cos \Theta_m) - \frac{\Gamma_2 \Gamma}{4\pi} \sum_{m=1}^N \log(1 + \cos \Theta_m). \end{aligned} \quad (3)$$

Note that the system has the invariant $\sum_{m=1}^N \cos \Theta_m$ due to the symmetry with respect to the rotation around the pole.

The N -ring configuration is represented by

$$\Theta_m = \theta_0, \quad \Psi_m = \frac{2\pi m}{N}, \quad m = 1, 2, \dots, N. \quad (4)$$

The stability of the N -ring on the sphere has been studied in many papers[2, 3, 4, 11, 14, 17, 21]. The periodic orbits observed in the N -vortex points problem have become an important theme recently[13, 21, 23]. Soulié and Tokieda[23] and Laurent-Polz[13] gave algebraic methods to find periodic orbits and heteroclinic connections by reducing the system with its symmetry. In the present article, we give another reduction method based on the linear stability analysis in order to show the existence of high-dimensional heteroclinic and homoclinic connections embedded in the phase space $\mathbb{P}_N$.

Now, we review the results of the paper [21], which are required to describe our method. Substituting (4) into the equations (1) and (2), we obtain

$$\dot{\Theta}_m = 0, \quad \dot{\Psi}_m = V_0(N),$$

in which

$$V_0(N) = \frac{\Gamma_1 - \Gamma_2}{4\pi \sin^2 \theta_0} + \frac{(\Gamma_1 + \Gamma_2 + 2\pi) \cos \theta_0}{4\pi \sin^2 \theta_0} - \frac{1}{2N} \frac{\cos \theta_0}{\sin^2 \theta_0}.$$

Hence, it is a steady solution in the spherical coordinates rotating in the longitudinal direction with the constant speed $V_0(N)$. When we perturb the steady solution,

$$\Theta_m(t) = \theta_0 + \epsilon \theta_m(t), \quad \Psi_m(t) = \frac{2\pi m}{N} + V_0(N)t + \epsilon \varphi_m(t), \quad \epsilon \ll 1, \quad (5)$$

then we have the linearized equations of $O(\epsilon)$ for the perturbations:

$$\dot{\theta}_m = \frac{1}{2N \sin \theta_0} \sum_{j \neq m}^N \frac{\varphi_m - \varphi_j}{1 - \cos \frac{2\pi}{N}(m-j)}, \quad (6)$$

$$\dot{\varphi}_m = \frac{1}{2N \sin^3 \theta_0} \sum_{j \neq m}^N \frac{\theta_m - \theta_j}{1 - \cos \frac{2\pi}{N}(m-j)} + B_N \theta_m. \quad (7)$$

The parameter B_N is represented by

$$B_N = \frac{1 + \cos^2 \theta_0}{2N \sin^3 \theta_0} - \frac{\kappa_1(1 + \cos^2 \theta_0)}{2 \sin^3 \theta_0} - \frac{\kappa_2 \cos \theta_0}{2 \sin^3 \theta_0}. \quad (8)$$

in which κ_1 and κ_2 are the equivalent parameters to Γ_1 and Γ_2 defined by

$$\kappa_1 = \frac{\Gamma_1 + \Gamma_2 + 2\pi}{2\pi}, \quad \kappa_2 = \frac{\Gamma_1 - \Gamma_2}{\pi}.$$

The eigenvalues of the linearized equations (6) and (7) are given explicitly as follows[21].

Theorem 1. For $p = 0, 1, \dots, N-1$, the eigenvalues $\lambda_p^\pm$ are represented by

$$\lambda_p^\pm = \pm \sqrt{\xi_p \eta_p}, \quad (9)$$

in which

$$\xi_p = \frac{p(N-p)}{2N \sin \theta_0}, \quad \eta_p = \frac{p(N-p)}{2N \sin^3 \theta_0} + B_N.$$

The expression (9) indicates that $\lambda_0^\pm = 0$ and $\lambda_p^\pm = \lambda_{N-p}^\pm$. Hence, when $N = 2M$, there exist zero eigenvalues $\lambda_0^\pm = 0$, double eigenvalues $\lambda_p^\pm$ for $p = 1, \dots, M-1$, and simple eigenvalues $\lambda_M^\pm$. When $N = 2M + 1$, we have zero eigenvalues $\lambda_0^\pm = 0$ and double eigenvalues $\lambda_p^\pm$ for $p = 1, \dots, M$. Hence, the multiplicity of the largest eigenvalues is different.

In addition, the theorem also provides the stability condition for the eigenvalues.

Proposition 2. For $p = 1, 2, \dots, M$, the eigenvalues $\lambda_p^\pm$ are neutrally stable if $\eta_p < 0$, or equivalently

$$\frac{pN - p^2}{N} + \frac{1 + \cos^2 \theta_0}{N} < \kappa_1(1 + \cos^2 \theta_0) + \kappa_2 \cos \theta_0 \equiv \kappa. \quad (10)$$

Since the eigenvalues satisfy the following order due to (9),

$$(\lambda_1^\pm)^2 < (\lambda_2^\pm)^2 < \cdots < (\lambda_M^\pm)^2, \quad (11)$$

the stability of the N -ring is determined by that of the largest eigenvalues.

As was studied in [21], when the number of the vortex points is even, each of the simple largest eigenvalues $\lambda_M^\pm$ has just one corresponding eigenvector, which allows us to reduce the equations (1) and (2) to an integrable system in the two-dimensional space spanned by the eigenvectors, and thus to study the global motion of the perturbed unstable N -ring. However, it is the double eigenvalue $\lambda_M^\pm$ that becomes unstable when the odd N -ring loses its stability. Hence, the unstable manifold corresponding to $\lambda_M^\pm$ has the two-dimensional tangent space, and thus the motion of the perturbed N -ring could be more complicated for odd case. Actually, numerical computation of the 3-ring at the equator[22] pointed out the existence of a high-dimensional heteroclinic structure in the phase space resulting in a non-trivial recurrent evolution. In the present paper, we confirm the numerical computation by a projection method.

This paper consists of five sections. In §2, we introduce a Hamiltonian projection method, which makes it possible to project the iso-surface of the Hamiltonian on two-dimensional spaces spanned by the eigenvectors corresponding to the pair of the unstable and stable eigenvalues. Applying the projection method to the 3-ring and the 5-ring problems in §3 and §4 respectively, we show the existence of high-dimensional homoclinic and heteroclinic manifolds. Conclusion is given in the last section.

2 A Hamiltonian projection method

Our projection method starts with the explicit representations of the eigenvectors corresponding to $\lambda_p^\pm$, whose proof is given in [21].

Theorem 3. *The eigenvectors $\vec{\phi}_p^\pm$ and $\vec{\psi}_p^\pm$ corresponding to the eigenvalues $\lambda_p^\pm$ for $p = 0, \dots, M$ are given by*

$$\begin{aligned} \vec{\psi}_p^\pm = & {}^t \left(\sqrt{\xi_p}, \sqrt{\xi_p} \cos \frac{2\pi}{N} p, \dots, \sqrt{\xi_p} \cos \frac{2\pi}{N} (N-1)p, \right. \\ & \left. \pm \sqrt{\eta_p}, \pm \sqrt{\eta_p} \cos \frac{2\pi}{N} p, \dots, \pm \sqrt{\eta_p} \cos \frac{2\pi}{N} (N-1)p \right), \end{aligned} \quad (12)$$

$$\begin{aligned} \vec{\phi}_p^\pm = & {}^t \left(0, \sqrt{\xi_p} \sin \frac{2\pi}{N} p, \dots, \sqrt{\xi_p} \sin \frac{2\pi}{N} (N-1)p, \right. \\ & \left. 0, \pm \sqrt{\eta_p} \sin \frac{2\pi}{N} p, \dots, \pm \sqrt{\eta_p} \sin \frac{2\pi}{N} (N-1)p \right). \end{aligned} \quad (13)$$

Noting that the eigenvectors have pure imaginary components if the eigenvalues $\lambda_p^\pm$ are neutrally stable, namely $\eta_p < 0$, we define the inner product of the two vectors $\vec{x} = {}^t (x_1, x_2, \dots, x_N)$ and $\vec{y} = {}^t (y_1, y_2, \dots, y_N)$ by

$$(\vec{x}, \vec{y}) \equiv \sum_{m=1}^N x_m y_m^*,$$

in which y^* denotes the complex conjugate of y . Then, we have the following lemma.

Lemma 4. *The eigenvalues $\vec{\psi}_p^\pm$ and $\vec{\phi}_p^\pm$ satisfy*

$$\begin{aligned} (\vec{\psi}_p^\pm, \vec{\phi}_q^\pm) &= 0, & (\vec{\psi}_p^\pm, \vec{\psi}_q^\pm) &= (\vec{\phi}_p^\pm, \vec{\phi}_q^\pm) = 0 \text{ for } p \neq q, \\ (\vec{\psi}_p^+, \vec{\psi}_p^-) &= (\vec{\phi}_p^+, \vec{\phi}_p^-) = \frac{N}{2}(\xi_p - |\eta_p|), & |\vec{\psi}_p^\pm|^2 &= |\vec{\phi}_p^\pm|^2 = \frac{N}{2}(\xi_p + |\eta_p|). \end{aligned}$$

Proof: It follows from the formula $\sum_{m=0}^{N-1} \omega_p^m = \frac{1-\omega_p^N}{1-\omega_p} = 0$ for $\omega_p = e^{\frac{2\pi i p}{N}}$ that

$$\sum_{m=0}^{N-1} \cos \frac{2\pi m p}{N} = \sum_{m=1}^{N-1} \sin \frac{2\pi m p}{N} = 0, \quad p \in \mathbb{Z}. \quad (14)$$

Hence, we have

$$\begin{aligned} (\vec{\psi}_p^+, \vec{\phi}_q^\pm) &= (\sqrt{\xi_p \xi_q} \pm \sqrt{\eta_p} \sqrt{\eta_q^*}) \sum_{m=1}^{N-1} \cos \frac{2\pi m p}{N} \sin \frac{2\pi m q}{N} \\ &= \frac{1}{2} (\sqrt{\xi_p \xi_q} \pm \sqrt{\eta_p} \sqrt{\eta_q^*}) \sum_{m=1}^{N-1} \left(\sin \frac{2\pi m(p+q)}{N} - \sin \frac{2\pi m(p-q)}{N} \right) = 0. \end{aligned}$$

and for $p \neq q$

$$\begin{aligned} (\vec{\psi}_p^+, \vec{\psi}_q^\pm) &= (\sqrt{\xi_p \xi_q} \pm \sqrt{\eta_p} \sqrt{\eta_q^*}) \sum_{m=0}^{N-1} \cos \frac{2\pi m p}{N} \cos \frac{2\pi m q}{N} \\ &= \frac{1}{2} (\sqrt{\xi_p \xi_q} \pm \sqrt{\eta_p} \sqrt{\eta_q^*}) \sum_{m=0}^{N-1} \left(\cos \frac{2\pi m(p+q)}{N} + \cos \frac{2\pi m(p-q)}{N} \right) = 0. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} (\vec{\psi}_p^+, \vec{\psi}_p^\pm) &= (\xi_p \pm |\eta_p|) \sum_{m=0}^{N-1} \cos^2 \frac{2\pi m p}{N} \\ &= \frac{1}{2} (\xi_p \pm |\eta_p|) \sum_{m=0}^{N-1} \left(\cos \frac{4\pi m p}{N} + 1 \right) = \frac{N}{2} (\xi_p \pm |\eta_p|). \end{aligned}$$

In the same way, $(\vec{\phi}_p^+, \vec{\phi}_p^\pm) = \frac{N}{2} (\xi_p \pm |\eta_p|)$. $\square$

In what follows, we assume that the number of the vortex points is odd. Then, since the eigenvectors $\vec{\psi}_p^\pm$ and $\vec{\phi}_p^\pm$ for $p = 1, \dots, M$ are linearly independent, they form the basis of the $4M$ -dimensional subspace of $\mathbb{R}^{2N}$. Basic idea of the Hamiltonian projection method is to express the motion of the N -ring by the linear combination of the eigenvectors, so two more vectors are required, which are simply provided in the following lemma. The proof is straightforward in view of (14).

Lemma 5. *Let $\vec{\zeta}^\pm$ be defined by*

$$\vec{\zeta}^\pm = \frac{1}{\sqrt{2N}} {}^t (1, 1, \dots, 1, \pm 1, \pm 1, \dots, \pm 1).$$

Then, they satisfy $(\vec{\psi}_p^\pm, \vec{\zeta}^\pm) = 0$, $(\vec{\phi}_p^\pm, \vec{\zeta}^\pm) = 0$ and $(\vec{\zeta}^+, \vec{\zeta}^-) = 0$.

Now, let the position of the vortex points $\vec{x} = (\Theta_1, \Theta_2, \dots, \Theta_N, \Psi_1, \Psi_2, \dots, \Psi_N) \in \mathbb{P}_N$ be represented by the following linear combination of the eigenvectors and the complementary vectors.

$$\vec{x} = \vec{x}_0 + \sum_{p=1}^M (a_p \vec{\psi}_p^+ + b_p \vec{\psi}_p^- + c_p \vec{\phi}_p^+ + d_p \vec{\phi}_p^-) + e \vec{\zeta}^+ + f \vec{\zeta}^-, \quad (15)$$

in which $a_p, b_p, c_p, d_p, e, f \in \mathbb{C}$, and $\vec{x}_0$ denotes the steady odd N -ring configuration, which is represented by

$$\vec{x}_0 = \left(\underbrace{\theta_0, \theta_0, \dots, \theta_0}_N, 0, \underbrace{\frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2\pi M}{N}}_M, \underbrace{-\frac{2\pi M}{N}, \dots, -\frac{4\pi}{N}, -\frac{2\pi}{N}}_M \right). \quad (16)$$

Then, it follows from Lemma 4 and Lemma 5 that the coefficients are represented by

$$\begin{aligned} a_p &= \frac{1}{2N\xi_p} (\vec{x} - \vec{x}_0, \vec{\psi}_p^+ + \vec{\psi}_p^-) + \frac{1}{2N|\eta_p|} (\vec{x} - \vec{x}_0, \vec{\psi}_p^+ - \vec{\psi}_p^-), \\ b_p &= \frac{1}{2N\xi_p} (\vec{x} - \vec{x}_0, \vec{\psi}_p^+ + \vec{\psi}_p^-) - \frac{1}{2N|\eta_p|} (\vec{x} - \vec{x}_0, \vec{\psi}_p^+ - \vec{\psi}_p^-), \\ c_p &= \frac{1}{2N\xi_p} (\vec{x} - \vec{x}_0, \vec{\phi}_p^+ + \vec{\phi}_p^-) + \frac{1}{2N|\eta_p|} (\vec{x} - \vec{x}_0, \vec{\phi}_p^+ - \vec{\phi}_p^-), \\ d_p &= \frac{1}{2N\xi_p} (\vec{x} - \vec{x}_0, \vec{\phi}_p^+ + \vec{\phi}_p^-) - \frac{1}{2N|\eta_p|} (\vec{x} - \vec{x}_0, \vec{\phi}_p^+ - \vec{\phi}_p^-), \\ e &= (\vec{x} - \vec{x}_0, \vec{\zeta}^+), \quad f = (\vec{x} - \vec{x}_0, \vec{\zeta}^-). \end{aligned} \quad (17)$$

Therefore, the constraint conditions for the projection of the evolution of the vortex points on the two-dimensional phase spaces spanned by $\vec{\psi}_p^\pm$, $\vec{\phi}_p^\pm$ and $\vec{\zeta}^\pm$ are equivalent to

$$\begin{aligned} \vec{\psi}_p^\pm : \quad & a_q = b_q = 0 \ (q \neq p), \quad c_q = d_q = 0 \ (\forall q), \quad e = f = 0, \\ \vec{\phi}_p^\pm : \quad & a_q = b_q = 0 \ (\forall q), \quad c_q = d_q = 0 \ (q \neq p), \quad e = f = 0, \\ \vec{\zeta}^\pm : \quad & a_q = b_q = 0 \ (\forall q), \quad c_q = d_q = 0 \ (\forall q). \end{aligned} \quad (18)$$

Substituting (12), (13) and (16) into (18), we have the constraint conditions for the projection.

Proposition 6. *The motion of the odd N -ring projected on the two-dimensional phase space spanned by $\vec{\psi}_p^\pm$ satisfies the following constraint conditions;*

$$\begin{aligned} \sum_{k=1}^N \Theta_k \cos \frac{2\pi q}{N} (k-1) &= 0, \quad \sum_{k=1}^N \Psi_k \cos \frac{2\pi q}{N} (k-1) = 0, & (q \neq p), \\ \sum_{k=2}^N \Theta_k \sin \frac{2\pi q}{N} (k-1) &= 0, \quad \sum_{k=2}^N \Psi_k \sin \frac{2\pi q}{N} (k-1) = 2 \sum_{k=1}^M \frac{2\pi}{N} k \sin \frac{2\pi q}{N} k, & (\forall q), \\ \sum_{k=1}^N \Theta_k &= N\theta_0, \quad \sum_{k=1}^N \Psi_k = 0. \end{aligned} \quad (19)$$

Proposition 7. *The motion of the odd N -ring projected on the two-dimensional*

phase space spanned by $\vec{\phi}_p^\pm$ satisfies the following constraint conditions;

$$\begin{aligned}
\sum_{k=1}^N \Theta_k \cos \frac{2\pi q}{N}(k-1) &= 0, & \sum_{k=1}^N \Psi_k \cos \frac{2\pi q}{N}(k-1) &= 0, & (\forall q), \\
\sum_{k=2}^N \Theta_k \sin \frac{2\pi q}{N}(k-1) &= 0, & \sum_{k=2}^N \Psi_k \sin \frac{2\pi q}{N}(k-1) &= 2 \sum_{k=1}^M \frac{2\pi}{N} k \sin \frac{2\pi q}{N} k, & (q \neq p), \\
\sum_{k=1}^N \Theta_k &= N\theta_0, & \sum_{k=1}^N \Psi_k &= 0.
\end{aligned} \tag{20}$$

Proposition 8. *The motion of the odd N -ring projected on the two-dimensional phase space spanned by $\vec{\zeta}^\pm$ satisfies the following constraint conditions; For $1 \leq q \leq M$, they are*

$$\begin{aligned}
\sum_{k=1}^N \Theta_k \cos \frac{2\pi q}{N}(k-1) &= 0, & \sum_{k=1}^N \Psi_k \cos \frac{2\pi q}{N}(k-1) &= 0, \\
\sum_{k=2}^N \Theta_k \sin \frac{2\pi q}{N}(k-1) &= 0, & \sum_{k=2}^N \Psi_k \sin \frac{2\pi q}{N}(k-1) &= 2 \sum_{k=1}^M \frac{2\pi}{N} k \sin \frac{2\pi q}{N} k.
\end{aligned} \tag{21}$$

Since the perturbed N -ring evolves in the iso-surface of the Hamiltonian embedded in $\mathbb{P}_N$ and hardly grows in the directions of the neutrally stable eigenvectors, we project the iso-surface of the Hamiltonian on the planar phase space spanned by the eigenvectors corresponding to the pair of the unstable and stable eigenvalues with the above constraint conditions. Then we observe the topological structure of the unstable and the stable manifolds in the projected spaces. Note that if $(\lambda_j^\pm)^2 > 0 > (\lambda_{j-1}^\pm)^2$ holds for an integer j , then the unstable and stable manifolds are embedded in the $4(M-j+1)$ -dimensional phase space spanned by $\vec{\psi}_p^\pm$ and $\vec{\phi}_p^\pm$ for $p = j, \dots, M$ due to (11). In the following sections, we apply the Hamiltonian projection method to the 3-ring and 5-ring to show the existence of high-dimensional heteroclinic and homoclinic connections in the iso-surfaces of the Hamiltonian.

Finally, we mention the projection of the iso-surface of the Hamiltonian on the special subspace spanned by $\vec{\zeta}^\pm$. It is easy to rewrite the constraint conditions in Proposition 8 in simpler form. The proof is straightforward due to (14).

Corollary 9. *The constraint conditions (21) are equivalent to*

$$\Theta_1 = \Theta_2 = \dots = \Theta_N, \tag{22}$$

$$\Psi_j = \begin{cases} \Psi_1 + \frac{2\pi}{N}(j-1), & 2 \leq j \leq M, \\ \Psi_1 - \frac{2\pi}{N}(N-j), & M+1 \leq j \leq N. \end{cases} \tag{23}$$

Imposing the constraint conditions (22) on the Hamiltonian (3), we have the projected Hamiltonian $H_{\vec{\zeta}^\pm}$:

$$\begin{aligned}
H_{\vec{\zeta}^\pm} &= -\frac{\Gamma^2}{8\pi} \sum_{m=1}^N \sum_{j \neq m}^N \log(1 - \cos^2 \Theta_1 - \sin^2 \Theta_1 \cos(\Psi_m - \Psi_j)) \\
&\quad - \frac{\Gamma_1 \Gamma}{4\pi} N \log(1 - \cos \Theta_1) - \frac{\Gamma_2 \Gamma}{4\pi} N \log(1 + \cos \Theta_1).
\end{aligned}$$

Since $\Psi_m - \Psi_j$ is constant from (23), the projected Hamiltonian is independent of the variables Ψ_m . Hence, the contour of the projected Hamiltonian is equivalent to

a straight line, $\Theta_1 = \text{Constant}$, on which the orbit of the vortex points is projected. In addition, the constraint conditions (22) and (23) indicate that when the orbit of the vortex points intersects the $\vec{\zeta}^\pm$ -plane, the vortex points form the N -ring configuration. In other words, if the distance between the orbit and the $\vec{\zeta}^\pm$ -plane, defined by

$$\begin{aligned} L_\zeta^{(N)} &= \left| \vec{x} - \vec{x}_0 - e\vec{\zeta}^+ - f\vec{\zeta}^- \right| \\ &= \sqrt{\frac{N}{2} \sum_{p=1}^M \{ \xi_p (|a_p + b_p^*|^2 + |c_p + d_p^*|^2) + |\eta_p| (|a_p - b_p^*|^2 + |c_p - d_p^*|^2) \}}, \end{aligned}$$

vanishes, the configuration of the vortex points becomes the N -ring.

3 Heteroclinic manifold in the motion of 3-ring

Then $N = 3$, the constraint conditions in Proposition 6 and Proposition 7 are equivalent to

$$\vec{\psi}_1^\pm : \begin{cases} \Theta_2 - \Theta_3 &= 0, \\ \Theta_1 + \Theta_2 + \Theta_3 &= 3\theta_0, \\ \Psi_2 - \Psi_3 &= \frac{4}{3}\pi, \\ \Psi_1 + \Psi_2 + \Psi_3 &= 0, \end{cases} \quad \vec{\phi}_1^\pm : \begin{cases} 2\Theta_1 - \Theta_2 - \Theta_3 &= 0, \\ \Theta_1 + \Theta_2 + \Theta_3 &= 3\theta_0, \\ 2\Psi_1 - \Psi_2 - \Psi_3 &= 0, \\ \Psi_1 + \Psi_2 + \Psi_3 &= 0. \end{cases}$$

Solving the equations, we have the relations between the variables as follows.

$$\vec{\psi}_1^\pm : \begin{cases} \Theta_1 = 3\theta_0 - 2\Theta_2, & \Theta_3 = \Theta_2, \\ \Psi_1 = -2\Psi_2 + \frac{4}{3}\pi, & \Psi_3 = \Psi_2 - \frac{4}{3}\pi, \end{cases} \quad \vec{\phi}_1^\pm : \begin{cases} \Theta_1 = \theta_0, & \Theta_3 = 2\theta_0 - \Theta_2, \\ \Psi_1 = 0, & \Psi_3 = -\Psi_2. \end{cases}$$

Substituting these relations into (3), we have the following projected Hamiltonians;

$$\begin{aligned} H_{\vec{\psi}_1^\pm} &= -\frac{\Gamma^2}{4\pi} \log \left(1 - \cos(3\theta_0 - 2\Theta_2) \cos \Theta_2 - \sin(3\theta_0 - 2\Theta_2) \sin \Theta_2 \cos \left(3\Psi_2 - \frac{4}{3}\pi \right) \right) \\ &\quad -\frac{\Gamma^2}{4\pi} \log \left(1 - \cos(3\theta_0 - 2\Theta_2) \cos \Theta_2 - \sin(3\theta_0 - 2\Theta_2) \sin \Theta_2 \cos \left(3\Psi_2 - \frac{2}{3}\pi \right) \right) \\ &\quad -\frac{\Gamma^2}{4\pi} \log \left(1 - \cos^2 \Theta_2 + \frac{1}{2} \sin^2 \Theta_2 \right) \\ &\quad -\frac{\Gamma_1 \Gamma}{4\pi} \log(1 - \cos(3\theta_0 - 2\Theta_2))(1 - \cos \Theta_2)^2 \\ &\quad -\frac{\Gamma_2 \Gamma}{4\pi} \log(1 + \cos(3\theta_0 - 2\Theta_2))(1 + \cos \Theta_2)^2, \end{aligned}$$

and

$$\begin{aligned} H_{\vec{\phi}_1^\pm} &= -\frac{\Gamma^2}{4\pi} \log(1 - \cos \theta_0 \cos \Theta_2 - \sin \theta_0 \sin \Theta_2 \cos \Psi_2) \\ &\quad -\frac{\Gamma^2}{4\pi} \log(1 - \cos \theta_0 \cos(2\theta_0 - \Theta_2) - \sin \theta_0 \sin(2\theta_0 - \Theta_2) \cos \Psi_2) \\ &\quad -\frac{\Gamma^2}{4\pi} \log(1 - \cos \Theta_2 \cos(2\theta_0 - \Theta_2) - \sin \Theta_2 \sin(2\theta_0 - \Theta_2) \cos 2\Psi_2) \\ &\quad -\frac{\Gamma_1 \Gamma}{4\pi} \log(1 - \cos \theta_0)(1 - \cos \Theta_2)(1 - \cos(2\theta_0 - \Theta_2)) \\ &\quad -\frac{\Gamma_2 \Gamma}{4\pi} \log(1 + \cos \theta_0)(1 + \cos \Theta_2)(1 + \cos(2\theta_0 - \Theta_2)). \end{aligned}$$

We plot the contour lines of $H_{\vec{\psi}_1^\pm}$ and $H_{\vec{\phi}_1^\pm}$ in the (Ψ_2, Θ_2) -plane to see the global structure of the iso-surfaces of the Hamiltonian. When plotting the contours, we must note that the variable Θ_2 moves in the range of $[0, \pi] \cap [\frac{1}{2}(3\theta_0 - \pi), \frac{3}{2}\theta_0]$ for $H_{\vec{\psi}_1^\pm}$, and $[0, \pi] \cap [2\theta_0 - \pi, 2\theta_0]$ for $H_{\vec{\phi}_1^\pm}$ respectively, due to $0 \leq \Theta_1, \Theta_2, \Theta_3 \leq \pi$.

According to Proposition 2, the stability of the largest eigenvalues $\lambda_1^\pm$ changes at $\kappa = 1 + \frac{1}{3}\cos^2\theta_0$. Hence, we show the contour plot of the projected Hamiltonians for the unstable case, $\kappa < 1 + \frac{1}{3}\cos^2\theta_0$. Figure 1 shows the contour plots of $H_{\vec{\psi}_1^\pm}$ and $H_{\vec{\phi}_1^\pm}$ for $(\Gamma_1, \Gamma_2) = (-0.2\pi, -0.2\pi)$ for the 3-ring at the equator, $\theta_0 = \frac{\pi}{2}$. In Figure 1 (a) and (b), the unstable and the stable manifolds departing from $(\Psi_2, \Theta_2) = (\frac{2}{3}\pi, \frac{\pi}{2})$, which corresponds to the original 3-ring configuration $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{2}{3}\pi, -\frac{2}{3}\pi) \in \mathbb{P}_3$, connect to another mirror symmetric 3-ring configuration $(\Psi_2, \Theta_2) = (\frac{4}{3}\pi, \frac{\pi}{2})$, namely $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, -\frac{2}{3}\pi, \frac{2}{3}\pi)$. Hence, the unstable and the stable manifolds have the heteroclinic connection. This confirms the existence of the heteroclinic structure for the 3-ring at the equator indicated numerically in [22].

In order to observe the evolution of the perturbed 3-ring configuration quantitatively, we consider the distance $L_\zeta^{(3)}$. Since $\eta_1 > 0$ when the 3-ring is linearly unstable, all the coefficients a_1, b_1, c_1 and d_1 are real. Hence, the distance $L_\zeta^{(3)}$ is explicitly given by

$$\begin{aligned} L_\zeta^{(3)} &= \sqrt{\frac{3}{2}\xi_1((a_1 + b_1)^2 + (c_1 + d_1)^2) + \frac{3}{2}\eta_1((a_1 - b_1)^2 + (c_1 - d_1)^2)} \\ &= \sqrt{\frac{(2\Theta_1 - \Theta_2 - \Theta_3)^2}{6} + \frac{(\Theta_2 - \Theta_3)^2}{2} + \frac{(2\Psi_1 - \Psi_2 - \Psi_3)^2}{6} + \frac{(\Psi_2 - \Psi_3 - \frac{4}{3}\pi)^2}{2}}. \end{aligned}$$

Here, instead of $L_\zeta^{(3)}$, we introduce the following distance $l_\zeta^{(3)}$:

$$l_\zeta^{(3)} = \sqrt{\frac{(2\Theta_1 - \Theta_2 - \Theta_3)^2}{6} + \frac{(\Theta_2 - \Theta_3)^2}{2} + \frac{1}{6}\sin^2\frac{2\Psi_1 - \Psi_2 - \Psi_3}{4} + \frac{1}{2}\sin^2\frac{\Psi_2 - \Psi_3 - \frac{4}{3}\pi}{4}}.$$

If $l_\zeta^{(3)} = 0$, then $2\Theta_1 - \Theta_2 - \Theta_3 = \Theta_2 - \Theta_3 = 0$, $2\Psi_1 - \Psi_2 - \Psi_3 = 4n\pi$ and $\Psi_2 - \Psi_3 - \frac{4}{3}\pi = 4m\pi$ for $m, n \in \mathbb{Z}$, which leads to $L_\zeta^{(3)} = 0$ because Ψ_i is 2π -periodic. On the other hand, when the vortex points form the mirror symmetric 3-ring configuration, we have $l_\zeta^{(3)} = \frac{\sqrt{6}}{4}$. Therefore, if $l_\zeta^{(3)} = 0$ or $\frac{\sqrt{6}}{4}$, the three vortex points form the 3-ring configuration.

The initial configuration of the three vortex points are given by

$$(\Theta_1(0), \Theta_2(0), \Theta_3(0), \Psi_1(0), \Psi_2(0), \Psi_3(0)) = \vec{x}_0 + \epsilon(\mu\vec{\psi}_1^+ + (1 - \mu)\vec{\phi}_1^+), \quad (24)$$

in which $\epsilon = 10^{-5}$ is the small amplitude of perturbation. The parameter μ ($0 \leq \mu \leq 1$) denotes the ratio of the initial unstable direction, which is required since the unstable manifold is tangent to the plane spanned by $\vec{\phi}_1^+$ and $\vec{\psi}_1^+$.

We show the evolutions of the perturbed 3-ring for various μ . First, Figure 2 (a) shows the evolutions of $\cos\Theta_1$ and $\cos\Theta_2$ when the initial unstable direction is $\mu = 0.4$. Note that $\cos\Theta_3$ is automatically determined due to the invariant $\sum_{m=1}^3 \cos\Theta_m = 0$. They show periodic behavior and enter in the vicinity of the zero-line periodically, at which the three vortices are located at the equator. On the other hand, the evolution of the distance $l_\zeta^{(3)}$ in Figure 2 (b) shows that it approaches the zero when both $\cos\Theta_1$ and $\cos\Theta_2$ get into the neighborhood of the zero line. Hence, the three vortex points return to the original 3-ring configuration periodically.

Second, we plot in Figure 3 the evolutions of (a) $\cos\Theta_1$ and $\cos\Theta_2$, and (b) the distances $l_\zeta^{(3)}$ for $\mu = 1.0$. The evolutions of $\cos\Theta_1$ and $\cos\Theta_2$ repeat not periodic

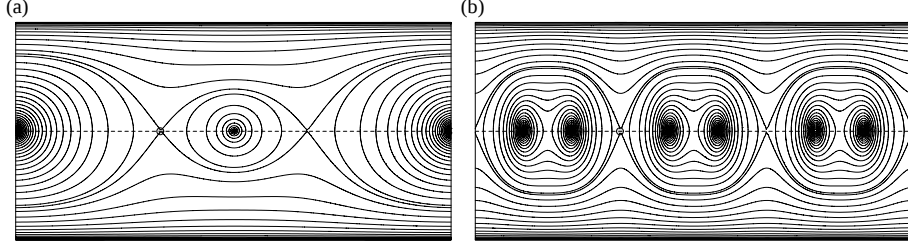


Figure 1: Contour plots of the projected Hamiltonians (a) $H_{\vec{\phi}_1^\pm}$ and (b) $H_{\vec{\psi}_1^\pm}$ in (Ψ_2, Θ_2) -plane, when the 3-ring is located at the equator for $(\Gamma_1, \Gamma_2) = (-0.2\pi, -0.2\pi)$. The circle in the figures denotes the steady 3-ring.

but similar pattern, and the distance $l_\zeta^{(3)}$ passes in the neighborhood of 0 or $\frac{\sqrt{6}}{4}$ when both $\cos \Theta_1$ and $\cos \Theta_2$ enter in the vicinity of the zero-line. Therefore, the three vortex points evolve recurrently in the sense that they regularly form either of the original 3-ring or the mirror symmetric 3-ring.

The two examples above demonstrate that the iso-surface of the Hamiltonian has the heteroclinic connection. On the other hand, since the orbit of the three vortex points is a one-dimensional curve in the neighborhood of the four dimensional heteroclinic manifold, it does not necessarily return to the 3-ring configuration regularly. Actually, as we see in Figure 4 showing the evolutions of (a) $\cos \Theta_1$ and $\cos \Theta_2$, and (b) the distance $l_\zeta^{(3)}$ for $\mu = 0.7$, the distance $l_\zeta^{(3)}$ occasionally approaches near zero or $\frac{\sqrt{6}}{4}$, but their evolutions look irregular. Hence, whether or not the perturbed 3-ring evolves regularly depends on the initial direction of the disturbance.

The heteroclinic manifold is still observed when the 3-ring is located at another latitude. Figure 5 shows the contour plots of (a) $H_{\vec{\phi}_1^\pm}$ and (b) $H_{\vec{\psi}_1^\pm}$, and (c) the evolution of the distance $l_\zeta^{(3)}$ with the initial direction $\mu = 0.2$ when the 3-ring is located at $\theta_0 = \frac{\pi}{4}$ and the strengths of the pole vortices are given by $(\Gamma_1, \Gamma_2) = (-0.3\pi, -0.4\pi)$. It clearly indicates that the perturbed 3-ring evolves almost periodically with returning to the 3-ring configuration, which supports the existence of the heteroclinic connection.

4 Homoclinic manifold in the motion 5-ring

When $N = 5$, solving the constraint conditions in Proposition 6 and 7, we have the following relations between the variables:

$$\vec{\psi}_2^\pm : \begin{cases} \Theta_1 = \sqrt{5}\theta_0 + (1 - \sqrt{5})\Theta_2, \\ \Theta_3 = \frac{1}{2}(5 - \sqrt{5})\theta_0 - \frac{1}{2}(3 - \sqrt{5})\Theta_2, \\ \Theta_4 = \frac{1}{2}(5 - \sqrt{5})\theta_0 - \frac{1}{2}(3 - \sqrt{5})\Theta_2, \\ \Theta_5 = \Theta_2, \\ \Psi_1 = (1 - \sqrt{5})(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_3 = \frac{4}{5}\pi - \frac{1}{2}(3 - \sqrt{5})(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_4 = -\frac{4}{5}\pi - \frac{1}{2}(3 - \sqrt{5})(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_5 = \Psi_2 - \frac{4}{5}\pi, \end{cases} \quad \vec{\psi}_1^\pm : \begin{cases} \Theta_1 = -\sqrt{5}\theta_0 + (1 - \sqrt{5})\Theta_2, \\ \Theta_3 = \frac{1}{2}(5 + \sqrt{5})\theta_0 - \frac{1}{2}(3 + \sqrt{5})\Theta_2, \\ \Theta_4 = \frac{1}{2}(5 + \sqrt{5})\theta_0 - \frac{1}{2}(3 + \sqrt{5})\Theta_2, \\ \Theta_5 = \Theta_2, \\ \Psi_1 = (1 + \sqrt{5})(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_3 = \frac{4}{5}\pi - \frac{1}{2}(3 + \sqrt{5})(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_4 = -\frac{4}{5}\pi - \frac{1}{2}(3 + \sqrt{5})(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_5 = \Psi_2 - \frac{4}{5}\pi, \end{cases}$$

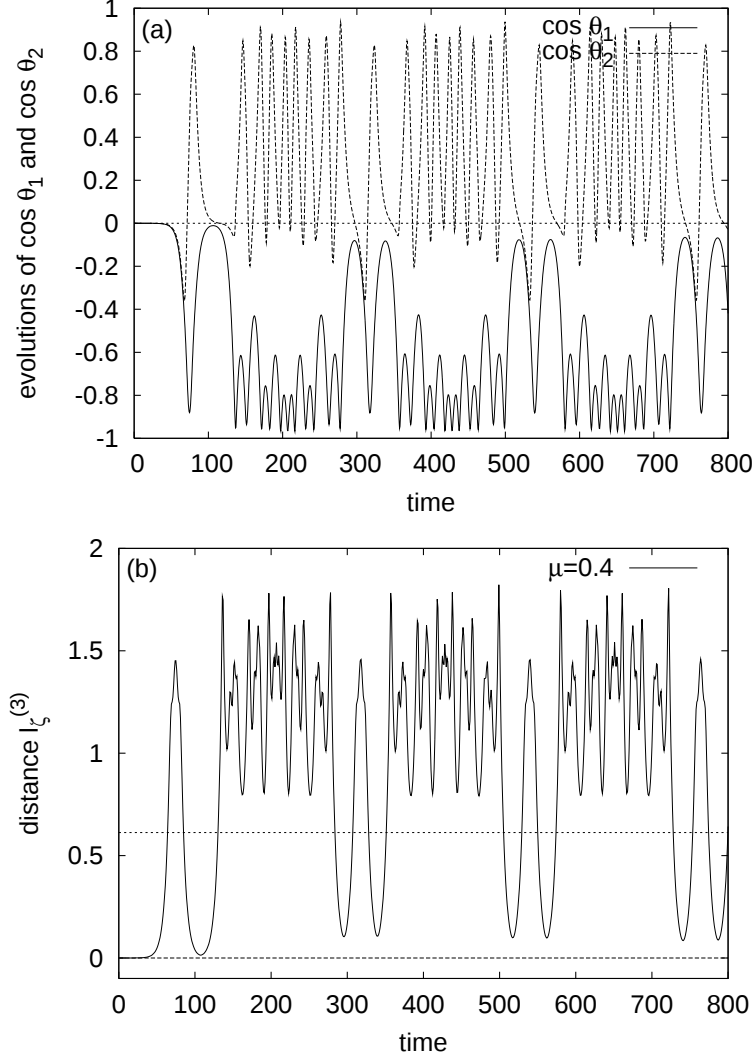


Figure 2: Evolutions of (a) $\cos \Theta_1$ and $\cos \Theta_2$ and (b) the distance $l_\zeta^{(3)}$ for the perturbed unstable 3-ring at the equator when $(\Gamma_1, \Gamma_2) = (-0.2\pi, -0.2\pi)$. The initial unstable direction is $\mu = 0.4$. The distance approaches zero periodically.

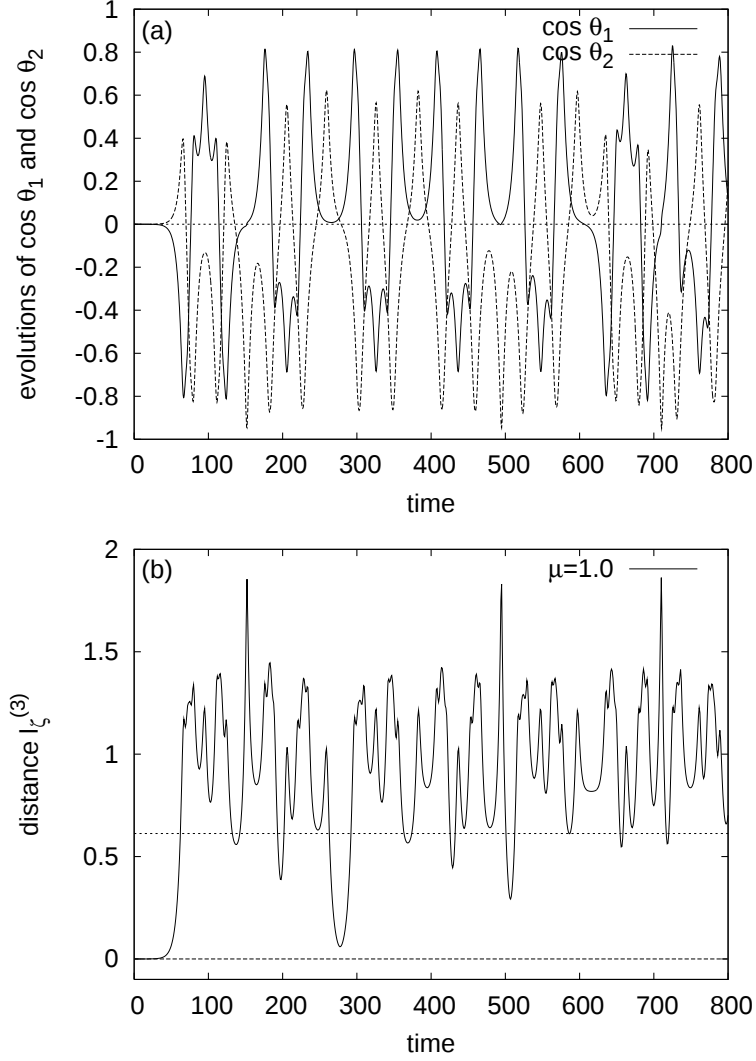


Figure 3: Evolutions of (a) $\cos \Theta_1$ and $\cos \Theta_2$ and (b) the distance $l_\zeta^{(3)}$ for the perturbed unstable 3-ring at the equator when $(\Gamma_1, \Gamma_2) = (-0.2\pi, -0.2\pi)$. The initial unstable direction is $\mu = 1.0$. The evolutions repeat the similar patten and the distance becomes 0 or $\frac{\sqrt{6}}{4}$ when both of the evolutions pass in the neighborhood of the zero-line.

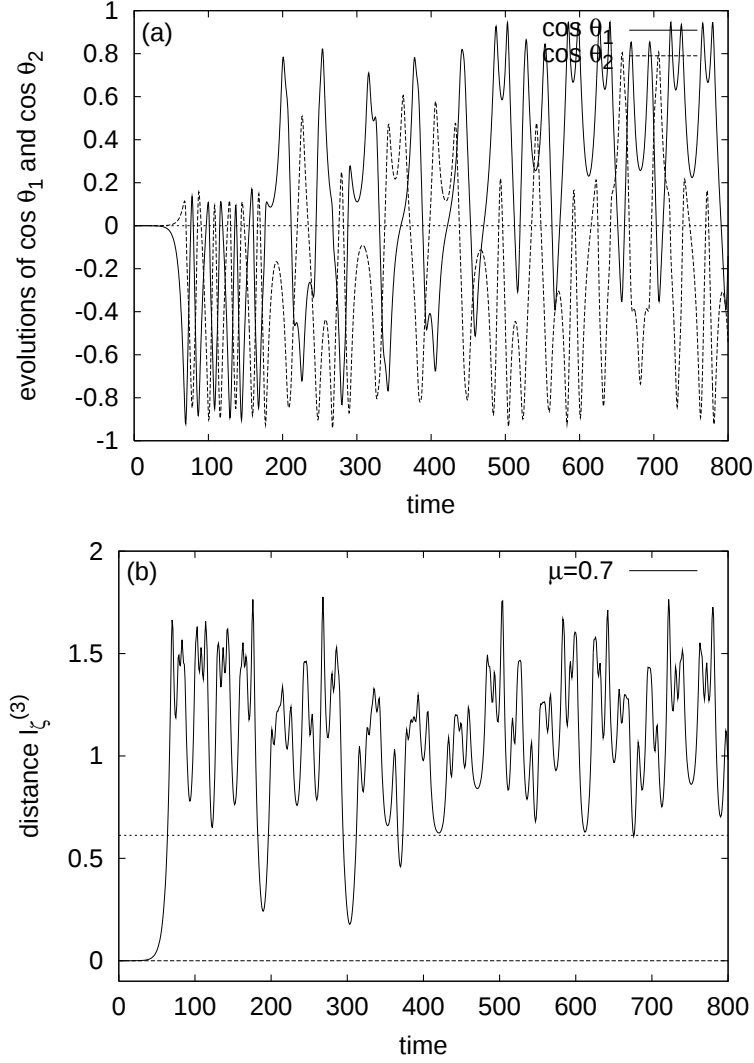


Figure 4: Evolutions of (a) $\cos \Theta_1$ and $\cos \Theta_2$ and (b) the distance $l_\zeta^{(3)}$ for the perturbed unstable 3-ring at the equator when $(\Gamma_1, \Gamma_2) = (-0.2\pi, -0.2\pi)$. The initial unstable direction is $\mu = 0.7$. The distance occasionally approaches 0 or $\frac{\sqrt{6}}{4}$, but the evolutions look irregular.

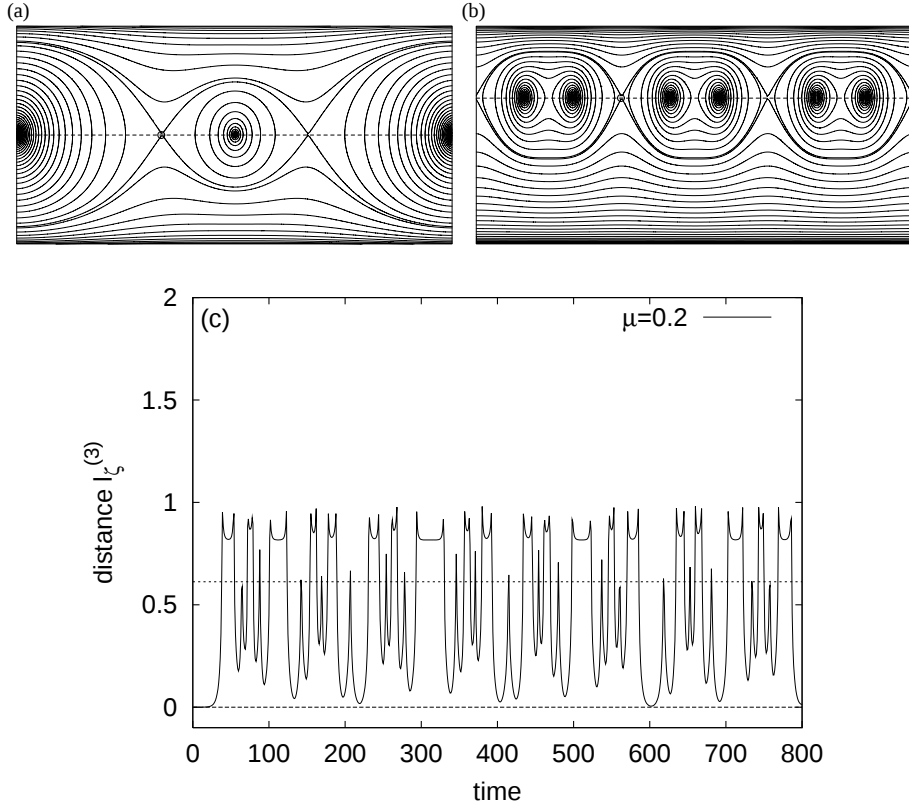


Figure 5: Contour plots of the projected Hamiltonians (a) $H_{\vec{\phi}_1^\pm}$ and (b) $H_{\vec{\psi}_1^\pm}$ in (Ψ_2, Θ_2) -plane, when the unstable 3-ring is located at $\theta_0 = \frac{\pi}{4}$ and $(\Gamma_1, \Gamma_2) = (-0.3\pi, -0.4\pi)$. The circles in the figures denote the steady 3-ring. (c) Evolution of the distance $l_\zeta^{(3)}$ for $\mu = 0.2$.

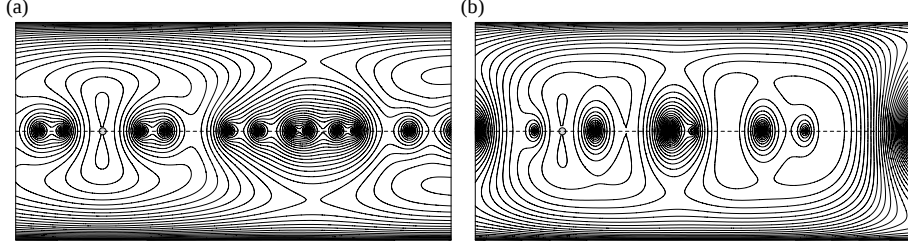


Figure 6: Contour plots of the projected Hamiltonians (a) $H_{\vec{\phi}_2^\pm}$ and (b) $H_{\vec{\psi}_2^\pm}$ in (Ψ_2, Θ_2) - plane, when the 5-ring is located at the equator for $(\Gamma_1, \Gamma_2) = (0.2\pi, 0.2\pi)$. The circles in the figures denote the steady 5-ring.

and

$$\vec{\phi}_2^\pm : \begin{cases} \Theta_1 = \theta_0, \\ \Theta_3 = \theta_0 - \frac{1}{2}(\sqrt{5} + 1)(\Theta_2 - \theta_0), \\ \Theta_4 = \theta_0 + \frac{1}{2}(\sqrt{5} + 1)(\Theta_2 - \theta_0), \\ \Theta_5 = 2\theta_0 - \Theta_2, \\ \Psi_1 = 0, \\ \Psi_3 = \frac{4}{5}\pi - \frac{1}{2}(\sqrt{5} + 1)(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_4 = -\frac{4}{5}\pi + \frac{1}{2}(\sqrt{5} + 1)(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_5 = -\Psi_2, \end{cases} \quad \vec{\phi}_1^\pm : \begin{cases} \Theta_1 = \theta_0, \\ \Theta_3 = \theta_0 + \frac{1}{2}(\sqrt{5} - 1)(\Theta_2 - \theta_0), \\ \Theta_4 = \theta_0 - \frac{1}{2}(\sqrt{5} - 1)(\Theta_2 - \theta_0), \\ \Theta_5 = 2\theta_0 - \Theta_2, \\ \Psi_1 = 0, \\ \Psi_3 = \frac{4}{5}\pi + \frac{1}{2}(\sqrt{5} - 1)(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_4 = -\frac{4}{5}\pi - \frac{1}{2}(\sqrt{5} - 1)(\Psi_2 - \frac{2}{5}\pi), \\ \Psi_5 = -\Psi_2. \end{cases}$$

Substituting them into the Hamiltonian (3), we obtain the projected Hamiltonians $H_{\vec{\psi}_2^\pm}$, $H_{\vec{\psi}_1^\pm}$, $H_{\vec{\phi}_2^\pm}$ and $H_{\vec{\phi}_1^\pm}$. Linear stability analysis for $N = 5$ indicates that the largest eigenvalue λ_2^+ becomes unstable at $\kappa = \frac{1}{5}(7 + \cos^2 \theta_0)$ and λ_1^+ becomes unstable at $\kappa = \frac{1}{5}(5 + \cos^2 \theta_0)$. Thus we plot the contour lines of the projected Hamiltonians $H_{\vec{\psi}_2^\pm}$ and $H_{\vec{\phi}_2^\pm}$ in (Θ_2, Ψ_2) -plane when only the largest eigenvalue λ_2^+ is unstable, namely $\frac{1}{5}(5 + \cos^2 \theta_0) < \kappa < \frac{1}{5}(7 + \cos^2 \theta_0)$. On the other hand, when both of the eigenvalues λ_2^+ and λ_1^+ are unstable, i.e. $\frac{1}{5}(5 + \cos^2 \theta_0) > \kappa$, we plot the contour lines of $H_{\vec{\psi}_2^\pm}$, $H_{\vec{\phi}_2^\pm}$, $H_{\vec{\psi}_1^\pm}$ and $H_{\vec{\phi}_1^\pm}$. We note again that we have to restrict the range of Θ_2 by using the constraint conditions, due to $0 \leq \Theta_i \leq \pi$ for all $i = 1, \dots, 5$.

Figure 6 shows the contour plot of the projected Hamiltonians $H_{\vec{\phi}_2^\pm}$ and $H_{\vec{\psi}_2^\pm}$ in (Θ_2, Ψ_2) -plane for $(\Gamma_1, \Gamma_2) = (0.2\pi, 0.2\pi)$, in which the largest eigenvalue λ_2^+ becomes unstable and $\lambda_1^\pm$ are neutrally stable. The projected unstable and stable manifolds departing from the 5-ring return to the original position, which indicates that the iso-surface of the Hamiltonian has the homoclinic connection in $\mathbb{P}_5$.

When both of the eigenvalues λ_1^+ and λ_2^+ become unstable, we plot the iso-surface of the projected Hamiltonians (a) $H_{\vec{\phi}_2^\pm}$, (b) $H_{\vec{\psi}_2^\pm}$, (c) $H_{\vec{\phi}_1^\pm}$ and (d) $H_{\vec{\psi}_1^\pm}$ in Figure 7. In this case, the unstable and the stable manifolds no longer correspond to each other, nor do they connect to another 5-ring configurations. Hence, there exist no heteroclinic and homoclinic connections in the phase space, which indicates that the perturbed 5-ring never returns to the 5-ring configuration.

5 Summary and remark

We have investigated the motion of the N -ring on the sphere in the presence of the pole vortices. Since linear stability analysis of the N -ring gave the explicit representation of the eigenvectors $\lambda_p^\pm$ and their linearly independent eigenvectors $\psi_p^\pm$

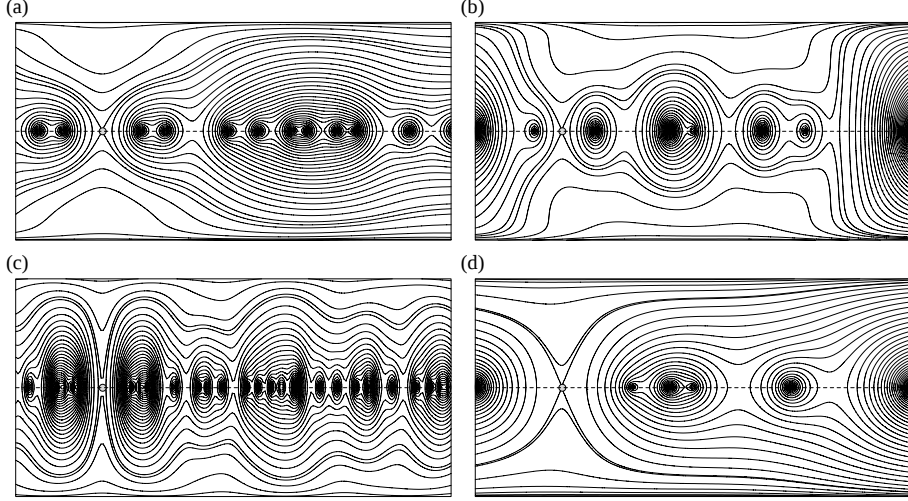


Figure 7: Contour plots of the projected Hamiltonians (a) $H_{\vec{\phi}_2^\pm}$, (b) $H_{\vec{\psi}_2^\pm}$, $H_{\vec{\phi}_1^\pm}$ and $H_{\vec{\psi}_1^\pm}$ in (Ψ_2, Θ_2) -plane, when the 5-ring is located at the equator for $(\Gamma_1, \Gamma_2) = (-0.1\pi, -0.1\pi)$. The circles in the figures denote the steady 5-ring.

and $\vec{\phi}_p^\pm$ for $p = 1, \dots, M$, it is possible to express the orbit of the vortex points in the phase space $\mathbb{P}_N$ as the linear combination of the eigenvectors. The linear combination leads us to obtain the projection conditions, with which we can reduce the $2N$ -dimensional dynamical system to the collection of the two-dimensional systems spanned by the pair of the unstable and stable eigenvectors. Thus projecting the iso-surface of the Hamiltonian to the restricted phase spaces, we observe the global structure of the Hamiltonian, in which the orbit of the vortex points exists. Applying the projection method to the 3-ring, we find the iso-surface of the 3-ring becomes the four-dimensional heteroclinic manifold. Accordingly, when the unstable 3-ring is perturbed slightly, it evolves in the iso-surface near the heteroclinic structure, which results in the non-trivial recurrent evolution as we have observed in the numerical computation [22]. However, since the co-dimension of the orbit in the heteroclinic manifold is high, the unstable 3-ring does not always show the recurrent or periodic evolution. It can be irregular depending on the initial direction of the perturbation. In the similar manner, we apply the projection method to the 5-ring and obtain the existence of the homoclinic connection.

Finally, we remark the application of the projection method to the case of even vortex points which has already been studied in [21]. As we stated in the introduction, the multiplicity of the largest eigenvalues $\lambda_M^\pm$ is different from the odd case. That is to say, the largest eigenvalues $\lambda_M^\pm$ have just one eigenvectors $\vec{\psi}_M^\pm (= \vec{\phi}_M^\pm)$. In [21], we successfully reduced the $2N$ -dimensional dynamical system to the planar system spanned by $\vec{\psi}_M^\pm$ from the alternately pairing symmetry that the eigenvectors have. On the other hand, the orbit of the vortex points $\vec{x}(t)$ can be similarly expressed by the linear combination of the eigenvectors $\vec{\phi}_p^\pm$, $\vec{\psi}_p^\pm$ for $p = 1, \dots, M-1$, $\vec{\psi}_M^\pm$ and the complementary vectors $\vec{\zeta}^\pm$ as follows:

$$\vec{x} = \vec{x}_0 + a_M \vec{\psi}_M^+ + b_M \vec{\psi}_M^- + \sum_{p=1}^{M-1} (a_p \vec{\psi}_p^+ + b_p \vec{\psi}_p^- + c_p \vec{\phi}_p^+ + d_p \vec{\phi}_p^-) + e \vec{\zeta}^+ + f \vec{\zeta}^-.$$

The constraint conditions to project the orbit on the two-dimensional subspace

spanned by $\vec{\psi}_M^\pm$ are provided by solving $a_p = b_p = c_p = d_p = 0$ for $p = 1, \dots, M-1$, and $e = f = 0$. Thus, in principle, the Hamiltonian projection method is also applicable to the case of even vortex points.

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