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HORO-TIGHT IMMERSIONS OF S^1

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Abstract. We characterize horo-tight immersions into $\mathbb{D}^m$ in terms of a family of real valued functions parametrized by $\mathbb{S}^{m-1}$. By means of such functions we provide an elementary proof that horo-tightness and tightness are equivalent properties in the class of immersions from S^1 into hyperbolic space.

What are the horo-tight immersions of spheres? This question was proposed by Thomas E. Cecil and Patrick J. Ryan in [5, pg 236]. We address this paper to the simplest case of that question, namely the horo-tight immersions of a simple closed curve.

The concept of horo-tightness was introduced in [4], whose main subjets are tight and taut immersions into hyperbolic space.

One may regard tightness, tautness and horo-tightness as generalizations of Euclidean tightness. An immersion into $\mathbb{R}^m$ is called Euclidean tight if every non-degenerate linear height function has the minimum number of critical points required by the Morse inequalities. The linear height functions in any Euclidean space are parametrized by the sphere of unit vectors. On the other hand, in a Lorentz space there are three types of spheres, each one formed entirely by spacelike, timelike or lightlike vectors. Such trichotomy gives rise to the concepts of tightness, tautness and horo-tightness respcetively.

In §2 we observe that, in the Poincaré model, the radii of the hyperhorospheres tangent to its boundary at a fixed point is an interesting function. Because we can characterize horo-tightnes by requiring that every non-degenerate such function has the minimum number of critical points.

The main result (Theorem 3.5), state that an immersion into the hyperbolic space of a simple closed curve is horo-tight if and only if it is tight. Then, in codimension greater than 1, tightness, tautness and horo-tightness are all equivalent concepts in the class of such immersions.

1. Notation and definitions

We begin with the definition of perfect functions.

A Morse function φ on a manifold M is called a *perfect function* if

(1) $M_r(\varphi) = \{x \in M \mid \varphi(x) \leqslant r\}$ is compact for all real r,
(2) there exists a field $\mathbf{F}$ such that for all $r \in \mathbb{R}$ and all integers k, the number of critical points of φ of index k which lie in $M_r(\varphi)$ is equal to the kth Betti number of $M_r(\varphi)$ over $\mathbf{F}$.

As observed in [4] there are manifolds which do not admit perfect funcitons.

Let $\mathbb{R}^{m+1} = \{(x_0, x_1, \ldots, x_m) \mid x_i \in \mathbb{R}\}$ be an $(m+1)$-dimensional vector space. For any $x = (x_0, x_1, \ldots, x_m)$ and $y = (y_0, y_1, \ldots, y_m)$ the *Lorentz inner product* of x and y is defined

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by $\mathfrak{b}(x,y) = -x_0y_0 + x_1y_1 + \cdots + x_my_m$. The pair $(\mathbb{R}^{m+1}, \mathfrak{b})$ is called the $(m+1)$-*Lorentz space* and is denoted by $\mathbb{R}^{m+1}_1$. A non-zero vector $x \in \mathbb{R}^{m+1}_1$ is called spacelike, timelike or lightlike if $\mathfrak{b}(x,x) > 0, \mathfrak{b}(x,x) < 0$ or $\mathfrak{b}(x,x) = 0$ respectively.

We use two models of hyperbolic space, $\mathbb{H}^m$ and $\mathbb{D}^m$. The space $\mathbb{H}^m$, a sphere of timelike vectors, is given by

$$\mathbb{H}^m = \{x \in \mathbb{R}^{m+1}_1 \mid \mathfrak{b}(x,x) = -1, x_0 \geqslant 1\},$$

on which $\mathfrak{b}$ restricts to a Riemannian metric of constant sectional curvature -1. Let $\mathbb{D}^m$ be the open unit ball at the origin of $\mathbb{R}^m = \{\mathbf{0}\} \times \mathbb{R}^m$. The stereographic projection $P : \mathbb{D}^m \to \mathbb{H}^m$ given by

$$P(x) = -e_0 + 2\frac{x + e_0}{1 - \mathfrak{b}(x,x)}$$

induces on $\mathbb{D}^m$ a Riemannian metric $\mathfrak{g}$ such that $(\mathbb{D}^m, \mathfrak{g})$ is isometric to $\mathbb{H}^m$. This is the well known Poincaré ball model.

We now define spheres of spacelike and lightlike vectors in $\mathbb{R}^{m+1}_1$: the *de Sitter m-space* $\mathbb{S}^m_1 = \{x \in \mathbb{R}^{m+1}_1 \mid \mathfrak{b}(x,x) = 1\}$ and the *lightcone* $(m-1)$-*sphere* $\mathbb{S}^{m-1}_+ = \{x \in \mathbb{R}^{m+1}_1 \mid \mathfrak{b}(x,x) = 0, x_0 = 1\}$.

An immersion $f : M \to \mathbb{H}^m$ is called *horo-tight* if every singular function $\mathcal{H}_v(x) = -\mathfrak{b}(f(x), v)$, $v \in \mathbb{S}^{m-1}_+$, is either degenerate or a perfect function. The function $L_h(x) = \ln(-\mathfrak{b}(x,v))$, $x \in \mathbb{H}^m$ measures the distance from x to the hyperhorosphere $h = \{x \in \mathbb{H}^m \mid \mathfrak{b}(x,v) = -1\}$.

An immersion $f : M \to \mathbb{H}^m$ is called tight if, for every $\sigma \in \mathbb{S}^m_1$, the function

$$x \mapsto L_\pi(x) = \sinh^{-1}(-\mathfrak{b}(x,\sigma))$$

is either degenerate or a pefect function. Recall that $L_\pi(x)$ is the distance from x to the hiperplane $\pi = \{x \in \mathbb{H}^m \mid \mathfrak{b}(x,\sigma) = 0\}$.

An immersion $f : M \to \mathbb{H}^m$ is called taut if, for every $p \in \mathbb{H}^m - f(M)$, the function

$$x \mapsto L_p(x) = (\cosh^{-1}(-\mathfrak{b}(x,p)))^2$$

is either degenerate or a perfect function. Recall that $L_p(x)$ is the square of the distance in $\mathbb{H}^m$ from p to x.

The above three concepts are invariant under isometries of $\mathbb{H}^m$, this follows from the characterization of such isometries, see for instance [6]. Most of the published work on these subjects are contained in the papers [2, 3, 4]. Next we list some results of [4] which will be used latter. First, we observe from Theorem 2.1 that if $f : M \to \mathbb{D}^m$ is a taut immersion in the Euclidean sense then $P \circ f : M \to \mathbb{H}^m$ is tight and taut. Next, Corollary 4.2 says that an immersion $f : M \to \mathbb{D}^m$ is tight if and only if $\Phi \circ f : M \to \mathbb{D}^m$ is tight in the Euclidean sense for every isometry Φ of $(\mathbb{D}^m, g)$. Finally, for an immersion $f : M \to \mathbb{H}^m$ of a compact manifold, Theorem 5.1 says that tightness or tautness implies horo-tightness.

2. Horo-tightness in $\mathbb{D}^m$

Definition 2.1. An immersion $f : M \to \mathbb{D}^m$ is *horo-tight* if $P \circ f : M \to \mathbb{H}^m$ is horo-tight.

Let $f : M^n \to \mathbb{D}^m$ be an immersion. Let $\mathbb{S}^{m-1}$ denote the sphere of unit vectors at the origin in $\mathbb{R}^m$. For $w \in \mathbb{S}^{m-1}$, let $\mathcal{R}_w : \mathbb{D}^m \to \mathbb{R}$ be the function defined by

$$R_w(p) = 1 - \frac{1 - \langle p,p\rangle}{2(1 - \langle p,w\rangle)} \tag{2.1}$$

where $\langle\,,\rangle$ is the Euclidean inner product. Geometrically $\mathcal{R}_w(p)$ is the radius of the unique hyperhorosphere containing p which is tangent to $\mathbb{S}^{m-1}$ at w. The natural restriction of R_w to M is a real-valued function.

We shall compare the singularities of the functions $\mathcal{R}_w$ and $\mathcal{H}_v$. If $(x_1,\dots,x_n)$ are local coordinates in a neighborhood of $x\in M$, then

$$\frac{\partial}{\partial x_i}\mathcal{R}_w=-\frac{(1-\langle f,f\rangle)\langle f_i,w\rangle-2(1-\langle f,w\rangle)\langle f_i,f\rangle}{2(1-\langle f,w\rangle)^2}$$

where $f_i=\frac{\partial}{\partial x_i}f$, also, for $v=(1,w)\in\mathbb{S}^{m-1}_+$ and for

$$\mathcal{H}_v(x)=-\mathfrak{b}(P\circ f(x),v)=-\frac{2\langle f(x),w\rangle-\langle f(x),f(x)\rangle-1}{1-\langle f(x),f(x)\rangle}$$

we have

$$\frac{\partial}{\partial x_i}\mathcal{H}_v=-2\frac{(1-\langle f,f\rangle)\langle f_i,w\rangle-2(1-\langle f,w\rangle)\langle f_i,f\rangle}{(1-\langle f(x),f(x)\rangle)^2}.$$

Thus $\mathcal{R}_w$ and $\mathcal{H}_v$, $v=(1,w)$, have the same critical points and, at such points, the Hessian of $\mathcal{R}_w$ equals that of $\mathcal{H}_v$ up to a positive scalar factor. Since $M_r(\mathcal{R}_w)=M_{\frac{r}{1-r}}(\mathcal{H}_v)$ for $0<r<1$, we have the following:

Proposition 2.2. *An immersion $f:M\to\mathbb{D}^m$ is horo-tight if and only if every singular $\mathcal{R}_w$ is either degenerate or a perfect function.*

As a direct result we have

Corollary 2.3. *The natural inclusion $i:N\to\mathbb{D}^m$ of any umbilic submanifold N of $(\mathbb{D}^m,\mathfrak{g})$ is a horo-tight embedding.*

3. Horo-tight immersions of S^1

Let S^1 denote a manifold homeomorphic to the unitary circle. In this section we characterize the horo-tight immersions of S^1 into hyperbolic space. As the next lemma says, we can work with embeddings $S^1\hookrightarrow\mathbb{H}^m$.

Lemma 3.1. *Every horo-tight immersion $f:S^1\to\mathbb{H}^m$ is an embedding.*

Proof. Suppose $f(x_1)=f(x_2)=p$ for two distinct points in S^1. Let $\mathbb{R}_w$ be a Morse function such that the hyperhorosphere $h_s=\{y\in\mathbb{H}^m\mid R_w(y)=s\}$ intersects $f(S^1)$ at p and q, $p\neq q$. It follows from the horo-tightness of f that s is a critical value of R_w and therefore R_w has at least three critical points. A contradiction. □

In dimension 1 the two-piece-property with respect to hyperhorospheres may be rephrased as follows:

Lemma 3.2. *Let $f:S^1\to\mathbb{D}^m$ be a horo-tight immersion. Then for any $w\in\mathbb{S}^{m-1}$ either $\mathcal{R}_w$ is constant or the equation $\mathcal{R}_w(x)=c$ has at most two roots.*

We treat separetely the one codimensional case of the main result.

Proposition 3.3. *An immersion $f:S^1\to\mathbb{D}^2$ is horo-tight only if it is tight.*

Proof. Let us assume that $\Phi \circ f$ is not tight in the Euclidean sense for some isometry Φ of $(\mathbb{D}^m, \mathfrak{g})$. It follows that $\alpha = \Phi \circ f$ is a non-convex curve. Hence there exists a tangent line r at some point p such that none of its closed half-planes contains $\Phi \circ f(S^1)$ entirely.

Now let h be a tangent horo-circle of $\Phi \circ f$ at p which passes trough the interior of $\Phi \circ f(S^1)$. See Figure 1. It follows that h intersects $\Phi \circ f(S^1)$ in at least three points. Since h certainly does not contain $\Phi \circ f(S^1)$ we have a contradiction with Lemma 3.2. Therefore $\Phi \circ f$ is tight in the Euclidean sense for every isometry Φ of $(\mathbb{D}^m, \mathfrak{g})$ and hence the result follows from Theorem 4.2 of [4]. □

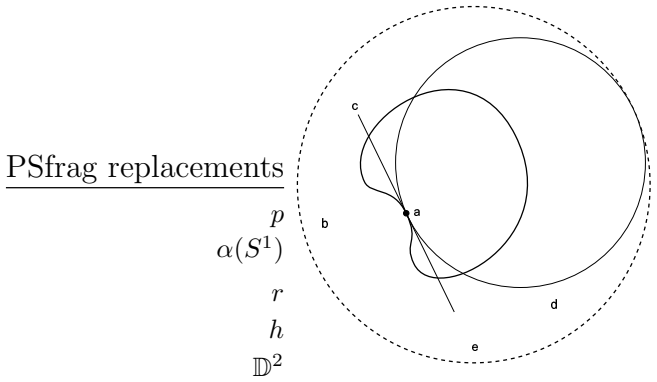

FIGURE 1. Horo-tightness implies convexity.

The second part of the main theorem follows from the lemma below.

Lemma 3.4. *Let $f : S^1 \to \mathbb{D}^m, m \geqslant 3$, be an immersion whose image contains the origin of $\mathbb{R}^m$. Then there exists a hyperhorosphere which intersects $f(S^1)$ in at least three points.*

Proof. We have to find $w \in \mathbb{S}^{m-1}$ and $0 < c < 1$ such that the equation $R_w(x) = c$ has three distinct roots. Since $R_w(\mathbf{0}) = 1/2$ for all $w \in \mathbb{S}^{m-1}$ and $\mathbf{0} \in f(S^1)$, it is sufficient to determine non-zero vectors $f(x), f(y)$ such that the linear system

$$\left\{ \begin{array}{rcl} \langle f(x), w\rangle & = & \langle f(x), f(x)\rangle \\ \langle f(y), w\rangle & = & \langle f(y), f(y)\rangle \end{array} \right. \tag{3.1}$$

has a solution in $\mathbb{S}^{m-1}$. Let $\pi_u \subset \mathbb{R}^m$ be the hyperplane given by the equation $\langle X, u\rangle = \langle u, u\rangle$. For small enough $f(x)$, the hyperplane $\pi_{f(x)}$ contains a non-zero vector $f(y) \neq f(x)$. Thus any $w \in \pi_{f(x)} \cap \pi_{f(y)}$ is a solution of the system (3.1). □

We now proof the main result.

Theorem 3.5. *An immersion $f : S^1 \to \mathbb{D}^m$ is horo-tight if and only if it is tight.*

Proof. We have shown that horo-tightness of $S^1 \hookrightarrow \mathbb{D}^2$ implies tightness, so let us assume $m \geqslant 3$. We will actually prove that f is taut and tight. In view of Theorem 2.1 of [4], it is enough to show that f is taut in the Euclidean sense.

For any point $p \in \mathbb{H}^m$ one can find an isometry $\Phi : \mathbb{H}^m \to \mathbb{H}^m$ such that $\Phi(p) = e_0$, see [6]. Hence we may assume that $\mathbf{0} \in f(S^1)$ and thus, by Lemmas 3.4 and 3.2 there exists a hyperhorosphere h which contains $f(S^1)$. For almost all hyperplanes π in $\mathbb{R}^m$ we have a

hyperhorosphere h' such that $\pi \cap h = h' \cap h$. Since f is horo-tight it is also tight in the Euclidean sense. The result now follows from Theorem 2.1 of [1].

The converse is a particular case of Theorem 5.1 of [4]. □

From the above proof we see that tautness, tightness, and horo-tightness are equivalent concepts in the class of immersions from S^1 into $\mathbb{D}^m, m \geqslant 3$. Whether the above results hold for higher dimensional spheres is an interesting question which may be studied elsewhere.

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