Congruence of Siegel modular forms and special values of their standard zeta functions

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Abstract In this paper, we consider the relation between the special values of the standard zeta functions and the congruence of cuspidal Hecke eigenforms with respect to $Sp_n(\mathbf{Z})$. In particular, we propose a conjecture concerning the congruence of Saito-Kurokawa lift, and prove it under certain condition. Furthermore, we give exact values of the standard zeta function for cuspidal Hecke eigenforms with respect to $Sp_2(\mathbf{Z})$.

1 Introduction

For a cuspidal Hecke eigenform f of weight k with respect to $Sp_n(\mathbf{Z})$, let $L(f, s, \underline{\mathrm{St}})$ be the standard zeta function of f. Let m be a positive integer such that $\rho(n) \leq m \leq k-n$ and $m \equiv n \bmod 2$, where $\rho(n) = 3$, or 1 accorrding as $n \equiv 1 \bmod 4$ and $n \geq 5$, or not. Then the value $\dfrac{L(f, m, \underline{\mathrm{St}})}{< f, f > \pi^{-n(n+1)/2+nk+(n+1)m}}$ belongs to $\mathbf{Q}(f)$ if all the Fourier coefficients of f belong to $\mathbf{Q}(f)$, where $< f, f >$ is the Petersson product and $\mathbf{Q}(f)$ is the field over $\mathbf{Q}$ generated by all Hecke eigenvalues (cf. [B2], [Mi]). In this paper, we consider the relation between the denominator of these values and the congruence of Hecke eigenvalues of cusp forms. This type of problem was first considered by Doi and Hida [D-H] in terms of special values of Rankin-Selberg zeta functions in the elliptic modular case. In Section 5, we give a generalization of their result in terms of the special values of standard zeta functions in the Siegel modular form case (cf. Theorems 5.2 and 5.3). The main tool for proving our main results is the pullback formula for Siegel Eisenstein series due to Böcherer [B1], [B2], which we will review in Section 4. This formula has been already used to prove to algebraicity of the special values of the standard zeta functions stated above. However, to complete the proof of our main results, we have to consider the integrality of the Eisenstein series acted by a certain differential operators. We will discuss this integrality in Sections 2 and 3. Furthermore, to formulate our main result reasonably, we have to consider a normalization of the standard zeta values because we have no normalization of Hecke eigenforms in case $n \geq 2$ unlike the elliptic modular case. We discuss this normalization of the standard zeta values in Section 5. Furthermore, in Section 6, we propose a conjecture concerning the congruence of Saito-Kurokawa lift, and prove it under certain condition. In Section 7, we give an explicit formula for this value in terms of Hecke eigenvalues of f and some other elementary quantities in case $n = 2$. This type of formula has been given in elliptic modular cases in [Ka2]. To get our formula in this paper, we need an explicit form of differential operators on the space of Siegel modular forms of degree 4, and an explicit formula for local Siegel series for a half-integral matrix of degree 4. As for the former, the generating function of the differential operators has been given in [I], and by a direct but rather elaborate computation we can get an explicit form of them. As for the latter, we can compute it by using the explicit formula in [Ka1] in principle. However, in this paper, we show a trick which enables us to reduce the computation of local Siegel series of degree 4 to that of degree 2. Finally, in Section 8, we give some numerical examples which support the conjecture in Section 6.

Notation. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. In particular put $M_n(R) = M_{nn}(R)$. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m = 0$ or $n = 0$. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset $X \bmod aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A \underset{R'}{\sim} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

2 Fourier coefficients of Siegel-Eisenstein series

For a complex number x put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$. For a subring K of $\mathbf{R}$ put

$$GSp_n(K)^+ = \{M \in GL_{2n}(K) \mid J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0\},$$

and

$$Sp_n(K) = \{M \in GSp_n(K)^+ \mid J_n[M] = J_n\},$$

where $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$. Furthermore, put

$$\Gamma^{(n)} = Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbf{R})^+$ and $Z \in \mathbf{H}_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function f on $\mathbf{H}_n$ we define $f|_k M$ as

$$(f|_k M)(Z) = \det(M)^{k/2} j(M, Z)^{-k} f(M(Z)).$$

A function f on $\mathbf{H}_n$ is called a C^∞-modular form of weight k with respect to $\Gamma^{(n)}$ if it satisfies the following conditions:

(i) f is a C^∞-function on $\mathbf{H}_n$;

(ii) $(f|_kM)(Z) = f(Z)$ for any $M \in \Gamma^{(n)}$;

We call a C^∞-modular form f a holomorphic modular form if
(i) f is holomorphic on $\mathbf{H}_n$;
(ii) if $n = 1$, for any $\alpha > 0$, $f(z)$ is bounded on the set $\{x + \sqrt{-1}y \mid y \geq \alpha\}$ for each $\alpha > 0$.

We denote by $\mathfrak{M}_k(\Gamma^{(n)})$ (resp. $\mathfrak{M}_k^\infty(\Gamma^{(n)})$ the space of holomorphic (resp. C^∞-) modular forms of weight k with respect to $\Gamma^{(n)}$. For a modular form f of weight k with respect to $\Gamma^{(n)}$, let

$$f(Z) = \sum_{A \in \mathcal{H}(\mathbf{Z})_{\geq 0}} a_f(A)\mathbf{e}(\mathrm{tr}(AZ)),$$

be the Fourier expansion of $f(Z)$, where tr denotes the trace of a matrix. We call $f(Z)$ a cusp form if $a_f(A) = 0$ unless A is positive-definite. We denote by $\mathfrak{S}_k(\Gamma^{(n)})$ the submodule of $\mathfrak{M}_k(\Gamma^{(n)})$ consisting of cusp forms. Let dv denote the invariant volume element on $\mathbf{H}_n$ defined by $dv = \det(\mathrm{Im}(Z))^{-n-1} \wedge_{1 \leq j \leq l \leq n} (dx_{jl} \wedge dy_{jl})$. Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{jl}) + \sqrt{-1}(y_{jl})$ with real matrices (x_{jl}) and (y_{jl}). For two C^∞-modular forms f and g of weight k with respect to $\Gamma^{(n)}$ we define the Petersson scalar product $< f, g >$ by

$$< f, g >= \int_{\Gamma^{(n)} \backslash \mathbf{H}_n} f(Z)\overline{g(Z)} \det(\mathrm{Im}(Z))^k dv,$$

provided the integral converges.

For a positive integer k we define the Siegel Eisenstein series $E_{n,k}(Z, s)$ of degree n as

$$E_{n,k}(Z, s) = \zeta(1-k-2s) \prod_{i=1}^{[n/2]} \zeta(1-2k-4s+2i) \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} j(M, Z)^{-k} (\det(\mathrm{Im}(M(Z))))^s$$

($Z \in \mathbf{H}_n, s \in \mathbf{C}$), where $\zeta(*)$ is Riemann's zeta function, and $\Gamma_\infty^{(n)} = \{\begin{pmatrix} * & * \\ 0_{n,n} & * \end{pmatrix} \in \Gamma^{(n)}\}$. Then $E_{n,k}(Z, s)$ is holomorphic at $s = 0$ as a function of s, and $E_{n,k}(Z, 0)$ is holomorphic as a function of Z unless $k = (n+2)/2 \equiv 2 \bmod 4$, or $k = (n + 3)/2 \equiv 2 \bmod 4$ (cf. [Sh4], [W]). From now on we assume that $E_{n,k}(Z,0)$ is holomorphic as a function of Z, and write $E_{n,k}(Z) = E_{n,k}(Z,0)$. To see the Fourier expansion of $E_{n,k}(Z,0)$, for a half-integral matrix B of degree n over $\mathbf{Z}$, we define the Siegel series $b(B,s)$ by

$$b(B,s) = \sum_{R \in S_n(\mathbf{Q})/S_n(\mathbf{Z})} \mathbf{e}(\mathrm{tr}(BR))\mu(R)^{-s},$$

where $\mu(R) = [R\mathbf{Z}^n + \mathbf{Z}^n : \mathbf{Z}^n]$. Furthermore we put

$$\Gamma_n(s) = \prod_{j=1}^{n} \pi^{j/2}\Gamma(s-(j-1)/2),$$

where $\Gamma(s)$ is Gamma function. For a p-adic number x put $\mathbf{e}_p(x) = \exp(2\pi\sqrt{-1}\tilde{x})$, where $\tilde{x}$ denotes a rational number such that $\tilde{x} - x \in \mathbf{Z}_p$. To investigate the Siegel series, for a prime number p and a half-integral matrix B of degree n over $\mathbf{Z}_p$ define the local Siegel series $b_p(B,s)$ by

$$b_p(B,s) = \sum_{R \in S_n(\mathbf{Q}_p)/S_n(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(BR))\mu_p(R)^{-s},$$

where $\mu_p(R) = [R\mathbf{Z}_p^n + \mathbf{Z}_p^n : \mathbf{Z}_p^n]$. Then we easily see that for a half-integral matrix B of degree n over $\mathbf{Z}$ we have

$$b(B,s) = \prod_p b_p(B,s).$$

Let m, n be non-negative integers such that $m \ge n \ge 1$. For $A \in \mathcal{H}_m(\mathbf{Z}_p)$ and $B \in S_n(\mathbf{Q}_p)$ define the local density $\alpha_p(A,B)$ and the primitive local density $\beta_p(A,B)$ by

$$\alpha_p(A,B) = 2^{\delta_{mn}} \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \# \mathcal{A}_e(A,B),$$

and

$$\beta_p(A,B) = 2^{\delta_{mn}} \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \# \mathcal{B}_e(A,B),$$

where δ_{mn} is Knonecker's delta.

$$\mathcal{A}_e(A,B) = \{X \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p) \mid A[X] - B \in p^e \mathcal{H}_n(\mathbf{Z}_p)\},$$

and

$$\mathcal{B}_e(A,B) = \{X \in \mathcal{A}_e(A,B) \mid \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}(X) = n\}.$$

We define $\chi_p(a)$ for $a \in \mathbf{Q}_p \backslash \{0\}$ as follows;

$$\chi_p(a) = \begin{cases} +1 & \text{if } \mathbf{Q}_p(\sqrt{a}) = \mathbf{Q}_p \\ -1 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic unramified} \\ 0 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic ramified.} \end{cases}$$

For a half-integral matrix B of even degree n define $\xi_p(B)$ by

$$\xi_p(B) = \chi_p((-1)^{n/2} \det B).$$

Let $B \in \mathcal{H}_n(\mathbf{Z})_{>0}$ with n even. Then we can write $(-1)^{n/2} 2^n \det B = \mathfrak{d}_B \mathfrak{f}_B^2$ with $\mathfrak{d}_B$ a fundamental discriminant and $\mathfrak{f}_B \in \mathbf{Z}_{>0}$. Furthermore, let $\chi_B = (\frac{\mathfrak{d}_B}{*})$ be the Kronecker character corresponding to $\mathbf{Q}(\sqrt{(-1)^{n/2} \det B})/\mathbf{Q}$. We note that we have $\chi_B(p) = \xi_p(B)$ for any prime p. Let $H_k = \overbrace{H \perp ... \perp H}^{k}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

For a non-degenerate half-integral matrix B of degree n over $\mathbf{Z}_p$ define a polynomial $\gamma_p(B; X)$ in X by

$$\gamma_p(B; X) = \begin{cases} (1-X)\prod_{i=1}^{n/2}(1-p^{2i}X^2)(1-p^{n/2}\xi_p(B)X)^{-1} & \text{if } n \text{ is even} \\ (1-X)\prod_{i=1}^{(n-1)/2}(1-p^{2i}X^2) & \text{if } n \text{ is odd} \end{cases}.$$

Then the following lemma is well known (e.g. [Ki1], Lemma 1)

Lemma 2.1. *For a non-degenerate half-integral matrix B of degree n over $\mathbf{Z}_p$ there exists a unique polynomial $F_p(B, X)$ in X over $\mathbf{Z}$ with constant term 1 such that*

$$b_p(B, s) = \gamma_p(B; p^{-s}) F_p(B; p^{-s}).$$

Furthermore for any positive integer $k \geq n/2$ and a half integral matrix A of degree $2k$ over $\mathbf{Z}_p$ such that $2A$ is unimodular, we have

$$\alpha_p(A, B) = F_p(B, \xi_p(A) p^{-k}) \gamma_p(B, \xi_p(A) p^{-k})$$

and, in particular,

$$\alpha_p(H_k, B) = F_p(B, p^{-k}) \gamma_p(B, p^{-k}).$$

Remark. For an element $B \in \mathcal{H}_n(\mathbf{Z}_p)$ of rank $m \geq 0$, there exists an element $\tilde{B} \in \mathcal{H}_m(\mathbf{Z}_p) \cap GL_m(\mathbf{Q}_p)$ such that $B \sim \tilde{B} \perp O_{n-m}$. We note that $b_p(\tilde{B}, s)$ does not depend on the choice of $\tilde{B}$ (cf. [Ki1]). Thus we write this as $b_p^{(m)}(B, s)$. Furthermore, $F_p(\tilde{B}, X)$ does not depend on the choice of $\tilde{B}$. Then we put $F_p^{(m)}(B, X) = F_p(\tilde{B}, X)$. For an element $B \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}$ of rank $m \geq 0$, there exist an element $\tilde{B} \in \mathcal{H}_m(\mathbf{Z})_{>0}$ such that $B \sim \tilde{B} \perp O_{n-m}$. Then by the above remark $b(\tilde{B}, s)$ does not depend on the choice of $\tilde{B}$. Thus we write this as $b^{(m)}(B, s)$. Furthermore, $\det \tilde{B}$ does not depend on the choice of B. Thus we put $\det^{(m)} B = \det \tilde{B}$. Similarly, we write $\chi_B^{(m)} = \chi_{\tilde{B}}$ if m is even.

Now for a semi-positive definite half-integral matrix B of degree $2n$ and of rank m, we put

$$c_{2n,l}(B) = 2^{[(m+1)/2]} \prod_p F_p^{(m)}(B, p^{l-m-1})$$

$$\times \begin{cases} \prod_{i=m/2+1}^{n} \zeta(1+2i-2l) L(1+m/2-l, \chi_B^{(m)}) & \text{if } m \text{ is even} \\ (-1)^{(m^2-1)/8} \prod_{i=(m+1)/2}^{n} \zeta(1+2i-2l) & \text{if } m \text{ is odd} \end{cases}.$$

Here we make the convention $F_p^{(m)}(B, p^{l-m-1}) = 1$ and $L(1+m/2-l, \chi_B^{(m)}) = \zeta(1-l)$ if $m = 0$. Then we have

Theorem 2.2. *Under the same assumption as above, we have*

$$E_{2n,l}(Z) = \sum_{B \in \mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}} c_{2n,l}(B) \mathbf{e}(\mathrm{tr}(BZ)).$$

Remark. $c_{2n,l}(B)$ is a rational number, and the prime divisor of its denominator is not greater than $(2l-1)!$. This is a weaker version of Böcherer's result in [B3].

3 Differential operators

In this section, following [B-S], [I], we introduce some differential operators acting on the space of modular forms. Let $X = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq d}$ be a matrix

of variables, and for $1 \leq i, j \leq m$, put $\Delta_{i,j} = \sum_{\nu=1}^{d} \frac{\partial^2}{\partial x_{i\nu} \partial x_{j\nu}}$. A polynomial $P(X)$ in X is called pluriharmonic if $\Delta_{ij} P = 0$ for any $1 \leq i, j, \leq m$. Take a polynomial mapping $P(X_1, X_2)$ from $M_{n,2l}(\mathbf{C}) \times M_{n,2l}(\mathbf{C})$ to $\mathbf{C}$ such that
D-1. $P(X_1, X_2)$ is pluriharmonic for each X_i $(i = 1, 2)$.
D-2. $P(X_1 g, X_2 g) = P(X_1, X_2)$ for any $g \in O(2l)$
D-3. $P(a_1 X_1, a_2 X_2) = (\det a_1)^\nu (\det a_2)^\nu P(X_1, X_2)$ for $a_1, a_2 \in GL(n, \mathbf{C})$.
Assume that $l \geq n$. Then there exists a unique polynomial mapping $Q(W)$ from $S_{2n}(\mathbf{C})$ to $\mathbf{C}$ s.t $P(X_1, X_2) = Q(\begin{pmatrix} X_1 {}^t X_1 & X_1 {}^t X_2 \\ X_2 {}^t X_1 & X_2 {}^t X_2 \end{pmatrix})$. We note that $\deg Q = n\nu$. Let $Z = (z_{ij})_{1 \leq i,j \leq 2n}$ be a matrix of variables with $z_{ij} = z_{ji}$, and we write $\frac{\tilde{\partial}}{\partial z_{ij}} = \frac{(1+\delta_{ij})}{2} \frac{\partial}{\partial z_{ij}}$, and $(\frac{\partial}{\partial Z}) = (\frac{\tilde{\partial}}{\partial z_{ij}})_{1 \leq i,j \leq 2n}$. For $f \in C^\infty(\mathbf{H}_{2n})$ we define $\mathcal{D}_Q(f)$ and $\tilde{\mathcal{D}}_Q(f)$ by

$$\mathcal{D}_Q(f) = Q(\frac{\partial}{\partial Z})(f)$$

and

$$\tilde{\mathcal{D}}_Q(f) = \mathcal{D}_Q(f)_{Z_{12}=0},$$

where we write $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^t Z_{12} & Z_2 \end{pmatrix}$ with $Z_1, Z_2 \in \mathbf{H}_n$ and $Z_{12} \in M_n(\mathbf{C})$. We consider the action of the above operators on the Fourier series. Let $A = (a_{ij}) \in \mathcal{H}_{2n}$. Then we have $\mathbf{e}(\mathrm{tr}(AZ)) = \exp(2\pi\sqrt{-1}(\sum_{\alpha,\beta=1}^{2n} a_{\alpha\beta} z_{\alpha\beta}))$. Then we have

$$\frac{\tilde{\partial}}{\partial_{\alpha\beta}}(\mathbf{e}(\mathrm{tr}(AZ))) = 2\pi\sqrt{-1} a_{\alpha\beta} \mathbf{e}(\mathrm{tr}(AZ)).$$

Thus we have

$$\mathcal{D}_Q(\mathbf{e}(\mathrm{tr}(AZ))) = (2\pi\sqrt{-1})^{n\nu} Q(A) \mathbf{e}(\mathrm{tr}(AZ)).$$

Now let $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^t Z_{12} & Z_2 \end{pmatrix} \in \mathbf{H}_{2n}$ as above, and $f(Z) = \sum_{A \in \mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}} a(A) \mathbf{e}(\mathrm{tr}(AZ))$. Then we have

$$\tilde{\mathcal{D}}_Q(f)(Z_1, Z_2)$$
$$= (2\pi\sqrt{-1})^{n\nu} \sum_{A_1, A_2 \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} \mathbf{e}(\mathrm{tr}(A_1 Z_1 + A_2 Z_2)) \sum_{R \in M_n(\mathbf{Z})} Q(\begin{pmatrix} A_1 & \frac{1}{2}R \\ \frac{1}{2}{}^t R & A_2 \end{pmatrix}) a(\begin{pmatrix} A_1 & \frac{1}{2}R \\ \frac{1}{2}{}^t R & A_2 \end{pmatrix}).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp_n(\mathbf{R})^+$ put $\gamma^{\uparrow} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\gamma^{\downarrow} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$. We define the mapping ι from $Sp_n(\mathbf{R}) \times Sp_n(\mathbf{R})$ to $Sp_{2n}(\mathbf{R})$ by

$$\iota : Sp_n(\mathbf{R}) \times Sp_n(\mathbf{R}) \ni (\gamma_1, \gamma_2) \hookrightarrow \gamma_1^{\uparrow}\, \gamma_2^{\downarrow} \in Sp_{2n}(\mathbf{R}).$$

Furthermore, for a function $f : \mathbf{H}_n \times \mathbf{H}_n \longrightarrow \mathbf{C}$, $\gamma_1, \gamma_2 \in Sp_n(\mathbf{R})$ we define

$$f|_l(\gamma_1, \gamma_2)(Z_1, Z_2) = j(\gamma_1, Z_1)^{-l} j(\gamma_2, Z_2)^{-l} f(\gamma_1(Z_1), \gamma_2(Z_2)).$$

Then we have

Theorem 3.1 ([I])

$$\tilde{\mathcal{D}}_Q(f)|_{l+\nu}(\gamma_1, \gamma_2) = \tilde{\mathcal{D}}_Q(f|_l \iota(\gamma_1, \gamma_2))$$

Now we apply the above theorem to the modular forms. For a subspace $\mathfrak{M}$ of $\mathfrak{M}_l^{\infty}(\Gamma^{(n)})$ let $\mathfrak{M} \otimes \mathfrak{M} = \{\sum_{i,j} a_{ij} f_i(Z_1) f_j(Z_2)$ (finite sum); $f_i, f_j \in \mathfrak{M}, a_{ij} \in \mathbf{C}\}$. Put $C_q(s) = s(s+1/2)\cdots(s+(q-1)/2)$. We choose $Q = Q_{n,l}^{\nu}$ such that

$$\tilde{\mathcal{D}}_{Q_{n,l}^{\nu}}(\det Z_{12}^{\nu}) = (-1)^{n\nu} \prod_{\mu=1}^{\nu} (C_n(\mu/2) C_n(l - n + \nu - \mu/2)),$$

and put

$$\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu} = \tilde{\mathcal{D}}_{Q_{n,l}^{\nu}}.$$

This coincides with $\overset{\circ}{\mathfrak{D}}{}_{n,l}^{\nu}$ in [B-S]. Then by Theorem 3.1 we easily see

Theorem 3.2. *$\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu}$ maps $\mathfrak{M}_l^{\infty}(\Gamma^{(2n)})$ to $\mathfrak{M}_{l+\nu}^{\infty}(\Gamma^{(n)}) \otimes \mathfrak{M}_{l+\nu}^{\infty}(\Gamma^{(n)})$. Furthermore $\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu}$ maps $\mathfrak{M}_l(\Gamma^{(2n)})$ into $\mathfrak{M}_{l+\nu}(\Gamma^{(n)}) \otimes \mathfrak{M}_{l+\nu}(\Gamma^{(n)})$, and in particular if $\nu > 0$, its image is contained in $\mathfrak{S}_{l+\nu}(\Gamma^{(n)}) \otimes \mathfrak{S}_{l+\nu}(\Gamma^{(n)})$.*

4 Pullback formula

$\mathbf{L}_n = \mathbf{L}_{\mathbf{Q}}(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$ denote the Hecke ring over $\mathbf{Q}$ associated with the Hecke pair $(GSp_n(\mathbf{Q})^+, \Gamma^{(n)})$. For each integer m define an element $T(m)$ of $\mathbf{L}_n$ by

$$T(m) = \sum_{d_1,...,d_n,e_1,...,e_n} \Gamma^{(n)}(d_1 \perp ... \perp d_n \perp e_1 \perp ... \perp e_n)\Gamma^{(n)},$$

where $d_1, ..., d_n, e_1, ..., e_n$ run over all positive integer satisfying

$$d_i | d_{i+1},\ e_{i+1}|e_i\ (i = 1, ..., n-1), d_n | e_n, d_i e_i = m\ (i = 1, ..., n).$$

Furthermore, for $i = 0, 1, ..., n$ and a prime number p not dividing N, put

$$T_i(p^2) = \Gamma^{(n)}(1_{n-i} \perp p1_i \perp p^2 1_{n-i} \perp p1_i)\Gamma^{(n)}.$$

As is well known, $\mathbf{L}_n$ is generated over $\mathbf{Q}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, ..., n)$. We denote by $\mathbf{L}_n'$ the subalgebra of $\mathbf{L}_n$ generated by over $\mathbf{Z}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, ..., n)$. Let $T = \Gamma^{(n)} M \Gamma^{(n)}$ be an element of $\mathbf{L}_n \otimes \mathbf{C}$. Write T as $T = \cup_\gamma \Gamma^{(n)}\gamma$ and for $f \in \mathfrak{M}_k(\Gamma^{(n)})$ define the Hecke operator $|_k T$ associated to T as

$$f|_k T = \det(M)^{k-(n+1)/2} \sum_\gamma f|_k \gamma.$$

We call this action the Hecke operator as usual (cf. [A].) If f is a eigenfunction of a Hecke operator $T \in \mathbf{L}_n \otimes \mathbf{C}$, we denote by $\lambda_f(T)$ its eigenvalue. We call $f \in \mathfrak{M}_k(\Gamma^{(n)})$ a Hecke eigenform if it is a common eigenfunction of all Hecke operators. Furthermore, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by eigenvalues of all $T \in \mathbf{L}_n$ as in Introduction. As is well known, $\mathbf{Q}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field K, let $\mathfrak{O}_K$ denote the ring of integers in K.

Theorem 4.1 *Let $k \geq n+1$. Let $f \in \mathfrak{S}_k(\Gamma^{(n)})$ be a common eigenfunction of all Hecke operators in $\mathbf{L}_n'$. Then $\lambda_f(T)$ belongs to $\mathfrak{O}_{\mathbf{Q}(f)}$ for any $T \in \mathbf{L}_n'$.*

The above theorem is known in case $n = 1, 2$ (cf. [Ku3]), and it seems more or less well known also for general n. But for the readers' convenience, we here give a proof to it. Let R be a subring of $\mathbf{C}$. Let $\mathfrak{S}_k(\Gamma^{(n)})(R)$ be the $\mathbf{Z}$-module consisting of elements of $\mathfrak{S}_k(\Gamma^{(n)})$ whose Fourier coefficients belong to R. It is known that we have $\mathfrak{S}_k(\Gamma^{(n)})(\mathbf{Z}) \otimes \mathbf{C} = \mathfrak{S}_k(\Gamma^{(n)})$ (cf. Shimura [Sh3]). Then Theorem 4.1 follows from the following proposition:

Proposition 4.2. *Assume that $k \geq n+1$. Let R be a subring of $\mathbf{C}$. Any $T \in \mathbf{L}_n'$ maps $\mathfrak{S}_k(\Gamma^{(n)})(R)$ to itself.*

Proof. Let

$$f(z) = \sum_{A>0} a(A)\mathbf{e}(\mathrm{tr}(Az))$$

be an element of $\mathfrak{S}_k(\Gamma^{(n)})(R)$. Then we have $a(A) \in R$ for any A. For a element $T \in \mathbf{L}_n'$ put

$$f|T(z) = \sum_{A>0} a|T(A)\mathbf{e}(\mathrm{tr}(Az))$$

Let p be a prime number. First let $T = T(p)$. Then we have

$$a|T(p)(A) = p^{nk-n(n+1)/2}$$

$$\times \sum_{d_1|d_2|\cdots|d_n|p} \sum_{D \in G_n \backslash G_n(d_1 \perp d_2 \perp \cdots d_n)G_n} (\det D)^{-k} a(p^{-1}A[{}^tD]),$$

where $G_n = GL_n(\mathbf{Z})$ (e.g. Exercise 4.2.10 of Andrianov [A].) Thus $a|T(p)(A)$ belongs to R for any A. Next for nonnegative integer m and integer l put

$$\beta_p(m,l) = \begin{cases} \prod_{i=1}^{l} \frac{p^{m-l+i}-1}{p^i-1} & \text{if } l \geq 1 \\ 1 & \text{if } l = 0 \\ 0 & \text{if } l < 0 \end{cases}$$

For $j = 0, 1, ..., n$ put

$$\hat{T}_j(p^2) = \sum_{t=0}^{j} \beta_p(n-t, j-t)T_t(p^2).$$

The matrix $(\beta_p(n-t+1, j-t))_{1 \leq j,t \leq n+1}$ is unimodular. Thus, to complete the proof, it suffices to show that $a|\hat{T}_j(p^2)(A) \in R$ for any $0 \leq j \leq n$ and $A > 0$. Now for a non-negative integers r_0, r_2 such that $r_0 + r_2 \leq n$, put

$$E(j, r_0, r_2) = kj - j(n+1) + k(r_2 - r_0) + r_0(n - r_2 + 1) + (j - r_0 - r_2)(j - r_0 - r_2 + 1)/2,$$

$$G_{n,r_0,r_2} = G_n \cap \begin{pmatrix} \mathbf{Z}^{r_0,r_0} & \mathbf{Z}^{r_0,n-r_0-r_2} & \mathbf{Z}^{r_0,r_2} \\ p\mathbf{Z}^{n-r_2-r_0,r_0} & \mathbf{Z}^{n-r_2-r_0,n-r_0-r_2} & \mathbf{Z}^{n-r_2-r_0,r_2} \\ p^2\mathbf{Z}^{r_2,r_0} & p\mathbf{Z}^{r_2,n-r_0-r_2} & \mathbf{Z}^{r_2,r_2} \end{pmatrix},$$

and

$$D_{r_0,r_2} = p^{-1}E_{r_0} \perp E_{n-r_0-r_2} \perp pE_{r_2}.$$

Then, by Theorem 4.1 of Hafner and Walling [H-W], we have

$$a|\hat{T}_j(p^2)(A)$$

$$= p^{n(k-n-1)} \sum_{\substack{0 \le r_0, r_2 \le n \\ r_0+r_2 \le j}} p^{E(j,r_0,r_2)} \sum_{D \in G_n \backslash G_{n,r_0,r_2}} \alpha(A, A[GD_{r_0,r_2}^{-1}])a(A[GD_{r_0,r_2}^{-1}]),$$

where $\alpha(A, A[GD_{r_0,r_2}^{-1}])$ is a certain integer determined by A and $A[GD_{r_0,r_2}^{-1}]$. (We note that the action of Hecke operators on the modular forms in their paper [H-W] is slightly different form ours. But the above statement is essentially the same as theirs.) Thus, $a|\hat{T}_j(p^2)(A)$ belongs to R, and we completes the proof.

Put

$$GSp_n(\mathbf{Q}_p) = \{M \in GL_{2n}(\mathbf{Q}_p); J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) \in \mathbf{Q}_p^{\times}\},$$

and let $\mathbf{L}_{np} = \mathbf{L}(GSp_n(\mathbf{Q}_p), GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q}_p), GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p))$. Now assume that f is a common eigenfunction of all Hecke operators, and for each prime number p, let $\alpha_0(p), \alpha_1(p), ..., \alpha_n(p)$ be the Satake parameters of $\mathbf{L}_{np}$ determined by f. We then define the standard L-function $L(f, s, \underline{\mathrm{St}})$ by

$$L(f, s, \underline{\mathrm{St}}) = \prod_p \prod_{i=1}^n \{(1-p^{-s})(1-\alpha_i(p)p^{-s})(1-\alpha_i(p)^{-1}p^{-s})\}^{-1}.$$

Let $E_{2n,l}(Z) = E_{2n,l}(Z, 0)$ be the Eisenstein series in Section 2. We then define $\mathcal{E}_{2n,l,k}(z_1, z_2)$ as

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = (-1)^{l/2+1}2^{-n}(2\pi\sqrt{-1})^{l-k}(l-n) \overset{\circ}{\mathcal{D}}{}_{n,l}^{k-l} (E_{2n,l})(z_1, z_2),$$

where $z_1, z_2 \in \mathbf{H}_n$. Let $f(z) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{>0}} a(A)\mathbf{e}(\mathrm{tr}(Az))$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$. For a positive integer $m \le k-n$ such that $m \equiv n \bmod 2$ put

$$\Lambda(f, m, \underline{\mathrm{St}}) = (-1)^{n(m+1)/2+1} 2^{-4kn+3n^2+n+(n-1)m+2}$$

$$\times \Gamma(m+1) \prod_{i=1}^{n} \Gamma(2k-n-i) \frac{L(f, m, \underline{\mathrm{St}})}{< f, f > \pi^{-n(n+1)/2+nk+(n+1)m}}.$$

We note that all the Fourier coefficients of $\tilde{\mathcal{E}}_{2n,l,k}(z_1, z_2)$ are rational and any prime divisor of its denominator is not greater than $(2l-1)!$. Then as a special case of [B2] we have

Theorem 4.3. *Let l, k and n be a positive integers such that $\rho(n) \le l \le k-n$, where $\rho(n)$ is the integer in Section 1. Let $f \in \mathfrak{S}_k(\Gamma^{(n)})$ be a common eigenfunction of all Hecke operators in $\mathbf{L}_n$. Then we have*

$$< f, \mathcal{E}_{2n,l,k}(*, -\bar{z})) >= \Lambda(f, l-n, \underline{\mathrm{St}}) f(z).$$

For two semi-positive definite half-integral matrices A_1, A_2 of degree n, put

$$\epsilon_{l,k}(A_1, A_2) = \sum_{A_2 - \frac{1}{4}A_1^{-1}[R] \ge 0} \tilde{c}_{2n,l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}) Q_{n,l}^{(k-l)}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}),$$

where $\tilde{c}_{2n,l}(A) = (-1)^{l/2+1} 2^{-n}(l-n) c_{2n,l}(A)$ for $A \in \mathcal{H}_{2n}(\mathbf{Z})_{\ge 0}$. Furthermore, for each semi-positive definite half-integral matrix A_1 put

$$\mathcal{F}_{l,k;A_1}(z_2) = \sum_{A_2 \in \mathcal{H}_n(\mathbf{Z})_{\ge 0}} \epsilon_{l,k}(A_1, A_2)\mathbf{e}(\mathrm{tr}(A_2 z_2)).$$

We note that $\mathcal{F}_{l,k;A_1}(z_2)$ belongs to $\mathfrak{M}_k(\Gamma^{(n)})$, and

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = \sum_{A_1 \in \mathcal{H}_n(\mathbf{Z})_{\ge 0}} \mathcal{F}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1 z_1)).$$

In particular, if $l < k$, $\mathcal{F}_{l,k;A_1}(z_2)$ belongs to $\mathfrak{S}_k(\Gamma^{(n)})$, and

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = \sum_{A_1 \in \mathcal{H}_n(\mathbf{Z})_{>0}} \mathcal{F}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1 z_1)).$$

Take a basis $\{f_i\}_{i=1}^{d_1}$ of $\mathfrak{S}_k(\Gamma^{(n)})$ consisting of Hecke eigenforms. Write

$$f_i(z) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{>0}} a_i(A)\mathbf{e}(\mathrm{tr}(Az)).$$

Now we compute the value $\Lambda(f, l, \underline{\mathrm{St}})$.

Theorem 4.4. *In addition to the above notation and the assumption, assume that $l \leq k-n-2$. Then for any positive definite half-integral matrix A_1 of degree n we have*

$$\mathcal{F}_{l+n,k;A_1}(z) = \sum_{i=1}^{d_1} \Lambda(f_i, l, \underline{\mathrm{St}}) a_i(A_1)\overline{f_i(-\bar{z})}.$$

Remark 2. Since $\mathcal{F}_{k,k;A_1}(z)$ is not a cusp form, the above formula does not hold for $l = k-n$. However, by modifying the above method, we can get a similar formula for this case.

5 Congruence of modular forms

In this section we consider the congruence between the Hecke eigenvalues of modular forms of the same weight. Let K be an algebraic number filed, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}_{(\mathfrak{P})}$ be the localization of $\mathfrak{O}$ at $\mathfrak{P}$ in K. Then we have the following lemma.

Lemma 5.1. *Let $f_1, \ldots\ldots, f_d$ be Hecke eigenforms in $\mathfrak{S}_k(\Gamma^{(n)})$ linearly independent over $\mathbf{C}$, and G an element of $\mathfrak{S}_k(\Gamma^{(n)})$. Let K be the composite field of $\mathbf{Q}(f_1), \mathbf{Q}(f_2), \ldots,$ and $\mathbf{Q}(f_d)$, and $\mathfrak{O} = \mathfrak{O}_K$. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}$. Assume that*

(1) *$c_G(A)$ belongs to $\mathfrak{O}_{(\mathfrak{P})}$ for any A, and $a_{f_1}(A_1) \in \mathfrak{O}_{(\mathfrak{P})}^*$ for some A_1.*

(2) *there exist $c_1, \ldots, c_d \in K$ such that $\mathrm{ord}_{\mathfrak{P}}(c_1) < 0$ and*

$$G(z) = \sum_{i=1}^{d} c_i f_i(z).$$

Then there exists $i \neq 1$ such that we have

$$\lambda_{f_i}(T) \equiv \lambda_{f_1}(T) \bmod \mathfrak{P}$$

for any $T \in \mathbf{L}'_n$.

Proof. The assertion for $d = 2$ has been proved in [Ku2], and the general case can easily be proved. But, for the readers' convenience, we here give a proof. By assumption, we have $d \geq 2$. We prove the assertion on d. First let $d = 2$. Then we have

$$G(z) = c_1 f_1(z) + c_2 f_2(z),$$

and

$$G|_k T(z) = c_1 \lambda_{f_1}(T) f_1(z) + c_2 \lambda_{f_2}(T) f_2(z)$$

for any $T \in \mathbf{L}'_n$. Thus we have

$$G|_k T(z) - \lambda_{f_2}(T) G(z) = c_1(\lambda_{f_1}(T) - \lambda_{f_2}(T)) f_1(z).$$

In particular, we have

$$c_{G|_k T}(A_1) - \lambda_{f_2}(T) a_G(A_1) = c_1(\lambda_{f_1}(T) - \lambda_{f_2}(T)) a_{f_1}(A_1).$$

Thus by assumption, we have

$$\mathrm{ord}_{\mathfrak{P}}(\lambda_{f_1}(T) - \lambda_{f_2}(T)) \geq \mathrm{ord}_{\mathfrak{P}}(c_1^{-1}) > 0.$$

This proves the assertion for $d = 2$. Let $d \geq 3$, and assume that the assertion holds for d. We have

$$G(z) = \sum_{i=1}^{d} c_i f_i(z).$$

If we have $\lambda_{f_1}(T) \equiv \lambda_{f_d}(T) \bmod \mathfrak{P}$ for any $T \in \mathbf{L}'_n$, the assertion holds. Otherwise, take $T \in \mathbf{L}'_n$ such that $\lambda_{f_1}(T) \not\equiv \lambda_{f_d}(T) \bmod \mathfrak{P}$. Then we have

$$G|_k T(z) = \sum_{i=1}^{d} c_i \lambda_{f_i}(T) f_i(z).$$

Thus we have

$$G|_k T(z) - \lambda_{f_d}(T) G(z) = \sum_{i=1}^{d-1} c_i(\lambda_{f_i}(T) - \lambda_{f_d}(T)) f_i(z).$$

All the Fourier coefficients of the left-hand side belong to $\mathfrak{O}_{(\mathfrak{P})}$, and $\mathrm{ord}_{\mathfrak{P}}(c_1(\lambda_{f_1}(T)-\lambda_{f_d}(T))) < 0$. Thus by the induction hypothesis, there exists an integer $2 \le i \le d-1$ satisfying the required condition.

Let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ and M be a subspace of $\mathfrak{S}_k(\Gamma^{(n)})$ stable under Hecke operators $T \in \mathbf{L}_n$. Assume that M is contained in $(\mathbf{C}f)^{\perp}$, where $(\mathbf{C}f)^{\perp}$ is the orthogonal complement of $\mathbf{C}f$ in $\mathfrak{S}_k(\Gamma^{(n)})$ with respect to the Petersson product. A prime ideal $\mathfrak{P}$ of $\mathfrak{O}_{\mathbf{Q}(f)}$ is called a congruence prime of f with respect to M if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \bmod \tilde{\mathfrak{P}}$$

for any $T \in \mathbf{L}_n'$, where $\tilde{\mathfrak{P}}$ is the prime ideal of $\mathfrak{O}_{\mathbf{Q}(f)\mathbf{Q}(g)}$ lying above $\mathfrak{P}$. If $M = (\mathbf{C}f)^{\perp}$, we simply call $\mathfrak{P}$ a congruence prime of f.

Now we consider the relation between the congruence primes and the standard zeta values. To consider this, we have to normalize the standard zeta value $\Lambda(f, l, \underline{\mathrm{St}})$ for a Hecke eigenform f because it is not uniquely determined by the system of Hecke eigenvalues of f. We note that there is no reasonable normalization of cuspidal Hecke eigenform in the higher degree case unlike the elliptic modular case. Thus we define the following quantities: for a Hecke eigenform f in $\mathfrak{S}_k(\Gamma^{(n)})$, by multiplying a suitable constant c we may assume all $c_f(A)'s$ are elements of $\mathbf{Q}(f)$ with bounded denominator. Let $\mathfrak{I}_f$ be the fractional ideal in $\mathbf{Q}(f)$ generated by all $c_f(A)'s$. Then $\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{I}_f^2$ is a fractional ideal in $\mathbf{Q}(f)$. We note that the value $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{I}_f)^2$ does not depend on the choice of c, where $N(\mathfrak{I}_f)$ is the norm of the ideal $\mathfrak{I}_f$. Furthermore, for a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, the order $\mathrm{ord}_{\mathfrak{P}}(\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{I}_f^2)$ does not depend on the choice of c either. In particular, if we assume the multiplicity one property for the Hecke eigenforms, these values are uniquely determined by the system of eigenvalues of f. Then by Theorem 4.4 and Lemma 5.1, we have

Theorem 5.2. *Let f be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$. Let l be a positive integer satisfying the condition in Theorem 4.4. Let $\mathfrak{P}$ be a prime ideal of $\mathfrak{O}$. Assume that* $\mathrm{ord}_{\mathfrak{P}}(\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{I}_f^2) < 0$ *and that it does not divide* $(2l-1)!$. *Then $\mathfrak{P}$ is a congruence prime of f. In particular, if a rational prime number p divides the denominator of $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{I}_f)^2$, then p is divided by some congruence prime of f.*

Now for a Hecke eigenform f in $\mathfrak{S}_k(\Gamma^{(n)})$, let $\mathfrak{T}_f$ denote the subspace of

$\mathfrak{S}_k(\Gamma^{(n)})$ spanned by all Hecke eigenforms with the same system of the Hecke eigenvalues as f.

Corollary. *In addition to the above notation and the assumption, assume that* $\mathfrak{S}_k(\Gamma^{(n)})$ *has the multiplicity one property. Then* $\mathfrak{P}$ *is a congruence prime of* f *with respect to* $\mathfrak{T}_f^{\perp}$. *In particular, if a rational prime number* p *divides the denominator of* $N_{\mathbf{Q}(f)}(\Lambda(f,l,\underline{\mathrm{St}}))N(\mathfrak{J}_f)^2$, *then* p *is divided by some congruence prime of* f *with respect to* $\mathfrak{T}_f^{\perp}$

Now a subspace M of $\mathfrak{S}_k(\Gamma^{(n)})$ is called non-splitting if there exists a Hecke eigenform f in M such that $M = < f^{\sigma} \mid \sigma \in \mathrm{Aut}(\mathbf{C}) >$. Clearly M is stable under the action of $\mathbf{L}_n$. Let $\mathbf{Z}[f]$ be a subring of $\mathfrak{O}_{\mathbf{Q}(f)}$ generated over $\mathbf{Z}$ by all Hecke eigenvalues of f. Then the different $\mathfrak{D}$ of $\mathbf{Z}[f]$ and the discriminant D of $\mathbf{Z}[f]$ does not depend on the choice of f, which will be denoted by $\mathfrak{D}_M$ and D_M, respectively. Then by the above result we easily see the following:

Theorem 5.3. *Let the notation and the assumption be as in Theorem 5.2. Let* M *be a non-splitting subspace of* $\mathfrak{S}_k(\Gamma^{(n)})$. *Let* f *be a Hecke eigenform in* M. *Let* $\mathfrak{D} = \mathfrak{D}_M$ *and* $D = D_M$. *Let* $\mathfrak{P}$ *be a prime ideal of* $\mathfrak{O}$. *Assume that* $\mathrm{ord}_{\mathfrak{P}}(\Lambda(f,l,\underline{\mathrm{St}})\mathfrak{J}_f^2\mathfrak{D}) < 0$ *and not dividing* $(2l-1)!$. *Then* $\mathfrak{P}$ *is a congruence prime of* f *with respect to* $M^{\perp}$. *In particular, if a rational prime number* p *divides the denominator of* $N_{\mathbf{Q}(f)}(\Lambda(f,l,\underline{\mathrm{St}}))N(\mathfrak{J}_f)^2D$, *then* p *is divided by some congruence prime of* f *with respect to* $M^{\perp}$.

6 A conjecture for the congruence of Saito-Kurokawa lift

Let $J_{k,1}^{\mathrm{cusp}}$ be the space of Jacobi cusp forms of weight k and of index 1 on $\Gamma^{(1)}$, and $\mathcal{V} : J_{k,1}^{\mathrm{cusp}} \longrightarrow \mathfrak{M}_k(\Gamma^{(2)})$ be the injection defined in Theorem 6.2 of Eichler and Zagier [E-Z]. Then $\mathcal{V}(J_{k,1}^{\mathrm{cusp}})$ is the Maass subspace of $\mathfrak{S}_k(\Gamma^{(2)})$, which will be denoted by $\mathfrak{S}_k(\Gamma^{(2)})^*$. Now we consider the congruence prime of the Saito-Kurokawa lift with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$. Let $f(z) = \sum_{m=1}^{\infty} a_f(m)\mathbf{e}(mz)$ be a Hecke eigenform in $\mathfrak{S}_{2k-2}(\Gamma^{(1)})$. For a Dirichlet character χ, let $L(f,s,\chi)$ be the Hecke L-function of f twisted by χ define by as follows:

$$L(f,s,\chi) = \sum_{m=1}^{\infty} a_f(m)\chi(m)m^{-s}.$$

In particular, if χ is the principal character we write $L(f,s,\chi)$ as $L(f,s)$. Put $\Omega_f^{(+)} = (2\pi\sqrt{-1})^{-1}L(f,1)$, and $\Omega_f^{(-)} = (2\pi\sqrt{-1})^{-2}L(f,2)$. For $j = \pm, l \in \mathbf{Z}_{>0}$, and a Dirichlet character χ such that $\chi(-1) = j(-1)^{l-1}$, put

$$\mathbf{L}(f,l,\chi) = \frac{(2\pi\sqrt{-1})^{-l}\Gamma(l)L(f,l,\chi)}{\Omega^{(j)}}.$$

In paricular, put $\mathbf{L}(f,l) = \mathbf{L}(f,l,\chi)$ if χ is the principal character. Furthemore, for positive integers $1 \le m, m' \le 2k-3$, put

$$C(f,m,m') = \frac{L(f,m)L(f,m')}{(2\pi\sqrt{-1})^{m+m'} < f,f >}.$$

Then it is well-known that $\mathbf{L}(f,l,\chi)$ belongs to the field $\mathbf{Q}(f)(\chi)$ generated over $\mathbf{Q}(f)$ by all the values of χ, and that $C(f,m,m') \in \mathbf{Q}(f)$ if $m-m'$ is odd (cf. Shimura [Sh2].)

Let $\hat{f}$ be the Saito-Kurokawa lift of f. Then we have

$$L(\hat{f},s,\underline{\mathrm{St}}) = \zeta(s)\prod_{i=1}^{2} L(f,s+k-i).$$

According to the numerical data, which will be given in Section 8, we expect that a prime of the numerator of algebraic part of a certain L-value of f is related to the congruence of $\hat{f}$. Now, from the above consideration and the numerical data in the last section, we would propose the following conjecture:

Conjecture. *Let $\mathfrak{P}$ be a prime ideal of $\mathbf{Q}(f)$ not dividing $(2k-1)!$. Then $\mathfrak{P}$ is a congruence prime of $\hat{f}$ with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$ if and only if $\mathfrak{P}$ divides the numerator of $\mathbf{L}(f,k)$.*

Remark 1. This is an analogue of the Doi-Hida-Ishii conjecture concerning the congruence primes of the Doi-Naganuma lifting [D-H-I]. We also note that this type of conjecture has been proposed by Harder [Ha] in the case of vector valued Siegel modular forms.

Remark 2. We can also formulate the above conjecture by putting $\Omega_f^- = \mathbf{L}(f,l)$ for an even positive integer $4 \le l \le k-4$. There might be another formulation of the conjecture in terms of the periods in Hida [Hi] or Stein [St1].

In this paper, we give some partial result for this conjecture.

Theorem 6.1. *Let the notation be as above. Let $\mathfrak{P}$ be a prime ideal of $\mathbf{Q}(f)$ not dividing $(2k-1)!$. Assume that*
(1) $\mathfrak{P}$ divides $\mathbf{L}(f,k)$
(2) $\mathfrak{P}$ is not a congruence prime of f
(3) $\mathfrak{P}$ does not divide $D\mathbf{L}(f,k-1,\chi_D)$ for some quadratic Dirichlet character corresponding to a fundamental discriminant $D<0$
(4) $\mathfrak{P}$ does not divide $C(f,2m+k-2,2m+k-1)\zeta(1-2m)$ for some integer $2 \le m \le k/2-2$.

Then $\mathfrak{P}$ is a congruence prime of $\hat{f}$ with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^\perp$.

Remark. We expect that the above assertion holds without assuming the conditions (2),(3), and (4).

Proof. By the above consideration, for $2 \le m \le k/2-2$ have

$$\Lambda(\hat{f},2m,\underline{\mathrm{St}}) = C_{k,m}\frac{\zeta(1-2m)L(f,2m+k-2)L(f,2m+k-1)}{\pi^{-3+2k+4m} < \hat{f},\hat{f} >},$$

where $C_{k,m}$ is a rational number whose numerator and denominator are not divisible by a prime number greater than $2k-2$. Let $\tilde{f}$ be an element of the Kohnen plus space $\mathfrak{S}_{k-1/2}(\Gamma_0(4))^+$ corresponding to f via the Shimura correspondence. Then by Kohnen and Skoruppa [K-S], we have

$$\frac{\Gamma(k)L(f,k)}{(2\pi)^k} = 3\cdot 2^{-k+3}\frac{< \hat{f},\hat{f} >}{< \tilde{f},\tilde{f} >}.$$

We also have $\mathfrak{J}_{\hat{f}} = \mathfrak{J}_{\tilde{f}}$. On the other hand, by Kohnen and Zagier [K-Z], we have

$$\frac{c_{\tilde{f}}(|D|)^2}{< \tilde{f},\tilde{f} >} = \frac{\Gamma(k-1)|D|^{k-3/2}L(f,k-1,\chi_D)}{\pi^{k-1} < f,f >}.$$

By the assumption (3), we have

$$\mathrm{ord}_{\mathfrak{P}}(\mathbf{L}(f,k-1,\chi_D)) = 0.$$

Thus by the assumptions (1), we have

$$\mathrm{ord}_{\mathfrak{P}}(\Lambda(\hat{f},2m,\underline{\mathrm{St}})\mathfrak{J}_{\hat{f}}^2)$$

$$= \mathrm{ord}_{\mathfrak{P}}(C(f,2m+k-2,m+k-1)\zeta(1-2m)\mathbf{L}(f,k)^{-1}) < 0$$

for some $2 \leq m \leq k/2-2$. Thus by Theorem 5.2 and the assumption (2), we easily see that $\mathfrak{P}$ is a congruence prime of $\hat{f}$ with respect to $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$.

Remark. We can formulate this type of conjecture for the congruence primes of the Ikeda lifting, which we will discuss in a subsequent paper.

7 Exact standard L-values in case $n=2$

In this section we obtain a useful formula for computing exact standard L-values in the case of degree 2. The following lemma can easily be proved (e.g. [Ka1, Proposition 2.2]).

Lemma 7.1. *Let $n = n_1 + n_2$ with n_1 even. Let $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p) \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p) \cap GL_{n_2}(\mathbf{Q}_p)$. Then for any $l \geq n$ we have*

$$\alpha_p(H_l, A_{11} \perp A_{22}) = \beta_p(H_l, A_{11})\alpha_p(H_{l-n_1} \perp (-A_{11}), A_{22}).$$

Proposition 7.2. *Let n_1 be an even integer. Let $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p) \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p)$. Let m be the rank of A_{22}. Then we have*

$$F_p^{(n_1+m)}(A_{11} \perp A_{22}, X) = F_p^{(m)}(A_{22}, \xi_p(A_{11})p^{n_1/2}X).$$

Proof. We may assume that A_{22} is non-degenerate. By Lemma 2.1 for any $l \geq n_1 + n_2$ we have

$$\alpha_p(H_l, A_{11} \perp A_{22}) = \gamma_p(A_{11} \perp A_{22}, p^{-l})F_p(A_{11} \perp A_{22}, p^{-l}).$$

By Lemma 7.1, we have

$$\alpha_p(H_l, A_{11} \perp A_{22}) = \beta_p(H_l, A_{11})\alpha_p(H_{l-n_1} \perp (-A_{11}), A_{22}).$$

Again by Lemma 2.1, we have

$$\alpha_p(H_{l-n_1} \perp (-A_{11}), A_{22}) = \gamma_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l})F_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l}).$$

Furthermore we have

$$\beta_p(H_l, A_{11}) = (1-p^{-l}) \prod_{i=1}^{n_1/2} (1-p^{2i-2l})(1-p^{n_1/2-l}\xi_p(A_{11}))^{-1}$$

(eg. [Ki2]), and by definition we have

$$\gamma_p(A_{11} \perp A_{22}, p^{-l}) = \beta_p(H_l, A_{11})\gamma_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l}).$$

Thus the assertion holds.

Corollary 1. *Let* $A = \begin{pmatrix} A_{11} & A_{12}/2 \\ {}^tA_{12}/2 & A_{22} \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbf{Z}_p) \cap GL_{n_1+n_2}(\mathbf{Q}_p)$ *with* $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p), A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p)$, *and* $A_{12} \in M_{n_1,n_2}(\mathbf{Z}_p)$. *Let* m *be the rank of* A. *Assume* $2A_{11} \in GL_{n_1}(\mathbf{Z}_p)$. *Then we have*

$$F_p^{(m)}(A, X) = F_p^{(m-n_1)}(A_{22} - \frac{1}{4}A_{11}^{-1}[A_{12}], \xi_p(A_{11})p^{n_1/2}X).$$

Corollary 2. *Let* n_1 *and* n_2 *be positive even integers. Let* $A = \begin{pmatrix} A_{11} & A_{12}/2 \\ {}^tA_{12}/2 & A_{22} \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbf{Z})_{>0}$ *with* $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z})_{>0}, A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z})_{>0}$, *and* $A_{12} \in M_{n_1,n_2}(\mathbf{Z})$. *Let* p_0 *be a prime number. Let* m *be the rank of* A. *Assume* $2A_{11} \in GL_{n_1}(\mathbf{Z}_p)$ *for any prime number* $p \neq p_0$, *and* $2A_{22} \in GL_{n_2}(\mathbf{Z}_{p_0})$. *Then we have*

$$\prod_p F_p^{(m)}(A, X) = F_{p_0}^{(m-n_2)}(A_{11} - \frac{1}{4}A_{22}^{-1}[{}^tA_{12}], \chi_{A_{22}}(p_0)p_0^{n_2/2}X)$$

$$\times \prod_{p \neq p_0} F_p^{(m-n_1)}(A_{22} - \frac{1}{4}A_{11}^{-1}[A_{12}], \chi_{A_{11}}(p)p^{n_1/2}X).$$

Now to make the computation in Section 8 easy, we give an explicit form of $F_p^{(1)}(A, X)$ and $F_p^{(2)}(A, X)$ in case $\deg A = 2$ (e.g. [Ka3].)

Proposition 7.3. *Let* $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$. *Put* $e = e_A =$ GCD(a_{11}, a_{12}, a_{22})

(1) *Assume* rank $A = 1$. *Then we have*

$$F_p^{(1)}(A, X) = \sum_{i=1}^{\mathrm{ord}_p(e_A)} (pX)^i.$$

(2) *Assume* $A > 0$. *Then we have*

$$F_p(A, X) = \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i} (p^3X^2)^j$$

$$-\chi_A(p)pX \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i-1} (p^3X^2)^j.$$

Now we give an explicit form of differential operator in the case of degree 2 due to Ibukiyama. Let

$$G_l(f_1, f_2, f_3; t)$$
$$= \frac{1}{R(f_1, f_2, f_3; t)^{(2l-5)/2}(\Delta_0(f_1, f_2, f_3; t)^2 - 4f_3t^2)^{1/2}},$$

where

$$\Delta_0(f_1, f_2, f_3; t) = 1 - f_1t + f_2t^2$$

and

$$R(f_1, f_2, f_3; t) = \Delta_0(f_1, f_2, f_3; t) + (\Delta_0(f_1, f_2, f_3; t)^2 - 4f_3t^2)^{1/2}/2.$$

Write

$$G_l(f_1, f_2, f_3; t) = \sum_{m=0}^{\infty} G_{l,m}(f_1, f_2, f_3)t^m,$$

and define a polynomial map $Q_{l,m}(\begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix})$ from $S_4(\mathbf{C})$ to $\mathbf{C}$ by

$$Q_{l,m}(\begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix}) = G_{l,m}(\det W_2, \det W_1 \det W_4, \det \begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix}),$$

where $W_1, W_4 \in S_2(\mathbf{C})$, and $W_2 \in M_n(\mathbf{C})$. Furthermore define a polynomial map $P_{l,m}(X_1, X_2)$ from $M_n(\mathbf{C}) \times M_n(\mathbf{C})$ to $\mathbf{C}$ by

$$P_{l,m}(X_1, X_2) = Q_{l,m}(\begin{pmatrix} X_1{}^tX_1 & X_1{}^tX_2 \\ X_2{}^tX_1 & X_2{}^tX_2 \end{pmatrix}).$$

Then by [I]

Proposition 7.4. *$P_{l,m}(X_1, X_2)$ satisfies the conditions D-1 $\sim$ D-3 in Section 3.*

Furthermore, by a direct but rather elaborate calculation we have

Proposition 7.5.

$$G_{l,m}(f_1, f_2, f_3) = \sum_{n=0}^{[m/2]} \begin{pmatrix} 2n + l - 5/2 \\ n \end{pmatrix} f_3^n$$

$$\times \sum_{\nu=0}^{[(m-2n)/2]} (-f_2)^\nu \begin{pmatrix} l + m - \nu - 5/2 \\ m - 2n - \nu \end{pmatrix} \begin{pmatrix} m - 2n - \nu \\ \nu \end{pmatrix} (2f_1)^{m-2n-2\nu}.$$

We note that $G_{l,m}(f_1, f_2, f_3) \in 2^{-m}\mathbf{Z}[f_1, f_2, f_3]$. Let

$$\mathcal{G}_{l,m} = G_{l,m}(\frac{1}{4}\det(\frac{\partial}{\partial z_{ij}})_{1 \le i \le 2, 3 \le j \le 4}, \det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1 \le i,j \le 2} \det(\frac{\tilde{\partial}}{\partial z_{ij}})_{3 \le i,j \le 4}, \det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1 \le i,j \le 4}),$$

where $\begin{pmatrix} s \\ m \end{pmatrix} = \dfrac{\prod_{i=1}^{m}(s - i + 1)}{m!}$. We note that

$$(\det(\frac{\partial}{\partial z_{ij}})_{1 \le i \le 2, 3 \le j \le 4})^m((z_{13}z_{24} - z_{14}z_{23})^m) = \Gamma(m+2)\Gamma(m+1).$$

Thus we have

$$\mathcal{G}_{l,m} = d_{l,m}\, \overset{\circ}{\mathcal{D}}{}^m_{2,l},$$

where

$$d_{l,m} = \frac{\Gamma(m+2)\Gamma(m+1) \begin{pmatrix} l + m - 5/2 \\ m \end{pmatrix}}{2^m \prod_{\mu=1}^{m} C_2(\mu/2) C_2(l - 2 + m - \mu/2)}.$$

Furthermore, put

$$\tilde{\Lambda}(f, l, \underline{\mathrm{St}}) = \frac{\begin{pmatrix} k - 5/2 \\ k - l - 2 \end{pmatrix}}{2^{3k-l-7}} \Gamma(l+1)\Gamma(k+l-2)\Gamma(k+l-1) \frac{L(f, l, \underline{\mathrm{St}})}{< f, f > (2\pi)^{2k+3l-3}},$$

and

$$\tilde{\mathcal{E}}_{4,l,k}(z_1, z_2) = d_{l,k-l}\mathcal{E}_{4,l,k}(z_1, z_2).$$

We note that

$$\tilde{\Lambda}(f, l, \underline{\mathrm{St}}) = d_{l+2,k-l-2}\Lambda(f, l, \underline{\mathrm{St}}).$$

Thus by Theorem 4.3, we have

$$< f, \tilde{\mathcal{E}}_{4,l+2,k}(*, -\bar{z})) >= \tilde{\Lambda}(f, l, \underline{\mathrm{St}})f(z)$$

for an even positive integer $l \le k-4$. Now for a positive definite half-integral matrices A_1 and A_2 of degree 2, let $\epsilon_{l,k}(A_1, A_2)$ be the one in Section 4, and put $\tilde{\epsilon}_{l,k}(A_1, A_2) = d_{l,k}\epsilon_{l,k}(A_1, A_2)$. Let p_0 be a prime number. Assume that $2A_1 \in GL_2(\mathbf{Z}_p)$ for any prime number $p \ne p_0$ and $2A_2 \in GL_2(\mathbf{Z}_{p_0})$. Then we have

$$\tilde{\epsilon}_{l,k}(A_1, A_2) = \sum_{R \in M_2(\mathbf{Z})} \tilde{c}_{4,l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})$$

$$\times G_{l,k-l}(\frac{1}{4}\det R, \det A_1 \det A_2, \det \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})),$$

where

$$\tilde{c}_{4,l}(A) = (-1)^{l/2+1}2^{-2}(l-2)$$

$$\times F_{p_0}^{(m-2)}(A_1 - \frac{1}{4}A_2^{-1}[{}^tR], \chi_{A_2}(p_0)p_0^{l-m}) \prod_{p \ne p_0} F_p^{(m-2)}(A_2 - \frac{1}{4}A_1^{-1}[R], \chi_{A_1}(p)p^{l-m})$$

$$\times \begin{cases} \prod_{i=m/2+1}^{2} \zeta(1+2i-2l)L(3-l, \chi_A^{(m)}) & \text{if } m = 2, 4 \\ \zeta(5-2l) & \text{if } m = 3, \end{cases}$$

for $A = \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}$. We note that $\tilde{\epsilon}_{l,k}(A_1, A_2)$ is rational number and any prime divisor of its denominator is not greater than $(2l-1)!$. We note that

$$\tilde{\mathcal{E}}_{4,l,k}(z_1, z_2) = \sum_{A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \sum_{A_2 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \tilde{\epsilon}_{l,k}(A_1, A_2)\mathbf{e}(\mathrm{tr}(A_1z_1 + A_2z_2)).$$

We note that $E_{4,l}(Z, 0)$ belongs to $\mathfrak{M}_l(\Gamma^{(4)})$ for an even integer $l \ge 4$.

Fix an $A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}$ and a prime number p. We define $\epsilon_{l,k}(i, A_1, A)$ as follows:

$$\epsilon_{l,k}(1, A_1, A) = \tilde{\epsilon}_{l,k}(A_1, A),$$

$$\epsilon_{l,k}(i,A_1,A) = \epsilon_{l,k}(i-1,A_1,pA) + p^{2k-3}\epsilon_{l,k}(i-1,A_1,A/p)$$
$$+p^{k-2} \sum_{D\in GL_2(\mathbf{Z})U_pGL_2(\mathbf{Z})/GL_2(\mathbf{Z})} \epsilon_{l,k}(i-1,A_1,A[D]/p),$$

where $U_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Let $\{f_j\}_{j=1}^d$ be a basis of $\mathfrak{S}_k(\Gamma^{(2)})$ consisting of Hecke eigenforms, and $\lambda_j = \lambda_{f_j}(T(p))$. Furthermore, let $\Phi(X) = \Phi_{T(p)}(X) = \sum_{i=0}^d b_{d-i}X^i$ be the characteristic polynomial of $T(p)$ in $\mathfrak{S}_k(\Gamma^{(2)})$. Then by the Hecke theory of Siegel modular forms (e.g. Andrianov [A]) we have

$$\epsilon_{l+2,k}(i,A_1,A) = \sum_{j=1}^d \lambda_j^{i-1}\tilde{\Lambda}(f_j,l,\underline{\mathrm{St}})a_j(A_1)a_j(A)$$

for any $A \in \mathcal{H}_2(\mathbf{Z})_{>0}$. Thus by Theorem 4.5 and Lemma 2.2 of Goto [Go], we have

Proposition 7.6. *In addition to the above assumption, let* $f = f_1, \lambda = \lambda_1, a(A_1) = a_1(A_1), a(A) = a_1(A), K = \mathbf{Q}(f)$ *and* $e_i = \epsilon_{l+2,k}(i,A_1,A)$. *Put* $\tilde{\Lambda}^*(f,l,\underline{\mathrm{St}}) = N_{K/\mathbf{Q}}(\tilde{\Lambda}(f,l,\underline{\mathrm{St}}))N(\mathfrak{I}_f)^2$. *Assume that* $\Phi'(\lambda) \neq 0$, *and* $a(A_1)a(A) \neq 0$. *Then for any positive even integer* $l \leq k-4$ *we have*

$$\tilde{\Lambda}^*(f,l,\underline{\mathrm{St}})$$
$$= N_{K/\mathbf{Q}}\left(\frac{\sum_{i=0}^d \sum_{j=i}^{d-1} e_{d-j}b_{j-i}\lambda^i}{\Phi'(\lambda)}\right)\frac{N(\mathfrak{I}_f)^2}{N_{K/\mathbf{Q}}(a(A_1)a(A))}.$$

8 Numerical examples and conjecture

We compute the special values of the standard zeta functions by using Mathematica. Let $\phi_{10,1}(\tau,z)$ and $\phi_{12,1}(\tau,z)$ be the Jacobi cusp forms in $J_{10,1}^{\mathrm{cusp}}$ and $J_{12,1}^{\mathrm{cusp}}$ in Page 40 of [E-Z], respectively. Here $\tau \in \mathbf{H}_1$ and $z \in \mathbf{C}$. Furthermore let $E_{1,k}(\tau)$ be the Eisenstein series of weight k with respect to $\Gamma^{(1)}$ defined in Section 2, and put $E_k(\tau) = \zeta(1-k)^{-1}E_{1,k}(\tau)$. Then it is well known that $E_4^a(\tau)E_6(\tau)^b\phi_{j,1}(\tau,z)$ $(a,b \geq 0, j = 10,12, 4a+6b+j = k)$ forms a basis of $J_{k,1}^{\mathrm{cusp}}$. Let $A_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}$.

(1) We have dim $\mathfrak{S}_{20}(\Gamma^{(2)}) = 3$, and dim $\mathfrak{S}_{38}(\Gamma^{(1)}) = 2$. Let f_1, f_2 be the basis of $\mathfrak{S}_{38}(\Gamma^{(1)})$ consisting of primitive forms. For $i = 1, 2$ let $\lambda_i = 48(-2025 + \sqrt{D})$ and $48(-2025 - \sqrt{D})$ with $D = 63737521$. Then λ_i is the eigenvalues of the $T(2)$ with respect to f_i. Then they satisfy the equation

$$X^2 + 194400X^2 - 137403408384 = 0,$$

and $\mathbf{Q}(f_i) = \mathbf{Q}(\lambda_i) = K$ with $K = \mathbf{Q}(\sqrt{D})$ (cf. [H-M]). Put $\theta_i = \lambda_i/96$. Then θ_i satisfies the following equation

$$g(X) := X^2 + 2025X - 14909224 = 0.$$

The discriminant of $g(X)$ is D. Thus the discriminant of $\mathbf{Q}(\sqrt{D})$ is D, and the ring of integers in $\mathbf{Q}(\sqrt{D})$ is $\mathbf{Z}(\theta_1)$. Let $h_1(\tau, z) = E_4(\tau)E_6(\tau)\phi_{10,1}(\tau, z)$, and $h_2(\tau, z) = E_4(\tau)^2\phi_{12,1}(\tau, z)$. These form a basis of $J_{20,1}^{\mathrm{cusp}}$. Put $g_i = \mathcal{V}h_i$ for $i = 1, 2$. Then these form a basis of $\mathfrak{S}_{20}(\Gamma^{(2)})^*$ whose A_0-th Fourier coefficient is 1. Furthermore for $i = 1, 2$ put

$$\hat{f}_i = -230g_1 + (-4862 - \theta_i)g_2.$$

Then $\hat{f}_i$ is the Saito-Kurokawa lift of f_i whose A_0-th Fourier coefficient is

$$a_{\hat{f}_i}(A_0) = -5092 - \theta_i.$$

We note that we have $\hat{f}_i = \chi_{20}^{(i)}/2$ for $i = 1, 2$, where $\chi_{20}^{(1)}$ and $\chi_{20}^{(2)}$ are the eigenforms in Kurokawa [Ku1]. Then we have

$$a_{\hat{f}_i}(A_1) = -10(4816 + \theta_i).$$

Furthermore we have

$$\lambda_{\hat{f}_i}(T(2)) = \lambda_i + 3 \cdot 2^{18}.$$

Then

$$N_{K/\mathbf{Q}}(a_{\hat{f}_i}(A_0)) = 2^2 \cdot 3^4 \cdot 5 \cdot 19 \cdot 23,$$

and

$$N_{K/\mathbf{Q}}(a_{\hat{f}_i}(A_1)) = -2^5 \cdot 3 \cdot 5^2 \cdot 23 \cdot 2659.$$

By a simple computation we have $N(\mathfrak{J}_{\hat{f}_i}) = 2^5 \cdot 3^2 \cdot 5 \cdot 23$.

Let $\Upsilon 20$ be the cuspidal Hecke eigenform in Skoruppa [Sk]. It is a unique (up to constant) Hecke eigenform in $\mathfrak{S}_{20}(\Gamma^{(2)})$ which is not a Saito-Kurokawa lift. We note that $\Upsilon 20 = \chi_{20}^{(3)}/2$, where $\chi_{20}^{(3)}$ is the Hecke eigenform in [Ku1]. Then $\hat{f}_1, \hat{f}_2$ and $\Upsilon 20$ form a basis of $\mathfrak{S}_{20}(\Gamma^{(2)})$. We have

$\mathfrak{J}_{\Upsilon 20} = 1$ and $a_{\Upsilon 20}(A_0) = 1$ and $a_{\Upsilon 20}(A_1) = 2^2$. Furthermore we have $\lambda_{\Upsilon 20}(T(2)) = -2^8 \cdot 3^2 \cdot 5 \cdot 73$. Thus by Proposition 7.6, we have the following tables:

l	$N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}})N(\mathfrak{J}_{\hat{f}_i})^2$
2	$3 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 29 \cdot 31/2^{58} \cdot D$
4	$3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 29 \cdot 31 \cdot 173 \cdot 443/2^{51} \cdot 23 \cdot D$
6	$3^2 \cdot 5 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 29 \cdot 31 \cdot 227 \cdot 1381069/2^{43} \cdot 23^2 \cdot D$
8	$3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 29 \cdot 31 \cdot 21347 \cdot 58169/2^{34} \cdot 23 \cdot D$
10	$3^4 \cdot 5^4 \cdot 7^2 \cdot 13^3 \cdot 17^2 \cdot 19 \cdot 31 \cdot 863 \cdot 3673 \cdot 3426433/2^{29} \cdot 23 \cdot D$
12	$3^2 \cdot 5 \cdot 7^2 \cdot 11 \cdot 17^2 \cdot 29 \cdot 37 \cdot 293 \cdot 691^2 \cdot 33721 \cdot 96875477/2^{16} \cdot 23 \cdot D$
14	$3^4 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13^2 \cdot 17 \cdot 29^2 \cdot 31 \cdot 467196139 \cdot 541368271/2^5 \cdot D$
16	$2^{13} \cdot 3^{10} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 29^2 \cdot 31^2 \cdot 67 \cdot 1699 \cdot 3617^2 \cdot 296551/D$

Here we note that the square of a prime factor of the l-th Bernoulli number appears in the numerator of $N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}})N(\mathfrak{J}_{\hat{f}_i})^2$.

l	$\tilde{\Lambda}(\Upsilon 20, l, \underline{\mathrm{St}})$
2	$3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31/2^{28}$
4	$-3^2 \cdot 5^2 \cdot 13 \cdot 23 \cdot 29 \cdot 31 \cdot 113/2^{25}$
6	$3^4 \cdot 5 \cdot 7 \cdot 29 \cdot 31 \cdot 7549/2^{17}$
8	$-3^3 \cdot 5 \cdot 7^2 \cdot \cdot 11 \cdot 29 \cdot 31 \cdot 37 \cdot 4861/2^{16}$
10	$-3 \cdot 5 \cdot 7 \cdot 31 \cdot 283 \cdot 617 \cdot 4098371/2^{13}$
12	$-3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 29 \cdot 31 \cdot 337 \cdot 91909/2^6$
14	$2^4 \cdot 3^4 \cdot 7^2 \cdot 13 \cdot 12893 \cdot 2166127$
16	$2^{11} \cdot 3^6 \cdot 5^3 \cdot 13 \cdot \cdot 23 \cdot 29 \cdot 347162819$

(2) We have $\dim \mathfrak{S}_{22}(\Gamma^{(2)}) = 4$, and $\dim \mathfrak{S}_{42}(\Gamma^{(1)}) = 3$. Let f_1, f_2, f_3 be the basis of $\mathfrak{S}_{42}(\Gamma^{(1)})$ consisting of primitive forms. For $i = 1, 2, 3$ let λ_i be the eigenvalues of the $T(2)$ with respect to f_i. Then they satisfy the equation

$$X^3 + 344688X^2 - 6374982426624X - 520435526440845312 = 0,$$

and $\mathbf{Q}(f_i) = \mathbf{Q}(\lambda_i)$ (cf. [H-M]). Put $\theta_i = \lambda_i/48$ for $i = 1, 2, 3$. Then θ_i is also an algebraic integer and satisfy the following equation:

$$g(X) := X^3 + 7181X^2 - 2766919456X - 4705905729536 = 0.$$

The discriminant of $g(X)$ is $-2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 1465869841 \cdot 578879197969$. Let $h_1(\tau, z) = E_4(\tau)^3\phi_{10,1}(\tau, z), h_2(\tau, z) = E_6(\tau)^2\phi_{10,1}(\tau, z)$, and $h_3(\tau, z) = E_4(\tau)E_6(\tau)\phi_{12,1}(\tau, z)$. Then these form a basis of $J_{22,1}^{\mathrm{cusp}}$. Put $g_i = \mathcal{V}h_i$ for $i = 1, 2, 3$. Then these form a basis of $\mathfrak{S}_{22}(\Gamma^{(2)})^*$ whose A_0-th Fourier coefficient is 1. Furthermore for $i = 1, 2, 3$ put

$$\hat{f}_i = 1155(435776+31\theta_i)g_1-220(4760624+79\theta_i)g_2+(286270336-605563\theta_i+\theta_i^2)g_3.$$

Then $\hat{f}_i$ is the Saito-Kurokawa lift of f_i whose A_0-th Fourier coefficient is

$$a_{\hat{f}_i}(A_0) = -257745664 - 421380\theta_i + \theta_i^2.$$

Then we have

$$a_{\hat{f}_i}(A_1) = 10(395073536 - 642480\theta_i + \theta_i^2)$$

and

$$a_{\hat{f}_i}(A_2) = -352(-3767171584 - 1827330\theta_i + \theta_i^2).$$

Furthermore we have

$$\lambda_{\hat{f}_i}(T(2)) = \lambda_i + 3 \cdot 2^{20}.$$

Then

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_0)) = -2^{14} \cdot 3^{13} \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13 \cdot 157 \cdot 1213$$

and

$$N_{K_i/\mathbf{Q}}(a_{\hat{f}_i}(A_1)) = -2^{24} \cdot 3^8 \cdot 5^5 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 157 \cdot 1447 \cdot 2437.$$

Let $\theta = \theta_1, \hat{f} = \hat{f}_1$, and $K = \mathbf{Q}(f_1)$. Put $\eta_2 = \theta(\theta + 13)/32, \eta_3 = (\theta^2 + 8)/9 + \theta, \eta_5 = (1 + \theta^2)/5 + \theta$, and $\eta_7 = (6 + 5\theta + \theta^2)/7 + 2\theta + 2$. Then by using, "round 2 method", we see that $< 1, \theta, \eta_2 >$ is 2-maximal in $\mathfrak{O}_K$ (cf. H. Cohen [C].) We have $[< 1, \theta, \eta_2 >:< 1, \theta, \theta^2 >] = 2^5$, and therefore the discriminant D of K is not divisible by 2. In the same manner, we see D is not divisible by $3 \cdot 5 \cdot 7$, and therefore, we have $D = 1465869841 \cdot 578879197969$ and $\mathfrak{O}_K =< 1, \theta, \eta_2, \eta_3, \eta_5, \eta_7 >_{\mathbf{Z}}$. We have

$$[\mathfrak{O}_K : \mathbf{Z}[1, \theta, \theta^2]] = 2^5 \cdot 3^2 \cdot 5 \cdot 7.$$

Put $R_1 = 1155(435776 + 31\theta_i), R_2 = -220(4760624 + 79\theta_i)$, and $R_3 = 286270336 - 605563\theta_i + \theta_i^2$. Then

$$[\mathbf{Z}[1, \theta, \theta^2] :< R_1, R_2, R_3 >] = 2^6 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^3 \cdot 13 \cdot 157,$$

and

$$[< R_1, R_2, R_3 >:< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >] = 2^8 3^4.$$

Furthermore we have $< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >\subset \mathfrak{I}_{\hat{f}}$. Thus we have $N(\mathfrak{I}_{\hat{f}}) = 2^{19} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 157$.

Let $\Upsilon 22$ be the cuspidal Hecke eigenform in Skoruppa [Sk]. It is a unique (up to constant) Hecke eigenform in $\mathfrak{S}_{22}(\Gamma^{(2)})$ which is not a Saito-Kurokawa lift. Then $\hat{f}_1, \hat{f}_2, \hat{f}_3$, and $\Upsilon 22$ form a basis of $\mathfrak{S}_{22}(\Gamma^{(2)})$ and $\mathbf{Q}(\hat{f}_i) = \mathbf{Q}(f_i)$. We have $\mathfrak{I}_{\Upsilon 22} = 1$ and $a_{\Upsilon 22}(A_0) = 1$ and $a_{\Upsilon 22}(A_1) = -2^2 \cdot 3$. Furthermore we have $\lambda_{\Upsilon 22}(T(2)) = -2^8 \cdot 3 \cdot 5 \cdot 577$. Thus by Proposition 7.6, we have the following tables:

l	$N_{K_i/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}})N(\mathfrak{I}_{\hat{f}_i})^2$
2	$3^9 \cdot 5^4 \cdot 7 \cdot 11^3 \cdot 13^5 \cdot 17^2 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37/2^{78} \cdot D$
4	$3^{12} \cdot 5^2 \cdot 7^4 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 151 \cdot 1601 \cdot 6551 \cdot 7951/2^{69} \cdot 1423 \cdot D$
6	$3^{12} \cdot 5^9 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 137 \cdot 809$ $\times 38029874887/2^{57} \cdot 7 \cdot 1423 \cdot D$
8	$3^9 \cdot 5 \cdot 7^5 \cdot 11 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 84521 \cdot 8947751$ $\times 699588169271/2^{41} \cdot 1423 \cdot D$
10	$3^{10} \cdot 5^9 \cdot 7^3 \cdot 11^4 \cdot 13^4 \cdot 17^2 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37^2$ $\times 1423469629 \cdot 27864526583393/2^{28} \cdot 1423 \cdot D$
12	$3^{12} \cdot 5 \cdot 11^2 \cdot 13 \cdot 17 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 691^3 \cdot 953$ $\times 243911 \cdot 4251563 \cdot 6617174324030971171/2^{10} \cdot 1423 \cdot D$
14	$2^6 \cdot 3^{12} \cdot 5^5 \cdot 7^8 \cdot 11^3 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37$ $\times 150197 \cdot 318467 \cdot 1465187 \cdot 13894099 \cdot 63630191/1423 \cdot D$
16	$2^{26} \cdot 3^{19} \cdot 5^5 \cdot 7^2 \cdot 11^3 \cdot 13^3 \cdot 19 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 3617^3$ $\times 1465869841 \cdot 2775014078857939 \cdot 22683897890722493/1423 \cdot D$
18	$2^{59} \cdot 3^{25} \cdot 5^{15} \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37^2$ $\times 43867^3 \cdot 365257 \cdot 13553776667/1423 \cdot D$

l	$\tilde{\Lambda}(\Upsilon 22; l, \underline{\mathrm{St}})$
2	$-3^3 \cdot 5 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37/2^{32}$
4	$-3^4 \cdot 5 \cdot 11 \cdot 13 \cdot 29 \cdot 31 \cdot 37 \cdot 103 \cdot 157/2^{27} \cdot 1423$
6	$-3^6 \cdot 11 \cdot 29 \cdot 31 \cdot 37^2 \cdot 485363/2^{24} \cdot 1423$
8	$-3^2 \cdot 29 \cdot 31 \cdot 37 \cdot 149 \cdot 3361493719/2^{18} \cdot 1423$
10	$-3^3 \cdot 5 \cdot 11 \cdot 37 \cdot 89 \cdot 1039 \cdot 2741 \cdot 3616027/2^{15} \cdot 1423$
12	$-3^4 \cdot 11^2 \cdot 31 \cdot 37 \cdot 421 \cdot 254725279909/2^8 \cdot 1423$
14	$-3^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 31 \cdot 37 \cdot 733 \cdot 2131 \cdot 82625047/2 \cdot 1423$
16	$-2^5 \cdot 3^7 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 30293340159041/1423$
18	$-2^{16} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 101 \cdot 439 \cdot 1049 \cdot 49991/1423$

In this case, we have

$$1423 = \mathfrak{P}_i \mathfrak{P}_i'$$

in $\mathfrak{O}_{\mathbf{Q}(f_i)}$, where $\mathfrak{P}_i =< \lambda_i + 967, 1423 >$ and $\mathfrak{P}_i' =< \lambda_i^2 + 778\lambda_i + 660, 1423 >$. We have deg $\mathfrak{P}_i = 1$ and deg $\mathfrak{P}_i' = 2$. Thus from the above table, $\mathfrak{P}_i$ is a congruence prime of $\hat{f}_i$. In fact we have

$$\lambda_{\hat{f}_i}(T(2)) \equiv \lambda_{\Upsilon 22}(T(2)) \bmod \mathfrak{P}_i.$$

On the other hand, according to the numerical table in Stein [St2], we have

$$|N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathbf{L}(f,22))| = \frac{11^3 \cdot 17 \cdot 1423}{2^{23} \cdot 3^{18} \cdot 5^{10} \cdot 13 \cdot 29 \cdot 31 \cdot 37^2 \cdot 137 \cdot 7481}.$$

This implies that Conjecture in Section 6 is true in this case. We will give other numerical examples in subsequent papers. Though we only treat the congruence of the Hecke eigenvalues of modular forms in this paper, we can also consider the congruence of the Fourier coefficients of them in some cases, and in particular, we can explain the examples of congruences of modular forms in Skoruppa [Sk] in terms of special values of L-functions, which we will also discuss in subsequent papers.

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