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A REMARK ON WEAK TYPE $(1,1)$ ESTIMATES OF HARDY-LITTLEWOOD MAXIMAL OPERATORS ON METRIC SPACES ACTING ON DIRAC MEASURES

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ABSTRACT. We consider weak type $(1,1)$ type estimates of Hardy-Littlewood maximal operators on a compact metric space with Radon measure, and also on a σ-compact metric space with Radon measure. We show that the analogus results with M. Trinidad Menarguez and F. Sorias' hold in these settings if we impose some conditions on metric measure spaces.

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1. Introduction

We treat in this paper weak type $(1,1)$ estimates of Hardy-Littlewood maximal operators on metric spaces acting on dirac measures. We prove that the weak type $(1,1)$ norm of the centered Hardy-Littelwood maximal operator acting on dirac measures are the same as that of the operator acting on integrable functions if we impose some conditions on the metric measure space considered. M. de Guzmán introduced the so-called discretization method ([2]) for the purpose of studying weak type $(1,1)$ estimates of Hardy-Littlewood maximal operators and singular integrals on Eucledian spaces. That is, to replace integrable functions by a finite sum of dirac measures in weak type $(1,1)$ estimates of these operators. He showed that if we have weak type $(1,1)$ estimates on Hardy-Littlewood maximal operators or Hilbert transform on a finite sum of dirac measures, then from that we can prove the weak type $(1,1)$ estimates of these operators on integrable functions (usual weak type $(1,1)$ estimates.) And he proved directly the weak type $(1,1)$ estimates of these operators on a finite sum of dirac measures. For the centered Hardy-Littlewood maximal operator in Eucledian space, we can preserve the operator norm by discretization, the fact which is shown by M.Trinidad Menarguez and F.Soria. Let us be more precise. Let ν be a (positive) finite sum of dirac deltas (which we also call simply as "dirac measures") in Eucledian

space. Then the centered Hardy-Littlewood maximal function of ν is defined as follows.

$$M\nu(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \nu.$$

Here, $B(x,r)$ is a Eucledian ball centered at $x \in \mathbf{R}^n$ and of radius $r > 0$, and $|\cdot|$ denotes the Lebesgue measure of sets in $\mathbf{R}^n$. M. Trinidad Menarguez and F. Soria's results ([1]) imply the following proposition.

Proposition 1.1. *Let f be an integrable function on $\mathbf{R}^n$ and let ν be a finite sum of dirac deltas. Let Mf be the centered Hardy-Littlewood maximal funciton of f. Let $M\nu$ be the centered Hardy-Littlewood maximal function of ν as is defined above. Then the following equality holds.*

$$\sup_{\lambda>0,\ f\not\equiv 0\in L^1(\mathbf{R}^n)} \frac{\lambda|\{x \mid Mf(x) > \lambda\}|}{\|f\|_1} = \sup_{\lambda>0,\ \nu:\text{dirac measures}} \frac{\lambda|\{x \mid M\nu(x) > \lambda\}|}{\|\nu\|_1}$$

Here $|\cdot|$ denotes the Lebesgue measure of subsets in $\mathbf{R}^n$ and $\|\nu\|_1$ denotes the total variation of ν.

This result was used to determin the weak type $(1,1)$ norm of the centered Hardy-Littlewood maximal operator on the real line. (cf. [5], [6].)

In this paper, we generalize this theorem to the fowllowing two theorems.

Theorem 1.2. *Let X be a compact metric space. Let μ be a Radon measure on X such that $\mu(B(x,r))$, where $B(x,r)$ is the ball centered at $x \in X$ and of radius $r > 0$, is continuous with the variable $r > 0$ for any $x \in X$. Let M be the centered Hardy-Littlewood maximal operator on X. Then the following equality where both sides are allowed to be infinite holds.*

$$\sup_{\lambda>0,\ f\not\equiv 0\in L^1(X)} \frac{\lambda|\{x \mid Mf(x) > \lambda\}|}{\|f\|_1} = \sup_{\lambda>0,\ \nu:\text{dirac measures}} \frac{\lambda|\{x \mid M\nu(x) > \lambda\}|}{\|\nu\|_1}$$

Here $|\cdot|$ denote the μ-measure of subsets in X and $\|\nu\|_1$ denotes the total variation of ν.

Theorem 1.3. *Let X be a σ-compact metric space. Let the function $g_{r,\delta}(x) := \mu(B(x, r+\delta))$ converges uniformly with respect to $x \in X$ to $g_{r,0}$ as δ tends to zero for any $r > 0$. Here δ is allowed to be take negative values. Let M be the centered Hardy-Littlewood maximal operator on X. Then the following equality where both sides are allowed to be infinite holds.*

$$\sup_{\lambda>0,\ f\not\equiv 0\in L^1(X)} \frac{\lambda|\{x \mid Mf(x) > \lambda\}|}{\|f\|_1} = \sup_{\lambda>0,\ \nu:\text{dirac measures}} \frac{\lambda|\{x \mid M\nu(x) > \lambda\}|}{\|\nu\|_1}$$

Here $|\cdot|$ denotes the μ-measure of subsets in X.

The latter theorem directly generalize the theorem by M. Trinidad Menarguez and F. Soria in the Eucledian spaces with Lebesgue measure. Althogh the proof of the theorem is similar to the proof by M. Trinidad Menarguez and F. Soria more or less, it cannot be paralleled since there is no algebraic structures on general metric spaces. Instead of algebraic strucures, we use the simple property of the metric "$d(x,y) = d(y,x)$", where d is the metric and x and y are any two points of the metirc space.

2. **Main results**

Since we cannot consider the convolution opertor in a general metric space, the complete analogue of M. Trinidad Menarguez and F. Soria's result cannot be established in this setting. We can, however, define maximal operators in a metric space, so concerning maximal operators, there would be possibilities to establish the anlogue of their result. In fact, we have the theorem below. The reader should notice the proof of the theorem basically follow the line of the arguments of M.Trinidad Menarguez and F.Soria ([1]). Before stating our theorem, we fix some notation. Let X be a metric space, and let μ be a Radon measure on X. Let ν be a finite sum of dirac deltas, that is the measure ν which can be written as $\nu = \Sigma_{k=1}^{L} \delta_{a_j}$. Here $\{a_j\} \subset N$ are not necessarily different points. We call these measures simply "dirac measures." Then we define $M\nu(x)$ as follows.

$$M\nu(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \nu.$$

Theorem 2.1. *Let X be a compact metric space. Let μ be a Radon measure on X such that $\mu(B(x,r))$ is continuous with the variable $r > 0$ for any $x \in X$. Let M be the centered Hardy-Littlewood maximal operator on X. Then the following equality where both sides are allowed to be infinite holds.*

$$(1) \quad \sup_{\lambda>0,\ f\not\equiv 0 \in L^1(X)} \frac{\lambda|\{x \mid Mf(x) > \lambda\}|}{\|f\|_1} = \sup_{\lambda>0,\ \nu:\text{dirac measures}} \frac{\lambda|\{x \mid M\nu(x) > \lambda\}|}{\|\nu\|_1}$$

equation

Here $\|\nu\|_1$ denotes the total variation of ν.

Equivalently, the weak type $(1,1)$ norm of the centered Hardy-Littlewood maximal operator actiing on integrable function in X is the same with that of the centered Hardy-Littlewood maximal operator actiing on dirac measures in X.

Proof. We first show that (R.H.S.) in (2.1) is greater than or equal to (L.H.S.) in (2.1). We set (R.H.S.) in (2.1) to be the numerical constant C.

We define $M_r f$ for any integrable functions $f \geq 0$ and $M_r\nu$ for any (positive) finite sum of dirac deltas ν as follows.

$$M_r f(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f d\mu,$$

$$M_r\nu(x) = \frac{1}{\mu(B(x,r))}\int_{B(x,r)} d\nu.$$

We also define $M_r(f-\nu)$ for any integrable functions $f \geq 0$ and any (positive) finite sum of dirac deltas ν as follows.

$$M_r(f-\nu)(x) = \frac{1}{\mu(B(x,r))}|\int_{B(x,r)} f d\mu - \int_{B(x,r)} d\nu|.$$

Let us number $\mathbf{Q}^+$ as $\{r_i\}_{i=1}^{i=+\infty}$. It will suffice to show that for each $I \in \mathbf{N}^+$ and for each $f \in L^1(N), f \geq 0$ and $\lambda > 0$,

$$|\{x \mid K_I^* f(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1,$$

where

$$K_I^* f(x) = \sup_{1\leq j\leq I} M_{r_j} f(x).$$

We define $K_I^*\nu(x)$ and $K_I^*(f-\nu)(x)$ in a similar way.

Let us first see that if $\nu = \Sigma_{k=1}^L c_k\delta_{a_k}$ with $c_k \in \mathbf{Q}^+$, we can immediately prove from our assumption that

$$|\{x \mid K_I^*\mu(x) > \lambda\}| \leq \frac{C}{\lambda}\Sigma_{k=1}^N c_k.$$

Let us consider the case $\nu = \Sigma_{k=1}^L c_k\delta_{a_k}$ with $c_k \in \mathbf{R}^+$. Fix $0 < \epsilon < \lambda$ and take for each k, some $c_k' \in \mathbf{Q}^+$ such that

$$0 < c_k - c_{k'} < \frac{\epsilon^2}{L\Sigma_{j=1}^I \sup_x |B(x,r_j)|}.$$

Then

$$|\{x \mid K_I^*\mu(x) > \lambda\}| \leq \frac{1}{\lambda-\epsilon}\Sigma_{k=1}^L c_k' + |A|$$

with

$$A = \{x \mid K_I^*(\Sigma_{k=1}^N (c_k - c_k')\delta_{a_k})(x) > \epsilon\}.$$

But

$$\begin{aligned} |A| &= |\{x \mid K_I^*(\Sigma_{k=1}^L (c_k - c_{k'})\delta_{a_k})(x) > \epsilon\}| \\ &\leq \Sigma_{j=1}^I |\{x \mid \Sigma_{k=1}^L (c_k - c_{k'})\chi_{B(a_k,r_j)}(x) > \epsilon\}| \\ &= \Sigma_{j=1}^I (\frac{1}{\epsilon}\Sigma_{k=1}^L (c_k - c_k')|B(a_k,r_j)|) \leq \epsilon \end{aligned}$$

Therefore, we have

$$|\{x \mid K_I^*\mu(x) > \lambda\}| \leq \frac{C}{\lambda}\Sigma_{k=1}^L c_k.$$

Let $f(x) = \Sigma_{k=1}^L c_k\chi_{A_k}$. If we consider the covering of X by $\frac{\delta}{2}$ radius balls, then, from the compactness of X we can divide each A_k into $A_k^1, A_k^2, \ldots, A_k^{q_k}$ such that $\mathrm{diam}(\mathrm{A_k^i}) < \delta$. The value of δ will be fixed later. Take a point $a_k^i \in A_k^i$ for each i, k. We define

$$\nu = \Sigma_{k=1}^L c_k(\Sigma_{i=1}^{q_k}|A_k^i|\delta_{a_k^i}).$$

f can be experessed as $f(x) = \Sigma_{k=1}^{L}\Sigma_{i=1}^{q_k} c_k \chi_{A_k^i}$. Let α be such that $0 < \alpha < \lambda$. We have

$$\begin{aligned}
&|\{x \mid K_I^* f(x) > \lambda\}| \\
&\leq |\{x \mid K_I^* \nu(x) > \lambda - \alpha\}| + |\{x \mid K_I^*(f-\nu)(x) > \alpha\}| \\
&\leq \frac{C}{\lambda - \alpha}\Sigma_{k=1}^{L} c_k |A_k| + \Sigma_{j=1}^{I} |\{x \mid M_{r_j}(f-\nu)(x) > \alpha\}| \\
&= \frac{C}{\lambda - \alpha}\Sigma_{k=1}^{L} c_k |A_k| \\
&+\Sigma_{j=1}^{I} |\{x \mid \frac{1}{|B(x,r_j)|}(|\int_{B(x,r_j)} f - \Sigma_{k=1}^{L} c_k (\Sigma_{i=1}^{q_k} |A_k^i| \chi_{B(a_k^i, r_j)}|) > \alpha\}|
\end{aligned}$$

The second term is equal to

$$\begin{aligned}
&\Sigma_{j=1}^{I} |\{x \mid \frac{\Sigma_{k=1}^{L}\Sigma_{i=1}^{q_k} c_k}{|B(x,r_j)|}(|\int_{A_k^i} \chi_{B(x,r_j)}(y)dy - |A_k^i| \chi_{B(a_k^i, r_j)}(x)|) > \alpha\}| \\
&\leq \Sigma_{j=1}^{I} \frac{\Sigma_{k=1}^{L}\Sigma_{i=1}^{q_k} c_k}{\alpha} \int_X (\frac{1}{|B(x,r_j)|}|\int_{A_k^i} \chi_{B(x,r_j)}(y)dy - |A_k^i| \chi_{B(a_k^i, r_j)}(x)|)dx \\
&= \Sigma_{j=1}^{I} \frac{\Sigma_{k=1}^{L}\Sigma_{i=1}^{q_k} c_k}{\alpha} \int_X (\frac{1}{|B(x,r_j)}|\int_{A_k^i} \chi_{B(x,r_j)}(y)dy - \int_{A_k^i} \chi_{B(a_k^i, r_j)}(x)dy|)dx \\
&= \Sigma_{j=1}^{I} \frac{\Sigma_{k=1}^{L}\Sigma_{i=1}^{q_k} c_k}{\alpha} \int_{A_k^i} (\int_X \frac{1}{|B(x,r_j)|}|\chi_{B(y,r_j)}(x) - \chi_{B(a_k^i, r_j)}(x)|dx)dy
\end{aligned}$$

We set

$$g_{j,\delta}(x) := |B(x, r_j + \delta)|.$$

Then $g_{j,\delta}$ increases and converges pointwise to $g_{j,0}$. as $\delta \to 0-$ and $g_{j,\delta}$ decreases and converges pointwise to $g_{j,0}$. as $\delta \to 0+$. Using Dini's theorem, we can see that the above convergence is uniform. From this fact, taking δ sufficiently small, the second term can be bounded by $\frac{I}{\alpha}\|f\|_1 \epsilon$. Letting first ϵ and then α go to zero, we have

$$|\{x \mid K_I^* f(x) > \lambda\}| \leq \frac{C}{\lambda}\Sigma_{k=1}^{L} c_k |A_k| \leq \frac{C}{\lambda}\|f\|_1.$$

Let $f(x)$ be an arbitrary nonnegative L^1 function. Then there exists nonnegative simple functions f_j increasing and converging to f. Since for each f_j we have

$$|\{x \mid K_I^* f_j(x) > \lambda\}| \leq \frac{C}{\lambda}\|f_j\|_1.$$

Letting $j \to +\infty$, we have

$$|\{x \mid K_I^* f(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

Thus we have the desired inequality.

We next show the converese inequality. We set (L.H.S.) in (2.1) to be the numerical constant C'.

Set $\nu = \Sigma_{k=1}^{L} \delta_{a_k}$. If we set $f_r(x) = \Sigma_{k=1}^{L} \frac{1}{|B(a_k,r)|} \chi_{B(a_k,r)}(x)$,

$$E_r = \{x \mid K_M^* f_r(x) > \lambda\}$$

and

$$E = \{x \mid \sup_{1 \leq j \leq I} |\Sigma_{k=1}^{L} \chi_{B(a_k,r_j)}(x)| > \lambda\},$$

then $|E_r| \leq \frac{C}{\lambda} \|f_r\|_1 = \frac{C'L}{\lambda}$ and we have $\chi_E(x) \leq \liminf_{n \to +\infty} \chi_{E_{\frac{1}{n}}}$.

By applying Fatou's lemma, we conclude

$$|E| \leq \int \liminf_{n \to +\infty} \chi_{E_{\frac{1}{n}}}(x) dx \leq \liminf_{n \to +\infty} |E_{\frac{1}{n}}| \leq \frac{C'L}{\lambda}.$$

Hence we have the converse inequality.

Remark 2.2. By checking the proof of the above theorem carefully, you may find that the condtion about the space and the measrue can be replaced by another condtion. In fact, we can have the following theorem.

Theorem 2.3. *Let X be a σ-compact metric space with Radon measure μ. Let the function $g_{r,\delta}(x) := \mu(B(x, r+\delta))$ converges uniformly with respect to $x \in X$ to $g_{r,0}$ as δ tends to zero for any $r > 0$. Here δ is allowed to take negative values. Let M be the centered Hardy-Littlewood maximal operator on X. Then the following equality where both sides are allowed to be infinite holds.*

$$\sup_{\lambda>0,\ f \not\equiv 0 \in L^1(X)} \frac{\lambda |\{x \mid Mf(x) > \lambda\}|}{\|f\|_1} = \sup_{\lambda>0,\ \nu:\text{dirac measures}} \frac{\lambda |\{x \mid M\nu(x) > \lambda\}|}{\|\nu\|_1}$$

Here $|\cdot|$ denotes the μ-measure of subsets in X.

The case which was mentioned in the introduction of the article, that is, the case that X is $\mathbf{R}^n$ and μ is Lebesgue measure is obviously inclueded in this theorem.

Remark 2.4. As we cannot approximate arbitrary finite measures by a sequence of integrable functions in some uniform way in arbitrary metric space which has no group structure in general, we cannot get the same kind of the result for arbitrary finite measure, that is the equalness between the weak $(1,1)$ norm of maximal functions acting on arbitrary measures and on integrable functions. However, one can prove that kind of results for maximal convolution operators on any locally compact group.(cf. [1].)

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