

HOKKAIDO UNIVERSITY

Title	Semimartingales from the Fokker-Planck equation
Author(s)	Mikami, Toshio
Citation	Hokkaido University Preprint Series in Mathematics, 724, 1-15
Issue Date	2005-06-01
DOI	https://doi.org/10.14943/83874
Doc URL	https://hdl.handle.net/2115/69532
Type	departmental bulletin paper
File Information	re724.pdf



Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

Semimartingales from the Fokker-Planck equation

Dedicated to Professor Wendell H. Fleming
on the occasion of his seventy seventh birthday

Toshio Mikami*
Hokkaido University

June 1, 2005

Abstract

We show the existence of a semimartingale of which one-dimensional marginal distributions are given by the solution of the Fokker-Planck equation with the p-th integrable drift vector ($p > 1$).

Keywords: stochastic control, marginal problem, Nelson process

1 Introduction.

Let $\mathcal{M}_1(\mathbf{R}^d)$ denote the complete separable metric space, with a weak topology, of Borel probability measures on $\mathbf{R}^d$ ($d \geq 1$).

Let $b : [0,1] \times \mathbf{R}^d \mapsto \mathbf{R}^d$ be measurable and $\{P_t(dx)\}_{0 \leq t \leq 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$, satisfy the following Fokker-Planck equation: for $f \in C_b^{1,2}([0,1] \times \mathbf{R}^d)$ and $t \in [0,1]$,

*Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan; mikami@math.sci.hokudai.ac.jp; phone & fax no. 81/11/706/3444; partially supported by the Grant-in-Aid for Scientific Research, No. 15340047, 15340051 and 16654031, JSPS.

$$\int_{\mathbf{R}^d} f(t,x)P_t(dx) - \int_{\mathbf{R}^d} f(0,x)P_0(dx) \tag{1.1}$$
$$= \int_0^t ds \int_{\mathbf{R}^d} \Big(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2}\triangle f(s,x) + < b(t,x), D_x f(s,x) > \Big) P_s(dx),$$

where $\triangle := \sum_{i=1}^d \partial^2/\partial x_i^2$, $D_x := (\partial/\partial x_i)_{i=1}^d$, and $< \cdot, \cdot >$ denotes the inner product in $\mathbf{R}^d$.

Inspired by Born's probabilistic interpretation of a solution to Schrödinger's equation, Nelson proposed the problem of the construction of a diffusion process $\{X(t)\}_{0\le t\le 1}$ for which the following holds (see [19]):

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + W(t) \quad (t \in [0,1]), \tag{1.2}$$
$$P(X(t) \in dx) = P_t(dx) \quad (t \in [0,1]), \tag{1.3}$$

where $\{W(t)\}_{0\le t\le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$-Wiener process.

The first result was given by Carlen [2] (see also [22]). It was generalized, by Mikami [12], to the case where the second order differential operator has a variable coefficient. The further generalization and almost complete resolution was made by Cattiaux and Léonard [3-6] (see also [1, 13, 14] for the related topics). But in these papers, they assumed that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^2 P_t(dx) < \infty \tag{1.4}$$

for some b for which (1.1) holds. This is called the **finite energy condition** for $\{P_t(dx)\}_{0\le t\le 1}$.

Remark 1.1 *It is known that b is not unique for $\{P_t(dx)\}_{0\le t\le 1}$ in (1.1) (see [12] or [3-6]).*

In this paper we consider Nelson's problem under a weaker assumption than (1.4): there exists $p > 1$ such that

$$\int_0^1 dt \int_{\mathbf{R}^d} |b(t,x)|^p P_t(dx) < \infty \tag{1.5}$$

for some b for which (1.1) holds. We call (1.5) the **generalized finite energy condition** for $\{P_t(dx)\}_{0\le t\le 1}$.

Let $L(t,x;u) : [0,1] \times \mathbf{R}^d \times \mathbf{R}^d \mapsto [0,\infty)$ be continuous and be convex in u. Let $\mathcal{A}$ denote the set of all $\mathbf{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0\le t\le 1}$ on a complete filtered probability space such that there exists a Borel measurable $\beta_X : [0,1] \times C([0,1]) \mapsto \mathbf{R}^d$ for which
(i) $\omega \mapsto \beta_X(t,\omega)$ is $\mathcal{B}(C([0,t]))_+$-measurable for all $t \in [0,1]$, where $\mathcal{B}(C([0,t]))$ denotes the Borel σ-field of $C([0,t])$,
(ii) $\{W_X(t) := X(t) - X(0) - \int_0^t \beta_X(s,X)ds\}_{0\le t\le 1}$ is a $\sigma[X(s) : 0 \le s \le t]$-Wiener process.
For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$V(P_0,P_1) := \inf\Big\{E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\Big| PX(t)^{-1} = P_t(t=0,1), X \in \mathcal{A}\Big\}, \tag{1.6}$$

$$\begin{aligned} & v(P_0,P_1) \\ := \ & \inf\Big\{\int_0^1\int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx)dt\Big| P(t,dx) = P_t(dx)(t=0,1), \\ & \{P(t,dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d), (b(t,x),P(t,dx)) \text{ satisfies (1.1)}\Big\}. \end{aligned} \tag{1.7}$$

In [12] where $u \mapsto L$ is quadratic, we proved and used the following:

$$V(P_0,P_1) = v(P_0,P_1). \tag{1.8}$$

Remark 1.2 *As a typical case, when $L = |u|^2$, the minimizer of $V(P_0,P_1)$ is known to be the h-path process for the space-time Brownian motion (see [7, 17] and the references therein). It is known that its zero-noise limit exists and is the unique minimizer of $T_M(P_0,P_1)$ (see [15, 18]).*

In this paper we prove (1.8) for a more general function L by the duality theorem for V. To make the point clearer, we describe [17] briefly. For P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$, put

$$\mathcal{V}(P_0,P_1) := \sup\Big\{\int_{\mathbf{R}^d}\varphi(1,y)P_1(dy) - \int_{\mathbf{R}^d}\varphi(0,x)P_0(dx)\Big\}, \tag{1.9}$$

where the supremum is taken over all classical solutions φ to the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} \frac{\partial \varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + H(t,x;D_x\varphi(t,x)) &= 0((t,x) \in (0,1)\times \mathbf{R}^d) \quad (1.10) \\ \varphi(1,\cdot) &\in C_b^\infty(\mathbf{R}^d) \end{aligned}$$

(see Lemma 3.1). Here for $(t,x,z) \in [0,1]\times\mathbf{R}^d\times\mathbf{R}^d$,

$$H(t,x;z) := \sup_{u\in\mathbf{R}^d}\{< z,u > -L(t,x;u)\}. \tag{1.11}$$

The following was proved in [17] and is called the duality theorem for the stochastic optimal control problem (1.6).

Theorem 1.1 (Duality Theorem) *Suppose that (A.1)-(A.4) in section 2 hold. Then for any P_0 and $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(P_0,P_1) = \mathcal{V}(P_0,P_1)(\in [0,\infty]). \tag{1.12}$$

Suppose in addition that $V(P_0,P_1)$ is finite. Then $V(P_0,P_1)$ has a minimizer and for any minimizer $\{X(t)\}_{0\le t\le 1}$ of $V(P_0,P_1)$,

$$\beta_X(t,X) = b_X(t,X(t)) := E[\beta_X(t,X)|(t,X(t))]. \tag{1.13}$$

Remark 1.3 *(1.12) can be considered as a counterpart in the stochastic optimal control theory of the duality theorem in the Monge-Kantorovich problem (see [10, 16, 20, 21] and the references therein).*

Using a similar result to (1.8) on small time intervals $\subset [0,1]$, we prove that for $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P}), \tag{1.14}$$

where

$$\mathbf{V}(\mathbf{P}) := \inf\Big\{E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\Big| PX(t)^{-1} = P_t(0\le t\le 1), X\in\mathcal{A}\Big\}, \tag{1.15}$$

$$\mathbf{v}(\mathbf{P}) := \inf\Big\{\int_0^1 dt \int_{\mathbf{R}^d} L(t,x;b(t,x))P_t(dx) | b \text{ satisfies (1.1)}\Big\}. \tag{1.16}$$

In particular, the existence of a minimizer of $\mathbf{V}(\mathbf{P})$ implies that of a semimartingale for which (1.2)-(1.3) hold. When $p = 2$ in (1.10), this semimartingale is Markovian. But we do not know if it is also true even when $1 < p < 2$. This is our future problem.

In section 2 we state our result which will be proved in section 4. Technical lemmas are given in section 3.

I would like to dedicate this paper to Professor Wendell H. Fleming on the occasion of his seventy seventh birthday. I would like to thank him for his constant encouragement since I was a student of his.

2 Main result.

In this section we state our result.

We state assumptions on L.

(A.1). There exists $p > 1$ such that

$$\liminf_{|u|\to\infty} \frac{\inf\{L(t,x;u) : (t,x) \in [0,1]\times\mathbf{R}^d\}}{|u|^p} > 0.$$

(A.2).

$$\Delta L(\varepsilon_1,\varepsilon_2) := \sup \frac{L(t,x;u) - L(s,y;u)}{1 + L(s,y;u)} \to 0 \quad \text{as } \varepsilon_1,\, \varepsilon_2 \to 0,$$

where the supremum is taken over all (t,x) and (s,y), $\in [0,1]\times\mathbf{R}^d$, for which $|t-s| \le \varepsilon_1$, $|x-y| < \varepsilon_2$ and all $u \in \mathbf{R}^d$.

(A.3). (i)$L(t,x;u) \in C^3([0,1]\times\mathbf{R}^d\times\mathbf{R}^d : [0,\infty))$,

(ii) $D_u^2 L(t,x;u)$ is positive definite for all $(t,x,u) \in [0,1]\times\mathbf{R}^d\times\mathbf{R}^d$,

(iii) $\sup\{L(t,x;o) : (t,x) \in [0,1]\times\mathbf{R}^d\}$ is finite,

(iv) $|D_x L(t,x;u)|/(1 + L(t,x;u))$ is bounded,

(v) $\sup\{|D_u L(t,x;u)| : (t,x) \in [0,1]\times\mathbf{R}^d, |u| \le R\}$ is finite for all $R > 0$.

(A.4). (i) $\Delta L(0,\infty)$ is finite, or (ii) $p = 2$ in (A.1).

Remark 2.1 *(i). (A.3, ii) implies that $L(t,x;u)$ is strictly convex in u. (ii). $(1+|u|^2)^{p/2}$ $(p > 1)$ satisfies (A.1)-(A.3) and (A.4,i).*

We state that (1.8) holds.

Theorem 2.1 *Suppose that (A.1)-(A.4) hold. Then for any* P_0 *and* $P_1 \in \mathcal{M}_1(\mathbf{R}^d)$,

$$V(P_0, P_1) = v(P_0, P_1)(\in [0, \infty]). \tag{2.1}$$

The following is our main result (see (1.15)-(1.16) for notations).

Theorem 2.2 *Suppose that (A.1)-(A.4) hold. Then*
(i) for any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$,

$$\mathbf{V}(\mathbf{P}) = \mathbf{v}(\mathbf{P})(\in [0, \infty]). \tag{2.2}$$

(ii) For any $\mathbf{P} := \{P_t(dx)\}_{0 \le t \le 1}, \subset \mathcal{M}_1(\mathbf{R}^d)$, *for which* $\mathbf{v}(\mathbf{P})$ *is finite, there exist a unique minimizer* $b_o(t,x)$ *of* $\mathbf{v}(\mathbf{P})$ *and a minimizer* $X, \in \mathcal{A}$, *of* $\mathbf{V}(\mathbf{P})$. *In particular, for any minimizer* $X, \in \mathcal{A}$, *of* $\mathbf{V}(\mathbf{P})$,

$$\beta_X(t, X) = b_o(t, X(t)) \tag{2.3}$$

and (1.2)-(1.3) with $b = b_o$ *hold.*

Remark 2.2 *(i). If* $\mathbf{v}(\mathbf{P})$ *is finite, then the generalized finite energy condition (1.5) holds from (A.1). (ii). If (A.1) and (A.4, ii) hold and* L *is convex in* u, *then one can easily show that (2.1)-(2.2) hold. Indeed, it is easy to show that the inequality "*$\ge$*" holds (see (3.6)-(3.7)). For any* $(P_t(dx), b(t,x))$ *for which (1.1) holds and* $\{P_t(dx)\}_{0 \le t \le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$, *one can construct a Markov process for which (1.2)-(1.3) hold (see [3, 4]). This implies the inequality "*$\le$*" in (2.1)-(2.2). Suppose in addition that (A.2) holds and that* L *is strictly convex in* u. *Then (2.3) holds, which can be proved in the same way as in the proof of Theorem 2.2.*

3 Lemmas.

In this section we give technical lemmas.

In the same way as to $\mathcal{A}$, we define the set of semimartingales $\mathcal{A}_t$ in $C([t,1])$. We recall the following result.

Lemma 3.1 ([8, p. 210, Remark 11.2]) *Suppose that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$, the HJB equation (1.10) with $\varphi(1,\cdot) = f$ has a unique solution φ, $\in C^{1,2}([0,1]\times\mathbf{R}^d) \cap C_b^{0,1}([0,1]\times\mathbf{R}^d)$, which can be written as follows:*

$$\begin{aligned}\varphi(t,x) &= \sup_{X\in\mathcal{A}_t}\Big\{E[f(X(1))|X(t)=x] \\ &\qquad -E\Big[\int_t^1 L(s,X(s);\beta_X(s,X))ds\Big|X(t)=x\Big]\Big\},\end{aligned} \tag{3.1}$$

where for the maximizer $X \in \mathcal{A}_t$, the following holds:

$$\beta_X(s,X) = D_z H(s,X(s);D_x\varphi(s,X(s))).$$

Fix $P_0 \in \mathcal{M}_1(\mathbf{R}^d)$. For $f \in C_b(\mathbf{R}^d)$, put

$$V^*(f) := \sup_{P\in\mathcal{M}_1(\mathbf{R}^d)}\Big\{\int_{\mathbf{R}^d} f(x)P(dx) - V(P_0,P)\Big\}, \tag{3.2}$$

$$v^*(f) := \sup_{P\in\mathcal{M}_1(\mathbf{R}^d)}\Big\{\int_{\mathbf{R}^d} f(x)P(dx) - v(P_0,P)\Big\}. \tag{3.3}$$

The following lemma plays a crucial role in the proof of Theorem 2.1.

Lemma 3.2 *(i) Suppose that (A.3, i, ii) hold. Then for any Q_0 and $Q_1 \in \mathcal{M}_1(\mathbf{R}^d)$,*

$$V(Q_0,Q_1) \geq v(Q_0,Q_1). \tag{3.4}$$

(ii) Suppose in addition that (A.1) and (A.3) hold. Then for any $f \in C_b^\infty(\mathbf{R}^d)$,

$$V^*(f) \geq v^*(f). \tag{3.5}$$

(Proof) We first prove (i). For $X \in \mathcal{A}$ for which $E[\int_0^1 L(t,X(t);\beta_X(t,X))dt]$ is finite and for which $PX(t)^{-1} = Q_t$ $(t = 0,1)$, $(b_X(t,x), P(X(t) \in dx))$ satisfies (1.1) with $(b(t,x),P_t(dx)) = (b_X(t,x),P(X(t)\in dx))$ (see (1.13) for notation). Indeed, for any $f \in C_b^{1,2}([0,1]\times\mathbf{R}^d)$ and $t \in [0,1]$, by Itô's formula,

$$
\begin{aligned}
&\int_{\mathbf{R}^d} f(t,x)P(X(t)\in dx) - \int_{\mathbf{R}^d} f(0,x)P(X(0)\in dx) \qquad (3.6)\\
=\ & E[f(t,X(t)) - f(0,X(0))]\\
=\ & \int_0^t ds E\Big[\frac{\partial f(s,X(s))}{\partial s} + \frac{1}{2}\triangle f(s,X(s)) + < \beta_X(s,X), D_x f(s,X(s)) >\Big]\\
=\ & \int_0^t ds E\Big[\frac{\partial f(s,X(s))}{\partial s} + \frac{1}{2}\triangle f(s,X(s)) + < b_X(s,X(s)), D_x f(s,X(s)) >\Big]\\
=\ & \int_0^t ds \int_{\mathbf{R}^d}\Big(\frac{\partial f(s,x)}{\partial s} + \frac{1}{2}\triangle f(s,x) + < b_X(s,x), D_x f(s,x) >\Big)P(X(s)\in dx).
\end{aligned}
$$

Hence, from Remark 2.1, (i), by Jensen's inequality,

$$
\begin{aligned}
& E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big] \qquad (3.7)\\
\geq\ & E\Big[\int_0^1 L(t,X(t);b_X(t,X(t)))dt\Big]\\
=\ & \int_0^1 dt \int_{\mathbf{R}^d} L(t,x;b_X(t,x))P(X(t)\in dx) \geq v(Q_0,Q_1).
\end{aligned}
$$

Next we prove (ii). For φ in (3.1) and $\{(b(t,x),P(t,dx))\}_{0\leq t\leq 1}$ for which $\{P(t,dx)\}_{0\leq t\leq 1}\subset \mathcal{M}_1(\mathbf{R}^d)$ and (1.1) with $P(0,dx)=P_0$ holds,

$$
\int_{\mathbf{R}^d} f(x)P(1,dx) - \int_{\mathbf{R}^d}\varphi(0,x)P_0(dx) \leq \int_0^1 dt\int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx). \qquad (3.8)
$$

Indeed, take $\psi\in C_o^\infty(\mathbf{R}^d:[0,\infty))$ for which $\psi(x)=1$ ($|x|\leq 1$) and $\psi(x)=0$ ($|x|\geq 2$), and put $\psi_R(x):=\psi(x/R)$ for $R>0$. Then from (1.6),

$$
\begin{aligned}
&\int_{\mathbf{R}^d}\psi_R(x)f(x)P(1,dx) - \int_{\mathbf{R}^d}\psi_R(x)\varphi(0,x)P(0,dx) \qquad (3.9)\\
=\ & \int_0^1 dt\int_{\mathbf{R}^d}\psi_R(x)\Big[\frac{\partial\varphi(t,x)}{\partial t} + \frac{1}{2}\triangle\varphi(t,x) + < b(t,x), D_x\varphi(t,x) >\Big]P(t,dx)\\
&\quad + \int_0^1 dt\int_{\mathbf{R}^d}\Big[< D_x\psi_R(x), D_x\varphi(t,x) > + \frac{1}{2}\triangle\psi_R(x)\varphi(t,x)\\
&\qquad\qquad + < b(t,x), D_x\psi_R(x) > \varphi(t,x)\Big]P(t,dx).
\end{aligned}
$$

Let $R \to \infty$. Then we obtain (3.8) from (1.10), (A.1) and Lemma 3.1.

Lemma 3.1 and (3.8) implies (ii). Indeed,

$$\begin{aligned} v^*(f) &= \sup\Big\{ \int_{\mathbf{R}^d} f(x)P(1,dx) - \int_0^1 dt \int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx)| \quad (3.10) \\ &\qquad P(0,dx) = P_0(dx), \{P(t,dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d), \\ &\qquad (b(t,x), P(t,dx)) \text{ satisfies (1.1).} \Big\} \\ &\le \int_{\mathbf{R}^d} \varphi(0,x)P_0(dx) \quad (\text{from (3.8)}) \\ &= \sup\Big\{ E\Big[f(X(1)) - \int_0^1 L(t,X(t);\beta_X(t,X))dt \Big]\Big| \\ &\qquad PX(0)^{-1} = P_0, X \in \mathcal{A} \Big\} \quad (\text{from Lemma 3.1}) \\ &= V^*(f). \square \end{aligned}$$

Let $(\Omega, \mathbf{B}, \{\mathbf{B}_t\}_{t\ge 0}, P)$ be a complete filtered probability space, X_o be a $(\mathbf{B}_0)$-adapted random variable, and $\{W(t)\}_{t\ge 0}$ denote a d-dimensional $(\mathbf{B}_t)$-Wiener process for which $W(0) = o$ (see e.g., [11]). For a $\mathbf{R}^d$-valued, $(\mathbf{B}_t)$-progressively measurable stochastic process $\{u(t)\}_{0\le t\le 1}$, put

$$X^u(t) = X_o + \int_0^t u(s)ds + W(t) \quad (t \in [0,1]). \tag{3.11}$$

Then the following is known.

Lemma 3.3 *Suppose that $E[\int_0^1 |u(t)|dt]$ is finite. Then $\{X^u(t)\}_{0\le t\le 1} \in \mathcal{A}$ and*

$$\beta_{X^u}(t, X^u) = E[u(t)|X^u(s), 0 \le s \le t] \tag{3.12}$$

(see [11, p. 270]). Besides, by Jensen's inequality,

$$E\Big[\int_0^1 L(t, X^u(t); u(t))dt\Big] \ge E\Big[\int_0^1 L(t, X^u(t); \beta_{X^u}(t, X^u))dt\Big]. \tag{3.13}$$

For $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ and $n \ge 1$, put

$$V_n(\mathbf{P}) := \inf\Big\{E\Big[\int_0^1 L(t,X(t);\beta_X(t,X))dt\Big]\Big| \quad (3.14)$$
$$PX(t)^{-1} = P_t\ \Big(t = \frac{i}{2^n}, i = 0,\cdots,2^n\Big), X \in \mathcal{A}\Big\},$$

$$v_n(\mathbf{P}) := \inf\Big\{\int_0^1 dt \int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx)\Big| \quad (3.15)$$
$$P(t,dx) = P_t(dx)\ \Big(t = \frac{i}{2^n}, i = 0,\cdots,2^n\Big),$$
$$\{P(t,dx)\}_{0\le t\le 1} \subset \mathcal{M}(\mathbf{R}^d), (b(t,x),P(t,dx)) \text{ satisfies (1.1)}\Big\}.$$

Then we have

Lemma 3.4 *Suppose that (A.1)-(A.4) hold. Then for any* $\mathbf{P} := \{P_t(dx)\}_{0\le t\le 1} \subset \mathcal{M}_1(\mathbf{R}^d)$ *and* $n \ge 1$*,*

$$v_n(\mathbf{P}) = V_n(\mathbf{P}). \quad (3.16)$$

(Proof) For $i = 0,\cdots,2^n-1$, put

$$V_{n,i}(\mathbf{P}) := \inf\Big\{E\Big[\int_0^{\frac{1}{2^n}} L(t,X(t);\beta_X(t,X))dt\Big]\Big| \quad (3.17)$$
$$PX(t)^{-1} = P_{t+\frac{i}{2^n}}\ \Big(t = 0, \frac{1}{2^n}\Big), X \in \mathcal{A}\Big\},$$

$$v_{n,i}(\mathbf{P}) \quad (3.18)$$
$$:= \inf\Big\{\int_0^{\frac{1}{2^n}} dt \int_{\mathbf{R}^d} L(t,x;b(t,x))P(t,dx)\Big|$$
$$P(t,dx) = P_{t+\frac{i}{2^n}}(dx)\ \Big(t = 0, \frac{1}{2^n}\Big), \{P(t,dx)\}_{0\le t\le\frac{1}{2^n}} \subset \mathcal{M}(\mathbf{R}^d),$$
$$(b(t,x),P(t,dx)) \text{ satisfies (1.1) on } [0,1/2^n]\}.$$

Then, from Theorem 2.1,

$$v_n(\mathbf{P}) = \sum_{i=0}^{2^n-1} v_{n,i}(\mathbf{P}) = \sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}). \tag{3.19}$$

Since $V_n(\mathbf{P}) \geq v_n(\mathbf{P})$ from (3.6)-(3.7), we only have to prove the following:

$$\sum_{i=0}^{2^n-1} V_{n,i}(\mathbf{P}) \geq V_n(\mathbf{P}). \tag{3.20}$$

Suppose that the left hand side of (3.20) is finite. For $i = 0, \cdots 2^n - 1$, take a minimizer $X_{n,i}$ of $V_{n,i}(\mathbf{P})$ (see Theorem 1.1), and put

$$P_{n,i} := PX_{n,i}\Big(\cdot - \frac{i}{2^n}\Big)^{-1} \quad \text{on } (C([\tfrac{i}{2^n}, \tfrac{i+1}{2^n}] : \mathbf{R}^d), \mathcal{B}(C([\tfrac{i}{2^n}, \tfrac{i+1}{2^n}] : \mathbf{R}^d))), \tag{3.21}$$

$$\begin{aligned} P_n\Big(dX|_{C([0,1]:\mathbf{R}^d)}\Big) \;&:=\; P_{n,0}\Big(dX|_{C([0,\frac{1}{2^n}]:\mathbf{R}^d)}\Big) \\ &\quad \times \Pi_{i=1}^{2^n-1} P_{n,i}\Big(dX|_{C([\frac{i}{2^n},\frac{i+1}{2^n}]:\mathbf{R}^d)}\Big| X_{n,i}\Big(\frac{i}{2^n}\Big) = X\Big(\frac{i}{2^n}\Big)\Big) \end{aligned} \tag{3.22}$$

on $(C([0,1] : \mathbf{R}^d), \mathcal{B}(C([0,1] : \mathbf{R}^d)))$. Under the completion of this measure, the coordinate process $\{X_n(t)\}_{0\leq t\leq 1}$ satisfies the following:

$$X_n(t) = X_n(0) + \sum_{i=0}^{2^n-1} \int_{\min(\frac{i}{2^n},t)}^{\min(\frac{i+1}{2^n},t)} b_{n,i}\Big(s - \frac{i}{2^n}, X_n(s)\Big)ds + W_{X_n}(t) \quad (0 \leq t \leq 1), \tag{3.23}$$

where $b_{n,i}$ denotes the drift vector of $X_{n,i}$ (see Theorem 1.1). In particular, $PX_n(t)^{-1} = P_t$ $(t = i/2^n, i = 0, \cdots, 2^n)$, which implies (3.20). □

4 Proofs.

In this section we prove our results given in section 2.

When $L = |u|^2$, the following proof extremely simplifies that of [12, Lemma 2.5].

(Proof of Theorem 2.1). Lemma 3.2, (i) and the following complete the proof:

$$
\begin{aligned}
& v(P_0, P_1) \qquad (4.1)\\
\geq\ & \sup_{f \in C_b^\infty(\mathbf{R}^d)} \Big\{ \int_{\mathbf{R}^d} f(x) P_1(dx) - v^*(f) \Big\} \quad \text{(from (3.3))} \\
\geq\ & \sup_{f \in C_b^\infty(\mathbf{R}^d)} \Big\{ \int_{\mathbf{R}^d} f(x) P_1(dx) - V^*(f) \Big\} \quad \text{(from Lemma 3.2, (ii))} \\
=\ & V(P_0, P_1) \quad \text{(from Theorem 1.1 (see (3.10))).} \square
\end{aligned}
$$

(Proof of Theorem 2.2). We first prove (i). From (3.6)-(3.7), $\mathbf{V}(\mathbf{P}) \geq \mathbf{v}(\mathbf{P})$. Therefore we only have to show that

$$
\mathbf{v}(\mathbf{P}) \geq \mathbf{V}(\mathbf{P}). \qquad (4.2)
$$

Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then, from Lemma 3.4,

$$
\mathbf{v}(\mathbf{P}) \geq v_n(\mathbf{P}) = V_n(\mathbf{P}) \qquad (4.3)
$$

and X_n constructed in (3.23) is a minimizer of $V_n(\mathbf{P})$.

Let b_n denote the drift vector of $\{X_n(t)\}_{0 \leq t \leq 1}$. It is easy to see that $\{(X_n(t), \int_0^t b_n(s, X_n(s))ds) : t \in [0,1]\}_{n \geq 1}$ is tight in $C([0,1] : \mathbf{R}^{2d})$ from (A.1) (see [22, Theorem 3] or [9]). Take a weakly convergent subsequence $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))ds) : t \in [0,1]\}_{k \geq 1}$ such that

$$
\begin{aligned}
& \liminf_{n \to \infty} E\Big[\int_0^1 L(t, X_n(t); b_n(s, X_n(s)))dt \Big] \qquad (4.4)\\
=\ & \lim_{k \to \infty} E\Big[\int_0^1 L(t, X_{n_k}(t); b_{n_k}(s, X_{n_k}(s)))dt \Big].
\end{aligned}
$$

Let $\{(X(t), A(t))\}_{t \in [0,1]}$ denote the limit of $\{(X_{n_k}(t), \int_0^t b_{n_k}(s, X_{n_k}(s))ds) : t \in [0,1]\}_{k \geq 1}$ as $k \to \infty$. Then $\{X(t) - X(0) - A(t)\}_{t \in [0,1]}$ is a $\sigma[X(s) : 0 \leq s \leq t]$-Wiener process and $\{A(t)\}_{t \in [0,1]}$ is absolutely continuous (see [22, Theorem 5] or [9]). We can also prove, in the same way as in the proof of [14, (3.17)], the following: from (4.3)-(4.4), (A.2) and (A.3, ii) (see Remark 2.1, (i)),

$$
\mathbf{v}(\mathbf{P}) \ \geq\ \limsup_{n \to \infty} E\Big[\int_0^1 L(t, X_n(t); b_n(t, X_n(t)))dt \Big] \qquad (4.5)
$$

$$
\begin{aligned}
&\geq \liminf_{n\to\infty} E\Big[\int_0^1 L(t, X_n(t); b_n(t, X_n(t)))dt\Big] \\
&\geq E\Big[\int_0^1 L\Big(t, X(t); \frac{dA(t)}{dt}\Big)dt\Big] \\
&\geq \tilde{E}\Big[\int_0^1 L\Big(t, X(t); \beta_X(t, X)\Big)dt\Big] \quad \text{(from Lemma 3.3)} \\
&\geq \mathbf{V}(\mathbf{P}).
\end{aligned}
$$

Here $\tilde{E}$ denotes the mean value by the completion of $PX(\cdot)^{-1}$ and we used the fact that $P(X(t) \in dx) = P_t(dx)$ for all $t \in [0, 1]$. Indeed,

$$
P(X(t) \in dx) = \lim_{n\to\infty} P\Big(X\Big(\frac{[2^n t]}{2^n}\Big) \in dx\Big) \quad \text{weakly},
$$

$$
P\Big(X\Big(\frac{[2^n t]}{2^n}\Big) \in dx\Big) = P_{\frac{[2^n t]}{2^n}}(dx) \to P_t(dx) \quad \text{as } n \to \infty \text{ weakly}.
$$

Next we prove (ii). Suppose that $\mathbf{v}(\mathbf{P})$ is finite. Then (2.2) and (4.5) show the existence of a minimizer X of $\mathbf{V}(\mathbf{P})$. In the same way as in (3.7), Theorem 2.2, (i) and the strict convexity of $u \mapsto L(t, x; u)$ (see Remark 2.1, (i)) imply that $\beta_X(t, X) = b_X(t, X(t))$ and $b_X(t, x)$ is a minimizer of $\mathbf{v}(\mathbf{P})$.

Let b_1 and b_2 be minimizers of $\mathbf{v}(\mathbf{P})$. Then for any $\lambda \in (0, 1)$, $\lambda b_1(t, x) + (1 - \lambda)b_2(t, x)$ satisfies (1.1), and

$$
\begin{aligned}
&\mathbf{v}(\mathbf{P}) \qquad (4.6) \\
\leq\ &\int_0^1 \int_{\mathbf{R}^d} L(t, x; \lambda b_1(t, x) + (1 - \lambda)b_2(t, x))P_t(dx) \\
\leq\ &\lambda \int_0^1 \int_{\mathbf{R}^d} L(t, x; b_1(t, x))P_t(dx) + (1 - \lambda) \int_0^1 \int_{\mathbf{R}^d} L(t, x; b_2(t, x))P_t(dx) \\
=\ &\mathbf{v}(\mathbf{P}).
\end{aligned}
$$

The strict convexity of $u \mapsto L(t, x; u)$ implies the uniqueness of a minimizer of $\mathbf{v}(\mathbf{P})$.□

References

[1] Blumenthal, R.M. and Corson, H.H., 1972. On continuous collections of measures, in: Le Cam, L., Neyman, J., Scott, E.L. (Eds.), Proc. 6th

Berkeley Sympos. Math. Statist. Probab. **2**, Berkeley 1970/1971, Univ. California press, Berkeley, pp. 33-40.

[2] Carlen, E.A., 1984. Conservative diffusions, Commun. Math. Phys. 94, 293-315.

[3] Cattiaux, P. and Léonard, C., 1994. Minimization of the Kullback information of diffusion processes, Ann. Inst. H. Poincar Probab. Statist. 30, 83-132.

[4] Cattiaux, P. and Léonard, C., 1995. Correction to: "Minimization of the Kullback information of diffusion processes" [Ann. Inst. H. Poincar Probab. Statist. 30 (1994), no. 1, 83–132], Ann. Inst. H. Poincar Probab. Statist. 31, 705–707.

[5] Cattiaux, P. and Léonard, C., 1995. Large deviations and Nelson processes, Forum Math. 7, 95–115.

[6] Cattiaux, P. and Léonard, C., 1996. Minimization of the Kullback information for some Markov processes, in: Azema, J., Emery, M., Yor, M. (Eds.), Séminaire de Probabilités, XXX, Lecture Notes in Math., Vol. 1626, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, pp. 288–311.

[7] Doob, J.L., 1984. Classical potential theory and its probabilistic counterpart, Springer - Verlag, Berlin, Heidelberg, New York, Tokyo.

[8] Fleming, W.H. and Soner, H.M., 1993. Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.

[9] Jacod, J. and Shiryaev, A.N., 1987. Limit theorems for stochastic processes, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.

[10] Kellerer, H.G., 1984. Duality theorems for marginal problems, Z. Wahrsch. Verw. Gebiete 67, 399–432.

[11] Liptser, R.S. and Shiryaev, A.N., 1977. Statistics of random processes I, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.

[12] Mikami, T., 1990. Variational processes from the weak forward equation, Commun. Math. Phys. 135, 19-40.

[13] Mikami, T., 1999. Markov marginal problems and their applications to Markov optimal control, in: McEneaney, W. M., Yin, G. G., Zhang, Q. (Eds.), Stochastic Analysis, Control, Optimization and Applications, A Volume in Honor of W. H. Fleming, Birkhäuser, Boston, pp. 457-476.

[14] Mikami, T., 2002. Optimal control for absolutely continuous stochastic processes and the mass transportation problem, Elect. Comm. in Probab. 7, 199-213.

[15] Mikami, T., 2004. Monge's problem with a quadratic cost by the zero-noise limit of h-path processes, Probab. Theory Related Fields 129, 245-260.

[16] Mikami, T., 2004. A Simple Proof of Duality Theorem for Monge-Kantorovich Problem, Hokkaido University preprint series, #677.

[17] Mikami, T. and Thieullen, M., 2004. Duality Theorem for Stochastic Optimal Control Problem, Hokkaido University preprint series, #652.

[18] Mikami, T. and Thieullen, M., 2005. Optimal Transportation Problem by Stochastic Optimal Control, Hokkaido University preprint series, #690.

[19] Nelson, E., 1967. Dynamical theories of Brownian motion, Princeton University Press, Princeton.

[20] Rachev, S.T. and Rüschendorf, L., 1998. Mass transportation problems, Vol. I: Theory, Vol. II: Application, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.

[21] Villani, C., 2003. Topics in Optimal Transportation, Amer. Math. Soc., Providence, RI.

[22] Zheng, W. A., 1985. Tightness results for laws of diffusion processes application to stochastic mechanics, Ann. Inst. Henri Poincaré 21, 103-124.