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| Title            | Circular surfaces                                                                 |
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| Citation         | Hokkaido University Preprint Series in Mathematics, 730, 1-20                     |
| Issue Date       | 2005                                                                              |
| DOI              | <a href="https://doi.org/10.14943/83880">https://doi.org/10.14943/83880</a>       |
| Doc URL          | <a href="https://hdl.handle.net/2115/69538">https://hdl.handle.net/2115/69538</a> |
| Type             | departmental bulletin paper                                                       |
| File Information | re730.pdf                                                                         |



# CIRCULAR SURFACES

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ABSTRACT. A circular surface is a one-parameter family of standard circles in  $\mathbf{R}^3$ . In this paper some corresponding properties of circular surfaces with classical ruled surfaces are investigated. Singularities of circular surfaces are also studied.

## 1. INTRODUCTION

One of the principal purpose of the classical differential geometry is the study of some classes of surfaces with special properties in  $\mathbf{R}^3$  such as developable surfaces, ruled surfaces or minimal surfaces etc. Recently there appeared several articles on the study of ruled surfaces (containing developable surfaces) [7, 8, 9, 11, 12, 13, 14, 15, 18, 27]. A *ruled surface* is a smooth one-parameter family of lines. Since lines are the simplest curves in  $\mathbf{R}^3$ , ruled surfaces might have simple structures. Therefore, it might be considered that almost all interesting properties had been known until the middle of the 20th century. It is, however, paid attention in several area again (especially, the architecture and the computer aided design etc (see [5, 24]).

In this paper we study smooth one-parameter family of standard circles with fixed radius. Such the surface is called a *circular surface with constant radius*. We only consider the case when the radius is constant, so that we simply call a *circular surface* instead of the above. We call each circle a generating circle. Like as ruled surfaces, circular surfaces might be also important subjects in several area. Moreover, the third named author has been interested in surfaces which contains many circles [17, 20, 21, 28, 29, 30]. A circular surface is a typical surface with such the property. One of the examples of circular surfaces is the canal surface (tube) of a space curve. The generic singularities of canal surfaces have been studied by Porteous [22]. However, there are no systematic studies on circular surfaces so far as we know. In this paper we study geometric properties and singularities of circular surfaces with constant radius comparing with those of ruled surfaces. There is a curve on a ruled surface with an important properties which is called a *striction curve* (cf., Section 2.1 or [6, 11]). There exists a unique striction curve on a “generic” ruled surface. The singularities of a ruled surface are located on the striction curve. We also consider curves on a circular surface with the similar properties as the striction curve in Section

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*Date:* June 6, 2005.

2000 Mathematics Subject classification. Primary 53A05, Secondary 57R45, 58Kxx.

Key Words and Phrases. circular surfaces, curvatures, singularities

3. In this case there are exactly two such curves on a “generic” circular surface and intersections with each circle are antipodal points each other (cf., Proposition 3.3). We call such curves the *striction curves* of a circular surface. The singularities of a circular surface are also located on the striction curves. The developable surfaces are ruled surfaces with the vanishing Gaussian curvature. It has been classically known that the developable surface is a part of a cylinder, a cone, the tangent developable of a space curve or a surface which is smoothly connected these three surfaces. The developable surface is also characterized by the properties that each ruling is a line of curvature except at umbilic points or singular points. Therefore we consider that an analogous property of circular surfaces to the developable surfaces is that each generating circle is a line of curvature except at umbilic points or singular points. We classify such circular surfaces into a canal surface, a sphere, a special kind of surfaces or a surface which is smoothly connected these three surfaces (cf., Theorem 4.1). The third surface in the classification is called a *roller coaster surface* which is analogous to the tangent developable of a space curve.

It has been shown in [11, 13] that singularities of generic ruled surfaces are only cross caps. In Section 6 we have also show that generic singularities of general circular surfaces are only cross caps. It follows from the classifications of tangent developable surfaces [7, 18, 27] that generic singularities of developable surfaces are cuspidal edges, swallow tails or cuspidal cross caps (cf., [13]). On the other hand, singularities of generic tangent developables are cuspidal edges or cuspidal cross caps ([4]). According to these classifications, we also consider classifications of singularities of analogous circular surfaces. Porteous [22] classified singularities of generic canal surfaces into cuspidal edges or swallowtails. In Section 5 we classify singularities of generic roller coaster surfaces into cuspidal edges, swallowtails or cuspidal cross caps. We also have another class of circular surfaces analogous to tangent developable surfaces in Section 4 (cf., Definition 4.5) which we call tangent circular surfaces. We have an example of tangent circular surfaces with both of cuspidal cross caps and cross caps or both of swallowtails and cross caps on a generating circle. We remark that ruled surfaces do not have such properties. Therefore, it might be said that the geometric structure of circular surfaces is much richer than that of ruled surfaces.

## 2. PRELIMINARIES

In this section we define the notion of circular surfaces. Let  $I$  be an open interval.

**Definition 2.1.** A *circular surface* is (the image of) a map

$$V : I \times \mathbf{R}/2\pi\mathbf{Z} \longrightarrow \mathbf{R}^3$$

defined by

$$V(t, \theta) = V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}(t, \theta) = \gamma(t) + r(t) (\cos \theta \mathbf{a}_1(t) + \sin \theta \mathbf{a}_2(t)), \quad (2.1)$$

where  $\gamma, \mathbf{a}_1, \mathbf{a}_2 : I \longrightarrow \mathbf{R}^3$  and  $r : I \longrightarrow \mathbf{R}_{>0}$ . We assume that  $\langle \mathbf{a}_1, \mathbf{a}_1 \rangle = \langle \mathbf{a}_2, \mathbf{a}_2 \rangle = 1$ ,  $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = 0$  for all  $t \in I$  where  $\langle \cdot, \cdot \rangle$  denotes the canonical inner product on  $\mathbf{R}^3$ . We call  $\gamma$  a *base curve* and a pair of two curves  $\mathbf{a}_1, \mathbf{a}_2$  a *director frame*. The standard circles  $\theta \mapsto \gamma(t) + r(\cos \theta \mathbf{a}_1(t) + \sin \theta \mathbf{a}_2(t))$  are called *generating circles*.

If  $r(t)$  is constant, we call it a *circular surface with constant radius*. In this paper, we stick to circular surfaces with constant radius unless otherwise stated, so that we simply call them *circular surfaces* here. In general, circular surfaces have singularities. We investigate singularities of circular surfaces. For a map germ  $f : (\mathbf{R}^n, \mathbf{0}) \longrightarrow (\mathbf{R}^p, \mathbf{0})$ ,  $\mathbf{0} \in \mathbf{R}^n$  is a *singular point* of  $f$  if  $\text{rank}(df)(\mathbf{0}) < \min\{n, p\}$ . Let  $S(f)$  denotes the set of singular points in  $\mathbf{R}^n$  of  $f$ . and we call  $f(S(f))$  the *singular loci*. In this paper we consider some properties of circular surfaces compared with those of classical ruled surfaces.

Throughout in this paper, maps are always  $C^\infty$  class.

**2.1. A quick review on the theory of classical ruled surfaces (cf., [13]).** In this subsection we briefly review the classical theory of ruled surfaces. For further details, see [6, 11, 13].

**Definition 2.2.** A *ruled surface* is (the image of) a map  $F : I \times \mathbf{R} \longrightarrow \mathbf{R}^3$  defined by

$$F(t, u) = F_{(\gamma, \delta)}(t, u) = \gamma(t) + u\delta(t), \quad (2.2)$$

where  $\delta : I \longrightarrow S^2 = \{\mathbf{x} \in \mathbf{R}^3 \mid \|\mathbf{x}\| = 1\}$ . We call  $\gamma$  a *base curve* and  $\delta$  a *director curve*. The straightlines  $u \mapsto \gamma(t) + u\delta(t)$  are called *rulings*.

For a ruled surface  $F_{(\gamma, \delta)}$ , if  $\delta$  is constant, then the ruled surface  $F_{(\gamma, \delta)}$  is a *cylinder*. According to this fact, a ruled surface is said to be *noncylindrical* if  $\delta'$  never vanishes. Then we have the following lemma ([6], Section 17).

**Lemma 2.3.** (1) Let  $F_{(\gamma, \delta)}$  be a noncylindrical ruled surface. Then there exists a smooth curve  $\sigma : I \longrightarrow \mathbf{R}^3$  such that  $\text{Image } F_{(\gamma, \delta)} = F_{(\sigma, \delta)}$  and  $\langle \sigma'(t), \delta'(t) \rangle = 0$ . The curve  $\sigma(t)$  is called the *striction curve* of  $F_{(\gamma, \delta)}$ .

(2) The striction curve of a noncylindrical ruled surface  $F_{(\gamma, \delta)}$  does not depend on the choice of the base curve  $\gamma$ .

(3) Let  $F_{(\gamma, \delta)}$  be a cylinder (i.e.,  $\delta$  is a constant vector). Then any curve  $\sigma(t) = \gamma(t) + u(t)\delta$  satisfies  $\text{Image } F_{(\gamma, \delta)} = F_{(\sigma, \delta)}$  and  $\langle \sigma'(t), \delta'(t) \rangle = 0$ . That is to say, a striction curve.

Furthermore, we have the following lemma [11].

**Lemma 2.4.** Let  $F_{(\sigma, \delta)}$  be a noncylindrical ruled surface whose base curve  $\delta$  is the striction curve. Then the singular set is given as follows:

$$S(F_{(\sigma, \delta)}) = \{(t, 0) \in \mathbf{R} \times \{0\} \mid \sigma'(t) = 0 \text{ or } \sigma'(t) \parallel \delta(t)\}.$$

This means that the striction curve of a ruled surface contains singular loci.

The previous research of ruled surfaces and its generalizations, the notion of striction curves and its analogies play principal roles. For a line congruence, it is a two-parameter family of straightlines of  $\mathbf{R}^3$ , the analogous notion of the striction curve is called the *focal surface* of a line congruence. The singular set of the line congruence is coincide with the focal surface (see [10]). A one-parameter family of  $n$ -planes of  $\mathbf{R}^{2n}$  is called an  $(n, 1, 2n)$ -ruled manifold in [25]. The notion of the striction curve can be also defined for such a class of submanifolds and it is equal to the singular loci. Moreover, the behavior of striction curves characterizes the type of singularity (see [25]).

Recently, there appeared several articles concerning on special classes of ruled surfaces as follows:

- Developable surfaces (cylinders, cones, tangent developable surfaces) [4, 7, 13, 18, 27],
- Principle normal surfaces, Bi-normal surfaces [13]
- Darboux developable surfaces ( $\gamma = \mathbf{b}$ ,  $\delta = \mathbf{e}$ ) [9, 13]
- Rectifying developable surfaces ( $\delta = \mathbf{D}$ ,  $\mathbf{D} = (\tau/k)\mathbf{e} + k\mathbf{b}$ ) [9]
- Focal surfaces of space curves [22]

In this paper, we consider some classes of circular surfaces which have corresponding properties with the above classes of ruled surfaces.

### 3. STRICION CURVES FOR CIRCULAR SURFACES

Striction curves of ruled surfaces are important for the study of singularities of ruled surfaces. In this section we study an analogous notion of striction curves of ruled surfaces for circular surfaces.

For a circular surface  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}(t, \theta)$ , vectors  $\{\mathbf{a}_1(t), \mathbf{a}_2(t), \mathbf{a}_3(t) = \mathbf{a}_1 \times \mathbf{a}_2\}$  form an orthonormal frame of  $\mathbf{R}^3$  which is called a *base frame* of the circular surface. We set

$$\begin{aligned} c_1(t) &= \mathbf{a}'_1(t) \cdot \mathbf{a}_2(t) = -\mathbf{a}'_2(t) \cdot \mathbf{a}_1(t) \\ c_2(t) &= \mathbf{a}'_1(t) \cdot \mathbf{a}_3(t) = -\mathbf{a}'_3(t) \cdot \mathbf{a}_1(t) \\ c_3(t) &= \mathbf{a}'_2(t) \cdot \mathbf{a}_3(t) = -\mathbf{a}'_3(t) \cdot \mathbf{a}_2(t). \end{aligned} \tag{3.1}$$

Then we have the following Frenet-Serre type formula:

$$\begin{pmatrix} \mathbf{a}'_1(t) \\ \mathbf{a}'_2(t) \\ \mathbf{a}'_3(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) \\ -c_1(t) & 0 & c_3(t) \\ -c_2(t) & -c_3(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_1(t) \\ \mathbf{a}_2(t) \\ \mathbf{a}_3(t) \end{pmatrix}. \tag{3.2}$$

By straightforward calculations, we can show that  $V$  has a singularity at  $(t, \theta)$  if and only if

$$\begin{cases} \mathbf{a}_3 \cdot (\gamma' + r \cos \theta \mathbf{a}'_1 + r \sin \theta \mathbf{a}'_2) = 0 \end{cases} \tag{3.3}$$

$$\begin{cases} \gamma' \cdot (\cos \theta \mathbf{a}_1 + \sin \theta \mathbf{a}_2) = 0, \end{cases} \tag{3.4}$$

where we write

$$\gamma'(t) = \lambda(t)\mathbf{a}_1(t) + \mu(t)\mathbf{a}_2(t) + \delta(t)\mathbf{a}_3(t). \quad (3.5)$$

Now we define the notion of striction curves of circular surfaces.

**Definition 3.1.** A curve

$$\sigma(t) = \gamma(t) + r(\cos \theta(t)\mathbf{a}_1(t) + \sin \theta(t)\mathbf{a}_2(t))$$

on a circular surface  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}(t, \theta)$  is a *striction curve* if  $\sigma$  satisfies

$$\sigma'(t) \cdot (\cos \theta(t)\mathbf{a}_1(t) + \sin \theta(t)\mathbf{a}_2(t)) = 0 \quad (3.6)$$

for all  $t \in I$ .

This condition is analogous to the condition for striction curves of ruled surfaces (cf., Lemma 2.3). For a circular surface  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}(t, \theta)$ , if  $\gamma'(t) \cdot \mathbf{a}_1(t) = \gamma'(t) \cdot \mathbf{a}_2(t) = 0$  for any  $t \in I$ , generating circles are always located on the normal planes of the curve  $\gamma$ . Such a surface has been known as a *canal surface* [2, 22]. In this case any curves on the surface transverse to generating circles satisfy the condition of striction curves. Therefore the class of canal surfaces is an analogous class to the class of cylindrical surfaces in ruled surfaces. We also define the notion of non-canal circular surfaces analogous to that of non-cylindrical ruled surfaces.

**Definition 3.2.** A circular surface  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}(t, \theta)$  is called *non-canal* at  $t \in I$  if  $\gamma, \mathbf{a}_1, \mathbf{a}_2$  satisfy

$$\gamma'(t) \cdot \mathbf{a}_1(t) \neq 0 \text{ or } \gamma'(t) \cdot \mathbf{a}_2(t) \neq 0 \quad (3.7)$$

We call  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}$  a *non-canal* if it is non-canal at any  $t \in I$ .

It has been known that canal surfaces can be regarded as wave fronts in the theory of Legendrian singularities (cf., [1]). Moreover, only cuspidal edges and swallowtails (cf., §5) are singularities of canal surfaces in generic [22]. For non-canal circular surface, we have the following proposition.

**Proposition 3.3.** *For a non-canal circular surface  $V(t, \theta)$ ,  $V$  has exactly two striction curves and intersections with each generating circle are antipodal points each other.*

*Proof.* (1) For a circular surface  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}$ , we take a function  $\theta(t)$  who satisfies  $\lambda(t) \cos \theta(t) + \mu(t) \sin \theta(t) = 0$ . We put

$$\sigma(t) = \gamma(t) + r(\cos \theta(t)\mathbf{a}_1(t) + \sin \theta(t)\mathbf{a}_2(t))$$

then  $\sigma$  satisfies the condition of striction curve. If  $\theta_2(t)$  also satisfies  $\lambda(t) \cos \theta_2(t) + \mu(t) \sin \theta_2(t) = 0$  then  $\theta_2(t) = \theta(t)$  or  $\theta_2(t) = \theta(t) + \pi$  holds.  $\square$

We can observe that the assertions of the above proposition correspond to those of Lemmas 2.3 and 2.4 for ruled surfaces.

## 4. TANGENT CIRCULAR SURFACES

A ruled surface is developable if and only if the Gaussian curvature  $K$  vanishes at any regular point, this is equivalent to that all of rulings are lines of curvature except at umbilic points or singular points. We study circular surfaces that all of generating circles are lines of curvature. We have the following classification theorem.

**Theorem 4.1.** *Let  $V$  be a circular surface such that all of generating circles are lines of curvature except at umbilical points or singular points. Then  $V$  is a part of a sphere, a canal surface, a surface represented by the following formula:*

$$R(t, \theta) = \gamma(t) + r(\cos \theta \mathbf{e} + \sin \theta (\cos \varphi(t) \mathbf{n}(t) + \sin \varphi(t) \mathbf{b}(t))), \quad (4.1)$$

where  $-\varphi(t)$  is a primitive function of the torsion  $\tau(t)$  of the space curve  $\gamma$ , or a surface which is smoothly connected these three surfaces. We remark that  $t$  is the arc-length parameter of  $\gamma$ .

We call this surface a *roller coaster surface*

It is easy to show that singularities of roller coaster surfaces coincide with striction curves  $R(t, \pm\pi/2)$ . We prove this theorem later. On the other hand, it is well known that a developable surfaces is a cone, a cylinder or a tangent developable surface. If we compare those classifications of developable surfaces and circular surfaces, there exists an interesting correspondence between the lists of classifications. Spheres correspond to cones, canal surfaces to cylinder and roller coaster surfaces to tangent developable surfaces. Although canal surfaces are well known, we cannot find the notion of roller coaster surfaces in any context so far as we know. Therefore, we study detailed properties of roller coaster surfaces here.

**4.1. Roller coaster surfaces.** We have the following theorem for roller coaster surfaces.

**Theorem 4.2.** *For a roller coaster surface*

$$R(t, \theta) = \gamma(t) + r(\cos \theta \mathbf{e} + \sin \theta (\cos \varphi(t) \mathbf{n}(t) + \sin \varphi(t) \mathbf{b}(t))),$$

(1) *Two principle curvatures  $\lambda_1$  and  $\lambda_2$  are given as follows:*

$$\lambda_1 = \frac{-k \sin \varphi}{(1 + r^2 k^2 \sin^2 \varphi)^{1/2}}, \quad \lambda_2 = -\frac{r^2 k^3 \sin^3 \varphi \cos \theta + k(\sin \varphi \cos \theta - r\tau \cos \varphi) + rk' \sin \varphi}{(1 + r^2 k^2 \sin^2 \varphi)^{3/2} \cos \theta}. \quad (4.2)$$

*In particular, the principle direction of  $\lambda_1$  points the direction of the generating circle and this curvature is constant along the generating circle.*

(2) *Gaussian curvature  $K$  and mean value curvature  $H$  are given as follows:*

$$K = \frac{k \sin \varphi (r^2 k^3 \sin^3 \varphi \cos \theta + k(\sin \varphi \cos \theta - r\tau \cos \varphi) + rk' \sin \varphi)}{(1 + r^2 k^2 \sin^2 \varphi)^2 \cos \theta}. \quad (4.3)$$

$$H = -\frac{2r^2 k^3 \sin^3 \varphi \cos \theta + k(2 \sin \varphi \cos \theta - r\tau \cos \varphi) + rk' \sin \varphi}{2(1 + r^2 k^2 \sin^2 \varphi)^{3/2} \cos \theta}. \quad (4.4)$$

(3) Flat roller coaster surfaces are subsets of planes.

(4) Minimal roller coaster surfaces are subsets of planes.

(5) Two striction curves coincide singular locus and their curvatures  $k_1(\theta = -\pi/2), k_2(\theta = \pi/2)$  and torsions  $\tau_1, \tau_2$  are given as follows:

$$\begin{aligned} k_1 &= \frac{k}{|1 + rk \cos \varphi|} & \tau_1 &= \frac{\tau}{1 + rk \cos \varphi} \\ k_2 &= \frac{k}{|1 - rk \cos \varphi|} & \tau_2 &= \frac{\tau}{1 - rk \cos \varphi}, \end{aligned}$$

where  $k$  and  $\tau$  are curvature and torsion of  $\gamma$ .

*Proof.* By

$$R_t \times R_\theta(t, \theta) = r \cos \theta \left( rk \cos \theta \sin \varphi \mathbf{e} + (-1 + rk \sin \theta \cos \varphi) \sin \varphi \mathbf{n} + (\cos \varphi + rk \sin \theta \sin^2 \varphi) \mathbf{b} \right),$$

we put

$$\begin{aligned} \nu(t, \theta) &= \left( rk \cos \theta \sin \varphi \mathbf{e} + (-1 + rk \sin \theta \cos \varphi) \sin \varphi \mathbf{n} \right. \\ &\quad \left. + (\cos \varphi + rk \sin \theta \sin^2 \varphi) \mathbf{b} \right) (1 + r^2 k^2 \sin^2 \varphi)^{-1/2}. \end{aligned} \quad (4.5)$$

Then  $\nu$  gives a well-defined unit normal vector. We can write fundamental values concretely as follows:

$$\begin{aligned} E &= r^2 k^2 \cos^2 \theta + (rk \cos \varphi \sin \theta - 1)^2, \\ F &= r(rk \cos \varphi - \sin \theta), \\ G &= r^2, \\ L &= \left( k \sin \varphi + r(k^2 \sin \theta \sin 2\varphi - \cos \theta \cos \varphi \tau) + k' \cos \theta \sin \varphi \right. \\ &\quad \left. + r^2 k^3 \sin \varphi (\cos^2 \theta + \cos^2 \varphi \sin^2 \theta) \right) (1 + r^2 k^2 \sin^2 \varphi)^{-1/2}, \\ M &= rk \sin \varphi (-rk \cos \theta + \sin \theta) (1 + r^2 k^2 \sin^2 \varphi)^{-1/2} \quad \text{and} \\ N &= r^2 k \sin \varphi (1 + r^2 k^2 \sin^2 \varphi)^{-1/2}. \end{aligned} \quad (4.6)$$

Since  $GM - FN = 0$ , we get (1) and (2) by easy calculation. Moreover, we can conclude that  $\lambda_1$  points the direction of generating circle. For the claim (3), we assume that  $\tau$  is not identically zero. Suppose  $R$  is a flat surface. Then we have  $\hat{K}(t, \theta) = r^2 k^3 \sin^3 \varphi \cos \theta + k(\sin \varphi \cos \theta - r\tau \cos \varphi) + rk' \sin \varphi \equiv 0$ . From  $\hat{K}(t, \theta) + (\partial^2 / \partial \theta^2) \hat{K}(t, \theta) = 0$ , we get  $k\tau \cos \varphi - k' \sin \varphi = 0$  for any  $t$ , this is equivalent to that  $k' \equiv \tau \equiv 0$  this means that  $\tau$  is an identically zero function. By (4.3), we have  $\sin \varphi \equiv 0$ . A roller coaster surface satisfying this condition is a part of a plane. Which proves (3). We can prove (4) by the same method. Since  $((\partial / \partial t) R(t, \theta)) \cdot (R(t, \theta) - \gamma(t)) = r \cos \theta$ , striction curves of a roller coaster surface are  $R(t, \pm\pi/2)$  (cf. (3.6)). Formulae (5) is straightforward calculations of curvatures and torsions of space curves  $t \mapsto R(t, \pm\pi/2)$ .  $\square$

**4.2. Proof of Theorem 4.1.** We assume  $\gamma$  is non-singular and  $t$  is an arc-length parameter.

All generating circles are line of curvature if and only if

$$\frac{\partial}{\partial \theta} \left( \frac{V_t \times V_\theta}{\|V_t \times V_\theta\|} \right) // V_t$$

This is equivalent to

$$\begin{cases} (rc_3 \sin \theta + rc_2 \cos \theta + \delta)(\delta(\mu \cos \theta - \lambda \sin \theta) + r(\mu c_2 - \lambda c_3)) &= 0 \\ (\mu \sin \theta + \lambda \cos \theta)(\delta(\mu \cos \theta - \lambda \sin \theta) + r(\mu c_2 - \lambda c_3)) &= 0. \end{cases}$$

Then we have

$$\delta(\mu \cos \theta - \lambda \sin \theta) + r(\mu c_2 - \lambda c_3) = 0 \quad (4.7)$$

for all non-singular point  $t$ . Singular points on a generating circle are at most two points, we differentiate (4.7) by  $\theta$ . Then we have

$$\delta(-\mu \sin \theta - \lambda \cos \theta) = 0. \quad (4.8)$$

This is equivalent to  $\lambda(t) = \mu(t) = 0$  or  $\delta(t) = 0$  for all  $t$ . The condition  $\lambda(t) = \mu(t) = 0$  means  $V$  is a canal surface and satisfies the condition of theorem. If we assume  $\delta(t) = 0$  then  $\mathbf{e} \in \langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ . So we rechoose director frame  $\{\mathbf{e}, \tilde{\mathbf{a}}_2\}$ . By  $\tilde{\mathbf{a}}_2 \in \langle \mathbf{n}, \mathbf{b} \rangle$ ,  $\tilde{\mathbf{a}}_2(t) = \cos \varphi(t) \mathbf{n}(t) + \sin \varphi(t) \mathbf{b}(t)$ .

Now by (4.7), we have  $\mu c_2 - \lambda c_3 = 0$ . By a calculation, we have  $\varphi'(t) + \tau(t) = c_3 = 0$ . This completes the proof.

**4.3. Examples of roller coaster surfaces.** In this section, we give some examples of roller coaster surfaces. Let  $\mathbf{e}, \mathbf{n}, \mathbf{b}$  are standard Frenet frame, curvature and torsion of each  $\gamma$ .

**Example 4.3.** Let  $\gamma(t) = (\cos t, 2 \sin t, 0)$ ,  $\phi(t) = -\int \tau dt + \pi/4$ ,  $d(t) = \cos \phi(t) + \sin \phi(t)$  and

$$R(t, \theta) = \gamma(t) + \cos \theta \mathbf{e}(t) + \sin \theta \mathbf{d}(t).$$

Then  $R$  is a roller coaster surface. We can observe cuspidal edge singularity along  $\gamma$  and its anti-podal loci.

**Example 4.4.** Let  $\gamma(t) = (\cos t/\sqrt{2}, t/\sqrt{2}, \sin t/\sqrt{2})$ ,  $\phi(t) = -\int \tau dt$ ,  $d(t) = \cos \phi(t) + \sin \phi(t)$  and

$$R(t, \theta) = \gamma(t) + 5(\cos \theta \mathbf{e}(t) + \sin \theta \mathbf{d}(t))/\sqrt{3}.$$

Then  $R$  is a roller coaster surface. We can observe cuspidal edge, swallowtails and cuspidal cross caps.

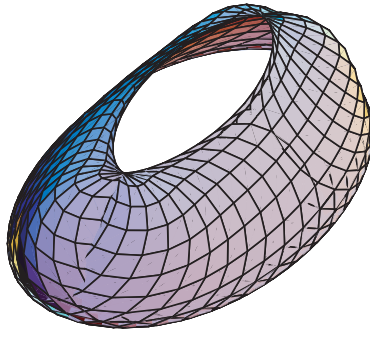


FIGURE 1. Roller coaster surface

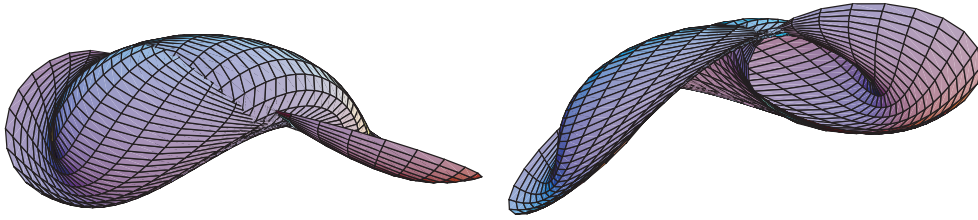


FIGURE 2. Roller coaster surface

**4.4. General tangent circular surfaces.** A tangent developable surface is a ruled surface that all of rulings are tangent lines of a space curve except at umbilic points and singular points. On the other hand, the corresponding circular surface (i.e., a roller coaster surface) is a circular surface that all of generating circles are tangent to a space curve and satisfies the condition that  $\varphi(t)$  is a primitive function of torsion  $\tau(t)$  of the space curve  $\gamma$  (see Theorem 4.1). If we only pay attention to the property of tangent developable surfaces that all of rulings are tangent lines of a space curve, we define a circular surface that satisfy all of generating circles are tangent to a space curve without any additional conditions.

**Definition 4.5.** A *tangent circular surface*  $T(s, \theta)$  is defined to be

$$T(s, \theta) = \xi(s) + r\mathbf{h}_\xi(s) + r(\cos \theta \mathbf{e}_\xi(s) + \sin \theta \mathbf{h}_\xi(s)) \quad (4.9)$$

where, we put  $\mathbf{h}_\xi(s) = \cos \psi(s)\mathbf{n}_\xi(s) + \sin \psi(s)\mathbf{b}_\xi(s)$ . When  $\mathbf{h}_\xi = \mathbf{b}$ , we call  $T$  tangent-bi-normal surfaces

**Example 4.6.** We put  $\gamma_1(t) = (\cos t, \sin t, t^2)$ ,  $\gamma_2(t) = (t^2, t^3, t^4)$ ,  $a_1(t) = (2, 3t, 4t^2)/A$  and  $a_2(t) = (12t(1+t), -8, -6)/5A$  then  $T_1(t, \theta) = \gamma_1 + \mathbf{b} + \cos \theta \mathbf{e} + \sin \theta \mathbf{b}$  and  $T_2(t, \theta) = \gamma_2 + a_2 + \cos \theta a_1 + \sin \theta a_2$  give examples of general tangent circular surfaces, where  $A = \sqrt{4 + 9t^2 + 16t^4}$ . This surface near  $t = 0$  is depicted in Figure 3

These examples show that singularities of  $T_1$  are CCR at  $(0, -\pi/2)$  and CR at  $(0, \pi/2)$ , moreover  $T_2$  are SW at  $(0, -\pi/2)$  and CR at  $(0, \pi/2)$ . There are quite different singularity appear on same generating circles. This situation is not appear for ruled surfaces, so that we might say circular

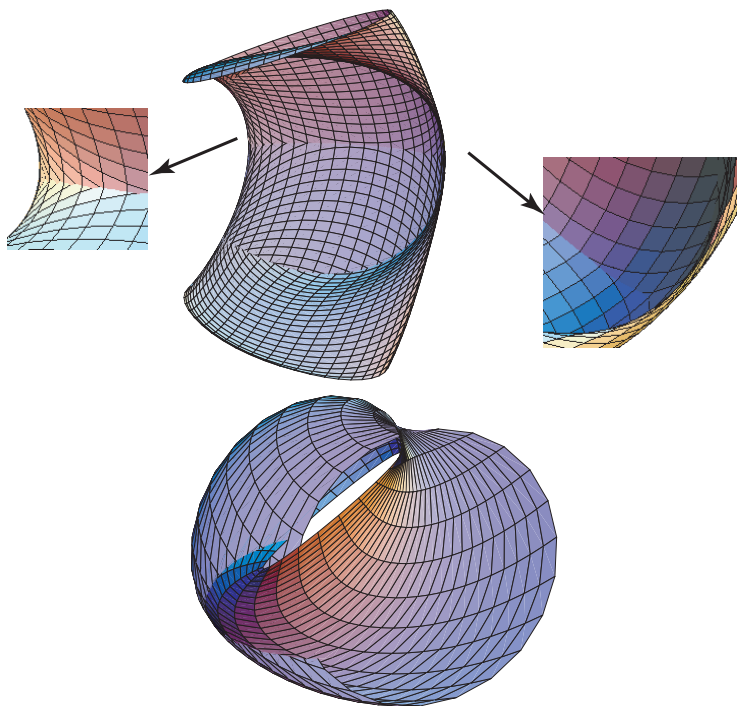


FIGURE 3. General tangent circular surfaces

surfaces have much more rich geometry than ruled surfaces. Singularities of tangent circular surfaces are studied in the section 5, meanings of CR, CCR, SW are also written in section 5.

## 5. SINGULARITIES OF CIRCULAR SURFACES

In this section we study conditions which characterize the singularity of circular surfaces. For two map germs  $f_1, f_2 : (\mathbf{R}^n, \mathbf{0}) \rightarrow (\mathbf{R}^p, \mathbf{0})$ , they are  $\mathcal{A}$ -equivalent if there exist diffeomorphism germs  $\phi : (\mathbf{R}^n, \mathbf{0}) \rightarrow (\mathbf{R}^n, \mathbf{0})$  and  $\psi : (\mathbf{R}^p, \mathbf{0}) \rightarrow (\mathbf{R}^p, \mathbf{0})$  such that  $\psi \circ f_1 = f_2 \circ \phi$ .

A map germ  $(x, y) \rightarrow (x^2, xy, y)$  at  $\mathbf{0}$  is called *cross cap*,  $(x, y) \rightarrow (x^2, x^3, y)$  at  $\mathbf{0}$  is called *cuspidal edge*,  $(x, y) \rightarrow (3x^4 + x^2y, 4x^3 + 2xy, y)$  at  $\mathbf{0}$  is called *swallowtail* and  $(x, y) \rightarrow (x^2, x^3y, y)$  at  $\mathbf{0}$  is called *cuspidal cross cap*(see figure 4). These are abbreviated as CR, CE, SW and CCR.

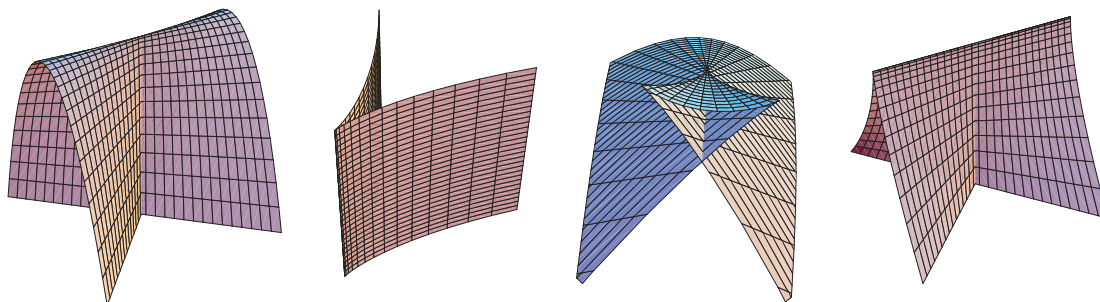


FIGURE 4. Cross cap, cuspidal edge swallowtail and cuspidal cross cap.

**5.1. Singularities of circular surfaces.** In this subsection, we study a necessary and sufficient condition that a circular surface germ is  $\mathcal{A}$ -equivalent to the cross cap. We have the following theorem.

**Theorem 5.1.** *A circular surface  $V$  is cross cap at  $(t_0, \theta_0)$  if and only if  $\delta(t_0) = -r(c_2(t_0) \cos \alpha + c_3(t_0) \sin \alpha)$ ,  $\theta_0 = \alpha$  and if  $\sin \alpha \neq 0$ ,*

$$\begin{aligned} & r\lambda_0\mu_0(-c'_2(t_0)\lambda_0 + c_2(t_0)\lambda'_0 + c'_3(t_0)\mu_0 - c_3(t_0)\mu'_0) \\ & - \lambda_0^2 \left( -r\lambda_0c'_3(t_0) + rc_2(t_0)\mu'_0 + \sqrt{\lambda_0^2 + \mu_0^2} \delta'_0 \right) \\ & - \mu_0^2 \left( r\mu_0c'_2(t_0) - rc_3(t_0)\lambda'_0 + \sqrt{\lambda_0^2 + \mu_0^2} \delta'_0 \right) \neq 0 \end{aligned}$$

holds, if  $\sin \alpha = 0$ ,

$$rc'_2(t_0)\mu_0 - rc_3(t_0)\lambda'_0 + |\mu_0|\delta'_0 \neq 0$$

holds. Here,  $\alpha$  is an angle satisfying  $\mu_0 \sin \alpha + \lambda_0 \cos \alpha = 0$  and we abbreviate  $(t_0)$  like as  $\lambda_0 = \lambda(t_0)$ .

*Proof.* If  $\sin \alpha \neq 0$ , take

$$v = \frac{\partial}{\partial t} + \frac{\lambda_0 - rc_1(t_0) \sin \alpha}{r \sin \alpha} \frac{\partial}{\partial \theta}$$

then  $dV(v)(t_0, \theta_0) = 0$ . By the characterization of cross cap, if

$$\begin{aligned} & \det \left( \frac{\partial}{\partial \theta} V_0, \left( \frac{\partial^2}{\partial t \partial \theta} + \frac{\lambda_0 - rc_1(t_0) \sin \alpha}{r \sin \alpha} \frac{\partial^2}{\partial \theta^2} \right) V_0, \right. \\ & \left. \left( \frac{\partial^2}{\partial t^2} + 2 \frac{\lambda_0 - rc_1(t_0) \sin \alpha}{r \sin \alpha} \frac{\partial^2}{\partial t \partial \theta} + \frac{(\lambda_0 - rc_1(t_0) \sin \alpha)^2}{r^2 \sin^2 \alpha} \frac{\partial^2}{\partial \theta^2} \right) V_0 \right) \end{aligned}$$

are linearly independent, then  $V$  at  $(t_0, \theta_0)$  is cross cap. By a computation, we get the conclusion. We can apply the same method to the case when  $\sin \alpha = 0$ .  $\square$

**5.2. Singularities of roller coaster surfaces.** In this subsection, we study singularities of roller coaster surfaces.

**5.2.1. Wave front surfaces.** A map  $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$  is called a (*wave*) *front* if it is the projection of a Legendrian immersion  $L_f : \mathbf{R}^2 \longrightarrow T_1^* \mathbf{R}^3$  into the unit cotangent bundle and  $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$  is called a *pseudo-front* if it is the projection of an isotropic map into the unit cotangent bundle. Cuspidal cross cap is not a front but a pseudo-front.

Let  $(U; u, v)$  be a domain in  $\mathbf{R}^2$  and  $f : U \longrightarrow \mathbf{R}^3$  a front. Identifying the unit cotangent bundle with the unit tangent bundle  $T_1 \mathbf{R}^3 \sim \mathbf{R}^3 \times S^2$ , there exists a unit vector field  $\nu : U \longrightarrow S^2$  such that the Legendrian lift  $L_f$  is expressed as  $\langle f, \nu \rangle$ . Since  $L_f = \langle f, \nu \rangle$  is Legendrian,

$$\langle df, \nu \rangle = 0 \text{ and } \langle \nu, \nu \rangle = 1$$

hold. Then there exists a function  $\lambda$  such that

$$f_u(u, v) \times f_v(u, v) = \lambda(u, v) \nu(u, v)$$

where  $\times$  denotes the exterior product in  $\mathbf{R}^3$  and  $f_u = \partial f / \partial u$ , for example. Obviously,  $(u, v) \in U$  is a singular point of  $f$  if and only if  $\lambda(u, v) = 0$ .

**Definition 5.2.** A singular point  $p \in \mathbf{R}^2$  of a front  $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$  is *non-degenerate* if  $d\lambda \neq 0$  holds at  $p$ .

By the implicit function theorem, for a non-degenerate singular point  $p$ , the singular set is parameterized by a smooth curve  $c : (-\varepsilon, \varepsilon) \longrightarrow \mathbf{R}^2$  in a neighborhood of  $p$ . Since  $p$  is non-degenerate, any  $c(t)$  is non-degenerate for sufficiently small  $t$ . Then there exists a unique direction  $\eta(t) \in T_{c(t)}U$  up to scalar multiplication such that  $df(\eta(t)) = 0$  for each  $t$ . We call  $c'(t)$  the *singular direction* and  $\eta(t)$  the *null-direction*. For further details in these notation, see [16].

It has been known the generic singularities of fronts  $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$  are cuspidal edges and swallowtails [1]. In [16] it has been shown the following useful criteria for cuspidal edge and swallowtails:

**Proposition 5.3** ([16], Proposition 1.3). *For a non-degenerate front  $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^3$  with singularity at  $\mathbf{0}$ .*

- *$f$  at  $\mathbf{0}$  is  $\mathcal{A}$ -equivalent to the cuspidal edge if and only if  $\det(c'(0), \eta(0)) \neq 0$ .*
- *$f$  at  $\mathbf{0}$  is  $\mathcal{A}$ -equivalent to the swallowtail if and only if*

$$\det(c'(0), \eta(0)) = 0 \quad \text{and} \quad \left. \frac{d}{dt} \det(c'(t), \eta(t)) \right|_0 \neq 0.$$

By using this criterion, we have the following characterizations that a roller coaster surface has cuspidal edges or swallowtails.

**Proposition 5.4.** (1) *A roller coaster surface  $V$  is a pseudo-front at any point  $(t, \theta)$ .*

(2) *A roller coaster surface  $V$  is a front near  $(t, \pm\pi/2)$  if and only if*

$$k\tau \cos \varphi - k' \sin \varphi \neq 0. \tag{5.1}$$

(3) *If a roller coaster surface is a front at a singular point, then the point is non-degenerate.*

*Proof.* In the formula of (4.5),  $A(t, \pm\pi/2)$  never vanish, so  $(R, \nu)$  gives a isotropic lift, we have

(1). By a direct calculation,

$$\begin{pmatrix} R_t & \nu_t \\ R_\theta & \nu_\theta \end{pmatrix} (t, \pm\pi/2) = \begin{pmatrix} 1 - r \cos \varphi(t) k(t) & 0 & 0 & \nu_t \\ -r & 0 & 0 & -rk(t) \sin \varphi(t) / \sqrt{A(t, \pm\pi/2)} & 0 & 0 \end{pmatrix},$$

$$\nu_t = \left( -\frac{k(t) (-1 + r \cos \varphi(t) k(t)) \sin \varphi(t)}{A(t, \pm\pi/2)^{1/2}}, \right.$$

$$\left. \frac{(\cos \varphi(t) + rk(t) \sin^2 \varphi(t)) B(t)}{A(t, \pm\pi/2)^{3/2}}, \frac{(1 - r \cos \varphi(t) k(t)) \sin \varphi(t) B(t)}{A(t, \pm\pi/2)^{3/2}} \right)$$

and

$$B(t) = \left( (-1 - r^2 k(t)^2 \sin^2 \varphi(t)) \tau(t) + r \sin \varphi(t) k'(t) - (1 - r \cos \varphi(t) k(t) + r^2 k(t)^2 \sin^2 \varphi(t)) \varphi'(t) \right),$$

this matrix is of full rank if and only if  $k\tau \cos \varphi - k' \sin \varphi \neq 0$  holds. Finally, we show (3). By the formula of  $\nu$ , we have

$$\lambda_t(t, \pm\pi/2) = 0, \quad \lambda_\theta(t, \pm\pi/2) = \mp r \left( \sqrt{1 + r^2 k(t)^2 \sin^2 \varphi(t)} \right).$$

This never vanish, so we have (3).  $\square$

From Proposition 5.3, we have following theorem.

**Theorem 5.5.** (1) *A roller coaster surface  $V$  is cuspidal edge at  $(t, \pm\pi/2)$  if and only if*

$$k\tau \cos \varphi - k' \sin \varphi \neq 0. \text{ and } 1 \pm rk \cos \varphi \neq 0 \quad (5.2)$$

*holds.*

(2) *A roller coaster surface  $V$  is swallowtail at  $(t, \pm\pi/2)$  if and only if*

$$k\tau \cos \varphi - k' \sin \varphi \neq 0, \quad 1 \pm rk \cos \varphi = 0, \text{ and } k' \cos \varphi + k\tau \sin \varphi \neq 0 \quad (5.3)$$

*holds.*

*Proof.* Clearly, singular curves are  $\{(t, \pm\pi/2)\}$ , and the null-directions are  $(r, \pm 1 - rk \cos \varphi)$ . By Proposition 5.3, we have the theorem.  $\square$

Here, we give geometric meanings of the point that roller coaster surface is front

By Proposition 5.4 (2), the condition  $k\tau \cos \varphi - k' \sin \varphi = 0$  characterizes that the roller coaster surface is not a front at the point. Under this condition that one of the vector  $\mathbf{h}$  generated by the direction frame is parallel to  $k' \mathbf{n} + k\tau \mathbf{b}$ . In this subsection we consider the geometric meaning of this vector. We assume that  $\gamma'(t) \neq 0$  and  $\gamma'(t) \times \gamma''(t) \neq 0$ .

From Bouquet's formula,  $\gamma$  is expanded

$$\gamma(t) = \gamma(0) + t\mathbf{e}(t) + t^2 k'(0) \mathbf{n}(0)/2 + t^3/6 (e(0) + k'(0) \mathbf{n}(0) + k(0) \tau(0) \mathbf{b}(0)) + O(t^3)$$

near  $t = 0$ , where  $O(t^3)$  denotes higher order part. Then a projected curve  $c$  of  $\gamma$  to normal plane of  $\gamma$  is expressed by

$$c(t) = t^2 k(0) \mathbf{n}(0)/2 + t^3/6 (k'(0) \mathbf{n}(0) + k(0) \tau(0) \mathbf{b}(0)) + O(t^3).$$

We put

$$\mathbf{v} = \lim_{\varepsilon \rightarrow 0} \mathbf{v}(\varepsilon) = \lim_{\varepsilon \rightarrow 0} (c(-\varepsilon), c(\varepsilon)) / |(c(-\varepsilon), c(\varepsilon))|.$$

Then, by elementary calculations, we can show that this vector is independent on the choice of both of the parameter  $t$  and parallel vectors to  $k' \mathbf{n} + k\tau \mathbf{b}$ .

5.2.2. *Conditions of cuspidal cross cap.* In this subsection, we give a condition of a circular surface to be a cuspidal cross cap. We need the following lemma.

**Lemma 5.6** ([26]). *A surface  $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^3, 0)$  has corank one singular point at origin and singular curve  $S(f)$  is regular curve. Let  $\xi$  denotes this curve and set  $\xi'(0) = (1, 0)$  and  $f(\xi'(0)) = (0, 0, 1)$ . Null direction of  $f$  are  $(1, 0)$ . We write intersection curves of  $f$  and planes  $(x, y, \varepsilon)_{\varepsilon \in (-\delta, \delta)}$ ,  $c_\varepsilon(s)$ . If  $c_\varepsilon$  have 3/2-cusp except for  $\varepsilon = 0$ , 5/2 cusp at  $\varepsilon = 0$  and  $d/d\varepsilon \det(c'', c''')|_{\varepsilon=0} \neq 0$  then  $f$  at 0 is equivalent to cuspidal cross cap.*

The condition  $d/d\varepsilon \det(c'', c''')|_{\varepsilon=0} \neq 0$  means that curves degenerate to 5/2-cusp at origin when  $\varepsilon = 0$ , but this degeneration is non-degenerate (see figures 5 also 4). This result was first



FIGURE 5. Cut curves near cuspidal cross cap.

pointed out by Porteous [23], see also [26]. Using this lemma, we can show the following theorem.

**Theorem 5.7.** (1) *For a roller coaster surface  $R$ , point  $(t_0, \pm\pi/2)$  is  $\mathcal{A}$ -equivalent to the cuspidal cross cap if every following hold.*

- $1 \mp r \cos \phi_0 \neq 0$ .
- $k_0 \tau_0 \cos \phi_0 - k'_0 \sin \phi_0 = 0$ .
- If  $\sin \phi_0 \neq 0$  then

$$\begin{aligned} & \tau_0 \left[ r^2 (2k'_0 \tau_0 \cos \phi_0 - k''_0 \sin \phi_0) \right. \\ & \quad + \left( \sin \phi_0 + r^2 (\tau_0^2 \sin \phi_0 \mp 3k'_0 \tau_0 + \tau'_0 \cos \phi_0) \pm r^3 (k'_0 \tau_0 + k''_0 \cos \phi_0 \sin \phi_0) \right) k_0 \\ & \quad \pm r \cos \phi_0 \left( -2 \sin \phi_0 + r^2 (\tau_0^2 \sin \phi_0 - \tau'_0 \cos \phi_0) \right) k_0^2 \\ & \quad \left. + r^2 k_0^3 \sin \phi_0 \mp 2r^3 k_0^4 \cos \phi_0 \sin^3 \phi_0 + r^4 k_0^5 \cos^2 \phi_0 \sin^3 \phi_0 \right] \neq 0. \end{aligned} \tag{5.4}$$

If  $\sin \phi_0 = 0$  then  $k'_0 \tau'_0 \neq 0$ .

•

$$\begin{aligned}
& \mp 16k_0 \cos \phi_0 + 16rk_0^2 \cos 2\phi_0 \\
& \mp 8r^2 \left( 4 \sin \phi_0 (k_0 \tau_0^2 - k_0'') - k_0^3 \cos 3\phi_0 + (4k_0 \tau_0' - k_0^3 + 8k_0' \tau_0) \cos \phi_0 \right) \\
& + 8r^3 \left( -k_0^4 - 2k_0^4 \cos 4\phi_0 - k_0^2 \tau_0^2 - 2k_0^2 \tau_0^2 \sin 2\phi_0 + 8k_0 k_0' \tau_0 + 2k_0 k_0' \tau_0 \sin 2\phi_0 \right. \\
& \quad + 6(k_0')^2 \sin 2\phi_0 + 4k_0^2 \tau_0' + k_0^2 \tau_0' \sin 2\phi_0 + k_0 k_0'' - 4k_0 k_0'' \sin 2\phi_0 \\
& \quad \left. + k_0 \cos 2\phi_0 (k_0^3 - 4k_0' \tau_0 - k_0 (\tau_0^2 - 4\tau_0') + k_0'') \right) \\
& \mp r^4 k_0 \left( -16k_0^2 \tau_0^2 (\sin \phi_0 + \sin 3\phi_0) - 32k_0 k_0' \tau_0 (\sin \phi_0 + \sin 3\phi_0) + 24(k_0')^2 (\sin \phi_0 + \sin 3\phi_0) \right. \\
& \quad - 8k_0^2 \tau_0' (\sin \phi_0 - \sin 3\phi_0) - 8k_0 k_0'' (\sin \phi_0 + \sin 3\phi_0) \\
& \quad - 2 \cos \phi_0 (k_0^4 + 20(k_0')^2 + 4k_0^2 (4\tau_0^2 - 3\tau_0') - 4k_0 k_0'') \\
& \quad \left. + \cos 3\phi_0 (9k_0^4 - 24(k_0')^2 + 8k_0^2 (2\tau_0^2 + \tau_0') + 8k_0 (-4\tau_0 k_0' + k_0'')) - 7k_0^4 \cos 5\phi_0 \right) \\
& + 2r^5 k_0^2 \sin \phi_0 \left( k_0 (k_0^3 \sin 5\phi_0 + 4 \cos \phi_0 (6k_0' \tau_0 - k_0 \tau_0' + \cos 2\phi_0 (-10k_0' \tau_0 + k_0 \tau_0'))) \right. \\
& \quad \left. - 2 \sin \phi_0 (k_0^4 - 11k_0^2 \tau_0^2 - 6(k_0')^2 + k_0 k_0'') - \sin 3\phi_0 (k_0^4 + 10k_0^2 \tau_0^2 - 12(k_0')^2 + 2k_0 k_0'') \right) \neq 0.
\end{aligned} \tag{5.5}$$

where,  $k_0 = k(t_0)$  for example.

(2) For a tangent-bi-normal surface  $T$ , point  $(t_0, -\pi/2)$  is  $\mathcal{A}$ -equivalent to the cuspidal cross cap if

- $(3 + 2r^2 k_0^2) \tau_0 + 2rk_0' = 0$ ,
- $-2k_0(2 + 3r^2 \tau_0^2) - 3k_0^3(r^2 + 2r^4 \tau_0^2) + 3r^3 k_0^2 \tau_0' + 3r(2\tau_0' + rk_0'') \neq 0$  and
- $2(-1 + r)r^2 k_0^3 - 2r^3 k_0^4 + k_0(-1 + r - 4r^4 \tau_0 k_0') - k_0^2(r + 2r^4 \tau_0') - r^2(3\tau_0' + 2rk_0'') \neq 0$

holds.

*Proof.* We assume  $\mathbf{e}_0 = (0, 0, 1)$ ,  $\mathbf{n}_0 = (1, 0, 0)$  and  $\mathbf{b}_0 = (0, 1, 0)$ . We write

$$R(t, \theta) = (R^1(t, \theta), R^2(t, \theta), R^3(t, \theta))$$

by this coordinates. We put a function  $g(t, \theta, z) = R^3(t, \theta) - z$ . Since  $\partial g / \partial \theta(t_0, \pm\pi/2) = \mp r \neq 0$ , by the implicit function theorem, there exists a  $C^\infty$ -function  $\theta = \theta(t, z)$  such that  $R^3(t, \theta(t, z)) = z$  holds. Then by

$$t = \tilde{t}, \quad \theta = \theta(\tilde{t}, z),$$

$(\tilde{t}, z)$  gives a new local coordinate system. From now on, we use the local coordinate system  $(\tilde{t}, z)$ . However for notational simplicity, we drop the overhead tilde's and write  $(\tilde{t}, z)$  just as  $(t, z)$ . We now set

$$\hat{R}(t, z) = (\hat{R}^1(t, z), \hat{R}^2(t, z), \hat{R}^3(t, z)) = R(t, \theta(t, z))$$

then  $\hat{R}^3(t, z) = z$  holds and we put  $c_z(t) = (\hat{R}^1(t, z), \hat{R}^2(t, z))$ . By Lemma 5.6, if  $c_0(t)$  has a 5/2-cusp at  $t = t_0$  and

$$\frac{d}{dz} \det \left( \frac{d^2}{dt^2} c_z(t), \frac{d^3}{dt^3} c_z(t) \right) \Big|_{(t_0, 0)} \neq 0$$

then  $R$  is a cuspidal cross cap. We have

$$\begin{aligned}\frac{dc_z(t)}{dt} &= ((R_{tt} + 2\theta'R_{t\theta} + (\theta')^2 R_{\theta\theta} + \theta''R_\theta) \cdot \mathbf{n}_0, (R_{tt} + 2\theta'R_{t\theta} + (\theta')^2 R_{\theta\theta} + \theta''R_\theta) \cdot \mathbf{b}_0), \\ \frac{d^2c_z(t)}{dt^2} &= ((R_{ttt} + 3\theta'R_{tt\theta} + 3(\theta')^2 R_{t\theta\theta} + 3\theta''R_{t\theta} + 3\theta'\theta''R_{\theta\theta} + \theta'''R_\theta) \cdot \mathbf{n}_0, \\ &\quad (R_{ttt} + 3\theta'R_{tt\theta} + 3(\theta')^2 R_{t\theta\theta} + 3\theta''R_{t\theta} + 3\theta'\theta''R_{\theta\theta} + \theta'''R_\theta) \cdot \mathbf{b}_0), \\ \theta'(t, \theta(t, z)) &= -\frac{R_t \cdot \mathbf{e}_0}{R_\theta \cdot \mathbf{e}_0}, \\ \theta''(t, \theta(t, z)) &= -\frac{(R_{tt} + 2\theta'R_{t\theta} + (\theta')^2 R_{\theta\theta}) \cdot \mathbf{e}_0}{R_\theta \cdot \mathbf{e}_0} \text{ and} \\ \theta'''(t, \theta(t, z)) &= -\frac{(R_{ttt} + 3\theta'R_{tt\theta} + 3(\theta')^2 R_{t\theta\theta} + 3\theta''R_{t\theta} + 3R_{\theta\theta}\theta'\theta'' + (\theta')^3 R_{\theta\theta\theta}) \cdot \mathbf{e}_0}{R_\theta}\end{aligned}$$

and so on. Here, we omit  $(t, \theta(t, z))$  and write  $\theta' = d\theta/dt$ . By a straightforward calculation, we get the formulae. By the same method, we also have (2).  $\square$

## 6. GENERIC CLASSIFICATIONS

In the space of ruled surfaces, there exists an open and dense subset such that for any ruled surface belongs to this set, a map germ at any point is a immersion or  $\mathcal{A}$ -equivalent to the cross cap (see [11]). We say this situation that *generic singularities* of ruled surfaces are only cross caps. Generic singularities of other classes of ruled surfaces are studied as follows:

- Developable surfaces(CE, SW, CCR)[7], cylinders(NS), cones, tangent developable surfaces(CE, CCR)[4].
- Principal normal surfaces(CR)[13], Bi-normal surfaces(NS).
- Darboux developable surfaces(CE, SW, CCR)[13].
- Rectifying developable(CE, SW)[9].

Here, NS means the non-singular point. In this section we consider the generic singularities of circular surfaces.

Let us define a subset  $M$  of  $\mathbf{R}^3 \times \mathbf{R}^3$  by

$$M = \{(Y, Z) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid \|Y\| = \|Z\| = 1, Y \cdot Z = 0\}.$$

Then  $M$  is a three dimensional manifold. We consider the space of  $C^\infty$ -mappings

$$C^\infty(I, \mathbf{R}^3 \times M) = \{(\gamma, (\mathbf{a}_1, \mathbf{a}_2)) \mid \gamma \in C^\infty(I, \mathbf{R}^3), (\mathbf{a}_1, \mathbf{a}_2) \in C^\infty(I, M)\}$$

equipped with Whitney  $C^\infty$ -topology. We consider this space as the *space of circular surfaces*. Then we have the following theorem.

**Theorem 6.1.** *There exists an open and dense subset  $\mathcal{O} \subset C^\infty(I, \mathbf{R}^3 \times M)$  such that for any  $(\gamma, (\mathbf{a}_1, \mathbf{a}_2)) \in \mathcal{O}$  and any  $(t, \theta) \in I \times \mathbf{R}/2\pi\mathbf{Z}$ ,  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2, r)}$  is non-canal and the map germ at  $(t, \theta)$  is an immersion or  $\mathcal{A}$ -equivalent to the cross cap.*

First of all, we remark that it is very hard to show that the condition of the cross cap which we have already given in Theorem 5.1 is a generic condition for circular surfaces. Therefore we give another proof here.

We define a map  $H : \mathbf{R}^3 \times M \times S^1 \longrightarrow \mathbf{R}^3$  by  $H(X, (Y, Z), \theta) = X + r(\cos \theta Y + \sin \theta Z)$ .

Then we have the following proposition.

**Proposition 6.2.** *For any fixed  $\theta \in I$ ,  $h_\theta : \mathbf{R}^3 \times M \longrightarrow \mathbf{R}^3$  is submersion, where  $h_\theta(X, Y, Z) = H(X, (Y, Z), \theta)$ .*

The space of circular surface is  $C^\infty(I, \mathbf{R}^3 \times M)$  equipped with Whitney  $C^\infty$  topology. Put a map  $j_1^k V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2)} : I \times S^1 \longrightarrow J^k(I, \mathbf{R}^3)$   $j_1^k V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2)}(t, \theta) = j^k h_\theta \circ (\gamma, (\mathbf{a}_1, \mathbf{a}_2))(t)$ . Since  $V(\gamma, \mathbf{a}_1, \mathbf{a}_2) = H \circ (\gamma, (\mathbf{a}_1, \mathbf{a}_2)) \times 1_{S^1}$ , we have the following lemma(c.f.,[31]).

**Lemma 6.3.** *For any submanifold  $W \subset J^k(I, \mathbf{R}^3)$ , the set*

$$T_W = \{(\gamma, \mathbf{a}_1, \mathbf{a}_2) \in C^\infty(I, \mathbf{R}^3 \times M) | j_1^k V_{(\gamma, (\mathbf{a}_1, \mathbf{a}_2))} \text{ is transverse to } W\}$$

*is a residual subset of  $C^\infty(I, \mathbf{R}^3 \times M)$ . Furthermore, if  $W$  is a closed set then  $T_W$  is an open set.*

For a circular surface  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2)}$ , we can regard the parameter  $\theta$  as a parameter of unfolding. By Lemma 6.3 and a same method as in Izumiya and Takeuchi [11], we have the following theorem.

**Theorem 6.4.** *There is an open and dense subset  $\mathcal{O}_2 \subset C^\infty(I, \mathbf{R}^3 \times M)$  such that for any  $(\gamma, \mathbf{a}_1, \mathbf{a}_2) \in \mathcal{O}_2$ ,  $F_{(\gamma, \mathbf{a}_1, \mathbf{a}_2)}$  has only cross cap as singularities.*

Since non canal surfaces are generic, we have the following theorem. This theorem asserts that generic singularities of circular surfaces are only cross cap.

**Theorem 6.5.** *There is an open and dense subset  $\mathcal{O} \subset C^\infty(I, \mathbf{R}^3 \times M)$  such that for any  $(\gamma, \mathbf{a}_1, \mathbf{a}_2) \in \mathcal{O}$ ,  $V_{(\gamma, \mathbf{a}_1, \mathbf{a}_2)}$  is non-canal and  $F$  has only cross cap as singularities.*

**6.1. Generic singularities of roller coaster surfaces.** Roller coaster surfaces are determined by a space curve  $\gamma$  and initial condition of  $\varphi$ . So, the set  $RC = \mathbf{R} \times C^\infty(I, \mathbf{R}_{>0} \times \mathbf{R})$  coincides the set of all roller coaster surfaces. We call  $RC = \mathbf{R} \times C^\infty(I, \mathbf{R}_{>0} \times \mathbf{R})$  equipped the Whitney  $C^\infty$ -topology the *space of roller coaster surfaces*. Then we have the following theorem.

**Theorem 6.6.** *There exists an open and dense subset  $\mathcal{O}_{RC} \subset RC$  such that for any  $V \in \mathcal{O}_{RC}$  and any  $(t, \theta)$ , the map germ of  $V$  at  $(t, \theta)$  is an immersion, a cuspidal edge, a swallowtail or a cuspidal cross cap.*

*Proof.* We define submanifolds

$$Q_1^\pm = \{j^2(k, \phi) | 1 \mp rk \cos \phi = 0\}, \quad Q_2 = \{j^2(k, \phi) | k' \cos \phi + k\tau \sin \phi = 0\}.$$

$Q_1^\pm \cap Q_2$  are closed submanifolds with codimension 2. Next, we set

$$Q_3 = \{j^2(k, \phi) \in J | k\tau \cos \phi - k' \sin \phi = 0\}.$$

Then  $Q_1^\pm \cap Q_3$  are closed submanifolds with codimension 2. Put

$$T = \{j^2(k, \phi) | \tau = 0\}, \quad S = \{j^2(k, \phi) | k' \sin \phi = 0\}.$$

Then these are closed subsets, so that  $J = J^2(I, \mathbf{R}_{>0} \times \mathbf{R}) \setminus (Q_1^\pm \cap Q_3) \cup (T \cap S)$  are manifolds.

If we put

$$Q_4^\pm = \{j^2(k, \phi) \in J | (5.4)_\pm = 0\} \quad \text{and} \quad Q_5^\pm = \{j^2(k, \phi) \in J | (5.5)_\pm = 0\},$$

then  $Q_3 \cap Q_4^\pm$  and  $Q_3 \cap Q_5^\pm$  are closed submanifolds in  $J$  with codimension 2. We will prove this fact later. Then, a dense subset

$$\begin{aligned} \mathcal{O}_{RC} = \{ & (k, \phi) \in C^\infty(I, \mathbf{R}_{>0} \times \mathbf{R}) | j^2(k, \phi) \text{ is transverse to} \\ & Q_1^\pm \cap Q_2, \quad Q_1^\pm \cap Q_3, \quad T \cap S, \quad Q_3 \cap Q_4^\pm \text{ and } Q_3 \cap Q_5^\pm \} \end{aligned}$$

has desired property. Since  $(Q_1^\pm \cap Q_2) \cup (Q_1^\pm \cap Q_3) \cup (T \cap S) \cup (Q_3 \cap Q_4^\pm) \cup (Q_3 \cap Q_5^\pm)$  is closed stratified set,  $\mathcal{O}_{RC}$  is an open set.  $\square$

We now prove the following lemma.

**Lemma 6.7.** *Sets  $Q_3 \cap Q_4^\pm$  and  $Q_3 \cap Q_5^\pm$  are closed submanifolds in  $J$  with codimension 2.*

*Proof.* We define two maps  $\Phi_l : J^2(I, \mathbf{R}_{>0} \times \mathbf{R}) \longrightarrow \mathbf{R}^2$  ( $l = 1, 2$ ) by

$$\begin{aligned} \phi_1(j^2(k, \Phi)) &= (\text{the left side hand of (5.1), the left side hand of (5.4)}_\pm) \text{ and} \\ \phi_2(j^2(k, \Phi)) &= (\text{the left side hand of (5.1), the left side hand of (5.5)}_\pm). \end{aligned}$$

We calculate the derivatives of  $\Phi_l$  with respect to the coordinates of  $J^2(I, \mathbf{R}_{>0} \times \mathbf{R})$  corresponding to the first and second order derivatives of the  $k$  and  $\phi$ . Then the derivatives of (5.1) with respect to first order derivatives of the  $k$  and  $\phi$  coincides with

$$(k \cos \phi, \sin \phi)$$

this is not zero at any  $\phi$ . Moreover, the derivatives of  $(5.4)_\pm$  with respect to second order derivatives of the  $k$  and  $\phi$  coincide with

$$(-r^2 \tau \sin \phi (1 \mp rk \cos \phi), r^2 k \tau \cos \phi (1 \mp rk \cos \phi))$$

and  $(5.5)_\pm$  of it is

$$\left( -4 \sin \phi \mp rk (\mp 1 + \cos 2\phi + (-2 \pm rk \sin \phi) \sin 2\phi), 4 \cos \phi \mp rk (2 + 2 \cos 2\phi + \sin 2\phi \pm 2rk \sin^3 \phi) \right).$$

Now, we assumed that  $1 \mp rk \cos \phi \neq 0$  and  $\tau \neq 0$ , these are not equal to zero any  $\phi$ . So,  $(0, 0)$  is a regular value of  $\phi_l$  and  $\phi_1^{-1}((0, 0)) = Q_3 \cap Q_4^\pm$  and  $\phi_2^{-1}((0, 0)) = Q_3 \cap Q_5^\pm$  hold. This proves the assertion.  $\square$

**6.2. Generic singularities of tangent-bi-normal circular surfaces.** Tangent-bi-normal-circular surfaces are determined by a space curve  $\gamma$  and  $r$ . So, the set  $EB = \mathbf{R}_{>0} \times C^\infty(I, \mathbf{R}_{>0} \times \mathbf{R})$  coincides the set of all tangent-bi-normal-circular surfaces.

We call  $EB = \mathbf{R}_{>0} \times C^\infty(I, \mathbf{R}_{>0} \times \mathbf{R})$  equipped the Whitney  $C^\infty$ -topology the space of roller coaster surfaces. Then by theorem 5.7 and transversality theorem, we have the following theorem.

**Theorem 6.8.** *There exists an open and dense subset  $\mathcal{O}_{EB} \subset EB$  such that for any  $V \in \mathcal{O}_{EB}$  and any  $(t, \theta)$ , the map germ of  $V$  at  $(t, \theta)$  is an immersion, a cross cap, a cuspidal edge.*

From the characterization of cross cap,  $T$  at  $(t, \pi/2)$  is  $\mathcal{A}$ -equivalent to the cross cap if and only if  $\tau(t) = 0$ ,  $\tau'(t) \neq 0$ . The proof is using same method to proof of theorem 6.6, we omit it. Cuspidal edges and cuspidal cross caps appear along the tangent curves. Cross caps appear on the antipodal striction curve of that curves.

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