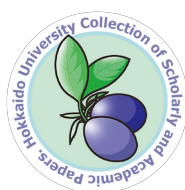




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Title	Ground State Energy of the Polaron in the Relativistic Quantum Electrodynamics
Author(s)	Sasaki, Itaru
Citation	Hokkaido University Preprint Series in Mathematics, 737, 1-8
Issue Date	2005
DOI	https://doi.org/10.14943/83887
Doc URL	https://hdl.handle.net/2115/69545
Type	departmental bulletin paper
File Information	re737.pdf



Ground State Energy of the Polaron in the Relativistic Quantum Electrodynamics

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July 30, 2005

Abstract

We consider the polaron model in the relativistic quantum electrodynamics(QED). We prove that the ground state energy of the model is finite for all values of the fine-structure constant and the ultraviolet cutoff Λ . Moreover we give an upper bound and a lower bound of the ground state energy.

Key words: relativistic QED; ground state energy, polaron model.

1 Introduction and Main Results

We consider the relativistic quantum electrodynamics(QED) for a fixed total momentum — the polaron model of the relativistic QED. The Hamiltonian, which describes a Dirac particle minimally coupled to the quantized radiation field, commutes with the total momentum operator and has a direct integral decomposition with respect to the total momentum operator([1], [2]). Each fibre in this direct integral decomposition is just the Hamiltonian of the polaron we consider. The Hilbert space of the polaron model is defined by

$$\mathcal{F} := \mathbb{C}^4 \otimes \mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\})), \quad (1)$$

where

$$\mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\})) := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n (L^2(\mathbb{R}^3 \times \{1, 2\})) \right] \quad (2)$$

is the photon Fock space ($\bigotimes_s^n$ denotes n -fold symmetric tensor product). For a closable operator T on $L^2(\mathbb{R}^3 \times \{1, 2\})$ we denote by $d\Gamma_b(T)$, the second quantization operator of T (see [4]). Let $a(f)$, $f \in L^2(\mathbb{R}^3 \times \{1, 2\})$ be the annihilation operator on the photon Fock space. For a function $g_j \in L^2(\mathbb{R}^3 \times \{1, 2\})$, $j = 1, 2, 3$, we set

$$A_j := a(g_j) + a(g_j)^*, \quad j = 1, 2, 3. \quad (3)$$

Let $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ be the 4×4 -Dirac matrices, i.e., $\{\alpha_i, \alpha_j\} = 2\delta_{i,j}$, $\{\alpha_i, \beta\} = 0$, $\beta^2 = 1$, $i, j = 1, 2, 3$. Here $\{A, B\} := AB + BA$. For three objects a_1, a_2, a_3 we set $\mathbf{a} = (a_1, a_2, a_3)$, and write $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^3 a_j b_j$, provided that $a_j b_j$ and $\sum_{j=1}^3 a_j b_j$ are defined.

The Hamiltonian of the polaron model we consider is

$$H(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + M\beta + d\Gamma_b(\omega) - \boldsymbol{\alpha} \cdot d\Gamma_b(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A}, \quad (4)$$

where $\mathbf{p} \in \mathbb{R}^3$ is the fixed total momentum, $M \geq 0$ is the mass of the Dirac particle, $q \in \mathbb{R}$ is a constant proportional to the fine-structure constant, and $\omega = |\mathbf{k}|$ is the 1-photon Hamiltonian ($\mathbf{k} \in \mathbb{R}^3$). Note that we omit the symbol $\otimes$ between the Hilbert space for the Dirac matrices $\mathbb{C}^4$ and the photon Fock space $\mathcal{F}_b(L^2(\mathbb{R}^3 \times \{1, 2\}))$. The most important example of $\{g_j\}_{j=1}^3$ is of the form

$$f_j(\mathbf{k}, r) := \frac{\chi_\Lambda(\mathbf{k})}{|\mathbf{k}|^{1/2}} e_j^{(r)}(\mathbf{k}), \quad (5)$$

where the measurable functions $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$ are the polarization vectors:

$$\mathbf{k} \cdot \mathbf{e}^{(r)}(\mathbf{k}) = 0, \quad \mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{e}^{(s)}(\mathbf{k}) = \delta_{r,s}, \quad \text{a.e. } \mathbf{k} \in \mathbb{R}^3, \quad r, s = 1, 2, \quad (6)$$

and $\chi_\Lambda(\mathbf{k})$ is the characteristic function of the ball $\{\mathbf{k} \in \mathbb{R}^3 \mid |\mathbf{k}| < \Lambda\}$, $\Lambda > 0$.

We define

$$E_0(\mathbf{p}) := \inf_{\substack{\Psi \in \text{Dom}(H(\mathbf{p})) \\ \|\Psi\|=1}} \langle \Psi, H(\mathbf{p})\Psi \rangle \quad (7)$$

the ground state energy of $H(\mathbf{p})$, where “Dom” means operator domain. We assume the following:

Hypothesis I. $g_j \in \text{Dom}(\omega^{-1/2}) \cap \text{Dom}(\omega)$, $\langle g_j, g_\ell \rangle \in \mathbb{R}$, $j, \ell = 1, 2, 3$.

It should be noted that it is highly non-trivial whether or not $E_0(\mathbf{p})$ is finite, because $H(\mathbf{p})$ contains the term $-\boldsymbol{\alpha} \cdot d\Gamma_b(\mathbf{k})$. This is the main problem discussed in the present paper.

We prove that the ground state energy $E_0(\mathbf{p})$ is finite under suitable conditions:

Theorem 1.1. *Assume Hypothesis I, and*

$$G(\mathbf{g}) := \inf_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \sum_{r=1,2} \int_{\mathbb{R}^3} \frac{|\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', r)|^2}{|\mathbf{k}| |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} d\mathbf{k}' < \infty. \quad (8)$$

Then, the ground state energy $E_0(\mathbf{p})$ is finite:

$$E_0(\mathbf{p}) > -\infty. \quad (9)$$

In particular, if $g_j = f_j$, $j = 1, 2, 3$, the ground state energy $E_0(\mathbf{p})$ is finite.

For a vector $u \in \mathbb{C}^4$, we set $a_j := \langle u, \alpha_j u \rangle_{\mathbb{C}^4}$, and

$$\mathcal{E}(\Lambda, u) := \mathbf{p} \cdot \mathbf{a} + M \langle u, \beta u \rangle + 4\pi\Lambda q^2 \frac{1 - |\mathbf{a}|^2}{|\mathbf{a}|} \log \left(\frac{1 + |\mathbf{a}|}{1 - |\mathbf{a}|} \right) - 4\pi\Lambda q^2.$$

In the physical case (i.e. the function g_j 's are given by (5)), the lower bound of $E_0(\mathbf{p}) + \sqrt{|\mathbf{p}|^2 + M^2}$ are proportional to Λ :

Theorem 1.2. *Let $g_j = f_j$, $j = 1, 2, 3$. Then*

$$C_1 \Lambda - \sqrt{|\mathbf{p}|^2 + M^2} \leq E_0(\mathbf{p}), \quad (10)$$

$$E_0(\mathbf{p}) \leq C_2(\Lambda) \quad (11)$$

where

$$C_1 := \inf_{\epsilon, \epsilon' > 0} \left\{ \epsilon |q| + 16\pi q^2 + \left(\epsilon' + \frac{1}{\epsilon'} \right) 4\pi q^2, \sqrt{\frac{4\pi |q|}{3\epsilon} + \left(1 + \frac{1}{\epsilon'} \right) 4\pi q^2} \right\}, \quad (12)$$

$$C_2(\Lambda) := \inf_{\substack{u \in \mathbb{C}^4 \\ \|u\|_{\mathbb{C}^4} = 1}} \mathcal{E}(\Lambda, u). \quad (13)$$

2 Proof of Theorem 1.1 and 1.2

Lemma 2.1. *Let A be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$. Let B be a symmetric operator with $\text{Dom}(A) \subset \text{Dom}(B)$ and*

$$\|B\Psi\| \leq \|A\Psi\|, \quad \Psi \in \text{Dom}(A). \quad (14)$$

Then, for all $\Psi \in D(A)$, $\langle \Psi, (A + B)\Psi \rangle \geq 0$.

Proof. By the Kato-Rellich theorem([4]), for all $\epsilon \in (-1, 1)$, $A + \epsilon B$ is self-adjoint and $A + \epsilon B \geq 0$. Therefore $\langle \Psi, (A + B)\Psi \rangle \geq 0$ for all $\Psi \in \text{Dom}(A)$. $\blacksquare$

By this lemma, it suffices to show that there exists a constant $E \geq 0$ such that

$$\|(\text{d}\Gamma_b(\omega) + E)\Psi\|^2 \geq \|\boldsymbol{\alpha} \cdot (\text{d}\Gamma_b(\mathbf{k}) + q\mathbf{A})\Psi\|^2, \quad \Psi \in \text{Dom}(\text{d}\Gamma_b(\omega)). \quad (15)$$

We use the following representation for $\boldsymbol{\alpha}$ -matrices:

$$\alpha_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix}, \quad j = 1, 2, 3,$$

with $(\sigma_1, \sigma_2, \sigma_3)$ being the Pauli matrices. Using anticommutation relation of α_1, α_2 and α_3 , we have

$$\|\boldsymbol{\alpha} \cdot (\text{d}\Gamma_b(\mathbf{k}) + q\mathbf{A})\Psi\|^2 = \sum_{j=1}^3 \|(\text{d}\Gamma_b(k_j) + qA_j)\Psi\|^2 - q\langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*]\Psi \rangle, \quad (16)$$

where $S_j = \sigma_j \oplus \sigma_j$, $j = 1, 2, 3$. The Hilbert space $\mathbb{C}^4 \otimes [\otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\})]$ is naturally embedded in $L^2(\mathbb{R}^3 \times \{1, 2\}; \mathbb{C}^4 \otimes [\otimes_s^{(n-1)} L^2(\mathbb{R}^3 \times \{1, 2\})])$. For a vector $\Psi \in \mathbb{C}^4 \otimes [\otimes_s^n L^2(\mathbb{R}^3 \times \{1, 2\})]$, we denote its value at point $(\mathbf{k}, r) \in \mathbb{R}^3 \times \{1, 2\}$ by $\Psi(\mathbf{k}, r, \cdot)$.

For $\Psi = (\Psi^{(n)})_{n=0}^\infty \in \mathcal{F}$, we define

$$a^{(r)}(\mathbf{k})\Psi := \left(\Psi^{(1)}(\mathbf{k}, r), \sqrt{2}\Psi^{(2)}(\mathbf{k}, r, \cdot), \dots, \sqrt{n}\Psi^{(n)}(\mathbf{k}, r, \cdot), \dots \right) \in \mathcal{F}, \quad \mathbf{k} \in \mathbb{R}^3, r = 1, 2, \quad (17)$$

a Fock space valued function. This operator $a^{(r)}(\mathbf{k})$ is the distributional kernel of the annihilation operator.

Lemma 2.2. *For all $\Psi \in \text{Dom}(\text{d}\Gamma_b(\omega))$ and $\epsilon > 0$, the following inequality holds:*

$$\left| q\langle \Psi, \mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*]\Psi \rangle \right| \leq |q|\epsilon \langle \Psi, \text{d}\Gamma_b(\omega)\Psi \rangle + \frac{|q|}{\epsilon} \langle \mathbf{g}, \omega \mathbf{g} \rangle \|\Psi\|^2, \quad (18)$$

where $\langle \mathbf{g}, \omega \mathbf{g} \rangle := \sum_{j=1}^3 \langle g_j, \omega g_j \rangle$.

Proof.

$$\begin{aligned}
\text{l.h.s of (18)} &= 2|q| \left| \text{Re} \int_{\mathbb{R}^3} \langle \Psi, -i\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}(\mathbf{k}, r)) a^{(r)}(\mathbf{k}) \Psi \rangle d\mathbf{k} \right| \\
&\leq 2|q| \int_{\mathbb{R}^3} \|\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}(\mathbf{k}, r)) \Psi\| \|a^{(r)}(\mathbf{k}) \Psi\| d\mathbf{k} \\
&= 2|q| \int_{\mathbb{R}^3} |\mathbf{k}|^{1/2} |\mathbf{g}(\mathbf{k}, r)| \cdot \| |\mathbf{k}|^{1/2} a^{(r)}(\mathbf{k}) \Psi \| \cdot \|\Psi\| d\mathbf{k} \\
&\leq 2|q| \langle \mathbf{g}, \omega \mathbf{g} \rangle^{1/2} \left[\int_{\mathbb{R}^3} \| |\mathbf{k}|^{1/2} a^{(r)}(\mathbf{k}) \Psi \|^2 \right]^{1/2} \|\Psi\| \\
&\leq |q| \epsilon \langle \Psi, d\Gamma_b(\omega) \Psi \rangle + \frac{|q|}{\epsilon} \langle \mathbf{g}, \omega \mathbf{g} \rangle \|\Psi\|^2,
\end{aligned}$$

where $\mathcal{J} := \sum_{r=1,2} \int$. ■

Lemma 2.3. *For all $\Psi \in \text{Dom}(d\Gamma_b(\omega))$ and $\epsilon > 0$, the following inequality holds:*

$$\langle \Psi, \mathbf{A}^2 \Psi \rangle \leq \left(2 + \epsilon + \frac{1}{\epsilon} \right) \langle \omega^{-1/2} \mathbf{g}, \omega^{-1/2} \mathbf{g} \rangle \langle \Psi, d\Gamma_b(\omega) \Psi \rangle + \left(1 + \frac{1}{\epsilon} \right) \langle \mathbf{g}, \mathbf{g} \rangle \|\Psi\|^2. \quad (19)$$

Proof.

$$\begin{aligned}
\langle \Psi, \mathbf{A}^2 \Psi \rangle &\leq \sum_{j=1}^3 \left[(1 + \epsilon) \|a(g_j) \Psi\|^2 + \left(1 + \frac{1}{\epsilon} \right) \|a(g_j)^* \Psi\|^2 \right] \\
&\leq \sum_{j=1}^3 \left[(1 + \epsilon) \| |\mathbf{k}|^{-1/2} g_j \|^2 \cdot \|d\Gamma_b(\omega)^{1/2} \Psi\|^2 \right. \\
&\quad \left. + \left(1 + \frac{1}{\epsilon} \right) \| |\mathbf{k}|^{-1/2} g_j \|^2 \cdot \|d\Gamma_b(\omega)^{1/2} \Psi\|^2 + \left(1 + \frac{1}{\epsilon} \right) \|g_j\|^2 \cdot \|\Psi\|^2 \right].
\end{aligned}$$
■

The following Lemma is the most important fact in the proof of Theorem 1.1:

Lemma 2.4. *For all $\Psi \in \text{Dom}(d\Gamma_b(\omega))$, the following inequality holds:*

$$\begin{aligned}
&\|d\Gamma_b(\omega) \Psi\|^2 - \sum_{j=1}^3 \|d\Gamma_b(k_j) \Psi\|^2 - q \langle d\Gamma_b(\mathbf{k}) \Psi, \mathbf{A} \Psi \rangle - q \langle \mathbf{A} \Psi, d\Gamma_b(\mathbf{k}) \Psi \rangle \\
&\geq -4q^2 G(\mathbf{g}) \langle \Psi, d\Gamma_b(\omega) \Psi \rangle - q \langle \Psi, (a(\mathbf{k} \cdot \mathbf{g})^* + a(\mathbf{k} \cdot \mathbf{g})) \Psi \rangle.
\end{aligned} \quad (20)$$

Proof. We define

$$F := \frac{\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)}{|\mathbf{k}| \cdot |\mathbf{k}| - \mathbf{k} \cdot \mathbf{k}'} \quad (21)$$

For all $\Psi \in \text{Dom}(\text{d}\Gamma_b(\omega))$, we have

$$\begin{aligned} \text{l.h.s of (20)} &= \iint_{\mathbb{R}^3} d\mathbf{k} \iint_{\mathbb{R}^3}' d\mathbf{k}' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}') \|(b - 2qF)a\Psi\|^2 \\ &\quad - q\langle \Psi, [a(\mathbf{k} \cdot \mathbf{g}) + a(\mathbf{k} \cdot \mathbf{g})^*] \Psi \rangle \\ &\quad - 4q^2 \iint_{\mathbb{R}^3} d\mathbf{k} \left[\iint_{\mathbb{R}^3}' d\mathbf{k}' (|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}')^{-1} |\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)|^2 \right] \|a\Psi\|^2, \end{aligned}$$

where $a := a^{(r)}(\mathbf{k})$, $b := a^{(\mu)}(\mathbf{k}')$, and $\mathfrak{f}' := \sum_{\mu=1,2} \int$. Since $|\mathbf{k}| \cdot |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}' \geq 0$, the inequality (20) holds. $\blacksquare$

Proof of Theorem 1.1. Using Lemmas 2.2 - 2.4, we get

$$\begin{aligned} &(\text{d}\Gamma_b(\omega) + E)^2 - \sum_{j=1}^3 (\text{d}\Gamma_b(k_j) + qA_j)^2 - q\mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*] \\ &\geq 2E\text{d}\Gamma_b(\omega) + E^2 - 4q^2 G(\mathbf{g})\text{d}\Gamma_b(\omega) - q[a(\mathbf{k} \cdot \mathbf{g})^* + a(\mathbf{k} \cdot \mathbf{g})] - |q|\text{d}\Gamma(\omega) \\ &\quad - |q|\langle \mathbf{g}, \omega \mathbf{g} \rangle - 4\langle \omega^{-1/2} \mathbf{g}, \omega^{-1/2} \mathbf{g} \rangle \text{d}\Gamma_b(\omega) - 2\langle \mathbf{g}, \mathbf{g} \rangle, \end{aligned}$$

in the sense of quadratic form on $\text{Dom}(\text{d}\Gamma_b(\omega))$. Since $a(\mathbf{k} \cdot \mathbf{g}) + a(\mathbf{k} \cdot \mathbf{g})^*$ is $\text{d}\Gamma_b(\omega)^{1/2}$ -bounded, for a large $E > 0$ we have

$$(\text{d}\Gamma_b(\omega) + E)^2 - \sum_{j=1}^3 (\text{d}\Gamma_b(k_j) + qA_j)^2 - q\mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{g}) + a(i\mathbf{k} \times \mathbf{g})^*] \geq 0.$$

By Lemma 2.1, for a large $E \geq 0$, we obtain

$$\text{d}\Gamma_b(\omega) - \boldsymbol{\alpha} \cdot \text{d}\Gamma_b(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A} \geq -E, \quad (22)$$

in the sense of quadratic form on $\text{Dom}(\text{d}\Gamma_b(\omega))$. This inequality implies that $E_0(\mathbf{p})$ is finite.

Next we show that $G(\mathbf{f}) < \infty$ if $g_j = f_j (j = 1, 2, 3)$. By the definitions of $\mathbf{e}^{(r)}(\mathbf{k})$, the vectors $\mathbf{k}/|\mathbf{k}|, \mathbf{e}^{(1)}(\mathbf{k}), \mathbf{e}^{(2)}(\mathbf{k})$ are the orthonormal basis of $\mathbb{C}^3$. Therefore

$$G(\mathbf{f}) = \sup_{\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}} \frac{1}{|\mathbf{k}|} \int_{\mathbb{R}^3} \frac{\chi_\Lambda(\mathbf{k}') d\mathbf{k}'}{|\mathbf{k}| |\mathbf{k}'| - \mathbf{k} \cdot \mathbf{k}'} \cdot \frac{1}{|\mathbf{k}'|} \left[|\mathbf{k}|^2 - \frac{(\mathbf{k} \cdot \mathbf{k}')^2}{|\mathbf{k}'|^2} \right] = \int_{\mathbb{R}^3} \frac{\chi_\Lambda(\mathbf{k}')}{|\mathbf{k}'|^2} d\mathbf{k}' = 4\pi\Lambda. \quad \blacksquare$$

Proof of Theorem 1.2. First we show (10). We set $g_j = f_j (j = 1, 2, 3)$. It is easy to see that

$$H(\mathbf{p}) \geq -\sqrt{|\mathbf{p}|^2 + M^2} + \text{d}\Gamma_b(\omega) - \boldsymbol{\alpha} \cdot \text{d}\Gamma_b(\mathbf{k}) - q\boldsymbol{\alpha} \cdot \mathbf{A}. \quad (23)$$

By the definition of $\mathbf{e}^{(r)}(\mathbf{k})$, we have $\mathbf{k} \cdot \mathbf{f}(\mathbf{k}, r) = 0$ ($\mathbf{k} \in \mathbb{R}^3, r = 1, 2$). Therefore, using Lemmas 2.2 – 2.4, we have

$$\begin{aligned} & (\mathrm{d}\Gamma_{\mathrm{b}}(\omega) + C_1\Lambda)^2 - (\mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k}) + q\mathbf{A})^2 - q\mathbf{S} \cdot [a(i\mathbf{k} \times \mathbf{f}) + a(i\mathbf{k} \times \mathbf{f})^*] \\ & \geq \left(2C_1\Lambda - 2|q|\epsilon\Lambda - 4q^2G(\mathbf{f}) - \left(2 + \epsilon' + \frac{1}{\epsilon'} \right) \langle \omega^{-1/2}\mathbf{f}, \omega^{-1/2}\mathbf{f} \rangle \right) \mathrm{d}\Gamma_{\mathrm{b}}(\omega) \\ & \quad + C_1^2\Lambda^2 - \frac{|q|}{2\epsilon\Lambda} \langle \mathbf{f}, \omega\mathbf{f} \rangle - \left(1 + \frac{1}{\epsilon'} \right) q^2 \langle \mathbf{f}, \mathbf{f} \rangle, \quad \epsilon, \epsilon' > 0. \end{aligned} \quad (24)$$

It is easy to see that $\langle \omega^{-1/2}\mathbf{f}, \omega^{-1/2}\mathbf{f} \rangle = 8\pi\Lambda$, $\langle \mathbf{f}, \omega\mathbf{f} \rangle = 8\pi\Lambda^3/3$, $\langle \mathbf{f}, \mathbf{f} \rangle = 4\pi\Lambda^2$. Hence, by the definition of C_1 , the left hand side of (24) is positive for suitable $\epsilon, \epsilon' > 0$. Thus, using Lemma 2.1 (and (16)), we have

$$H(\mathbf{p}) \geq -\sqrt{|\mathbf{p}|^2 + M^2} - C_1\Lambda. \quad (25)$$

For normalized vectors $u \in \mathbb{C}^4$, $\psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$ we define

$$\begin{aligned} a_j &:= \langle u, \alpha_j u \rangle, \quad h(\mathbf{a}) := \mathrm{d}\Gamma_{\mathrm{b}}(\omega - \mathbf{a} \cdot \mathbf{k}) - q\mathbf{a} \cdot \mathbf{A}, \\ \Psi &:= u \otimes \psi \in \mathcal{F}. \end{aligned}$$

Note that $\omega - \mathbf{a} \cdot \mathbf{k} \geq 0$ and $\omega - \mathbf{a} \cdot \mathbf{k}$ is injective as a multiplication operator. We have

$$\langle \Psi, H(\mathbf{p})\Psi \rangle = \mathbf{a} \cdot \mathbf{p} + M\langle u, \beta u \rangle + \langle \psi, h(\mathbf{a})\psi \rangle \quad (26)$$

Since $h(\mathbf{a})$ is a van Hove type Hamiltonian, we have

$$\begin{aligned} \inf \sigma(h(\mathbf{a})) &= -q^2 \|(|\mathbf{k}| - \mathbf{a} \cdot \mathbf{k})^{-1/2} \mathbf{a} \cdot \mathbf{f}\|^2 \\ &= -4\pi\Lambda q^2 + q^2(1 - |\mathbf{a}|^2) \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \frac{\chi_{\Lambda}(\mathbf{k})}{|\mathbf{k}|^2 - (\mathbf{a} \cdot \mathbf{k})^2} \\ &= -4\pi\Lambda q^2 + 2\pi\Lambda q^2(1 - |\mathbf{a}|^2) \frac{1}{|\mathbf{a}|} \log \left(\frac{1 - |\mathbf{a}|}{1 + |\mathbf{a}|} \right), \end{aligned}$$

where σ means the spectrum(e.g. [3]). Thus we have

$$E_0(\mathbf{p}) \leq \inf_{u \in \mathbb{C}^4; \|u\|=1} \inf_{\psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))} \langle \Psi, H(\mathbf{p})\Psi \rangle = \inf_{u \in \mathbb{C}^4; \|u\|=1} \mathcal{E}(u)$$

■

Acknowledgements

The author is grateful to Professor A. Arai of Hokkaido university for helpful comments.

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