



HOKKAIDO UNIVERSITY

Title	Global existence on nonlinear Schrödinger-IMBq equations
Author(s)	Cho, Yonggeun; Ozawa, Tohru
Citation	Hokkaido University Preprint Series in Mathematics, 744, 1-19
Issue Date	2005
DOI	https://doi.org/10.14943/83894
Doc URL	https://hdl.handle.net/2115/69552
Type	departmental bulletin paper
File Information	re744.pdf



GLOBAL EXISTENCE ON NONLINEAR SCHRÖDINGER-IMBQ EQUATIONS

YONGGEUN CHO AND TOHRU OZAWA

ABSTRACT. In this paper, we consider the Cauchy problem of Schrödinger-IMBq equations in $\mathbb{R}^n, n \geq 1$. We first show the global existence and blowup criterion of solutions in the energy space for the 3 and 4 dimensional system without power nonlinearity under suitable smallness assumption. Secondly the global existence is established to the system with p -powered nonlinearity in $H^s(\mathbb{R}^n), n = 1, 2$ for some $\frac{n}{2} < s < \min(2, p)$ and some $p > \frac{n}{2}$. We also provide a blowup criterion for $n = 3$ in Triebel-Lizorkin space containing BMO space naturally.

1. INTRODUCTION

We consider the Cauchy problem to the following system of equations (nonlinear Schrödinger-IMBq equations):

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = vu & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ \partial_t^2 v - \Delta v - \Delta \partial_t^2 v = \Delta(f(v) + |u|^2) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(0) = u_0, \quad (v(0), \partial_t v(0)) = (v_0, v_1) & \text{in } \mathbb{R}^n, \end{cases} \quad (1.1)$$

where u is a complex-valued function of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, v is a real-valued function of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and $f(v) = \lambda|v|^{p-1}v$ for a fixed real number λ . In this paper, we restrict our attention to positive time for simplicity since the case of negative time is treated analogously. The system is regarded as a substitute for the Zakharov system:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = vu, \\ \partial_t^2 v - \Delta v = \Delta|u|^2. \end{cases}$$

See [15, 16] for further details. Concerning the Zakharov system, see [8, 9, 20, 21, 22] for instance.

2000 *Mathematics Subject Classification.* 35Q53, 47J35.

Key words and phrases. Cauchy problem, Schrödinger-IMBq equations, global existence, blowup criterion.

The first author was supported by JSPS.

The local existence of system (1.1) was studied in [4] in $H^k, k \geq 5$ setting in multidimensional space and recently the global existence has been studied in [28] in H^2 setting in one space dimension.

The case $\lambda = 0$ for $1 \leq n \leq 4$ has been studied in [19], where the local well-posedness is proved in L^2 setting in space dimension $n \leq 4$ and global well-posedness is also proved in the energy class in space dimensions $n \leq 2$. Here the energy class stands for $H^1 \oplus L^2 \oplus \omega L^2$ for $(u, v, \partial_t v)$, where $v_1 \in \omega L^2$ means $\omega^{-1} v_1 \in L^2$ and $H^s = (1 - \Delta)^{-\frac{s}{2}} L^2$ is the usual Sobolev space with the norm $\|\psi\|_{H^s} = \|(1 - \Delta)^{\frac{s}{2}} \psi\|_{L^2}$. We use similar notation $v \in \omega^\alpha X$ to mean $\omega^{-\alpha} v \in X$ for a function space X and a nonnegative number α .

In this paper, we study the global existence of solutions to the system (1.1), extending results in [19, 28].

By Duhamel's principle, (1.1) is rewritten as

$$\begin{aligned} u(t) &= U(t)u_0 - i \int_0^t U(t-t')(vu)(t') dt', \\ v(t) &= (\partial_t K)(t)v_0 + K(t)v_1 + \int_0^t K(t-t')\omega^2(f(v) + |u|^2)(t') dt', \end{aligned} \tag{1.2}$$

where $U(t) = e^{i(t/2)\Delta}$, $K(t) = \omega^{-1} \sin t\omega$, $\partial_t K(t) = \cos \omega t$ and $\omega = (-\Delta)^{\frac{1}{2}}(1 - \Delta)^{-\frac{1}{2}}$. The second integral equation of (1.2) is also written as

$$\begin{aligned} \begin{pmatrix} v(t) \\ \omega^{-1} \partial_t v(t) \end{pmatrix} &= V(t) \begin{pmatrix} v_0 \\ \omega^{-1} v_1 \end{pmatrix} \\ &\quad + \int_0^t V(t-t') \begin{pmatrix} 0 \\ \omega(f(v) + |u|^2)(t') \end{pmatrix} dt', \end{aligned} \tag{1.3}$$

where

$$V(t) = \exp \left(t \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \right) = \begin{pmatrix} \cos t\omega & \sin t\omega \\ -\sin t\omega & \cos t\omega \end{pmatrix}.$$

The local and global existence results for IMBq equation, namely, $u = 0$, can be found in [5, 6, 14, 25, 26, 27] and the references therein.

In [19], the Strichartz estimates for Schrödinger evolution group U for (1.2) and conservation laws were the basic tools for the local or global well-posedness in case that $\lambda = 0$.

The Strichartz estimate on U can be stated as follows: For any admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$,

$$\begin{aligned} \|U(\cdot)\phi\|_{L_T^q L^r} &\leq C\|\phi\|_{L^2}, \\ \left\| \int_0^t U(t-t')F(t') dt' \right\|_{L_T^q L^r} &\leq C\|F\|_{L_T^{\tilde{q}} L^{\tilde{r}}}, \end{aligned} \quad (1.4)$$

where C is independent of $T > 0$ and $\|G\|_{L_T^q L^r} = \|G\|_{L_t^q(0, T; L_x^r)}$. Here we say the pair (q, r) is *admissible*, if $\frac{1}{q} + \frac{n}{2r} = \frac{n}{4}$, $2 \leq q, r \leq \infty$ and $(q, r, n) \neq (2, \infty, 2)$. The estimate (1.4) hold even when $T = \infty$. We use the notation $\|u\|_{L^q L^r} = \|u\|_{L_t^q(0, \infty; L^r)}$ when $T = \infty$. For the details of Strichartz estimate, see [12].

The solution (u, v) with sufficient regularity for the system (1.1) or (1.2) satisfies the basic physical laws, L^2 and energy conservations. Let $[0, T]$ be a existence time interval. Then for all $t \in [0, T]$

$$\|u(t)\|_{L^2} = \|u(0)\|_{L^2}, \quad (1.5)$$

$$\begin{aligned} E(t) &\equiv \frac{1}{2}(\|\nabla u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\omega^{-1}\partial_t v(t)\|_{L^2}^2) \\ &\quad + (v(t), |u(t)|^2) + \frac{\lambda}{p+1}\|v(t)\|_{L^{p+1}}^{p+1} \\ &= E(0). \end{aligned} \quad (1.6)$$

Using the standard regularizing argument for $f(v)$, the conservation laws can be shown with $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus \omega L^2$ for $\lambda = 0$ and $H^1 \oplus H^1 \oplus H^1$ for $\lambda \neq 0$. See for instance [2, 9].

The first result is an extension of the global existence in [19] for the case of $n = 1, 2$ to the case $n = 3, 4$.

Theorem 1.1. *Let $n = 3$. Assume that $\lambda = 0$. Then*

- (i) *There exists a constant ε_0 such that for any $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus \omega L^2$ with*

$$(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1}\|_{L^2})\|u_0\|_{L^2} \leq \varepsilon_0$$

the system (1.1) has a unique global solution (u, v) such that for any admissible pair (q, r)

$$\begin{aligned} u &\in C_b([0, \infty); H^1) \cap L_{loc}^q([0, \infty); L^r), \\ v &\in C_b^2([0, \infty); L^2), \quad \partial_t v \in C_b([0, \infty); \omega L^2). \end{aligned} \quad (1.7)$$

(ii) Let T^* be the maximal existence time to the Cauchy problem (1.1) with general initial data and it be finite. Then we have

$$\int_0^{T^*} (T^* - t) \|u(t)\|_{L^4}^2 dt = \infty. \quad (1.8)$$

Remark 1. $C_b^k([0, \infty))$, $k \geq 0$ is the space of bounded C^k functions on $[0, \infty)$.

Remark 2. The smallness condition in (i) of Theorem 1.1 is satisfied for data for the following forms:

- (1) $(u_0, v_0, v_1) = (\varepsilon\phi, \varepsilon\psi_0, \varepsilon\psi_1)$ with $(\phi, \psi_0, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$ and $\varepsilon > 0$ sufficiently small.
- (2) $(u_0, v_0, v_1) = (\phi_\varepsilon, \psi_0, \psi_1)$, where $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{2}}\phi(\varepsilon^{-1}x)$, $(\phi, \psi_1, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$, and $\varepsilon > 0$ sufficiently small.

Theorem 1.2. Let $n = 4$. Assume that $\lambda = 0$. Then

- (i) There exists a constant $\varepsilon_0 > 0$ such that for any $(u_0, v_0, v_1) \in H^1 \oplus L^2 \oplus \omega L^2$ with

$$\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1}v_1\|_{L^2} \leq \varepsilon_0$$

the system (1.1) has a unique global solution (u, v) satisfying (1.7) replaced by $L^q L^r$.

- (ii) There exist a pair of functions $(u_0^+, v_0^+, v_1^+) \in L^2 \times L^2 \times \omega L^2$ such that

$$\begin{aligned} \|u(t) - U(t)u_0^+\|_{L^2} &\rightarrow 0, \\ \|v(t) - \partial_t K(t)v_0^+ - K(t)v_1^+\|_{L^2} &\rightarrow 0, \\ \|\omega^{-1}(\partial_t v(t) + \omega^2 K(t)v_0^+ - \partial_t K(t)v_1^+)\|_{L^2} &\rightarrow 0 \end{aligned} \quad (1.9)$$

as $t \rightarrow +\infty$.

Remark 3. The smallness condition in Theorem 1.2 is satisfied for data of the following forms:

- (1) $(u_0, v_0, v_1) = (\varepsilon\phi, \varepsilon\psi_0, \varepsilon\psi_1)$ with $(\phi, \psi_0, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$ and $\varepsilon > 0$ sufficiently small.

- (2) $(u_0, v_0, v_1) = (\phi_\varepsilon, \varepsilon\psi_0, \varepsilon\psi_1)$, where $\phi_\varepsilon(x) = \varepsilon^2\phi(\varepsilon x)$, $(\phi, \psi_0, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$, and $\varepsilon > 0$ sufficiently small. Note that $\|\phi_\varepsilon\|_{L^2} = \|\phi\|_{L^2}$ and that the size of the L^2 norm may not be small.
- (3) $(u_0, v_0, v_1) = (\phi^\varepsilon, \psi_0^\varepsilon, \psi_1^\varepsilon)$, where $\phi^\varepsilon(x) = \varepsilon^{-a}\phi(\varepsilon^{-1}x)$, $\psi_j^\varepsilon(x) = \varepsilon^{-b}\psi_j(\varepsilon^{-1}x)$, $j = 0, 1$, $(\phi, \psi_0, \psi_1) \in H^1 \oplus L^2 \oplus \omega L^2$, $0 < a < 1$, $0 < b < 2$, and $\varepsilon > 0$ sufficiently small.

Remark 4. There is no result on the global existence or on blowup criteria for Zakharov system for $n = 3, 4$, except special problem (see [8, 10, 20, 21, 22]).

Now we consider the case of nonzero nonlinearity. For the simplicity of presentation, we assume that for some positive number s

$$(u_0, v_0, v_1) \in H^s \oplus H^s \oplus \omega H^s, \quad 1 \leq n \leq 3, \quad \lambda > 0, \quad 1 < p < \infty. \quad (1.10)$$

The second result is the following.

Theorem 1.3. (i) *If $n = 1$, $1 < p < \infty$ and $1 \leq s < \min(2, p)$, then there exists a unique global solution (u, v) satisfying*

$$u \in C([0, \infty); H^s), \quad v \in C^2([0, \infty); H^s), \quad \partial_t v \in C([0, \infty); \omega H^s). \quad (1.11)$$

(ii) *If $n = 2$, $1 < p \leq 3$ and $1 < s < \min(2, p)$, then there exists a unique global solution (u, v) satisfying (1.11).*

Remark 5. Part (i) is an extension of result in [28], where the global existence of solutions in H^2 is studied for odd integer $p \geq 3$.

Remark 6. In the case $n = 1$, we use the conservation laws for the control of L^∞ through the Sobolev embedding. But when $n = 2$, we cannot use such embedding any more. Instead, we use a version of Brezis-Gallouet-Wainger inequality in Sobolev space (see [18]). In view of these Sobolev inequalities, we can obtain a global existence in $H^s(\mathbb{R}^n)$ even for all $s > 1$, if $n = 1$ and p is odd interger, and if $n = 2$ and $p = 3$.

The next result is on the local existence and blowup criterion for $n = 3$.

Theorem 1.4. (i) If $n = 3$, $1 < p < \infty$ and $\frac{3}{2} < s < \min(2, p)$, then there exists a positive time T_* and unique solution (u, v) such that

$$u \in C([0, T_*]; H^s), \quad v \in C^2([0, T_*]; H^s), \quad \partial_t v \in C([0, T_*]; \omega H^s). \quad (1.12)$$

(ii) Let T^* be the maximal existence time of solution (u, v) to Cauchy problem (1.1) and it be finite. Then

$$\int_0^{T^*} \left(\|u(t)\|_{\dot{F}_{\infty, \infty}^0} + \|v(t)\|_{\dot{F}_{\infty, \infty}^0} \right)^{p-1} dt = \infty$$

for $\frac{3}{2} < p \leq 2$. Furthermore, if

$$(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1} v_1\|_{L^2}) \|u_0\|_{L^2}$$

is sufficiently small, then

$$\int_0^{T^*} \|u(t)\|_{\dot{F}_{\infty, \infty}^0}^{\frac{4(p-1)}{p+1}} dt = \infty$$

for $\frac{3}{2} < p \leq \frac{5}{3}$.

Here, $\dot{F}_{\infty, \infty}^0$ is the Triebel-Lizorkin space defined as follows. Let φ be a Littlewood-Paley function such that $\sum_{j \in \mathbb{Z}} \varphi\left(\frac{\xi}{2^j}\right) = 1$, if $\xi \neq 0$. Let Δ_j be a frequency projection operator such that $\widehat{\Delta_j \psi}(\xi) = \varphi(\xi/2^j) \widehat{\psi}(\xi)$, where $\widehat{\psi}$ is the Fourier transform of ψ .

$$\dot{F}_{\infty, \infty}^0 = \left\{ \psi \in \mathcal{S}' : \|\psi\|_{\dot{F}_{\infty, \infty}^0} \equiv \left\| \sup_{j \in \mathbb{Z}} |\Delta_j \psi| \right\|_{L^\infty} < \infty \right\}.$$

It should be noted that $\text{BMO} = \dot{F}_{\infty, 2}^0 \hookrightarrow \dot{F}_{\infty, \infty}^0$ (for the details, see [23, 24]).

Remark 7. The blowup criterion in Theorem 1.4 can be extended to some value of p such that $2 < p \leq 3$ for large data and $\frac{5}{3} < p \leq 5$ for small data. For the details, see Remark 10 below.

Remark 8. For the local existence in Theorems 1.3 and 1.4, the H^s -regularity on data for $s > \frac{n}{2}$ is used to control the L^∞ norm of u and v for estimation of the bilinear term vu and of the nonlinear term $\lambda|v|^{p-1}v$ via Sobolev embedding. But by deriving dispersive estimates on K and $\partial_t K$, we can control L^∞ without resort to the Sobolev embedding. For one dimensional argument, see [6] for instance. In the forthcoming paper, we will treat the multidimensional case.

If not specified, throughout this paper, we denote C by a generic constant varying line by line and depending only on the norms of initial data, s , λ , p , admissible pair (q, r) and absolute constant.

2. CASE $\lambda = 0$: PROOF OF THEOREMS 1.1 AND 1.2

In this section, we consider the global existence of the system with $n = 3, 4$ in case that $\lambda = 0$ based on the conservation laws (1.5) and (1.6). Since the local existence was studied in [19], we have only to consider global a priori estimates of solutions in the 3 and 4 dimensional energy space. That is to say, it suffices to show that for all $T > 0$

$$\|u\|_{L_T^\infty H^1} + \|v(t)\|_{L_T^\infty L^2} + \|\omega^{-1}v_t\|_{L_T^\infty L^2} \leq C. \quad (2.1)$$

2.1. Proof of Theorem 1.1.

2.1.1. *Global existence.* By the Hölder inequality, the standard Sobolev inequality $\|u\|_{L^6} \leq C_0 \|\nabla u\|_{L^2}$, and the L^2 conservation (1.5), we have

$$|(v, |u|^2)| \leq \|v\|_{L^2} \|u\|_{L^2}^{\frac{1}{2}} \|u\|_{L^6}^{\frac{3}{2}} \leq C_0^{\frac{3}{2}} \|u_0\|_{L^2}^{\frac{1}{2}} \|v\|_{L^2} \|\nabla u\|_{L^2}^{\frac{3}{2}}. \quad (2.2)$$

This implies upper and lower bounds on $E(0)$ in terms of Cauchy data in the energy space. Regarding lower bounds, we have

$$\begin{aligned} E(0) &= \frac{1}{2} (\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1}v_1\|_{L^2}^2) + (v_0, |u_0|^2) \\ &\geq \frac{1}{2} (\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1}v_1\|_{L^2}^2) \\ &\quad - \frac{1}{2} \|v_0\|_{L^2}^2 - \frac{C_0^3}{2} \|u_0\|_{L^2} \|\nabla u_0\|_{L^2}^3 \\ &\geq \frac{1}{2} (1 - C_0^3 \|u_0\|_{L^2} \|\nabla u_0\|_{L^2}) \|\nabla u_0\|_{L^2}^2 + \frac{1}{2} \|\omega^{-1}v_1\|_{L^2}^2 \\ &\geq 0, \end{aligned} \quad (2.3)$$

provided $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \leq C_0^{-3}$.

Now we set

$$M(t) = \frac{1}{2} (\|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\omega^{-1}\partial_t v\|_{L^2}^2).$$

Then the RHS of the last inequality in (2.2) is bounded by

$$C_0^{\frac{3}{2}} \|u_0\|_{L^2}^{\frac{1}{2}} M(t) \|\nabla u\|_{L^2}^{\frac{1}{2}} \leq 2^{\frac{1}{4}} C_0^{\frac{3}{2}} \|u_0\|_{L^2}^{\frac{1}{2}} M(t)^{\frac{5}{4}}.$$

By the energy conservation (1.6), this implies

$$\begin{aligned} M(t) &= E(t) - (v, |u|^2) \\ &= E(0) - (v, |u|^2) \\ &\leq E(0) + 2^{\frac{1}{4}} C_0^{\frac{3}{2}} \|u_0\|_{L^2}^{\frac{1}{2}} M(t)^{\frac{5}{4}}. \end{aligned} \tag{2.4}$$

We see from (2.4) that

$$M(t) \leq 5E(0), \tag{2.5}$$

provided

$$E(0)(2C_0^6 \|u_0\|_{L^2}^2) < \frac{1}{5} \left(\frac{4}{5}\right)^4. \tag{2.6}$$

The required a priori estimate (2.1) follows from (2.5) under smallness condition given by (2.3), (2.4) and (2.6).

An application of (2.5) is the Strichartz estimate of u . For any admissible pair (q, r) , using (1.4), we have

$$\|u\|_{L_T^q L^r} \leq CT^{\frac{1}{q}} \|u\|_{L_T^\infty H^1} \leq C(T),$$

where $C(T)$ is a constant depending only on C and T . Therefore, we deduce that $u \in L_{loc}^q L^r$.

2.1.2. Blow-up criterion. Assume that

$$\int_0^{T^*} (T^* - t) \|u(t)\|_{L^4}^2 dt < \infty. \tag{2.7}$$

Then let us observe from the equation (1.2) that

$$\|\partial_t v(t)\|_{L^2} \leq C(\|v_0\|_{L^2} + \|v_1\|_{L^2}) + C \int_0^t \|u(t')\|_{L^4}^2 dt'.$$

and hence that the finiteness of (2.7) implies that of $\|\partial_t v\|_{L^1(0, T^*; L^2)}$.

Now we assume that

$$\int_0^{T^*} \|\partial_t v\|_{L^2} dt \equiv M < \infty. \tag{2.8}$$

Then taking L^2 norm of $v(t) = v_0 + \int_0^t \partial_t v(t') dt'$, we have

$$\sup_{0 \leq t < T^*} \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + M. \tag{2.9}$$

By conservation laws (1.5), (1.6), and the estimates (2.2) and (2.9),

$$\begin{aligned} \frac{1}{2} \|\nabla u\|_{L^2}^2 &= E(t) - \frac{1}{2} (\|v\|_{L^2}^2 + \|\omega^{-1} \partial_t v\|_{L^2}^2) - (v, |u|^2) \\ &\leq E(0) + \|v\|_{L^2} \|u\|_{L^4}^2 \\ &\leq E(0) + (\|v_0\|_{L^2} + M) \|u_0\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{3}{2}}. \end{aligned} \quad (2.10)$$

This implies

$$\sup_{0 \leq t < T^*} \|\nabla u(t)\|_{L^2}^2 \leq CE(0) + CE(\|v_0\|_{L^2} + M)^4 \|u_0\|_{L^2}^2 \quad (2.11)$$

Moreover, we obtain

$$\begin{aligned} \|v\|_{L^2}^2 + \|\omega^{-1} \partial_t v\|_{L^2}^2 &= 2E(0) - \|\nabla u\|_{L^2}^2 - 2(v, |u|^2) \\ &\leq 2E(0) - \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|v\|_{L^2}^2 + 8\|u\|_{L^4}^4 \\ &\leq 2E(0) - \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|v\|_{L^2}^2 + 8C_0^3 \|u_0\|_{L^2} \|\nabla u\|_{L^2}^3. \end{aligned}$$

This implies

$$\begin{aligned} \sup_{0 \leq t < T^*} (\|v(t)\|_{L^2}^2 + \|\omega^{-1} \partial_t v\|_{L^2}^2) \\ \leq 4E(0) + 16C_0^3 \|u_0\|_{L^2} \left(\sup_{0 \leq t < T^*} \|\nabla u(t)\|_{L^2} \right)^3. \end{aligned} \quad (2.12)$$

Estimates (2.11) and (2.12) contradict the maximality of T^* .

2.2. Proof of Theorem 1.2.

2.2.1. *Global existence.* By the Hölder inequality and the standard Sobolev inequality $\|u\|_{L^4} \leq C_0 \|\nabla u\|_{L^2}$, we have

$$|(v, |u|^2)| \leq \|v\|_{L^2} \|u\|_{L^4}^2 \leq C_0^2 \|v\|_{L^2} \|\nabla u\|_{L^2}^2.$$

Therefore,

$$\begin{aligned} E(0) &= \frac{1}{2} (\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1} v_1\|_{L^2}^2) + (v, |u|^2) \\ &\leq \frac{1}{2} (\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|\omega^{-1} v_1\|_{L^2}^2) + C_0^2 \|v_0\|_{L^2} \|\nabla u_0\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} E(0) &\geq \frac{1}{2} (\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2) - \varepsilon C_0^2 \|v_0\|_{L^2} \|\nabla u_0\|_{L^2} \\ &\geq \frac{1 - \varepsilon C_0^2}{2} (\|\nabla u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2), \end{aligned}$$

provided $\|\nabla u_0\|_{L^2} \leq \varepsilon < C_0^{-2}$.

Now we set

$$M(t) = \frac{1}{2} (\|\nabla u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|\omega^{-1} \partial_t v(t)\|_{L^2}^2).$$

Then

$$\begin{aligned} M(t) &= E(0) - (v, |u|^2) \\ &\leq E(0) + C_0^2 \|v\|_{L^2} \|\nabla u\|_{L^2}^2 \\ &\leq E(0) + C_0^2 M(t) \|\nabla u\|_{L^2} \\ &\leq E(0) + 2^{\frac{1}{2}} C_0^2 M(t)^{\frac{3}{2}}, \end{aligned}$$

from which the required a priori estimate on M follows, provided $\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1} v_1\|_{L^2}$ is sufficiently small.

By the endpoint estimate $(q, r) = (2, 4)$ of (1.4), we have

$$\begin{aligned} \|u\|_{L_T^2 L^4} &\leq C \|u_0\|_{L^2} + C \|vu\|_{L_T^2 L^{\frac{4}{3}}} \\ &\leq C \|u_0\|_{L^2} + C \|v\|_{L_T^\infty L^2} \|u\|_{L_T^2 L^4}, \end{aligned}$$

where C is independent of $T > 0$. Since M may be taken sufficiently small by the smallness assumption on the data,

$$\|u\|_{L_T^2 L^2} \leq C \|u_0\|_{L^2},$$

where C is independent of $T > 0$. By Fatou's lemma, $u \in L^2 L^4$. Since $u \in L^\infty L^2$, for any admissible pair (q, r) , $\|u\|_{L^q L^r} \leq C \|u_0\|_{L^2}$.

2.2.2. *Scattering.* By the integral equation, we have

$$U(-t)u(t) - U(-s)u(s) = -i \int_s^t U(-t')(vu)(t') dt'.$$

By the endpoint Strichartz estimate, we have

$$\begin{aligned} \|U(-t)u(t) - U(-s)u(s)\|_{L^2} &= \left\| \int_s^t U(-t')(vu)(t') dt' \right\|_{L^2} \\ &\leq C \|vu\|_{L^2(s, t; L^{\frac{4}{3}})} \\ &\leq C \|v\|_{L^\infty L^2} \|u\|_{L^2(s, t; L^4)} \\ &\rightarrow 0 \end{aligned}$$

as $t > s \rightarrow +\infty$. This gives a unique asymptotic state $u_0^+ \in L^2$.

Similarly, the existence of unique asymptotic sates $(v_0^+, v_1^+) \in L^2 \oplus \omega L^2$ follows by the integral equation (1.3).

3. CASE $\lambda \neq 0$: PROOF OF THEOREMS 1.3 AND 1.4

3.1. Local existence. In this section, we discuss the local existence theory of the system (1.1) with general nonlinear term $f(v)$. To do this, let us first introduce a lemma for the nonlinear estimates.

Lemma 3.1. (See Lemma 3.4 in [7], Lemma 3.1 in [6] and appendices in [11])

(i) Let $f \in C^1(\mathbb{R}, \mathbb{R})$ with $f(0) = f'(0) = 0$ and assume that

$$|f'(t_1) - f'(t_2)| \leq C \begin{cases} |t_1 - t_2|^{p-1}, & \text{if } 1 \leq p \leq 2 \\ (|t_1|^{p-2} + |t_2|^{p-2})|t_1 - t_2|, & \text{if } p > 2 \end{cases} \quad (3.1)$$

for all $t_1, t_2 \in \mathbb{R}$. Let $0 \leq s < \min(2, p)$. Then f satisfies the estimate

$$\|f(v)\|_{\dot{H}^s} \leq C \|v\|_{L^\infty}^{p-1} \|v\|_{\dot{H}^s} \quad (3.2)$$

for any $v \in L^\infty \cap \dot{H}^s$.

(ii) Let $f \in C^k(\mathbb{R}, \mathbb{R})$ ($k \geq 2$) with

$$|f^{(j)}(t)| \leq C |t|^{p-j} \quad (3.3)$$

for all $0 \leq j \leq k \leq p$ and $t \in \mathbb{R}$. Then f satisfies the estimate

$$\|f(v)\|_{\dot{H}^s} \leq C \|v\|_{L^\infty}^{p-1} \|v\|_{\dot{H}^s} \quad (3.4)$$

for any s with $0 \leq s \leq k$ and any $v \in L^\infty \cap \dot{H}^s$.

(iii) For any $s \geq 0$, we have

$$\|uv\|_{\dot{H}^s} \leq C (\|u\|_{L^\infty} \|v\|_{\dot{H}^s} + \|v\|_{L^\infty} \|u\|_{\dot{H}^s}) \quad (3.5)$$

for any $v \in L^\infty \cap \dot{H}^s$

Using the lemma above, we obtain the following.

Proposition 3.2. Let $f \in C^k(\mathbb{R}, \mathbb{R})$ satisfy (3.1) for $k = 1$ and (3.3) for $k \geq 2$. Suppose that $(u_0, v_0, v_1) \in H^s \oplus H^s \oplus \omega H^s$ for $s < p$, if $k = 1$ and $s \leq k$, if $k \geq 2$. Then there exists a positive time T_* and unique solution (u, v) satisfying the regularity (1.12).

Proof of Proposition 3.2. Let us define nonlinear functionals N_1 and N_2 by

$$\begin{aligned} N_1(u, v)(x, t) &= U(t)u_0 - i \int_0^t U(t-t')(vu)(t') dt', \\ N_2(u, v)(x, t) &= (\partial_t K)(t)v_0 + K(t)v_1 + \int_0^t K(t-t')\omega^2(f(v) + |u|^2)(t') dt'. \end{aligned}$$

We also define a complete metric space $X_R(T)$ with metric d_T by

$$\begin{aligned} X_R(T) &= \{(u, v) : \|(u, v)\|_{X(T)} \equiv \|u\|_{L_T^\infty H^s} + \|v\|_{L_T^\infty H^s} \leq R\}, \\ d_T((u, v), (\tilde{u}, \tilde{v})) &= \|(u, v) - (\tilde{u}, \tilde{v})\|_{L_T^\infty L^2}. \end{aligned}$$

Then from Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{n}{2}$ and Lemma 3.1, we have for any $(u, v) \in X_R(T)$

$$\begin{aligned} \|N_1(t)\|_{H^s} &\leq \|u_0\|_{H^s} + C \int_0^t \|v(t')\|_{H^s} \|u(t')\|_{H^s} dt' \leq C + CR^2T, \\ \|N_2(t)\|_{H^s} &\leq \|v_0\|_{H^s} + \|\omega^{-1}v_1\|_{H^s} + C \int_0^t (\|v\|_{H^s}^p + \|u\|_{H^s}^2) dt' \\ &\leq C + C(M(R)R + R^2)T. \end{aligned}$$

If $R \geq 3C$ and $T < T_1$ for some small T_1 such that

$$CR^2T_1 + C(M(R)R + R^2)T_1 \leq C,$$

then $(N_1(u, v), N_2(u, v)) \in X_R(T)$.

On the other hand, if $(u, v), (\tilde{u}, \tilde{v}) \in X_R(T)$ and $T < T_2$ for some small T_2 , then from Lemma 3.1

$$\begin{aligned} &d_T((N_1(u, v), N_2(u, v)), (N_1(\tilde{u}, \tilde{v}), N_2(\tilde{u}, \tilde{v}))) \\ &\leq C \int_0^T (\|(u, v)\|_{X(t')} + \|(\tilde{u}, \tilde{v})\|_{X(t')}) d_{t'}((u, v), (\tilde{u}, \tilde{v})) dt' \\ &\quad + \int_0^T \left(\|v\|_{H^s}^{p-1} + \|\tilde{v}\|_{H^s}^{p-1} + \|u\|_{H^s} + \|\tilde{u}\|_{H^s} \right) d_{t'}((u, v), (\tilde{u}, \tilde{v})) dt' \\ &\leq (2CRT + 2M(R)T) d_T((u, v), (\tilde{u}, \tilde{v})) \\ &\leq \frac{1}{2} d_T((u, v), (\tilde{u}, \tilde{v})). \end{aligned}$$

Therefore by contraction mapping theorem, there exists a solution $(u, v) \in X_R(T_*)$ of (1.2), where $T_* = \min(T_1, T_2)$. The uniqueness follows immediately from the above argument. Using the original equation, the time regularity is readily obtainable. So we leave them to the readers. This completes the proof of proposition. $\square$

Remark 9. (1) Since s can be chosen to be greater than equal to 1, as stated in the introduction, the L^2 norm of the solution u and the energy of (u, v) are conserved up to the existence time of solution.

(2) If $f(v) = \lambda|v|^{p-1}v$ with $p > 1$, then f satisfies the condition (3.1).

(3) If $f(v) = \lambda v^k$ for some fixed $\lambda \in \mathbb{R}$ and integer $k \geq 2$, then we do not need the restriction on n , s and p for the local existence of solutions.

3.2. Proof of Theorem 1.3. Now we prove the first and second parts of main theorem. Since $\lambda > 0$, from the L^2 and energy conservation, we can get

$$\begin{aligned} \|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty L^{p+1}} &\leq C \quad \text{for } n = 1, \\ \|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty L^{p+1}} &\leq C(T) \quad \text{for } n = 2. \end{aligned}$$

These estimate follow from the observation that

$$\begin{aligned} |(v, |u|^2)| &\leq \frac{1}{4}\|v\|_{L^2}^2 + \frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^6, \quad \text{if } n = 1, \\ |(v, |u|^2)| &\leq \frac{1}{4}\|\nabla u\|_{L^2}^2 + C\|u\|_{L^2}^2\|v\|_{L^2}^2 \quad \text{if } n = 2 \end{aligned}$$

and the estimate $\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + \int_0^t \|\partial_t v(t')\|_{L^2} dt'$. For the details, see Section 4 of [19].

As in [26], from the regularity of the local solution we can observe that

$$\partial_t \left(|\partial_t v|^2 + v^2 + \frac{2\lambda}{p+1}|v|^{p+1} \right) = ((1-\Delta)^{-1}(v + f(v)) + \omega^2(|u|^2)) \partial_t v. \quad (3.6)$$

Here we note that $(1-\Delta)^{-1}$ is bounded from L^r to L^∞ for $1 \leq r \leq \infty$ if $n = 1$, for $1 < r \leq \infty$ if $n = 2$, and for $\frac{n}{2} < r \leq \infty$ if $n \geq 3$ and that ω^2 is bounded in L^r for any r with $1 \leq r \leq \infty$. Thus if $n = 1, 2$, then we have

$$\begin{aligned} &\|\partial_t v(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \lambda\|v(t)\|_{L^\infty}^{p+1} \\ &\leq C + C \int_0^t \|u\|_{L^\infty}^4 dt' + C \int_0^t \|(1-\Delta)^{-1}f(v)\|_{L^\infty}^2 dt' \\ &\quad + C \int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) dt' \\ &\leq C + C \int_0^t \|u\|_{L^\infty}^4 dt' + C \int_0^t \|v\|_{L^{p+1}}^{2p} dt' + C \int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) dt' \\ &\leq C + C \int_0^t \|u\|_{L^\infty}^4 dt' + CT\|v\|_{L_T^\infty L^{p+1}}^{2p} + C \int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) dt'. \end{aligned}$$

Hence Gronwall's inequality yields

$$\sup_{0 \leq t \leq T} \left(\|\partial_t v(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2 + \|v\|_{L^\infty}^{p+1} \right) \leq C(T) \int_0^T \|u\|_{L^\infty}^4 dt. \quad (3.7)$$

To obtain the global existence, we have only to prove that

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} + \|\omega^{-1} \partial_t v(t)\|_{H^s} \leq C(T) \quad (3.8)$$

for all $t \in [0, T]$ and any $T > 0$.

If $n = 1$, then by Sobolev embedding and energy conservation (1.6), we see that $\|u\|_{L^\infty([0, T] \times \mathbb{R})} \leq C$ and hence by (3.7), $\|v\|_{L^\infty([0, T] \times \mathbb{R})} \leq C(T)$. Therefore we have

$$\begin{aligned} \|u(t)\|_{H^s} &\leq C + C \int_0^t (\|v\|_{L^\infty} \|u\|_{H^s} + \|v\|_{H^s} \|u\|_{L^\infty}) dt' \\ &\leq C + C(T) \int_0^t (\|v\|_{H^s} + \|u\|_{H^s}) dt', \\ \|v(t)\|_{H^s} &\leq C + C \int_0^t (\|v\|_{L^\infty}^{p-1} \|v\|_{H^s} + \|u\|_{L^\infty} \|u\|_{H^s}) dt' \\ &\leq C + C(T) \int_0^t (\|v\|_{H^s} + \|u\|_{H^s}) dt', \\ \|\omega^{-1} \partial_t v(t)\|_{H^s} &\leq C + C(T) \int_0^t (\|v\|_{H^s} + \|u\|_{H^s}) dt'. \end{aligned} \quad (3.9)$$

Combining above three inequalities and using Gronwall's inequality, we get (3.8) for all $t \in [0, T]$.

Since the embedding $H^1 \hookrightarrow L^\infty$ does not hold for $n = 2$, instead we use the Brezis-Gallouet-Wainger inequality (for instance see [1] and [18]). More precisely, we use the following inequality

$$\|u(t)\|_{L^\infty} \leq C \|u(t)\|_{H^1} \left(1 + \log \left(1 + \frac{\|u(t)\|_{H^s}}{\|u(t)\|_{H^1}} \right) \right)^{\frac{1}{2}}. \quad (3.10)$$

Since $\|u(t)\|_{H^1} \leq C(T)$ for all $t \in [0, T]$, from (3.10) we have

$$\|u(t)\|_{L^\infty} \leq C(T) (1 + \log(1 + \|u(t)\|_{H^s}))^{\frac{1}{2}}. \quad (3.11)$$

Now using (3.7), (3.9) and the fact that $\frac{2(p-1)}{p+1} \leq 1$ for $p \leq 3$, we have

$$\begin{aligned} &\|u\|_{H^s} + \|v\|_{H^s} \\ &\leq C + C(T) \int_0^t (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s})) (\|v\|_{H^s} + \|u\|_{H^s}) dt'. \end{aligned}$$

Thus by Gronwall's inequality, we finally get the bound (3.8) for all $t \in [0, T]$. This completes the proof of (i) and (ii) of Theorem 1.3.

3.3. Proof of Theorem 1.4. Since the local existence was already established in the previous section, we consider only blowup criterion in this section.

Using (3.7), we first have for $1 < p \leq 2$

$$\begin{aligned} & \|u(t)\|_{H^s} + \|v(t)\|_{H^s} \\ & \leq C + \int_0^t (1 + \|u\|_{L^\infty} + \|v\|_{L^\infty})^{p-1} (\|u\|_{H^s} + \|v\|_{H^s}) dt'. \end{aligned} \quad (3.12)$$

Let us invoke the Brezis-Gallouet-Wainger inequality in Triebel-Lizorkin space. For any $s > \frac{3}{2}$,

$$\|\psi\|_{L^\infty} \leq C(1 + \|\psi\|_{\dot{F}_{\infty,\infty}^0} (1 + \log(1 + \|\psi\|_{H^s}))). \quad (3.13)$$

For the proof, see [3] and Remark 8 below.

Now we set

$$M(t) = \|u(t)\|_{\dot{F}_{\infty,\infty}^0} + \|v(t)\|_{\dot{F}_{\infty,\infty}^0}.$$

Then by (3.13), we obtain for all $t \in [0, T^*)$

$$\begin{aligned} & \|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leq C(T^*) \\ & + \int_0^t (1 + M(t'))^{p-1} (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s}))^{p-1} (\|u\|_{H^s} + \|v\|_{H^s}) dt'. \end{aligned}$$

Hence by Gronwall's inequality we have

$$\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leq C(T^*) \exp \left[C(T^*) \exp \left(C(T^*) \int_0^t M(t')^{p-1} dt' \right) \right].$$

Since left hand side of the above inequality tend to infinity as $t \rightarrow T^*$, we can obtain the first part of blowup criterion.

On the other hand, if $(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1}v_1\|_{L^2})\|u_0\|_{L^2}$ is small, then by the same argument as in the proof of part (i) of Theorem 1.1, we can obtain that

$$\|u\|_{L_T^\infty H^1} + \|v\|_{L_T^\infty L^{p+1}} \leq C(T)$$

for all $T < T^*$. Hence using (3.6), we have

$$\begin{aligned} & \|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \lambda \|v\|_{L^\infty}^{p+1} \\ & \leq C + C \int_0^t \|u\|_{L^\infty}^4 dt' + C \int_0^t \|v\|_{L^{pr}}^{2p} dt' + C \int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) dt'. \end{aligned}$$

For the third term, we use the estimate $\|(1 - \Delta)^{-1}f(v)\|_{L^\infty} \leq C\|f(v)\|_{L^r}$ for $\frac{3}{2} < r \leq \infty$. If $p < 2$, then we take $r = \frac{p+1}{p}$. If $2 \leq p \leq 5$, then we choose r such

that $\frac{p+1}{p} < r < \frac{p+1}{p-1}$ and hence

$$pr > p+1, \quad p \left(1 - \frac{p+1}{pr}\right) < 1. \quad (3.14)$$

Since $p+1 < pr < \infty$,

$$\|v\|_{L^{pr}} \leq \|v\|_{L^{p+1}}^\theta \|v\|_{L^\infty}^{1-\theta},$$

where $\theta = \frac{p+1}{pr}$. Thus by (3.14) and Young's inequality, we get

$$\begin{aligned} & \|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \lambda \|v\|_{L^\infty}^{p+1} \\ & \leq C(T) + CT \|u\|_{L^\infty}^4 + C \int_0^t (\|\partial_t v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2) dt'. \end{aligned}$$

By Gronwall's inequality, we obtain the estimate (3.7) and substituting this into (3.12), we can deduce from the fact $\frac{4(p-1)}{p+1} \leq 1$ for $p \leq \frac{5}{3}$ that

$$\begin{aligned} & \|u(t)\|_{H^s} + \|v(t)\|_{H^s} \leq C(T^*) \\ & + \int_0^t (1 + \|u\|_{\dot{F}_{\infty,\infty}^0})^{\frac{4(p-1)}{p+1}} (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s})) (\|u\|_{H^s} + \|v\|_{H^s}) dt'. \end{aligned}$$

Hence the Gronwall's inequality yields the second blowup criterion. For the case that $\frac{5}{3} < p \leq 5$, see Remark 10 below.

Remark 10. Concerning the Brezis-Gallouet-Wainger inequality in Triebel-Lizorkin space, let us introduce a slightly modified version. We first define a homogeneous Triebel-Lizorkin type space $\dot{\mathfrak{F}}_{\infty,q}^0 (0 < q < \infty)$ as follows.

$$\begin{aligned} \dot{\mathfrak{F}}_{\infty,q}^0 & \equiv \left\{ \psi : \psi = \sum_{j \in \mathbb{Z}} \Delta_j \psi_j \text{ in } \mathcal{S}' \text{ for some } \psi_j \in \mathcal{S}' \text{ with } \{\Delta_j \psi_j\}_{j \in \mathbb{Z}} \in L^\infty \ell^q \right\}, \\ \|\psi\|_{\dot{\mathfrak{F}}_{\infty,q}^0} & \equiv \inf_{\psi = \sum \Delta_j \psi_j} \|\{\Delta_j \psi_j\}\|_{L^\infty \ell^q}. \end{aligned} \quad (3.15)$$

If $q = \infty$, then we define $\dot{\mathfrak{F}}_{\infty,\infty}^0$ by $\dot{F}_{\infty,\infty}^0$. The usual Triebel-Lizorkin space $\dot{F}_{\infty,q}^0 (0 < q < \infty)$ is defined by

$$\begin{aligned} \dot{F}_{\infty,q}^0 & \equiv \left\{ \psi : \psi = \sum_{j \in \mathbb{Z}} \Delta_j \psi_j \text{ for some } \{\psi_j\} \in L^\infty \ell^q \right\}, \\ \|\psi\|_{\dot{F}_{\infty,q}^0} & \equiv \inf_{\psi = \sum \Delta_j \psi_j} \|\{\psi_j\}\|_{L^\infty \ell^q}. \end{aligned}$$

One can easily see that $\dot{B}_{\infty,q}^0 \hookrightarrow \dot{\mathfrak{F}}_{\infty,q}^0 \hookrightarrow \dot{F}_{\infty,q}^0$ for $q < \infty$, while in general it is likely that the converse inclusion $\dot{F}_{\infty,q}^0 \hookrightarrow \dot{\mathfrak{F}}_{\infty,q}^0$ is an open question.

If $\psi \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$, then

$$\begin{aligned}\psi &= \sum_{j < -N} \Delta_j \psi + \sum_{-N \leq j \leq N} \Delta_j \psi + \sum_{j > N} \Delta_j \psi \\ &\equiv \psi_- + \psi_0 + \psi_+.\end{aligned}$$

Revisiting the proof in [3], for the first and second terms, we obtain

$$|\psi_-(x)| + |\psi_+(x)| \leq C2^{-N} \|\psi\|_{H^s}. \quad (3.16)$$

On the other hand, as for ψ_0 , we can find $\psi_j \in \mathcal{S}'$ such that $\psi = \sum_j \Delta_j \psi_j$ and $\{\Delta_j \psi_j\} \in L^\infty \ell^q$, and hence

$$|\psi_0(x)| \leq |\tilde{\psi}_N(x)| + \sum_{-N+1 \leq j \leq N-1} |\Delta_j \psi_j(x)|,$$

where

$$\tilde{\psi}_N = \sum_{-N \leq j \leq N} \sum_{k \in \mathbb{Z}} \Delta_j \Delta_k \psi_k - \sum_{-N+1 \leq j \leq N-1} |\Delta_j \psi_j|.$$

Since the number of sum consisting of $\tilde{\psi}_N$ is finite, independently of N , and hence

$\|\tilde{\psi}_N\|_{L^\infty} \leq C \|\psi\|_{\dot{\mathfrak{F}}_{\infty, \infty}^0}$, we deduce from the definition (3.15) that

$$|\psi_0(x)| \leq C \|\psi\|_{\dot{\mathfrak{F}}_{\infty, \infty}^0} + N^{\frac{1}{q'}} \|\psi\|_{\dot{\mathfrak{F}}_{\infty, q}^0}. \quad (3.17)$$

Combining (3.16) and (3.17), we obtain

$$\|\psi\|_{L^\infty} \leq C2^{-N} \|\psi\|_{H^s} + N^{\frac{1}{q'}} \|\psi\|_{\dot{\mathfrak{F}}_{\infty, q}^0}.$$

Thus choosing the optimal N , we can obtain

$$\|\psi\|_{L^\infty} \leq C(1 + \|\psi\|_{\dot{\mathfrak{F}}_{\infty, q}^0} (1 + \log(1 + \|\psi\|_{H^s}))^{\frac{1}{q'}}). \quad (3.18)$$

Applying the estimate (3.18) to (3.7) and (3.12), we see that for $2 < p \leq 3$ and

$$q = \frac{p-1}{p-2}$$

$$\begin{aligned}&\|u(t)\|_{H^s} + \|v(t)\|_{H^s} \\ &\leq C(T^*) + \int_0^t (1 + \|u\|_{\dot{\mathfrak{F}}_{\infty, q}^0} + \|v\|_{\dot{\mathfrak{F}}_{\infty, q}^0})^{p-1} \\ &\quad \times (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s})) (\|u\|_{H^s} + \|v\|_{H^s}) dt'\end{aligned}$$

and hence

$$\int_0^{T^*} (\|u\|_{\dot{\mathfrak{F}}_{\infty, q}^0} + \|v\|_{\dot{\mathfrak{F}}_{\infty, q}^0})^{p-1} dt = \infty.$$

Similarly, for $\frac{5}{3} < p \leq 5$ and $q = \frac{4p-4}{3p-5}$

$$\begin{aligned} & \|u(t)\|_{H^s} + \|v(t)\|_{H^s} \\ & \leq C(T^*) + \int_0^t (1 + \|u\|_{\dot{\mathfrak{F}}_{\infty,q}^0})^{\frac{4(p-1)}{p+1}} (1 + \log(1 + \|u\|_{H^s} + \|v\|_{H^s})) \\ & \quad \times (\|u\|_{H^s} + \|v\|_{H^s}) dt'. \end{aligned}$$

Hence Gronwall's inequality yields the blowup criterion

$$\int_0^{T^*} \|u\|_{\dot{\mathfrak{F}}_{\infty,q}^0}^{\frac{4(p-1)}{p+1}} dt = \infty,$$

provided the maximal existence time T^* is finite and

$$(\|\nabla u_0\|_{L^2} + \|v_0\|_{L^2} + \|\omega^{-1}v_1\|_{L^2})\|u_0\|_{L^2}$$

is sufficiently small.

REFERENCES

- [1] H. Brezis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal. **4** (1980), 677–681.
- [2] T. Cazenave and F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s* , Nonlinear Anal. **14** (1990), 807–836.
- [3] D. Chae, *On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces*, Comm. Pure Appl. Math. **55** (2002), 654–678.
- [4] G. Chen, *Cauchy problem for multidimensional coupled system of nonlinear Schrödinger equation and generalized IMBq equation*, Comment. Math. Univ. Caroline **39** (1998), 15–38.
- [5] G. Chen, J. Xing and Z. Yang, *Cauchy problem for generalized IMBq equation with several variables*, Nonlinear Anal. **26** (1996), 1255–1270.
- [6] Y. Cho and T. Ozawa, *Remarks on modified improved Boussinesq equations in one space dimension*, Hokkaido Univ. Preprint Series in Math. #723.
- [7] J. Ginibre, T. Ozawa and G. Velo, *On the existence of the wave operators for a class of nonlinear Schrödinger equations*, Annales IHP - Physique théorique **60** (1994), 211–239.
- [8] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Func. Anal. **151** (1997), 384–436.
- [9] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations I, The Cauchy problem general case*, J. Func. Anal. **32** (1979), 1–32.
- [10] J. Ginibre and G. Velo, *Scattering theory for the Zakharov system*, preprint.
- [11] T. Kato, *On nonlinear Schrödinger equations II. H^s -solutions and unconditional well-posedness*, J. Anal. Math. **67** (1995), 281–306.
- [12] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.
- [13] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO with application to the Euler equations*, Commun. Math. Phys. **214** (2000), 191–200.
- [14] Y. Liu, *Existence and blow up of solutions of a nonlinear Pochhammer-Chree equation*, Indiana Univ. Math. J. **45** (1996), 797–816.

- [15] V.G. Makhankov, *On stationary solutions of the Schrödinger equation with a self-consistent potential satisfying Boussinesq's equation*, Physics Letters **50A** (1972), 42–44.
- [16] V. G. Makhankov, *Dynamics of classical solitons (in non-integrable systems)*, Physics Reports, Phys. Lett. C **35** (1978), 1–128.
- [17] M. Nakamura and T. Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces*, Reviews in Math. Phys. **9** (1997), 397–410.
- [18] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Func. Anal. **127** (1995), 259–269.
- [19] T. Ozawa and K. Tsutaya, *On the Cauchy problem for Schrödinger-improved Boussinesq equations*, Hokkaido Univ. Preprint Series in Math. #740; Adv. Stud. Pure Math., in press.
- [20] T. Ozawa and Y. Tsutsumi, *Global existence and asymptotic behavior of solutions for the Zakharov equations in three space dimensions*, Adv.Math. Sci. Appl. **3** (1993/94), 301–334.
- [21] A. Shimomura, *Scattering theory for Zakharov equations in three-dimensional space with large data*, Commun. Contemp. Math. **6** (2004), 881–899.
- [22] C. Sulem and P.L. Sulem, “The nonlinear Schrödinger equation,” Applied Mathematical Sciences, 139. Springer, New York, 1999.
- [23] H. Triebel, “Theory of function spaces,” Monographs in Mathematics, 78. Birkhäuser, Basel, 1983.
- [24] H. Triebel, “Theory of function spaces,” II. Monographs in Mathematics, 84. Birkhäuser, Basel, 1992.
- [25] S. Wang and G. Chen, *Existence and nonexistence of global solutions for generalized IMBq equation*, Nonlinear Anal. **36** (1999), 961–980.
- [26] S. Wang and G. Chen, *The Cauchy problem for the generalized IMBq equation in $W^{s,p}(\mathbb{R}^n)$* , J. Math. Anal. Appl. **266** (2002), 38–54.
- [27] S. Wang and G. Chen, *Small amplitude solutions of the generalized IMBq equation*, J. Math. Anal. Appl. **274** (2002), 846–866.
- [28] S. Wang and G. Chen, *Cauchy problem for the nonlinear Schrödinger-IMBq equations*, preprint.

DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, SAPPORO 060-0810, JAPAN

E-mail address: ygcho@math.sci.hokudai.ac.jp

E-mail address: ozawa@math.sci.hokudai.ac.jp