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SMALL DATA SCATTERING FOR THE KLEIN-GORDON EQUATION WITH CUBIC CONVOLUTION NONLINEARITY

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Abstract. We consider the scattering problem for the Klein-Gordon equation with cubic convolution nonlinearity. We give some estimates for the nonlinearity, and prove the existence of the scattering operator, which improves the known results in some sense. Our proof is based on the Strichartz estimates for the inhomogeneous Klein-Gordon equation.

1. Introduction

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = F_\gamma(u) \tag{1.1}$$

in space-time $\mathbb{R} \times \mathbb{R}^n$, where u is a real-valued or a complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}^n$. The nonlinearity $F_\gamma(u)$ is a cubic convolution term $F_\gamma(u) = -(V_\gamma * |u|^2)u$ with

$$|V_\gamma(x)| \le C|x|^{-\gamma}. \tag{1.2}$$

Here, $0 < \gamma < n$ and $*$ denotes the convolution in the space variables. The term $F_\gamma(u)$ is an approximative expression of the nonlocal interaction of specific elementary particles. Menzala and Strauss started to study this equation in [5].

In order to treat the scattering problem, we define the scattering operator for (1.1). First, we list some notation to give the definition. Let H^s be the usual Sobolev space $(1-\Delta)^{-s/2}L_2(\mathbb{R}^n)$ and let $H^{s,\sigma}$ be the weighted Sobolev space $(1-\Delta)^{-s/2}\langle x\rangle^{-\sigma} L_2(\mathbb{R}^n)$. A Hilbert space $X^{s,\sigma}$

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is denoted by $H^{s,\sigma} \oplus H^{s-1,\sigma}$. For a positive number δ and a Banach space A, we denote the set $\{a \in A; \|a|A\| \leq \delta\}$ by $B(\delta; A)$. Then the scattering operator is defined as the mapping $S : B(\delta; X^{s,\sigma}) \ni (f_-, g_-) \mapsto (f_+, g_+) \in X^{s,0}$ if the following condition holds for some $\delta > 0$:

For any $(f_-, g_-) \in B(\delta; X^{s,\sigma})$, there uniquely exist a time-global solution $u \in C(\mathbb{R}; H^s)$ of (1.1), and data $(f_+, g_+) \in X^{s,0}$ such that $u(t)$ approaches $u_\pm(t)$ in H^s as t tends to $\pm\infty$, where $u_\pm(t)$ are solutions of linear Klein-Gordon equations whose initial data are $(f_\pm, g_\pm)$, respectively.

We call that "$(S, X^{s,\sigma})$ is well-defined" if we can define the scattering operator $S : B(\delta; X^{s,\sigma}) \to X^{s,0}$ for some $\delta > 0$.

By Mochizuki [6], it is shown that if $n \geq 3$, $s \geq 1$, $\gamma < n$ and $2 \leq \gamma \leq 2s + 2$, then $(S, X^{s,0})$ is well-defined. By using the methods of Mochizuki and Motai [7] and Strauss [11], we see that if $n \geq 2$, $s \geq 1$, $4/3 < \gamma < 2$ and $\sigma > 1/3$, then $(S, X^{s,\sigma})$ is well-defined. In view of the condition of σ, there is a gap between the two cases $\gamma \geq 2$ and $\gamma < 2$. Our aim of this paper is to fill the gap. By using the Strichartz estimate for pre-admissible pair and the complex interpolation method for the weighted Sobolev space, we show that $(S, X^{s,\sigma})$ is well-defined if $4/3 < \gamma < 2$ and $\sigma > (2-\gamma)/2$, which improves the condition above.

In order to state our results, we give notation which will be used in this paper.

For $s \in \mathbb{R}$ and $(1/p, 1/q) \in [0,1] \times [0,1]$, let H^s_p be the Sobolev space $(1-\Delta)^{-s/2} L_p(\mathbb{R}^n)$ and let B^s_p be the Besov space $B^s_{p,2}(\mathbb{R}^n)$ (see, e.g., [1] for the definition of the Besov space). For $s \in \mathbb{R}$, we set $E^s[u](t) = \|\big(u(t), \partial_t u(t)\big)|X^{s,0}\|$. For $s_0 \in \mathbb{R}$ and $Q = (1/q, 1/r) \in [0,1] \times [0,1]$, $L(s_0, Q)$ denotes either $L_q(\mathbb{R}; H^{s_0}_r(\mathbb{R}^n))$ or $L_q(\mathbb{R}; B^{s_0}_r(\mathbb{R}^n))$. Put $J = (0,t)$ or $(-\infty, t)$ or (t, ∞), $\omega = \sqrt{1-\Delta}$ and $U(t) = \exp(\pm it\omega)$. For a Banach space A, $\mathcal{B}^0(\mathbb{R}; A)$ is the set of all A-valued, continuous and bounded functions on $\mathbb{R}$. Moreover, if f in $\mathcal{B}^0(\mathbb{R}; A)$ has its derivative, and if $\partial_t f \in \mathcal{B}^0(\mathbb{R}; A)$, then we write $f \in \mathcal{B}^1(\mathbb{R}; A)$. For $s \in \mathbb{R}$, $\mathcal{H}^s$ denotes $\mathcal{B}^0(\mathbb{R}; H^s) \cap \mathcal{B}^1(\mathbb{R}; H^{s-1})$ with the norm $\|u|\mathcal{H}^s\| = \|u|L(s, (0, 1/2))\| + \|\partial_t u|L(s-1, (0,1/2))\|$. Furthermore, we set

$$\underline{\mathcal{H}^s} = \big\{u \in \mathcal{H}^s; \text{ there exist } f, g \in \acute{\mathcal{S}}(\mathbb{R}^n) \text{ such that } u(t) = \cos t\omega f + \omega^{-1} \sin t\omega g, \omega^{-1}\partial_t u(t) \in \mathcal{H}^s\big\}.$$

We call $u = u(t,x)$ a free solution if $u \in \underline{\mathcal{H}^s}$ for some $s \in \mathbb{R}$. For a free solution u_0, $u \in \acute{\mathcal{S}}(\mathbb{R}^n)$ is said to be a u_0-solution if

$$u(t) = u_0(t) + \int_0^t \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau.$$

For $s, s_0 \in \mathbb{R}$ and $Q = (1/q, 1/r) \in [0,1]\times[0,1]$, we denote $L(s_0,Q)\cap\mathcal{H}^s$ and $L(s_0,Q)\cap\underline{\mathcal{H}^s}$ by $Z(s_0,s,Q)$ and $\underline{Z}(s_0,s,Q)$, respectively. Define $1/q_\varepsilon = 1/3-\varepsilon$ and $1/r_\theta = 1/2-(1+\theta)/3n$. Assume that $4/3 < \gamma < 2$. Then we can easily show that there exist sufficiently small $\varepsilon(\gamma) > 0$ and $\theta(\gamma) \in (0,1)$ such that

$$\frac{1}{6} < \frac{n}{2}\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big) < \frac{1}{q_{\varepsilon(\gamma)}} < n\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big),$$
$$\gamma = 2 - 2\Big\{\frac{2}{q_{\varepsilon(\gamma)}} - n\Big(\frac{1}{2} - \frac{1}{r_{\theta(\gamma)}}\Big)\Big\}.$$

For $Q_\gamma = (1/q_{\varepsilon(\gamma)}, 1/r_{\theta(\gamma)})$, we set

$$s(Q_\gamma) = \max\Big\{\frac{n+2}{n}\Big(1 - \frac{3}{q_{\varepsilon(\gamma)}}\Big), \frac{2-\gamma}{4}\Big\}.$$

We are now ready to state our main results.

Theorem 1.1. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $s \geq 1$, and put $s_\gamma = s(Q_\gamma)$, $Z = Z(s_\gamma + s - 1, s, Q_\gamma)$. Then there exist some positive numbers δ_0, δ_+, δ_- satisfying the following properties:*

(i) If $u_0 \in B(\delta_0; \underline{Z})$, then there uniquely exist $u \in Z$ and $u_+, u_- \in \underline{Z}$ such that u is a u_0-solution and we have

$$\lim_{t\to\pm\infty} E^s[u - u_\pm](t) = 0. \tag{1.3}$$

Moreover, the operators $\widetilde{V_\pm} : B(\delta_0;\underline{Z}) \ni u_0 \mapsto u_\pm \in \underline{Z}$ are well defined, injective and continuous.

(ii) If $u_\pm \in B(\delta_\pm;\underline{Z})$, then there uniquely exist $u \in Z$ and $u_0 \in \underline{Z}$ such that u is a $u_0 -$ solution and (1.3) holds.
Moreover, the operators $W_\pm : B(\delta_\pm;\underline{Z}) \ni u_\pm \mapsto u_0 \in \underline{Z}$ are well defined, injective and continuous.

(iii) The numbers $\delta_\pm$ satisfy $B(\delta_-;\underline{Z}) \subset B(\delta_0;\underline{Z})$, $W_-(B(\delta_-;\underline{Z})) \subset B(\delta_0;\underline{Z})$ and $B(\delta_+;\underline{Z}) \subset V_+ \circ W_-(B(\delta_-;\underline{Z}))$. In particular, the operator $\widetilde{S} = V_+ \circ W_- : B(\delta_-;\underline{Z}) \to \underline{Z}$ is well defined, injective and continuous.

The following result follows from Theorem 1.1.

Corollary 1.2. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $\sigma > (2-\gamma)/2$, $s \geq 1$ and put $s_\gamma = s(Q_\gamma)$, $Z = Z(s_\gamma + s - 1, s, Q_\gamma)$, $u_\star(t) = \cos t\omega f_\star +$*

$\omega^{-1}\sin t\omega f_\star$, *where* $\star$ *denotes either* $0, +$ *or* $-$. *Then there exist some positive numbers* η_0 *and* η_- *satisfying the following properties:*

(i) If $(f_0, g_0) \in B(\eta_0; X^{s,\sigma})$, *then there uniquely exist* $u \in Z$ *and* $(f_+, g_+), (f_-, g_-) \in X^{s,0}$ *such that* u *is a* $u_0 -$ *solution and (1.3) holds.*
Moreover, the operators $V_\pm : B(\eta_0; X^{s,\sigma}) \ni (f_0, g_0) \mapsto (f_\pm, g_\pm) \in X^{s,0}$ *are well defined, injective and continuous.*

(ii) If $(f_-, g_-) \in B(\eta_-; X^{s,\sigma})$, *there uniquely exist* $u \in Z$ *and* $(f_+, g_+) \in X^{s,0}$ *such that* u *satisfies*

$$u(t) = u_-(t) + \int_t^\infty \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau \tag{1.4}$$

and (1.3) holds.
Moreover, the scattering operator $S : B(\eta_-; X^{s,\sigma}) \ni (f_-, g_-) \mapsto (f_+, g_+) \in X^{s,0}$ *is well defined, injective and continuous.*

Remark 1. If V satisfies $V(x) = V(-x)$, and $0 < \gamma < 3$, then we can easily show small data global existence for (1.1) using the energy equality (see, e.g.,[5]) and an a priori estimate. However, the existence of the scattering operator is not known.

Remark 2. For recent results on the wave equation with a cubic convolution, see, e.g., Hidano [3] and Tsutaya [13].

2. Preliminaries

In this section, we prove the key lemma for our main results. We first state the generalized Hölder inequality in [10].

Proposition 2.1. *If* $s \geq 0$, $1 < p_j < \infty$, $j = 1, \cdots, 5$ *and* $1/p_1 = 1/p_2 + 1/p_3 = 1/p_4 + 1/p_5$, *then we have*

$$\|fg|H^s_{p_1}\| \lesssim \|f|H^s_{p_2}\|\|g|L_{p_3}\| + \|f|L_{p_4}\|\|g|H^s_{p_5}\|. \tag{2.1}$$

We need the Strichartz estimates proved by Nakamura and Ozawa [9] (see also [8]).

Proposition 2.2. *(i) If* $2/q_j = n(1/2 - 1/r_j)$, $2\rho_j = (n+2)(1/2 - 1/r_j)$, $2 \leq q_j, r_j \leq \infty$, $(q_j, r_j) \neq (2, \infty)$, $j = 1, 2$, *then we have*

$$\|\int_J U(t-\tau)h(\tau)d\tau|L_{q_1}B^{-\rho_1}_{r_1}\| \lesssim \|h|L_{q_2'}B^{\rho_2}_{r_2'}\|. \tag{2.2}$$

(ii) If $1/r_4' + 2/nq_4' = 1/r_3 + 2/nq_3 + 2/n$, $\max(0, 1/2 - 1/n) < 1/r_j < 1/2$, $0 < 1/q_j < n(1/2 - 1/r_j)$, $1/q_3 < 1/q_4'$, $\rho_3 + \rho_4 = (n +$

$2)(1/r_4' - 1/r_3)/2$, *then we have*

$$\| \int_J U(t-\tau)h(\tau)d\tau | L_{q_3} B^{-\rho_1}_{r_3} \| \lesssim \| h | L_{q_4'} B^{\rho_4}_{r_4'} \|. \tag{2.3}$$

We next state the estimates of the nonlinearity.

Lemma 2.3. *Assume that* $0 \le \rho \le \tilde{\rho}$, $0 < 1/\tilde{r} < 1/2 \le 1/r < 1$, $0 < \gamma < n$. *If there exist some* $\theta_j \in [0,1]$, $j = 1, 2$, *satisfying*

$$1 + \frac{1}{r} = \frac{\gamma}{n} + (\frac{1}{\tilde{r}} - \theta_1 \frac{\tilde{\rho} - \rho}{n}) + 2(\frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n}), \tag{2.4}$$

$$\frac{1}{\tilde{r}} - \theta_1 \frac{\tilde{\rho} - \rho}{n}, \frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n} > 0, \tag{2.5}$$

then we have

$$\| F(u) | H^{\rho}_r \| \lesssim \| u | H^{\tilde{\rho}}_{\tilde{r}} \|^3 \tag{2.6}$$

and

$$\| F(u) | B^{\rho}_r \| \lesssim \| u | B^{\tilde{\rho}}_{\tilde{r}} \|^3. \tag{2.7}$$

Proof. Put

$$\begin{aligned}
&\frac{1}{p_1} = -1 + \frac{\gamma}{n} + (\frac{1}{\tilde{r}} - \theta_1 \frac{\tilde{\rho} - \rho}{n}) + (\frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n}),\\
&\frac{1}{p_2} = \frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n}, \quad \frac{1}{p_3} = -1 + \frac{\gamma}{n} + 2(\frac{1}{\tilde{r}} - \theta_2 \frac{\tilde{\rho}}{n}),\\
&\frac{1}{p_4} = \frac{1}{\tilde{r}} - \theta_1 \frac{\tilde{\rho} - \rho}{n}, \quad \frac{1}{p_5} = \frac{1}{p_4} + \frac{1}{p_2}.
\end{aligned}$$

Then we have $1/r = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, $1 < r, p_j < \infty$, $j = 1, \cdots, 5$. By Proposition 2.1 and the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned}
\| F(f) | H^{\rho}_r \| &= \| (V * |f|^2) f | H^{\rho}_r \|\\
&\lesssim \| V * |f|^2 | H^{\rho}_{p_1} \| \| f | L_{p_2} \| + \| V * |f|^2 | L_{p_3} \| \| f | H^{\rho}_{p_4} \|,
\end{aligned}$$

$$\begin{aligned}
\| V * |f|^2 | H^{\rho}_{p_1} \| &\lesssim \| V * \omega^{\rho} |f|^2 | L_{p_1} \|\\
&\lesssim \| \omega^{\rho} |f|^2 | L_{p_5} \|\\
&= \| |f|^2 | H^{\rho}_{p_5} \|\\
&\lesssim \| f | H^{\rho}_{p_4} \| \| f | L_{p_2} \|
\end{aligned}$$

and

$$\| V * |f|^2 | L_{p_3} \| \lesssim \| f | L_{p_2} \|^2.$$

By the embedding $H^{\tilde{\rho}}_{\tilde{r}} \hookrightarrow L_{p_2}$ and $H^{\tilde{\rho}}_{\tilde{r}} \hookrightarrow H^{\rho}_{p_4}$, we obtain (2.6). From (2.6), we see that (2.7) holds since $H^{\rho}_r \hookrightarrow B^{\rho}_r$ and $B^{\tilde{\rho}}_{\tilde{r}} \hookrightarrow H^{\tilde{\rho}}_{\tilde{r}}$. □

Finally we state the key lemma to prove Theorem 1.1.

Lemma 2.4. *Assume that $n \geq 2$, $4/3 < \gamma < 2$, $s \geq 1$ and put $L = L(s_\gamma + s - 1, Q_\gamma)$. Then there exists some $\delta > 0$ satisfying as follows: If $u_0 \in B_\delta(L)$, then there uniquely exists $u \in L$ such that we have*

$$u(t) = u_0(t) + \int_J \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau, \tag{2.8}$$

$$\|u|L\| \leq \frac{4}{3}\|u_0|L\|, \tag{2.9}$$

$$\| \int_J U(t-\tau)F(u(\tau))d\tau|L(s-1,(0,1/2))\| \leq \frac{1}{3}\|u_0|L\|. \tag{2.10}$$

Proof. (Step I.) In order to show the existence of a time-global solution, we define the contraction mapping on the suitable complete metric space. Put $Y = B(\frac{4}{3}\|u_0|L\|; L)$ and $d(u,v) = \|u - v|L\|$. Then (Y,d) is a nonempty complete metric space. We define a mapping Φ by

$$\Phi : u \longmapsto u_0 + \int_J \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau.$$

By Proposition 2.2,(2), we have

$$\| \int_J \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau|L\| \lesssim \|F(u)|L(s_\gamma + s - 2 + \tilde{\rho}, (\frac{3}{q_\varepsilon}, \frac{1}{\tilde{r_\theta}}))\|.$$

Here (ε, θ) denotes $(\varepsilon(\gamma), \theta(\gamma))$ and $(\tilde{r_\theta}, \tilde{\rho})$ satisfies

$$\frac{1}{\tilde{r_\theta}} + \frac{6}{nq_\varepsilon} = \frac{1}{r_\theta} + \frac{2}{nq_\varepsilon} + \frac{2}{n},$$
$$\tilde{\rho} = \frac{n+2}{2}(\frac{1}{\tilde{r_\theta}} - \frac{1}{r_\theta}) = \frac{n+2}{2}(\frac{2}{n} - \frac{4}{nq_\varepsilon}) = \frac{n+2}{3n} + 2\frac{n+2}{n}\varepsilon.$$

Remark that $\max(0, 1/2 - 1/n) < 1/r_\theta$, $1/\acute{r_\theta} < 1/2$, $0 < 1/q_\varepsilon < n(1/2 - 1/r_\theta)$ and $0 < 1 - 3/q_\varepsilon < n(1/2 - 1/\acute{r_\theta})$ since

$$\frac{1}{2} < \frac{1}{\tilde{r_\theta}} = \frac{1}{r_\theta} - \frac{4}{nq_\varepsilon} + \frac{2}{n} = \frac{1}{2} + \frac{1-\theta}{3n} + \frac{4\varepsilon}{n} < \frac{1}{2} + \frac{1}{n},$$
$$0 < 1 - \frac{3}{q_\varepsilon} = 3\varepsilon < \frac{1-\theta}{3} + 4\varepsilon = n(\frac{1}{\tilde{r_\theta}} - \frac{1}{2}).$$

Since $\tilde{\rho} \leq 1$, we have

$$\|F(u)|L(s_\gamma + s - 2 + \tilde{\rho}, (\frac{3}{q_\varepsilon}, \frac{1}{\tilde{r_\theta}}))\| \lesssim \|F(u)|L(s_\gamma + s - 1, (\frac{3}{q_\varepsilon}, \frac{1}{\tilde{r_\theta}}))\|.$$

It follows from Lemma 2.3 that

$$\|F(u)|L(s_\gamma + s - 1, (\frac{3}{q_\varepsilon}, \frac{1}{\tilde{r_\theta}}))\| \lesssim \|u|L\|^3$$

since $1 + 1/\tilde{r_\theta} = \gamma/n + 3/r_\theta$.
(Step II.) We estimate the left hand side of (2.10). By Proposition 2.2,(2),

$$\| \int_J U(t-\tau)F(u(\tau))d\tau|L(s-1,(0,1/2))\| \lesssim \|F(u)|L(s-1+\dot{\rho}, (\frac{3}{q_\varepsilon}, \frac{1}{\dot{r_\theta}}))\|,$$

where

$$\frac{1}{\dot{r_\theta}} = \frac{1}{2} + \frac{2}{n}(1 - \frac{3}{q_\varepsilon}), \dot{\rho} = \frac{n+2}{n}(1 - \frac{3}{q_\varepsilon}).$$

Remark that $2(1-3/q_\varepsilon) = n(1/\dot{r_\theta} - 1/2)$ and $2\dot{\rho} = (n+2)(1/\dot{r_\theta} - 1/2)$. Put

$$\vartheta = \frac{1}{s - 1 + s_\gamma}\Big\{\frac{1}{q_\varepsilon} - \frac{n}{2}\big(\frac{1}{2} - \frac{1}{r_\theta}\big)\Big\}.$$

Then we have $0 \le \dot{\rho} \le s_\gamma$, $0 \le \vartheta \le 1$,

$$1 + \frac{1}{\dot{r_\theta}} = \frac{\gamma}{n} + \frac{1}{r_\theta} + 2(\frac{1}{r_\theta} - \vartheta\frac{s-1+s_\gamma}{n}),$$
$$\frac{1}{r_\theta} - \vartheta\frac{s-1+s_\gamma}{n} > 0.$$

Thus, by Lemma 2.3, we have

$$\|F(u)|L(s-1+\dot{\rho}, (\frac{3}{q_\varepsilon}, \frac{1}{\dot{r_\theta}}))\| \lesssim \|u|L\|^3.$$

(Step III.) We prove (2.9) and (2.10). By Steps I and II, we have

$$\|\Phi u|L\| \le \|u_0|L\| + C\|u|L\|^3,$$
$$\|\Phi u - \Phi v|L\| \le C\big(\|u|L\|^2 + \|v|L\|^2\big)\|u - v|L\|$$

and

$$\| \int_J U(t-\tau)F(u(\tau))d\tau|L(s-1,(0,1/2))\| \le C\|u|L\|^3$$

for all $u, v \in Y$. Thus, if we put $\|u_0|L\| \le \delta$ and $\delta \le 3/(8\sqrt{C})$, then we obtain

$$\|\Phi u|L\| \le (1 + \frac{64}{27}C\delta^2)\|u_0|L\| \le \frac{4}{3}\|u_0|L\|, \tag{2.11}$$

$$d(\Phi u, \Phi v) \le \frac{32}{9}C\delta^2\|u - v|L\| \le \frac{1}{2}d(u, v) \tag{2.12}$$

and

$$\| \int_J U(t-\tau)F(u(\tau))d\tau | L(s-1,(0,1/2))\| \leq \frac{64}{27}C\delta^2 \|u_0|L\| \leq \frac{1}{3}\|u_0|L\|. \tag{2.13}$$

By (2.11) and (2.12), Φ is a contraction mapping from (Y,d) into itself. Thus, there exists a unique fixed point $u \in Y$. By (2.13), we obtain (2.10). □

3. Proof of theorem 1.1

In this section, we give a proof of Theorem 1.1 dividing into five steps.

(Step I.) For all $u_0 \in B_\delta(\underline{Z}(s_\gamma + s - 1, s, Q_\gamma))$, by Lemma 2.4, there uniquely exists u_0-solution satisfying (2.9) and (2.10). We have

$$\begin{aligned}\|u|\mathcal{H}^s\| \leq & \|u_0|\mathcal{H}^s\| + \| \int_0^t U(t-\tau)F(u(\tau))d\tau | L(s-1,(0,1/2))\| \\ \leq & \|u_0|\mathcal{H}^s\| + \|u_0|L\|/3 \\ \leq & \|u_0|Z\|.\end{aligned}$$

Thus, $\|u|Z\| \leq 3\|u_0|Z\|$. Put

$$u_\pm = u_0 + \int_0^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau.$$

Then we have

$$\begin{aligned}\|u_\pm|Z\| \leq & \|u_0|Z\| + \|(\int_0^t + \int_t^{\pm\infty})\frac{\sin(t-\tau)\omega}{\omega}F(u(\tau))d\tau|Z\| \\ \leq & 5/3\|u_0|Z\|\end{aligned}$$

and $u_\pm \in \underline{\mathcal{H}^s}$.

(Step II.) It follows from Lemma 2.4 that

$$E[u-u_\pm](t) \lesssim \|\chi_{(t,\pm\infty)}u|L\| \to 0$$

as $t \to \pm\infty$.

Assume that there exist some $\tilde{u}_\pm \in \mathcal{H}^s$ such that $E[u-\tilde{u}_\pm](t) \to 0$ as $t \to 0$. By the energy equality, we have

$$E[u_\pm - \tilde{u}_\pm](0) \simeq E[u_\pm - \tilde{u}_\pm](t) \to 0$$

as $t \to 0$. Hence $u_\pm = \tilde{u}_\pm$. Thus, $V_\pm : B_\delta(\underline{Z}) \ni u_0 \mapsto u_\pm \in \underline{Z}$ is well defined.

(Step III.) It is clear to see $u_0 \mapsto u$ is injective. If we put $u \mapsto u_\pm$ and $\tilde{u} \mapsto u_\pm$, then we have

$$u_0 + \int_0^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau = \tilde{u}_0 + \int_0^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} F(\tilde{u}(\tau))d\tau.$$

Since u is u_0-solution, we have

$$u + \int_t^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau))d\tau = \tilde{u} + \int_t^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} F(\tilde{u}(\tau))d\tau.$$

Thus,

$$\begin{aligned}\|u-\tilde{u}|Z\| =& \| \int_t^{\pm\infty} \frac{\sin(t-\tau)\omega}{\omega} \big\{F(u)-F(\tilde{u})\big\} d\tau |Z\| \\ \leq& \frac{1}{2}\|u-\tilde{u}|Z\|\end{aligned}$$

if δ is sufficiently small. Hence $V_\pm$ is injective.
(Step IV.) For sufficiently small δ, we have

$$\|u-v|Z\| \leq \|u_0-v_0|Z\| + \frac{1}{2}\|u-v|Z\|,$$

$$\|u_\pm - v_\pm|Z\| \leq \|u_0-v_0|Z\| + \frac{1}{2}\|u-v|Z\|.$$

Thus, $u_\pm \to v_\pm$ in Z if $u_0 \to v_0$ in Z. Hence Theorem 1.1,(i) holds. Theorem 1.1,(ii) can be proved analogously.
(Step V.) We put $\delta_- = \delta_0/3$. Then we have $B_{\delta_-}(\underline{Z}) \hookrightarrow B_{\delta_0}(\underline{Z})$ and $W_-(B_{\delta_-}(\underline{Z})) \hookrightarrow B_{\delta_0}(\underline{Z})$ since $\|V_\pm(u_0)|Z\| \leq 3\|u_0|Z\|$ and $\|W_\pm(u_\pm)|Z\| \leq 3\|u_\pm|Z\|$. Put $\delta_+ = \delta_-/9 = \delta_0/27$. Then $V_- \circ W_+(B_{\delta_+}(\underline{Z})) \subset B_{\delta_+}(\underline{Z})$. Thus $B_{\delta_+}(\underline{Z}) \hookrightarrow V_+ \circ W_-(B_{\delta_-}(\underline{Z}))$. Hence we have completed the proof of Theorem 1.1.

4. Proof of corollary

It is sufficient to show the following lemma:

Lemma 4.1. *Assume that $n \geq 2$, $\max(0, 1/2-1/n) < 1/r < 1/2$ and $(n/2-n/r)/2 < 1/q < (n/2-n/r)$. Then we have*

$$\|U(\cdot)f|L_qL_r\| \lesssim \|f|H^{s,\sigma}\| \tag{4.1}$$

if

$$s > \frac{n+2}{2}(\frac{1}{2}-\frac{1}{r}) + \frac{n+2}{2n}\big\{\frac{2}{q} - n(\frac{1}{2}-\frac{1}{r})\big\}$$

and

$$\sigma > \frac{2}{q} - n(\frac{1}{2}-\frac{1}{r}).$$

Proof. We can choose $\vartheta \in (0,1)$ and sufficiently small $\delta > 0$ which satisfy

$$\frac{1}{q} = (1-\vartheta)\frac{n}{2}(\frac{1}{2}-\frac{1}{r}) + \vartheta\big\{n(\frac{1}{2}-\frac{1}{r}) - \delta\big\}.$$

By using the Strichartz estimate for the homogeneous Klein-Gordon equation prove by [4], we have

$$\|U(\cdot)f|L_{q_0}L_r\| \lesssim \|f|H^{(n+2)(1/2-1/r)/2}\|,$$

where $1/q_0 = (n/2 - n/r)/2$. On the other hand, by [2], we see that

$$\begin{aligned}\|U(t)f|L_r\| &\lesssim \langle t\rangle^{-(n/2-n/r)} \max\big\{\|f|H^{n(1/2-1/r)}\|, \|f|H_r^{(n+2)(1/2-1/r)}\|\big\} \\ &\lesssim \langle t\rangle^{-(n/2-n/r)} \|f|H^{s_1,\sigma_1}\|,\end{aligned}$$

where $s_1 = (n+2)(1/2-1/r)$, $\sigma_1 = n(1/2-1/r)+\delta$. Thus, it holds

$$\|U(\cdot)f|L_{q_1}L_r\| \lesssim \|f|H^{s_1,\sigma_1}\|,$$

where $1/q_1 = n(1/2-1/r)-\delta$. By (24) in [12], we have

$$[H^{s_1/2,0}, H^{s_1,\sigma_1}]_\vartheta = H^{(1+\vartheta)s_1/2,\vartheta\sigma_1}.$$

Here, $[\cdot,\cdot]$ denotes the complex interpolation method and

$$\vartheta = \big\{\frac{n}{2}(\frac{1}{2}-\frac{1}{r}) - \delta\big\}^{-1}\big\{\frac{1}{q} - \frac{n}{2}(\frac{1}{2}-\frac{1}{r})\big\}.$$

Hence, we obtain

$$\|U(\cdot)f|L_qL_r\| \lesssim \|f|H^{(1+\vartheta)s_1/2,\vartheta\sigma_1}\|.$$

We can put $s > (1+\vartheta)s_1/2$ and $\sigma > \vartheta\sigma_1$ with sufficiently small δ. Thus, (4.1) holds. □

By substituting $q = q_{\varepsilon(\gamma)}$ and $r = r_{\theta(\gamma)}$ for (4.1), we can easily see that Corollary 1.2 holds.

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