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A Scaling Limit for a General Class of Quantum Field Models and its Application to Nuclear Physics and Condensed Matter Physics

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Abstract

A scaling limit for the generalized spin-boson (GSB) model is considered. We derive a scaling limit of the Hamiltonian of the GSB model independently of whether or not the quantum scalar field has a mass.

Applying it to a model for the field of the nuclear force with isospin, we obtain an effective potential of the interaction between nucleons. Also, we get some applications to condensed matter physics.

Keywords: scaling limit, GSB model, effective potential, isospin, nuclear force, lattice spin system

AMS Subject Classification: 81Q10

1 Introduction

We consider a scaling limit of the Hamiltonian of the GSB model introduced in [2]. This is an abstract model of an interaction system consisting of particles and a quantized scalar field ([2]). Although a scaling limit for the GSB model has been considered in [2], some assumptions for the interaction part of the Hamiltonian are made. Applying a method used in [3], one can easily derive a scaling limit under assumptions different from those in [2]. These assumptions are kinds of commutativity conditions for operators in the interaction part. Indeed, in the case where the field quantum is spinless, these assumptions are satisfied. However, in the case where the field quantum has a spin or an isospin, these assumptions are not satisfied. In this paper, we investigate a scaling limit of the Hamiltonian in such a general case. To take a scalig limit of the Hamiltonian H of the GSB model, we introduce a parameter $\Lambda > 0$ in H so that we have a family $\{H(\Lambda)\}_\Lambda$ of self-adjoint operators on a Hilbert space. We

are interested in the limit of $H(\Lambda)$ as $\Lambda \to \infty$ in the sense of the unitary group $\{e^{-itH(\Lambda)}\}$ $(t \in \mathbb{R})$.

In concrete realizations of the abstract model, scaling limits yield effective Hamiltonians in the vacuum of the Bose field. These Hamiltonians describe quantum mechanical Hamiltonians with effective potentials caused by the interaction of particles with the Bose field.

We organize this paper as follows. In Section 2, we present abstract results on a scaling limit of a family of self-adjoint operators. We state the main results of the present paper in Section 3. Using the method of the abstract scaling limit theory in Section 2, we prove the main results in Section 4. In Section 5, we discuss some examples. Applying the main results, we derive effective potentials in these examples.

2 An Abstract Scaling Limit

In this section, we prove a theorem of a scaling limit of self-adjoint operators acting on the tensor product of two Hilbert spaces discussed in [1, 4] with a little modification for later use. To begin with, we introduce preliminary notations. The scalar product and the associated norm of a Hilbert space $\mathcal{L}$ are denoted by $\langle *, \cdot \rangle_{\mathcal{L}}$ and $\| \cdot \|_{\mathcal{L}}$, respectively; the scalar product is linear in $\cdot$, antilinear in $*$. The uniform operator norm of bounded operators is donoted by $\| \cdot \|$. If there is no danger of confusion, we omit the subscript $\mathcal{L}$ in $\langle *, \cdot \rangle_{\mathcal{L}}$ and $\| \cdot \|_{\mathcal{L}}$. Moreover, the domain, the spectrum and the resolvent set of an operator T are denoted by $D(T)$, $\sigma(T)$ and $\rho(T)$, respectively.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces and

$$\mathcal{X} := \mathcal{H} \otimes \mathcal{K}.$$

Let A (resp.B) be a non-negative self-adjoint operator in $\mathcal{H}$(resp. $\mathcal{K}$) with

$$\ker B \neq \{0\}.$$

Let P_B be the orthogonal projection from $\mathcal{K}$ onto $\ker B$. We suppose that a family of symmetric operators $\{C_\Lambda\}_{\Lambda > 0}$ in $\mathcal{X}$ admits the following conditions:

(i) For all $\varepsilon > 0$ there exists a constant $\Lambda(\varepsilon)$ such that for all $\Lambda > \Lambda(\varepsilon)$, $D(C_\Lambda) \supset D_{AB} := D(A \otimes I + I \otimes B) = D(A \otimes I) \cap D(I \otimes B)$ with

$$\|C_\Lambda \Phi\|_{\mathcal{X}} \leq \varepsilon \|(A \otimes I + \Lambda I \otimes B)\Phi\|_{\mathcal{X}} + b(\varepsilon)\|\Phi\|_{\mathcal{X}}, \quad \Phi \in D_{AB}, \tag{2.1}$$

where $b(\varepsilon) > 0$ is a constant independent of $\Lambda > \Lambda(\varepsilon)$.

(ii) There exists a symmetric operator C in $\mathcal{X}$ such that $D(C) \supset D(A) \otimes \ker B$, with, for all $z \in \mathbb{C} \setminus [0, \infty)$,

$$\text{s-}\lim_{\Lambda \to \infty} C_\Lambda (A \otimes I + \Lambda I \otimes B - z)^{-1} = C(A - z)^{-1} \otimes P_B. \tag{2.2}$$

The following theorem is fundamental in this paper.

Theorem 2.1 ([4, Proposition 2.1]) *Let operators A, B, C_Λ and C be as above. Then the following (a)-(c) hold.*
(a) For all $\Lambda > \Lambda_0$ with some Λ_0, the operator

$$K_\Lambda := A \otimes I + \Lambda I \otimes B + C_\Lambda$$

is self-adjoint on D_{AB} and bounded from below uniformly in $\Lambda > \Lambda_0$. Moreover, it is essentially self-adjoint on any core for $A \otimes I + I \otimes B$.
(b) The operator

$$K_\infty := A \otimes I + (I \otimes P_B)C(I \otimes P_B)$$

is self-adjoint on $D(A \otimes I)$ and bounded from below. Moreover, it is essentially self-adjoint on any core for $A \otimes I$.
(c) For all $z \in [\bigcap_{\Lambda>\Lambda_0} \rho(K_\Lambda)] \cap \rho(K_\infty)$,

$$s\text{-}\lim_{\Lambda\to\infty} (K_\Lambda - z)^{-1} = (K_\infty - z)^{-1}(I \otimes P_B). \tag{2.3}$$

Moreover, for all $t \in \mathbb{R}$,

$$s\text{-}\lim_{\Lambda\to\infty} e^{itK_\Lambda} = e^{itK_\infty}(I \otimes P_B). \tag{2.4}$$

Proof. Let

$$K_0(\Lambda) = A \otimes I + \Lambda I \otimes B.$$

Then, by (2.1), for each $\varepsilon > 0$, there exists $\Lambda_0 = \Lambda(\varepsilon)$ such that, for all $\Lambda > \Lambda_0$,

$$\|C_\Lambda \Phi\| \le \varepsilon \|K_0(\Lambda)\Phi\| + b(\varepsilon)\|\Phi\|, \qquad \Phi \in D_{AB}. \tag{2.5}$$

Hence, by the Kato-Rellich theorem (see [7]), the operator

$$K_\Lambda := K_0(\Lambda) + C_\Lambda$$

with $\Lambda > \Lambda_0$ is self-adjoint on D_{AB} and essentially self-adjoint on any core of $K_0(\Lambda)$. Further, K_Λ is bounded from below with

$$K_\Lambda \ge -\frac{b(\varepsilon)}{1-\varepsilon}, \tag{2.6}$$

where $0 < \varepsilon < 1$, and $\Lambda > \Lambda_0$. Therefore, the ground state energy E_Λ of K_Λ defined by $E_\Lambda := \inf \sigma(K_\Lambda)$ is bounded from below uniformly in $\Lambda > \Lambda_0$:

$$\epsilon_0 := \inf_{\Lambda>\Lambda_0} E_\Lambda > -\infty. \tag{2.7}$$

Thus, (a) follows.

By (2.5) and (2.6), we obtain

$$\|C_\Lambda (K_0(\Lambda) - z)^{-1}\Psi\| \le (\varepsilon|z| + b(\varepsilon))\|(K_0(\Lambda) - z)^{-1}\Psi\| + \varepsilon\|\Psi\|, \tag{2.8}$$

for all $z \in [\bigcap_{\Lambda>\Lambda_0} \rho(K_\Lambda)] \cap \rho(K_\infty)$. It is easy to show that

$$\text{s-}\lim_{\Lambda\to\infty} (K_0(\Lambda) - z)^{-1} = (A - z)^{-1} \otimes P_B. \tag{2.9}$$

By this fact and (2.2), taking $\Lambda \to \infty$, we have

$$\|C(A-z)^{-1} \otimes P_B \Psi\| \leq (\varepsilon|z| + b(\varepsilon))\|(A \otimes I - z)^{-1}\Psi\| + \varepsilon\|\Psi\|. \tag{2.10}$$

Hence, we have

$$\|C(I \otimes P_B)\Phi\| \leq \varepsilon\|A \otimes I\Phi\| + (2|z|\varepsilon + b(\varepsilon))\|\Phi\|, \quad \Phi \in D(A \otimes I). \tag{2.11}$$

This implies $(I \otimes P_B)C(I \otimes P_B)$ is infinitesimally small with respect to $A \otimes I$. Thus, by the Kato-Rellich theorem again, (b) holds.

To prove (c), let $|z| > 0$ be sufficiently large so that $z < \epsilon_0$. Iterating the second resolvent formula, we have

$$(K_\Lambda - z)^{-1} = \sum_{n=0}^{N} (-1)^n (K_0(\Lambda) - z)^{-1} {T_\Lambda}^n + R_{\rm N}(\Lambda), \tag{2.12}$$

where

$$T_\Lambda := C_\Lambda (K_0(\Lambda) - z)^{-1},$$

and

$$R_N(\Lambda) := (-1)^{N+1} (K_\Lambda - z)^{-1} {T_\Lambda}^{N+1}.$$

By (2.8),

$$\|T_\Lambda\| \leq 2\varepsilon + \frac{b(\varepsilon)}{|z|}. \tag{2.13}$$

Hence, taking $\varepsilon < 1/2$,

$$\|T_\Lambda\| < 1$$

uniformly in $\Lambda > \Lambda_0$, if $|z|$ is surfficiently large. Since we can easily prove

$$\|R_N(\Lambda)\| \leq \frac{\|T_\Lambda\|^{N+1}}{|\epsilon_0 - z|},$$

it follows that

$$(K_\Lambda - z)^{-1} = \sum_{n=0}^{\infty} (-1)^n (K_0(\Lambda) - z)^{-1} {T_\Lambda}^n \tag{2.14}$$

is norm convergent uniformly in $\Lambda > \Lambda_0$. Taking $\Lambda \to \infty$ and using (2.2), we get the desired result with $z < 0$, $|z|$ being sufficiently large. (For the details, see [1, Theorem 2.2].) The proof in the case where z is not real is similar to that in [1]. Formula (2.4) follows from (2.3) (see [1]). □

3 Definition of a Model and the Main Results

We consider a model of a quantum system $\mathcal{S}$ coupled to an abstract Bose field. We denote the Hilbert space of the system $\mathcal{S}$ by $\mathcal{H}$ which is taken to be an arbitary separable complex Hilbert space. In concrete realizations, $\mathcal{S}$ may be a system of quantum particles. We denote the one-boson Hilbert space by $\mathcal{K}$ which is taken to be an arbitary separable complex Hilbert space.

To describe the Bose field, one uses the Boson Fock space over $\mathcal{K}$:

$$\begin{aligned}\mathcal{F}_{\rm b}(\mathcal{K}) :&= \bigoplus_{n=0}^{\infty} \bigotimes_{\rm s}^{n} \mathcal{K} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \;\middle|\; n \geq 0, \quad \psi^{(n)} \in \bigotimes_{\rm s}^{n} \mathcal{K}, \quad \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\},\end{aligned}$$

where $\bigotimes_{\rm s}^{n} \mathcal{K}$ denotes the n-fold symmetric tensor product of $\mathcal{K}$ with $\bigotimes_{\rm s}^{0} \mathcal{K} := \mathbb{C}$. As is well known [7, §X.7], one of basic objects on $\mathcal{F}_{\rm b}(\mathcal{K})$ is the annihilation operator $a(f)(f \in \mathcal{K})$ which is a densely defined closed linear operator on $\mathcal{F}_{\rm b}(\mathcal{K})$ such that, for all $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n (f \otimes \psi^{(n-1)}), \qquad n \geq 1,$$

where S_n is the symmetrization operator on $\bigotimes^n \mathcal{K}$ ($S_n{}^* = S_n$, $S_n{}^2 = S_n$, $\bigotimes_s^n \mathcal{K} = S_n(\bigotimes^n \mathcal{K})$). The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)(g \in \mathcal{K})$ obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0 \tag{3.1}$$

for all $f, g \in \mathcal{K}$ on the dense subspace

$$\mathcal{F}_0(\mathcal{K}) := \{\psi \in \mathcal{F}_{\rm b}(\mathcal{K}) \mid \textit{there exists a number } n_0 \textit{ such that } \psi^{(n)} = 0 \textit{ for all } n \geq n_0\},$$

where $[X, Y] := XY - YX$.

Let

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \qquad f \in \mathcal{K},$$

which is called the Segal field operator. It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{K})$ [7, §X.7]. We denote its closure by the same symbol $\phi(f)$.

It follows from (3.1) that, for all $f, g \in \mathcal{K}$,

$$[\phi(f), \phi(g)] = i\Im\langle f, g \rangle \tag{3.2}$$

on $\mathcal{F}_0(\mathcal{K})$. Moreover we have

$$e^{i\phi(f)} e^{i\phi(g)} = e^{-i\Im\langle f,g\rangle} e^{i\phi(g)} e^{i\phi(f)}, \qquad f, g \in \mathcal{K}, \tag{3.3}$$

which are called the Weyl relations of $\{\phi(f) | f \in \mathcal{K}\}$ [7, §X.7].

For every self-adjoint operator T on $\mathcal{K}$, one can define a self-adjoint operator $d\Gamma(T)$, called the second quantization of T [6, p.302], by

$$d\Gamma(T) := \bigoplus_{n=0}^{\infty} T^{(n)},$$

with $T^{(0)} = 0$ and $T^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\overbrace{T}} \otimes \cdots \otimes I \right) \;\middle|\; \bigotimes_{\rm alg}^{n} D(T),$$

where I denotes identity and $\bigotimes_{\rm alg}$ algebraic tensor product. If T is non-negative, then so is $d\Gamma(T)$.

We assume that T is a non-negative, injective and self-adjoint operator on $\mathcal{K}$. Then, the free Hamiltonian of the Bose field is defined by

$$H_{\rm b} := d\Gamma(T)$$

acting on $\mathcal{F}_{\rm b}(\mathcal{K})$.

The Hilbert space of the coupled system of $\mathcal{S}$ and the Bose field is given by the tensor product

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\rm b}(\mathcal{K}).$$

Suppose that A is a self-adjoint operator on $\mathcal{H}$ and bounded from below, which denotes physically the Hamiltonian of the quantum system $\mathcal{S}$. Let B_j $(j = 1, \dots, J, J \in \mathbb{N})$ be a bounded self-adjoint operator on $\mathcal{H}$ and $g_j \in \mathcal{K}$ $(j = 1, \dots, J)$. As a total Hamiltonian of the coupled system, we take the following operator:

$$H := A \otimes I + I \otimes H_{\rm b} + g \sum_{j=1}^{J} B_j \otimes \phi(g_j), \tag{3.4}$$

where $g \in \mathbb{R}$ is a constant denoting the coupling constant of the system $\mathcal{S}$ and the Bose field.

To state main results of this paper, we need some assumptions.

(A.1) The vectors $g_j (j = 1, \dots, J)$ satisfy the following conditions:

$$g_j \in D(T^{-3/2}), \quad j = 1, \dots, J, \tag{3.5}$$

and

$$\langle g_j, g_k \rangle \ , \ \ \langle g_j, T^{-1} g_k \rangle \ , \ \ \langle T^{-1} g_j, T^{-1} g_k \rangle \ \in \mathbb{R}, \quad j, k = 1, \dots, J. \tag{3.6}$$

(A.2) There exists a dense subspace $\mathcal{D} \subset D(A)$ such that

$$B_j \mathcal{D} \subset \mathcal{D}, \ \ j = 1, \cdots, J. \tag{3.7}$$

(A.3) $[B_j, A]|\mathcal{D} (j = 1, \cdots, J)$ are bounded.

It follows from (3.3) and (3.6) that $\{\phi(g_j)\}_{j=1}^J$ and $\{\phi(iT^{-1}g_j)\}_{j=1}^J$ are the families of strongly commuting self-adjoint operators, respectively.

We introduce a scaled Hamiltonian by

$$H(\Lambda) := A \otimes I + \Lambda^2 I \otimes H_{\rm b} + g\Lambda H_{\rm I}, \ \ \Lambda > 0, \tag{3.8}$$

where

$$H_{\rm I} := \sum_{j=1}^{J} B_j \otimes \phi(g_j).$$

Let P_0 be the orthogonal projection onto $\ker H_{\rm b}$ which is the one-dimensional subspace generated by the Fock vacuum $\Omega := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\rm b}(\mathcal{K})$:

$$\ker H_{\rm b} = \{\alpha\Omega \mid \alpha \in \mathbb{C}\}.$$

Then, we obtain the following result which is one of the main theorems in this paper.

Theorem 3.1 *Assume (A.1)-(A.3). Let $z \in \mathbb{C}\backslash\mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large. Then,*

$$s\text{-}\lim_{\Lambda\to\infty}(H(\Lambda)-z)^{-1} = (A+E-z)^{-1}\otimes P_0, \tag{3.9}$$

where

$$E = -\frac{g^2}{2}\sum_{j,k}\langle T^{-1}g_j, g_k\rangle B_k B_j. \tag{3.10}$$

Moreover, for all $t \in \mathbb{R}$,

$$s\text{-}\lim_{\Lambda\to\infty} e^{itH(\Lambda)} = e^{it(A+E)}\otimes P_0. \tag{3.11}$$

If A is a bounded self-adjoint operator on $\mathcal{H}$, (A.2) and (A.3) hold. Therefore, Theorem 3.1 implies the following corollary:

Corollary 3.2 *Suppose that (A.1) holds and that A is bounded. Then, (3.9) holds for all $z \in \mathbb{C}\setminus\mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large. Moreover, (3.11) holds for all $t \in \mathbb{R}$.*

We can state a result of another type without the condition (A.3). To do this, we introduce some objects. We denote by $[B_j, A]$ the closure of $[B_j, A]|\mathcal{D}(j = 1, \cdots, J)$. Put $\mu_0 := \inf\sigma(A)$ and

$$\tilde{A} := A - \mu_0,$$

which is a non-negative self-adjoint operator on $\mathcal{H}$. We need the following assumption.

(A.4) $\mathcal{D}$ is a core for A and $[B_j, A]$ $(j = 1, \cdots, J)$ are $\tilde{A}^{1/2}$-bounded, i.e. $D(\tilde{A}^{1/2}) \subset D([B_j, A])$ and there exist constants $a_j, b_j \geq 0$ such that, for all $u \in D(\tilde{A}^{1/2})$,

$$\|[B_j, A]u\| \leq a_j \left\|\tilde{A}^{1/2}u\right\| + b_j\|u\|. \tag{3.12}$$

Moreover, $[B_j, A]|\mathcal{D}$ $(j = 1, \cdots, J)$ are commuting with $B_k (k = 1, \cdots, J)$ on $\mathcal{D}$.

Then, we obtain the following theorem:

Theorem 3.3 *Assume (A.1), (A.2) and (A.4). Then, (3.9) holds for all $z \in \mathbb{C}\setminus\mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large. Moreover, (3.11) holds for all $t \in \mathbb{R}$.*

4 Proof of the Main Results

The basic idea for the proof of Theorem 3.1 is to apply Theorem 2.1 to a unitary transformation of the scaled Hamiltonian $H(\Lambda)$. We need some lemmas.

Let

$$\mathcal{F}_{\rm fin}(T) = \mathcal{L}\{\Omega, a(f_1)^* \dots a(f_n)^*\Omega \mid n \geq 1, f_j \in D(T), j = 1, \dots, n\}$$

and

$$\mathcal{F}_T = \mathcal{H} \otimes_{\rm alg} \mathcal{F}_{\rm fin}(T),$$

where $\mathcal{L}\{\cdots\}$ denotes the subspace algebraically spanned by vectors in the set $\{\cdots\}$. We introduce a unitary operator as follows: It is well known or easy to prove (e.g., [5, Theorem 4.30]) that the operator

$$S := \sum_{j=1}^{J} B_j \otimes \phi(iT^{-1}g_j)$$

is essentially self-adjoint. We denote its closure by the same symbol S. Then, we can define the unitary operator $U(t)$ ($t \in \mathbb{R}$) generated by S;

$$U(t) := e^{itS}\ , \quad t \in \mathbb{R}. \tag{4.1}$$

Let

$$E_N(t) = \frac{(it)^N}{N!} U(t) \left[\mathrm{ad}^N(S) I \otimes H_{\mathrm{b}}\right] U(t)^{-1}, \quad t \in \mathbb{R},$$

where, for each X in an algebra $\mathcal{A}$, we inductively define

$$\mathrm{ad}^0(X)Y = Y, \quad \mathrm{ad}^n(X)Y = [X, \mathrm{ad}^{n-1}(X)Y], \quad Y \in \mathcal{A}, \quad n \geq 1. \tag{4.2}$$

Lemma 4.1 *For each $\Psi \in \mathcal{F}_T$, $N \geq 1$, and $t \in \mathbb{R}$, there exist $\xi_N(t)$ and $\eta_N(t) \in [-|t|, |t|]$ such that*

$$U(t) I \otimes H_{\mathrm{b}} U(t)^{-1} \Psi = \left[\sum_{n=0}^{N-1} \frac{(it)^n}{n!} \mathrm{ad}^n(S) I \otimes H_{\mathrm{b}} + E_N(\xi_N(t))\right] \Psi, \tag{4.3}$$

and

$$U(t) H_{\mathrm{I}} U(t)^{-1} \Psi = \left[H_{\mathrm{I}} + i \sum_{n=1}^{N-1} \frac{(it)^n}{n!} \mathrm{ad}^{n+1}(S) I \otimes H_{\mathrm{b}} + i E_N(\eta_N(t))\right] \Psi. \tag{4.4}$$

Proof. It is easy to see that for all $\Psi \in \mathcal{F}_T$ and $t \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \frac{|t|^n}{n!} \|\mathrm{ad}^n(S) I \otimes H_{\mathrm{b}} \Psi\| < \infty \tag{4.5}$$

and

$$U(t) I \otimes H_{\mathrm{b}} U(t)^{-1} \Psi = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathrm{ad}^n(S) I \otimes H_{\mathrm{b}} \Psi. \tag{4.6}$$

(See [3, 5].) By (4.2),

$$\mathrm{ad}^{N+n}(S) I \otimes H_{\mathrm{b}} = \mathrm{ad}^N(S) \left(\mathrm{ad}^n(S) I \otimes H_{\mathrm{b}}\right), \quad n,\ N \geq 0. \tag{4.7}$$

Let

$$F(t) = \left\langle \Phi, U(t) I \otimes H_{\mathrm{b}} U(t)^{-1} \Psi \right\rangle, \quad t \in \mathbb{R}, \tag{4.8}$$

where $\Phi \in \mathcal{F}_T$. By (4.5), (4.6),

$$F(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \left\langle \Phi, \mathrm{ad}^n(S) I \otimes H_{\mathrm{b}} \Psi \right\rangle, \tag{4.9}$$

and

$$|F(t)| \leq \|\Phi\| \left(\sum_{n=0}^{\infty} \frac{|t|^n}{n!} \|\mathrm{ad}^n(S) I \otimes H_{\mathrm{b}} \Psi\| \right). \tag{4.10}$$

Hence $F(t)$ is infinitely differentiable, and

$$F^{(N)}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \left\langle \Phi, \mathrm{ad}^{N+n}(S) I \otimes H_{\mathrm{b}} \Psi \right\rangle, \quad N \geq 0, \tag{4.11}$$

By (4.7) and the Taylor theorem, (4.3) holds.

It is easy to see that

$$\mathrm{ad}^1(S) I \otimes H_{\mathrm{b}} = -i H_{\mathrm{I}}. \tag{4.12}$$

Hence, in the same way as above we can prove (4.4). □

Applying Lemma 4.1 with $N = 2$, we obtain the following proposition:

Proposition 4.2 *Let*

$$E(t) = U(t) \left[\mathrm{ad}^2(S) I \otimes H_{\mathrm{b}} \right] U(t)^{-1}, \quad t \in \mathbb{R}. \tag{4.13}$$

Then, for each $\Psi \in \mathcal{F}_T$, there exist $\xi(t) \in [-|t|, |t|]$ such that

$$U(t) I \otimes H_{\mathrm{b}} U(t)^{-1} \Psi = \left[I \otimes H_{\mathrm{b}} + t H_{\mathrm{I}} - \frac{t^2}{2} E(\xi(t)) \right] \Psi, \tag{4.14}$$

and $\eta(t) \in [-|t|, |t|]$ such that

$$U(t) H_{\mathrm{I}} U(t)^{-1} \Psi = \left[H_{\mathrm{I}} - t E(\eta(t)) \right] \Psi. \tag{4.15}$$

Let α, $\beta \in \mathbb{C}$. Then, by (4.14), (4.15), we get the following:

$$\begin{aligned} &U(t) \left[\alpha I \otimes H_{\mathrm{b}} + \beta H_{\mathrm{I}} \right] U(t)^{-1} \Psi \\ &\quad = \left[\alpha I \otimes H_{\mathrm{b}} + (\alpha t + \beta) H_{\mathrm{I}} - \frac{\alpha t^2}{2} E(\xi(t)) - \beta t E(\eta(t)) \right] \Psi, \end{aligned} \tag{4.16}$$

for each $\Psi \in \mathcal{F}_T$.

In what follows, we consider the unitary transformation of $A \otimes I$ by $U(t)$. Assume (A.2) and set

$$\mathcal{D}_T := \mathcal{D} \otimes_{\mathrm{alg}} \mathcal{F}_{\mathrm{fin}}(T).$$

It is easy to see that if (A.3) or (A.4) holds,

$$U(t) \mathcal{D}_T \subset D(A \otimes I), \quad t \in \mathbb{R}.$$

Let

$$\delta A(t) = U(t) A \otimes I U(t)^{-1} - A \otimes I. \tag{4.17}$$

Then, if (A.3) or (A.4) holds, for each $\Psi \in \mathcal{D}_T$,

$$\begin{aligned} &U(t) \left[A \otimes I + \alpha I \otimes H_{\mathrm{b}} + \beta H_{\mathrm{I}} \right] U(t)^{-1} \Psi \\ &\quad = \left[A \otimes I + \alpha I \otimes H_{\mathrm{b}} + (\alpha t + \beta) H_{\mathrm{I}} + \delta A(t) - \frac{\alpha t^2}{2} E(\xi(t)) - \beta t E(\eta(t)) \right] \Psi. \end{aligned} \tag{4.18}$$

Applying (4.18) with

$$\alpha = \Lambda^2, \quad \beta = g\Lambda, \quad t = -\frac{g}{\Lambda}, \quad \Lambda > 0$$

we obtain the following proposition:

Proposition 4.3 *For all each $\Psi \in \mathcal{D}_T$, there exist $\xi(t)$ and $\eta(t) \in [-|t|, |t|]$ such that*

$$U(g/\Lambda)^{-1} H(\Lambda) U(g/\Lambda)\Psi = \left[A \otimes I + \Lambda^2 I \otimes H_{\rm b} + C_\Lambda\right]\Psi, \tag{4.19}$$

where

$$C_\Lambda = \delta A(-g/\Lambda) - \frac{g^2}{2} E(\xi(-g/\Lambda)) + g^2 E(\eta(-g/\Lambda)). \tag{4.20}$$

Hence, we apply Theorem 2.1 to the right hand side of (4.19). To do this, we prove the following proposition. We denote the closure of a closable operator L by $\bar{L}$.

Proposition 4.4 *Assume (A.1), (A.2) and either (A.3) or (A.4). For all $\varepsilon > 0$, there exists $\Lambda_0 = \Lambda_0(\varepsilon) > 0$ such that, for all $\Lambda \geq \Lambda_0$,*

$$\left\|\overline{C_\Lambda}\Psi\right\| \leq \varepsilon \left\|(\tilde{A} \otimes I + \Lambda^2 I \otimes H_{\rm b})\Psi\right\| + b(\varepsilon)\|\Psi\|, \quad \Psi \in D(A \otimes I) \cap D(I \otimes H_{\rm b}), \tag{4.21}$$

where $b(\varepsilon) > 0$ is a constant independent of $\Lambda \geq \Lambda_0$.

To prove proposition 4.4, we need some lemmas.

Lemma 4.5 *Assume (A.1) and that $v(t)$ $(t \in \mathbb{R})$ satisfies*

$$v(t) \in [-|t|, |t|]. \tag{4.22}$$

Then, for each $\varepsilon > 0$, there exists t_0 such that, for all $|t| \leq t_0$,

$$\left\|\overline{E(v(t))}\Psi\right\| \leq \frac{\varepsilon}{|t|^2}\|I \otimes H_{\rm b}\Psi\| + b(\varepsilon)\|\Psi\|, \quad \Psi \in D(I \otimes H_{\rm b}), \tag{4.23}$$

where $b(\varepsilon) > 0$ is a constant independent of t such that $|t| \leq t_0$.

Proof. By (4.12) and (4.13), we obtain the following inequality: For all $\Psi \in \mathcal{F}_T$,

$$\begin{aligned} \|E(t)\Psi\| \leq & \left\| \sum_{j,k} \left\langle T^{-1} g_j, g_k \right\rangle B_k B_j \otimes I U(t)^{-1} \Psi \right\| \\ & + \sum_{j,k} \|[B_j, B_k]\| \left\| I \otimes \phi(iT^{-1} g_j)\phi(g_k) U(t)^{-1} \Psi \right\|. \end{aligned} \tag{4.24}$$

By using the same way as in Lemma 4.1 and Propsition 4.2, one can easily show that there exists $\delta(t) \in [-|t|, |t|]$ $(t \in \mathbb{R})$ such that

$$\begin{aligned} & U(t) I \otimes \phi(iT^{-1} g_j)\phi(g_k) U(t)^{-1} \Psi \\ & = \Big\{ I \otimes \phi(iT^{-1} g_j)\phi(g_k) + itU(\delta(t))\big[\mathrm{ad}^1(S)(I \otimes \phi(iT^{-1} g_j)\phi(g_k))\big] U(\delta(t))^{-1} \Big\} \Psi. \end{aligned} \tag{4.25}$$

It follows from (3.6) that

$$\mathrm{ad}^1(S)(I\otimes\phi(iT^{-1}g_j)\phi(g_k)) = -i\sum_l \left\langle T^{-1}g_l, g_k\right\rangle B_l\otimes\phi(iT^{-1}g_j) \tag{4.26}$$

on $\mathcal{F}_T$. By putting (4.26) into (4.25), we obtain the following inequality:

$$\begin{aligned}
&\left\|I\otimes\phi(iT^{-1}g_j)\phi(g_k)U(v(t))^{-1}\Psi\right\| \\
&= \left\|U(v(t))I\otimes\phi(iT^{-1}g_j)\phi(g_k)U(v(t))^{-1}\Psi\right\| \\
&\le \left\|I\otimes\phi(iT^{-1}g_j)\phi(g_k)\Psi\right\| \\
&\qquad + |v(t)|\left(\sum_l \left|\left\langle T^{-1}g_l, g_k\right\rangle\right| \|B_l\|\right)\left\|I\otimes\phi(iT^{-1}g_j)U(\delta(v(t)))^{-1}\Psi\right\|
\end{aligned} \tag{4.27}$$

Note that $|v(t)|\le|t|$ and that $I\otimes\phi(iT^{-1}g_j)$ commutes with $U(t)$ on $\mathcal{F}_T$. Hence, we get, for all $\Psi\in\mathcal{F}_T$,

$$\begin{aligned}
&\left\|I\otimes\phi(iT^{-1}g_j)\phi(g_k)U(v(t))^{-1}\Psi\right\| \\
&\le \left\|I\otimes\phi(iT^{-1}g_j)\phi(g_k)\Psi\right\| + |t|\left(\sum_l \left|\left\langle T^{-1}g_l, g_k\right\rangle\right| \|B_l\|\right)\left\|I\otimes\phi(iT^{-1}g_j)\Psi\right\|
\end{aligned} \tag{4.28}$$

The following fact is well known: for all $f\in D(T^{-1/2})$ and $g\in D(T^{1/2})\cap D(T^{-1/2})$,

$$\|I\otimes\phi(f)\Psi\| \le \sqrt{2}\varepsilon\left\|T^{-1/2}f\right\|\|I\otimes H_{\mathrm{b}}\Psi\| + \left(\frac{\left\|T^{-1/2}f\right\|}{2\sqrt{2}\varepsilon} + \frac{\|f\|}{\sqrt{2}}\right)\|\Psi\|, \tag{4.29}$$

$$\|I\otimes\phi(f)\phi(g)\Psi\| \le c\|I\otimes H_{\mathrm{b}}\Psi\| + d\|\Psi\|, \tag{4.30}$$

where $\varepsilon>0$ is an arbitary constant and $c,d>0$ are some constants independent of Ψ. Hence, it follows that, for all $|t|\le 1$,

$$\begin{aligned}
&\left\|I\otimes\phi(iT^{-1}g_j)\phi(g_k)U(v(t))^{-1}\Psi\right\| \\
&\le \left[\sqrt{2}\left\|T^{-3/2}g_j\right\|\left(\sum_l \left|\left\langle T^{-1}g_l, g_k\right\rangle\right| \|B_l\|\right)\varepsilon + c\right]\|I\otimes H_{\mathrm{b}}\Psi\| \\
&+ \left[\left(\sum_l \left|\left\langle T^{-1}g_l, g_k\right\rangle\right| \|B_l\|\right)\left(\frac{\left\|T^{-3/2}g_j\right\|}{2\sqrt{2}\varepsilon} + \frac{\left\|T^{-1/2}g_j\right\|}{\sqrt{2}}\right) + d\right]\|\Psi\|.
\end{aligned} \tag{4.31}$$

By (4.24) and (4.28), we get

$$\|E(v(t))\Psi\| \le a(\varepsilon)\|I\otimes H_{\mathrm{b}}\Psi\| + b(\varepsilon)\|\Psi\|, \quad \Psi\in\mathcal{D}_T, \tag{4.32}$$

where

$$a(\varepsilon) = \sum_{j,k} \|[B_j, B_k]\| \left[\sqrt{2} \left\| T^{-3/2} g_j \right\| \left(\sum_l \left| \langle T^{-1} g_l, g_k \rangle \right| \|B_l\| \right) \varepsilon + c \right], \quad (4.33)$$

$$b(\varepsilon) = \sum_{j,k} \|[B_j, B_k]\| \left[\left(\sum_l \left| \langle T^{-1} g_l, g_k \rangle \right| \|B_l\| \right) \left(\frac{\left\| T^{-3/2} g_j \right\|}{2\sqrt{2}\varepsilon} + \frac{\left\| T^{-1/2} g_j \right\|}{\sqrt{2}} \right) \right]$$
$$+ d \sum_{j,k} \|[B_j, B_k]\| + \left\| \sum_{j,k} \langle T^{-1} g_j, g_k \rangle B_k B_j \right\|. \quad (4.34)$$

Let t_0 be a constant satisfying

$$\frac{a(\varepsilon)}{\varepsilon} = \frac{1}{{t_0}^2}. \quad (4.35)$$

We may assume that $0 < t_0 \leq 1$ without loss of generality, since $\varepsilon > 0$ can be taken to be sufficiently small. Then, for all $|t| \leq t_0$ and $\Psi \in \mathcal{F}_T$, (4.23) holds. Because $\mathcal{F}_T$ is a core for $I \otimes H_{\rm b}$, we obtain the desired result. □

In what follows, we estimate $\delta A(t)$ defined by (4.17). Let

$$c_s = \sum_{j=1}^{J} \|[B_j, A]\| \|T^s g_j\|, \qquad s \in \mathbb{R}.$$

Lemma 4.6 *Assume (A.1)-(A.3). Then, for all $\Psi \in D(I \otimes H_{\rm b})$ and $t \in \mathbb{R}$,*

$$\left\| \overline{\delta A(t)} \Psi \right\| \leq |t| \left(\sqrt{2} c_{-3/2} \left\| I \otimes {H_{\rm b}}^{1/2} \Psi \right\| + \frac{1}{\sqrt{2}} c_{-1} \|\Psi\| \right). \quad (4.36)$$

Proof. In the same way as in Lemma 4.1, one can easily prove the following fact : For each $\Psi \in \mathcal{F}_T$, there exists $\zeta(t) \in [-t, t] (t \in \mathbb{R})$ such that

$$U(t) A \otimes I U(t)^{-1} \Psi = \left\{ A \otimes I + itU(\zeta(t)) \left[\mathrm{ad}^1(S) A \otimes I \right] U(\zeta(t))^{-1} \right\} \Psi.$$

Hence,

$$\|\delta A(t) \Psi\| \leq |t| \sum_{j=1}^{J} \left\| [B_j, A] \otimes \phi(iT^{-1} g_j) U(\zeta(t)) \Psi \right\| \quad (4.37)$$

$$\leq |t| \sum_{j=1}^{J} \|[B_j, A]\| \left\| I \otimes \phi(iT^{-1} g_j) \Psi \right\|, \quad (4.38)$$

where we have used the strong commutativity of $I \otimes \phi(iT^{-1} g_j)$ and $U(t) (t \in \mathbb{R})$. It is well-known that

$$\|I \otimes \phi(f) \Psi\| \leq \sqrt{2} \left\| T^{-1/2} f \right\| \left\| I \otimes {H_b}^{1/2} \Psi \right\| + \frac{\|f\|}{\sqrt{2}} \|\Psi\|, \quad (4.39)$$

for all $f \in D(T^{-1/2})$. Applying (4.39) to (4.38), we obtain (4.36) for all $\Psi \in \mathcal{F}_T$. Since $\mathcal{F}_T$ is a core for $I \otimes H_b$, (4.36) extends to all $\Psi \in D(I \otimes H_b)$. □

Instead of (A.3) in Lemma 4.6, assume (A.4). We next estimate $\delta A(t)$ in this case. Let

$$d_s(u) = \sum_{j=1}^{J} u_j \left\| T^s g_j \right\|, \qquad s \in \mathbb{R},$$

where $u = (u_1, \cdots, u_J) \in \mathbb{R}^J$. We set

$$a := (a_1, \cdots, a_J), \qquad b := (b_1, \cdots, b_J),$$

where a_j and b_j are the constants in (A.4).

Lemma 4.7 *Assume that (A.1), (A.2) and (A.4). Then, for all $\Psi \in D(\tilde{A} \otimes I + I \otimes H_{\rm b})$,*

$$\left\| \overline{\delta A(t)} \Psi \right\| \leq |t| \left(a(\varepsilon_1, \varepsilon_2) \left\| (\tilde{A} \otimes I + I \otimes H_b) \Psi \right\| + b(\varepsilon_1, \varepsilon_2) \|\Psi\| \right), \tag{4.40}$$

where

$$\begin{aligned} a(\varepsilon_1, \varepsilon_2) &:= d_{-3/2}(a) + \varepsilon_1 d_{-1}(a) + \sqrt{2}\varepsilon_2 d_{-3/2}(b), \\ b(\varepsilon_1, \varepsilon_2) &:= \frac{1}{d_{-1}(a)} + \frac{1}{2\sqrt{2}\varepsilon_2} d_{-3/2}(b) + \frac{1}{\sqrt{2}} d_{-1}(b). \end{aligned}$$

with $\varepsilon_1, \varepsilon_2 > 0$ being arbitary constants.

Proof. In the same way as in Lemma 4.6, we can prove the inequality (4.37) under the assumption (A.1) and (A.2). By (A.1) and (A.4), we get

$$[B_j, A] \otimes \phi(iT^{-1} g_j) U(t) \Psi = U(t) [B_j, A] \otimes \phi(iT^{-1} g_j) \Psi$$

for all $\Psi \in \mathcal{D}_T$. Thus, we obtain

$$\|\delta A(t) \Psi\| \leq |t| \sum_{j=1}^{J} \| [B_j, A] \otimes \phi(iT^{-1} g_j) \Psi \| \tag{4.41}$$

for all $\Psi \in \mathcal{D}_T$. By (A.4) and (4.39), we can prove

$$\begin{aligned} &\left\| [B_j, A] \otimes \phi(iT^{-1} g_j) \Psi \right\| \\ &\quad \leq a_j \left(\sqrt{2} \left\| T^{-3/2} g_j \right\| \left\| \tilde{A}^{1/2} \otimes {H_{\rm b}}^{1/2} \Psi \right\| + \frac{\left\| T^{-1} g_j \right\|}{\sqrt{2}} \left\| \tilde{A}^{1/2} \otimes I \Psi \right\| \right) \\ &\qquad + b_j \left(\sqrt{2} \left\| T^{-3/2} g_j \right\| \left\| I \otimes {H_{\rm b}}^{1/2} \Psi \right\| + \frac{\left\| T^{-1} g_j \right\|}{\sqrt{2}} \|\Psi\| \right). \end{aligned}$$

It is easy to prove the following inequalities:

$$\begin{aligned} \left\| \tilde{A}^{1/2} \otimes {H_{\rm b}}^{1/2} \Psi \right\| &\leq \frac{1}{\sqrt{2}} \left\| (\tilde{A} \otimes I + I \otimes H_{\rm b}) \Psi \right\|, \\ \left\| \tilde{A}^{1/2} \otimes I \Psi \right\| &\leq \varepsilon_1 \left\| (\tilde{A} \otimes I + I \otimes H_{\rm b}) \Psi \right\| + \frac{1}{4\varepsilon_1} \|\Psi\|, \\ \left\| I \otimes {H_{\rm b}}^{1/2} \Psi \right\| &\leq \varepsilon_2 \left\| (\tilde{A} \otimes I + I \otimes H_{\rm b}) \Psi \right\| + \frac{1}{4\varepsilon_2} \|\Psi\|, \end{aligned}$$

for all $\Psi \in \mathcal{D}_T$. Thus, we obtain the desired result in the same way as in Lemma 4.5. □

Proof of Propotiton 4.4. In Lemma 4.5, we put for each $\varepsilon > 0$, $\Lambda_0 = |g|/t_0$. Then, for all $\Lambda > \Lambda_0$,

$$\left\| \overline{E(v(-g/\lambda))}\Psi \right\| \leq \varepsilon \left\| \Lambda^2 I \otimes H_{\rm b}\Psi \right\| + c(\varepsilon)\|\Psi\|, \quad \Psi \in D(I \otimes H_b),$$

where $c(\varepsilon) > 0$ is a constant independent of $\Lambda \geq \Lambda_0$. It is easy to see that $\Lambda^2 I \otimes H_{\rm b}$ and $I \otimes {H_{\rm b}}^{1/2}$ are $\tilde{A} \otimes I + \Lambda^2 I \otimes H_{\rm b}$-bounded. Thus, Lemma 4.5-4.7 imply the desired result. □

Propsition 4.4 and the Kaot-Rellich theorem imply the following theorem:

Theorem 4.8 *Assume (A.1), (A.2) and either (A.3) or (A.4). Then, $A \otimes I + \Lambda^2 I \otimes H_{\rm b} + \overline{C_\Lambda}$ is self-adjoint on $D(A \otimes I) \cap D(I \otimes H_{\rm b})$ and the following operater equality holds:*

$$U(g/\Lambda)^{-1} H(\Lambda) U(g/\Lambda) = A \otimes I + \Lambda^2 I \otimes H_{\rm b} + \overline{C_\Lambda}. \tag{4.42}$$

In what follows, we consider the limit of $\overline{C_\Lambda}\left(A \otimes I + \Lambda^2 I \otimes H_b\right)$ as $\Lambda \to \infty$. Let

$$C = \frac{g^2}{2} E(0) = \frac{g^2}{2} {\rm ad}^2(S) I \otimes H_{\rm b}.$$

Proposition 4.9 *Assume (A.1), (A.2) and either (A.3) or (A.4). Then, for all $z \in \mathbb{C} \setminus [\mu_0, \infty)$,*

$$s\text{-}\lim_{\Lambda \to \infty} \overline{C_\Lambda}\left(A \otimes I + \Lambda^2 I \otimes H_{\rm b} - z\right)^{-1} = C(A - z)^{-1} \otimes P_0. \tag{4.43}$$

Proof. We can write

$$\begin{aligned} &\overline{C_\Lambda}\left(A \otimes I + \Lambda^2 I \otimes H_b - z\right)^{-1} \\ &\quad = \overline{C_\Lambda}(A - z)^{-1} \otimes P_0 + \overline{C_\Lambda}\left(A \otimes I + \Lambda^2 I \otimes H_b - z\right)^{-1} I \otimes (I - P_0). \end{aligned}$$

By Lemma 4.6 and Lemma 4.7,

$$\text{s-}\lim_{t \to 0} \delta A(t)\Psi = 0, \qquad \Psi \in \mathcal{D}_T,$$

and it is easy to see that

$$\text{s-}\lim_{t \to 0} E(t)\Psi = C\Psi, \qquad \Psi \in \mathcal{F}_T.$$

Since $\overline{C_\Lambda}(A - z)^{-1} \otimes P_0$ is uniformly bounded in Λ, we obtain

$$\text{s-}\lim_{\Lambda \to \infty} \overline{C_\Lambda}(A - z)^{-1} \otimes P_0 = C(A - z)^{-1} \otimes P_0.$$

Hence we need only to prove the following fact:

$$\text{s-}\lim_{\Lambda \to \infty} \overline{C_\Lambda}\left(A \otimes I + \Lambda^2 I \otimes H_{\rm b} - z\right)^{-1} I \otimes (I - P_0) = 0. \tag{4.44}$$

By (4.21), we have

$$\begin{aligned}&\left\|\overline{C_\Lambda}(A\otimes I+\Lambda^2 I\otimes H_{\rm b}-z)^{-1}I\otimes(I-P_0)\Psi\right\|\\&\quad\le\varepsilon\|I\otimes(I-P_0)\Psi\|\\&\qquad+(\varepsilon|z|+b(\varepsilon))\left\|(A\otimes I+\Lambda^2 I\otimes H_{\rm b}-z)^{-1}I\otimes(I-P_0)\Psi\right\|.\end{aligned}$$

Thus, for all $\varepsilon>0$,

$$\limsup_{\Lambda\to\infty}\left\|\overline{C_\Lambda}(A\otimes I+\Lambda^2 I\otimes H_{\rm b}-z)^{-1}I\otimes(I-P_0)\Psi\right\|\le\varepsilon\|\Psi\|,$$

which implies (4.44). □

We are now ready to prove the main results: By Proposition 4.4 and Propotition 4.9, we can apply Theorem 2.1 to the right hand side of (4.42). Thus, we obtain

$$\text{s-}\lim_{\Lambda\to\infty}\left(A\otimes I+\Lambda^2 I\otimes H_{\rm b}+\overline{C_\Lambda}-z\right)^{-1}=(A\otimes I+C_\infty-z)^{-1}I\otimes P_0,\quad(4.45)$$

where

$$C_\infty:=(I\otimes P_0)C(I\otimes P_0).$$

It is easy to see that

$$\text{s-}\lim_{t\to0}U(t)=I.$$

Hence, it follows from (4.42) and (4.45) that

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)-z)^{-1}=(A\otimes I+C_\infty-z)^{-1}I\otimes P_0.\qquad(4.46)$$

By [1, Theorem 2.12], we need only to the prove that partial expectation of C with respect to Ω is equal to E defined by (3.10). For all u, $v\in\mathcal{H}$,

$$\langle v\otimes\Omega,Cu\otimes\Omega\rangle=-i\frac{g^2}{2}\sum_{j,k}\left\langle v\otimes\Omega,[B_j\otimes\phi(iT^{-1}g_j),B_k\otimes\phi(g_k)]u\otimes\Omega\right\rangle,$$

and

$$\begin{aligned}&\left\langle v\otimes\Omega,\left[B_j\otimes\phi(iT^{-1}g_j),B_k\otimes\phi(g_k)\right]u\otimes\Omega\right\rangle\\&\qquad=-i\left\langle T^{-1}g_j,g_k\right\rangle\left\langle v,\left(\frac{1}{2}[B_j,B_k]+B_kB_j\right)u\right\rangle\end{aligned}$$

Since, by $[B_j,B_k]=-[B_k,B_j]$,

$$\sum_{j,k}\left\langle T^{-1}g_j,g_k\right\rangle[B_j,B_k]=0,$$

we get

$$\langle v\otimes\Omega,Cu\otimes\Omega\rangle=\langle v,Eu\rangle.$$

Hence, it follows that

$$(I\otimes P_0)C(I\otimes P_0)=E\otimes P_0.$$

By this fact and (4.46), we obtain the desired result. □

5 Examples

In this section we apply the abstract theory in the last section, to some concrete models.

5.1 Interaction between Nucleons and Pions with Isospin

We consider a model of N nucleons interacting with a pion field. But, for simplicity, we assume that each nucleon has only two energy levels.

Let $\sigma_j, \tau_j (j = 1, 2, 3)$ be the Pauli matrices:

$$\sigma_1 = \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\sigma_j{}^{(i)} = 1_2 \otimes \cdots \otimes \overset{i\text{th}}{\breve{\sigma_j}} \otimes \cdots \otimes 1_2, \quad j = 1, 2, 3,$$

$$\tau_\alpha{}^{(i)} = 1_2 \otimes \cdots \otimes \overset{i\text{th}}{\breve{\tau_\alpha}} \otimes \cdots \otimes 1_2, \quad \alpha = 1, 2, 3,$$

where 1_2 is the 2×2 identity matrix. Physically, $\sigma^{(i)} = (\sigma_1{}^{(i)}, \sigma_2{}^{(i)}, \sigma_3{}^{(i)})$ and $\tau^{(i)} = (\tau_1{}^{(i)}, \tau_2{}^{(i)}, \tau_3{}^{(i)})$ denote the spin and the isospin of the ith particle, respectively. Set

$$\mathcal{H}_N := \left[\bigotimes^N \mathbb{C}^2\right] \bigotimes \left[\bigotimes^N \mathbb{C}^2\right].$$

If there is no danger of confusion, we denote the operators $\sigma_j{}^{(i)} \otimes (\bigotimes^N 1_2)$ and $(\bigotimes^N 1_2) \otimes \tau_\alpha{}^{(i)}$ acting on $\mathcal{H}_N$ by the same symbol $\sigma_j{}^{(i)}$ and $\tau_\alpha{}^{(i)}$, respectively. We denote by $\hbar$ the Planck constant divided by 2π. Put

$$B^{(i)}_{j,\alpha} := \frac{\hbar}{2} \sigma_j{}^{(i)} \tau_\alpha{}^{(i)}, \quad i = 1, \cdots, N, \ j, \alpha = 1, 2, 3,$$

which act on $\mathcal{H}_N$. It is straightforward to see that

$$\left[B^{(i)}_{j,\alpha}, B^{(l)}_{k,\beta}\right] = 0, \quad i \neq l, \ j, k, \alpha, \beta = 1, 2, 3. \tag{5.1}$$

By the anticommutativity of the Pauli matrices, it follows that, for $i = 1, \cdots, N$, $j, k, \alpha, \beta = 1, 2, 3$,

$$\left\{B^{(i)}_{j,\alpha}, B^{(i)}_{k,\beta}\right\} = \frac{\hbar^2}{4} \delta_{jk} \delta_{\alpha\beta}, \tag{5.2}$$

where $\{X, Y\} = XY + YX$ and δ_{ij} is Kronecker's delta. Let

$$\mathcal{K} = \bigoplus^3 L^2(\mathbb{R}^3)$$

and

$$\phi_\alpha(f) := \phi(f_\alpha), \qquad f \in L^2(\mathbb{R}^3),$$

where

$$f_\alpha := (\delta_{\alpha 1} f, \delta_{\alpha 2} f, \delta_{\alpha 3} f).$$

We denote by m and c the mass of a pion and the speed of light, respectively. Let

$$\omega(k) = \sqrt{\hbar^2 k^2 c^2 + m^2 c^4} \ (k \in \mathbb{R}^3),$$

where ω denotes a dispersion relation of one free pion. The function ω defines a multiplication operator on $\mathcal{K}$. We denote it by the same symbol ω:

$$\omega f := (\omega f_1, \omega f_2, \omega f_3), \quad f = (f_1, f_2, f_3) \in \mathcal{K}$$

with $f_i \in D(\omega)$. Let $H_{\rm b} = d\Gamma(\omega)$. Then, $H_{\rm b}$ represents the free Hamiltonian of the pion field. We denote by ρ the density of a nucleon, which is a real distribution satisfying $\widehat{\partial_j \rho}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, where $\widehat{\partial_j \rho}$ denotes the Fourier transform of $\partial_j \rho$. Let

$$g_j{}^{(i)} = -\frac{\sqrt{\hbar}}{\sqrt{(2\pi)^3 \omega}} \widehat{\hbar \partial_j \rho_i}, \tag{5.3}$$

where $\rho_i = \rho(\cdot - x_i)$ and $x_i \in \mathbb{R}^3$ indicates the coordinate of the jth nucleon. A Hamiltonian of spin-nucleons interacting with a pion field is defined by

$$H(\hbar, c) := \frac{\hbar}{2} \sum_{i=1}^{N} \sigma_3{}^{(i)} \otimes I + I \otimes H_{\rm b} + g \sum_{i=1}^{N} \sum_{1 \le j, \alpha \le 3} B_{j,\alpha}^{(i)} \otimes \phi_\alpha(g_j{}^{(i)}), \tag{5.4}$$

acting on $\mathcal{H}_N \otimes \mathcal{F}_{\rm b}(\mathcal{K})$, where $g \in \mathbb{R}$ is a coupling constant. Here we define the scaled Hamiltonian by

$$H(\Lambda) := \frac{1}{\Lambda^2} H(\Lambda^2 \hbar, \Lambda^2 c).$$

Then, we can write

$$H(\Lambda) = A \otimes I + \Lambda^2 I \otimes H_{\rm b} + g \Lambda H_{\rm I},$$

where

$$A = \frac{\hbar}{2} \sum_{i=1}^{N} \sigma_3{}^{(i)}, \quad H_{\rm I} = \sum_{i=1}^{N} \sum_{1 \le j, \alpha \le 3} B_{j,\alpha}^{(i)} \otimes \phi_\alpha(g_j{}^{(i)}).$$

Now we can prove the following theorem:

Theorem 5.1 *Suppose that*

$$\omega^{-3/2} g_j{}^{(i)} \in L^2(\mathbb{R}^3), \quad i = 1, \cdots, N, \quad j = 1, 2, 3.$$

Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large,

$$s\text{-}\lim_{\Lambda \to \infty} (H(\Lambda) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_0,$$

where

$$H_{\rm eff} = \frac{\hbar}{2} \sum_{i=1}^{N} \sigma_3{}^{(i)} + N E_0 + \sum_{1 \le i < l \le N} V_{i,l}$$

and

$$E_0 = -\frac{3}{2}\frac{g^2\hbar}{(2\pi)^3}\left(\frac{\hbar}{2}\right)^2\int_{\mathbb{R}^3}|\hbar k|^2\frac{|\hat{\rho}(k)|^2}{\omega(k)^2}dk,$$

$$V_{i,l} = -\frac{g^2\hbar}{(2\pi)^3}\sum_{\alpha=1}^{3}{\tau_\alpha}^{(i)}{\tau_\alpha}^{(l)}\int_{\mathbb{R}^3}\left(\frac{\hbar}{2}\sigma^{(i)}\cdot\hbar k\right)\left(\frac{\hbar}{2}\sigma^{(l)}\cdot\hbar k\right)\frac{|\hat{\rho}(k)|^2}{\omega(k)^2}e^{ik\cdot(x_l-x_i)}dk.$$

Moreover, for all $t\in\mathbb{R}$,

$$s\text{-}\lim_{\Lambda\to\infty}e^{itH(\Lambda)} = e^{itH_{\rm eff}}\otimes P_0.$$

Proof. Applying Corollary 3.2, we have

$$\begin{aligned}E &= -\frac{g^2}{2}\sum_{\alpha=1}^{3}\sum_{j,k,i,l}\left\langle\frac{{g_j}^{(i)}}{\sqrt{\omega}},\frac{{g_k}^{(l)}}{\sqrt{\omega}}\right\rangle{B_{k,\alpha}}^{(l)}{B_{j,\alpha}}^{(i)}\\ &= -\frac{g^2}{2}\sum_{\alpha=1}^{3}\left(L_1+L_2\right),\end{aligned}$$

where

$$\begin{aligned}L_1 &:= \sum_{j,k,i}\left\langle\frac{{g_j}^{(i)}}{\sqrt{\omega}},\frac{{g_k}^{(i)}}{\sqrt{\omega}}\right\rangle{B_{k,\alpha}}^{(i)}{B_{j,\alpha}}^{(i)},\\ L_2 &:= \sum_{j,k,i\neq l}\left\langle\frac{{g_j}^{(i)}}{\sqrt{\omega}},\frac{{g_k}^{(l)}}{\sqrt{\omega}}\right\rangle{B_{k,\alpha}}^{(l)}{B_{j,\alpha}}^{(i)}.\end{aligned}$$

By (5.2), we obtain

$$\begin{aligned}L_1 &= \sum_{j,i}\left\|\frac{{g_j}^{(i)}}{\sqrt{\omega}}\right\|^2{B_{j,\alpha}}^{(i)2} + \sum_{j<k,i}\left\langle\frac{{g_j}^{(i)}}{\sqrt{\omega}},\frac{{g_k}^{(i)}}{\sqrt{\omega}}\right\rangle\left\{{B_{k,\alpha}}^{(i)},{B_{j,\alpha}}^{(i)}\right\}\\ &= \left(\frac{\hbar}{2}\right)^2\sum_{j,i}\left\|\frac{{g_j}^{(i)}}{\sqrt{\omega}}\right\|^2\end{aligned}$$

By (5.1), it follows that

$$L_2 = 2\sum_{1\le i<l\le N}\sum_{j,k}\left\langle\frac{{g_j}^{(i)}}{\sqrt{\omega}},\frac{{g_k}^{(l)}}{\sqrt{\omega}}\right\rangle{B_{k,\alpha}}^{(l)}{B_{j,\alpha}}^{(i)}.$$

From (5.3) we get

$$\left\langle\frac{{g_j}^{(i)}}{\sqrt{\omega}},\frac{{g_k}^{(l)}}{\sqrt{\omega}}\right\rangle = \frac{\hbar}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{(\hbar k_j)(\hbar k_k)\,|\hat{\rho}(k)|^2}{\omega(k)^2}e^{ik\cdot(x_l-x_i)}dk,$$

and

$$\left\|\frac{{g_j}^{(i)}}{\sqrt{\omega}}\right\|^2 = \frac{\hbar}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{(\hbar k_j)^2\,|\hat{\rho}(k)|^2}{\omega(k)^2}dk.$$

Thus, we obtain the desired result. □

Remark 5.1 *Physically,* E_0 *and* $V_{i,l}$ *above are considered respectively as the self-energy of each nucleon and an effective potential of the force between two nucleons caused by the exchange of pions.*

5.2 A Lattice Spin System Interacting with a Bose Field

Let Λ be a finite set of the ν-dimentional lattice $\mathbb{Z}^\nu$ and consider the case where an N component spin $\mathbb{S} = (S^{(1)}, S^{(2)}, \cdots S^{(N)})$ sits on each site $i \in \Lambda$ and each component $S^{(n)}$ acts on $\mathbb{C}^s$ with $s \in \mathbb{N}$. The Hilbert space of this spin system is given by $\mathcal{H}_\Lambda = \bigotimes_{i\in\Lambda}\mathcal{H}_i$ with $\mathcal{H}_i = \mathbb{C}^s$, $i \in \Lambda$. The spin at site i is defined by $\mathbb{S}_i = (S_i^{(1)}, S_i^{(2)}, \cdots S_i^{(N)})$, ${S_i}^{(n)} = I \otimes \cdots \otimes S^{(n)} \otimes \cdots \otimes I$ with $S^{(n)}$ acting on $\mathcal{H}_i$. A Hamiltonian of the spin system interacting with a Bose field is given by

$$H_\Lambda := \left(- \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j \right) \otimes I + I \otimes H_{\rm b} + \alpha \sum_{i\in\Lambda}\sum_{n=1}^{N} {S_i}^{(n)} \otimes \phi({g_i}^{(n)}),$$

acting in $\mathcal{H}_\Lambda \otimes \mathcal{F}_b(L^2(\mathbb{R}^\nu))$, where $J_{ij} \in \mathbb{R}$, i, $j \in \Lambda$, are constants and ${g_i}^{(n)} \in L^2(\mathbb{R}^\nu)$, $i \in \Lambda$, $n = 1, \cdots, N$. Here, $\alpha \in \mathbb{R}$ is a coupling constant. This model is a general type of a lattice spin system interacting with a Bose field (see [2, 5]), which is a concrete realization of the abstract model H in (3.4) with the following choice:

$$\mathcal{H} = \mathcal{H}_\Lambda, \quad \mathcal{K} = L^2(\mathbb{R}^\nu), \quad g = \alpha$$
$$A = - \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j, \quad T = \omega, \quad B_j = {S_i}^{(n)}, \quad g_j = {g_i}^{(n)},$$

where $\omega : \mathbb{R}^\nu \to [0, \infty)$ is a Borel measurable function, almost everywhere finite with respect to the Lebesgue measure on $\mathbb{R}^\nu$, physically denoting the dispersion relation of a free boson in momentum representation. Let

$$H(\lambda) := \left(- \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j \right) \otimes I + \lambda^2 I \otimes H_{\rm b} + \alpha\lambda \sum_{i\in\Lambda}\sum_{n=1}^{N} {S_i}^{(n)} \otimes \phi({g_i}^{(n)}).$$

By applying Corollary 3.2, we obtain the following theorem:

Theorem 5.2 *Suppose that*

$$\omega^{-3/2}{g_i}^{(n)} \in L^2(\mathbb{R}^\nu), \quad i \in \Lambda, \quad n = 1, \cdots, N.$$

Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large,

$$s\text{-}\lim_{\lambda\to\infty} (H(\lambda) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_0,$$

where

$$H_{\rm eff} = - \sum_{(i,j)\subset\Lambda} J_{ij}\mathbb{S}_i \cdot \mathbb{S}_j + \sum_{i\in\Lambda} E_i + \sum_{i\neq j} V_{i,j}$$

and

$$E_i = -\frac{\alpha^2}{2} \sum_{n=1}^{N} \left\| \frac{{g_i}^{(n)}}{\sqrt{\omega}} \right\|^2, \quad V_{i,j} = -\frac{\alpha^2}{2} \sum_{n,m} \left\langle \frac{{g_i}^{(n)}}{\sqrt{\omega}}, \frac{{g_j}^{(m)}}{\sqrt{\omega}} \right\rangle {S_i}^{(n)} {S_j}^{(m)}.$$

Proof. Similar to the proof of Theorem 5.1. Note that the following anticommutation relations:

$$\{S_i^{(n)}, S_i^{(m)}\} = 2\delta_{nm}, \quad i = 1, \cdots, N.$$

□

Remark 5.2 *Physically, E_i and $V_{i,j}$ above are considered respectively as the self-energy of each spin and an effective interaction between two spins. In particular, the case where*

$$\nu = 3, \quad N = 3, \quad s = 2, \quad \omega(k) = |k|, \quad g_i{}^{(n)} = \widehat{\rho(\cdot - x_i)}/\sqrt{|k|}$$

is interesting. Here, x_i is the coordinate of a lattic point and ρ is a real distribution sutisfying $\hat{\rho}/|k|^{1/2}, \hat{\rho}/|k|^2 \in L^2(\mathbb{R}^3)$. This case is considered as a lattice spin system interacting with phonons.

5.3 A Model of a Fermi Field Interacting with a Bose Field

Let $\mathcal{F}_{\rm f}(\mathcal{L})$ be the fermion Fock space over the Hilbert space $\mathcal{L}$ and $\psi(f), f \in \mathcal{L}$, the fermion annihilation operators on $\mathcal{F}_f(\mathcal{L})$, which are bounded. We denote by $H_{\rm f}$ the second quantization operator of a self-adjoint operator T' acting on $\mathcal{L}$. Then, a Hamiltonian of a quantum system of a Fermi field interacting with a Bose field is given by

$$H := H_{\rm f} \otimes I + I \otimes H_{\rm b} + \alpha \sum_{j=1}^{J} \psi(f_j)^* \psi(f_j) \otimes \phi(g_j),$$

acting in $\mathcal{F}_{\rm f}(\mathcal{L}) \otimes \mathcal{F}_{\rm b}(K)$, where $f_j \in \mathcal{L}, j = 1, \cdots, J$ and $\alpha \in \mathbb{R}$ is a coupling constant. In the case $\mathcal{L} = L^2(\mathbb{R}^3; \mathbb{C}^2)$ and $\mathcal{K} = L^2(\mathbb{R}^3)$, this model may serve as a model of electrons interacting with phonons in a metal. (See [2].)

Let

$$H(\Lambda) = H_{\rm f} \otimes I + \Lambda^2 I \otimes H_{\rm b} + \alpha\Lambda \sum_{j=1}^{J} \psi(f_j)^* \psi(f_j) \otimes \phi(g_j).$$

Applying Theorem 3.1, we can prove the following theorem:

Theorem 5.3 *Suppose that (3.5),(3.6) and*

$$f_j \in D(T'), \qquad j = 1, \cdots, J.$$

Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ surfficiently large,

$$s\text{-}\lim_{\Lambda \to \infty} (H(\Lambda) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_0,$$

where

$$H_{\rm eff} = H_{\rm f} - \frac{\alpha^2}{2} \sum_{j,k} \left\langle g_j, T^{-1} g_k \right\rangle \psi(f_j)^* \psi(f_j) \psi(f_k)^* \psi(f_k).$$

Proof. It is well known or easy to see that, for $f \in D(T')$,

$$\psi(f)D(H_f) \subset D(H_f), \quad \psi(f)^* D(H_f) \subset D(H_f),$$

and

$$[H_f, \psi(f)^*] = \psi(T'f)^*, \quad [H_f, \psi(f)] = -\psi(T'f).$$

This implies (3.7) and $[\psi(f_j)^*\psi(f_j), H_f]$ are bounded. Thus, we obtain the desired result. □

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