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DECAY OF CORRELATIONS IN SUSPENSION SEMI-FLOWS OF ANGLE-MULTIPLYING MAPS

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ABSTRACT. We consider suspension semi-flows of angle-multiplying maps on the circle. Under a C^r generic condition on the ceiling function, we show that there exists an anisotropic Sobolev space[2] contained in the L^2 space such that the Perron-Frobenius operator for the time- t -map act on it and that the essential spectral radius of that action is bounded by the square root of the inverse of the minimum expansion rate. This leads to a precise description on decay of correlations, which extends the result of M. Pollicott[5].

1. INTRODUCTION

In this paper we study decay of correlations in suspension semi-flows of angle-multiplying maps on the circle. We consider Perron-Frobenius operators for the time- t -maps of the semi-flows and let them act on the anisotropic Sobolev spaces introduced in [2]. We will show, under a C^r generic condition on the ceiling function, that the essential spectral radius of the action is bounded by the square root of the inverse of the minimum expansion rate. This leads to a precise description on decay of correlations, which extends the result of M. Pollicott[5] on exponential decay.

Actually a prototype of the argument in this paper has been appeared in [1], where a class of volume-expanding hyperbolic endomorphisms were studied. In this paper, we apply essentially the same idea to analyze the time- t -maps of the suspension semi-flows.

We henceforth fix integers $\ell \geq 2$ and $r \geq 3$. Let $\tau : S^1 \rightarrow S^1$ be the angle-multiplying map on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ defined by $\tau(x) = \ell x$. Let $C_+^r(S^1)$ be the space of positive-valued C^r function on S^1 . For each $f \in C_+^r(S^1)$, the suspension semi-flow $\mathbf{T}_f = \{T_f^t : X_f \rightarrow X_f\}_{t \geq 0}$ of τ is defined by

$$X_f = \{(x, s) \in S^1 \times \mathbb{R} \mid 0 \leq s < f(x)\}$$

and

$$T_f^t(x, s) = (\tau^{n(x, s+t; f)}(x), s + t - f^{(n(x, s+t; f))}(x))$$

where $f^{(n)}(x) = \sum_{i=0}^{n-1} f(\tau^i(x))$ and $n(x, t; f) = \max\{n \geq 0 \mid f^{(n)}(x) \leq t\}$.

Let m be the normalization of the restriction of the standard Lebesgue measure on $S^1 \times \mathbb{R}$ to X_f . This is an ergodic invariant measure for $\mathbf{T}_f$. For $z = (x, s) \in X$ and $t \geq 0$, we put $E(z, t; f) = \ell^{n(x, s+t; f)}$, which is the expansion rate along the orbit of z up to time t . The minimum expansion rate of $\mathbf{T}_f$ is defined by

$$\lambda_{\min}(\mathbf{T}_f) = \lim_{t \rightarrow \infty} \left(\min_{z \in X} E(z, t; f) \right)^{1/t}.$$

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For C^1 functions ψ and φ supported on the interior X_f° of X_f , we consider the correlation $\mathbf{Cor}_t(\psi, \varphi) = \int \psi \cdot \varphi \circ T_f^t dm$ for $t \geq 0$. If $\mathbf{T}_f$ is mixing, we have

$$\lim_{t \rightarrow \infty} \mathbf{Cor}_t(\psi, \varphi) = \left(\int \psi dm \right) \left(\int \varphi dm \right).$$

The question is the rate of convergence in this limit. For simplicity, let us assume $\int \varphi dm = \int \psi dm = 0$. In [5], M. Pollicott showed, under a mild condition on f , that the rate is exponential: $|\mathbf{Cor}_t(\psi, \varphi)| < \text{const.} \exp(-\epsilon t)$ for some $\epsilon > 0$. Our results give a more precise asymptotic expansion under a C^r generic condition on f : For any real number $\rho > (\lambda_{\min}(\mathbf{T}_f))^{-1/2}$, there exists $\lambda_i \in \mathbb{C}$ with $\rho \leq |\lambda_i| < 1$ and integers $m_i \geq 0$ for $1 \leq i \leq k$ such that

$$(1) \quad \left| \mathbf{Cor}_t(\psi, \varphi) - \sum_{i=1}^k H_i(\psi, \varphi) \cdot t^{m_i} \lambda_i^t \right| \leq H_0(\psi, \varphi) \rho^t \quad \text{for } t \geq 0$$

for any $\psi \in C_0^1(X_f)$ and $\varphi \in L^2(X_f)$, where $C_0^1(X_f)$ is the space of C^1 functions supported on the interior X_f° and $H_i(\psi, \varphi)$ are coefficients that depend on ψ and φ .

In order to state the main results, we introduce a few more notations. The differential $(DT_f^t)_z$ of T_f^t at $z \in X_f$ is defined in the usual way if both z and $T^t(z)$ belong to X_f° and, otherwise, is defined by $(DT_f^t)_z = \lim_{\epsilon \rightarrow +0} (DT_f^t)_{z+(0,\epsilon)}$. Fix a real number $\ell^{-1} < \gamma_0 < 1$.[†] Put $\theta_f = (1/(\gamma_0 \ell - 1)) \max_{x \in S^1} |f'(x)|$ and

$$\mathbf{C}_f = \mathbf{C}(\theta_f) := \{(x, y) \in \mathbb{R}^2 \mid |y| \leq \theta_f |x|\}.$$

Then $\mathbf{C}_f$ is strictly invariant for $\mathbf{T}_f$ in the sense that $(DT_f^t)_z(\mathbf{C}_f) \subset \mathbf{C}(\gamma_0 \theta_f)$ for $z = (x, s) \in X_f$ and $t \geq f(x) - s$. We define, for $t \geq 0$,

$$\mathbf{m}(f, t) = \max_{z \in X} \max_{w \in (T_f^t)^{-1}(z)} \sum_{\zeta \notin w} \frac{1}{E(\zeta, t; f)} \leq 1$$

where $\sum_{\zeta \notin w}$ is the sum over $\zeta \in (T_f^t)^{-1}(z)$ s.t. $(DT_f^t)_\zeta(\mathbf{C}_f) \cap (DT_f^t)_w(\mathbf{C}_f) \neq \{0\}$. We define

$$\mathbf{m}(f) = \limsup_{t \rightarrow \infty} \mathbf{m}(f, t)^{1/t}.$$

The Perron-Frobenius operator $\mathcal{P}_f^t : L^1(X_f) \rightarrow L^1(X_f)$ is defined by

$$\mathcal{P}_f^t(u)(z) = \sum_{w \in (T_f^t)^{-1}(z)} \frac{u(w)}{\det(DT_f^t)_w}.$$

It holds $\mathbf{Cor}_t(\psi, \varphi) = \int \mathcal{P}_f^t \psi \cdot \varphi dm$. We shall prove the following theorems:

Theorem 1.1. *There exists a Hilbert space $C_0^1(X_f) \subset W_*(X_f) \subset L^2(X_f)$ such that $\mathcal{P}_f^t$ for large $t \geq 0$ is restricted to the bounded operator $\mathcal{P}_f^t : W_*(X_f) \rightarrow W_*(X_f)$ whose essential spectral radius is bounded by $\mathbf{m}(f)^{t/2}$.*

Theorem 1.2. *For each $\rho > 1$, there exists an open and dense subset $\mathcal{R}$ in $C_+^r(S^1)$ such that, if $f \in \mathcal{R}$, the corresponding semi-flow $\mathbf{T}_f = \{T_f^t\}$ is weakly mixing and satisfies $\mathbf{m}(f) \leq \rho \cdot \lambda_{\min}^{-1}(\mathbf{T}_f)$.*

From these theorems, we obtain the following corollary.

[†]It is certainly better to choose γ_0 close to 1 in the following.

Corollary 1.3. *For a C^r generic $f \in C_+^r(S^1)$, the semi-flow $\mathbf{T}_f = \{T_f^t\}$ is mixing and there exists a Hilbert space $C_0^1(X) \subset W_*(X) \subset L^2(X)$ such that $\mathcal{P}_f^t$ for large $t \geq 0$ is restricted to the bounded operator $\mathcal{P}_f^t : W_*(X) \rightarrow W_*(X)$ whose essential spectral radius is bounded by $\lambda_{\min}(\mathbf{T}_f)^{-t/2}$.*

The estimate (1) for C^r generic f is an immediate consequence of this corollary.

2. PROOF OF THEOREM 1.1

In the argument below, we consider the semi-flow $\mathbf{T}_f = \{T_f^t\}_{t \geq 0}$ for $f \in C_+^r(S^1)$. For simplicity, we will write T^t and $\mathcal{P}^t$ for T_f^t and $\mathcal{P}_f^t$ respectively.

2.1. Local charts on X_f . First we set up a system of local charts on X_f . Take $\eta > 0$ and $\delta > 0$ and set

$$R = (-\eta, \eta) \times (\delta, 2\delta) \subset Q = (-2\eta, 2\eta) \times (0, 3\delta).$$

For each $a = (x_0, s_0) \in X_f$ such that $[x_0 - 2\eta, x_0 + 2\eta] \times \{s_0\} \subset X_f$, we define

$$\kappa_a : Q \rightarrow X, \quad \kappa_a(x, s) = T^s(x_0 + x, s_0).$$

Take small η and δ so that κ_a is injective on Q whenever it is defined. Next take a finite subset $A \subset X$ so that κ_a for $a \in A$ are defined and that $\kappa_a(R)$ for $a \in A$ is a covering of X whose intersection multiplicity is bounded by an absolute constant (say 100). This is possible if we let the ratio δ/η be small.

2.2. Anisotropic Sobolev spaces. We recall the anisotropic Sobolev spaces[2]. For a cone $\mathbf{C} \subset \mathbb{R}^2$, we define $\mathbf{C}^* = \{v \in \mathbb{R}^2 \mid (v, u) = 0 \text{ for some } u \in \mathbf{C} \setminus \{0\}\}$. For two cones $\mathbf{C}, \mathbf{C}' \subset \mathbb{R}^2$, we write $\mathbf{C} \Subset \mathbf{C}'$ if the closure of $\mathbf{C}$ is contained in the interior of $\mathbf{C}'$ except for the origin. Let $C^r(R)$ be the set of C^r functions supported on R . Fix a C^∞ function $\chi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\chi(s) = 1$ for $s \leq 1$ and $\chi(s) = 0$ for $s \geq 2$. A *polarization* Θ is a combination $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ of closed cones $\mathbf{C}_\pm$ and C^∞ functions $\varphi_\pm : S^1 \rightarrow [0, 1]$ on the unit circle $S^1 \subset \mathbb{R}^2$ satisfying $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$ and

$$(2) \quad \varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \in S^1 \cap \mathbf{C}_+; \\ 0, & \text{if } \xi \in S^1 \cap \mathbf{C}_-, \end{cases} \quad \varphi_-(\xi) = 1 - \varphi_+(\xi).$$

For two polarizations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$, we write $\Theta < \Theta'$ if $\mathbb{R}^2 \setminus \mathbf{C}'_+ \Subset \mathbf{C}_-$. For a polarization $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$, an integer $n \geq 0$ and $\sigma \in \{\pm\}$, we define the C^∞ function $\psi_{\Theta, n, \sigma} : \mathbb{R}^2 \rightarrow [0, 1]$ by

$$\psi_{\Theta, n, \sigma}(\xi) = \begin{cases} \varphi_\sigma(\xi) \cdot (\chi(2^{-n}|\xi|) - \chi(2^{-n+1}|\xi|)), & \text{if } n \geq 1; \\ \chi(|\xi|)/2, & \text{if } n = 0. \end{cases}$$

For a function $u \in C^r(R)$, we define

$$u_{\Theta, n, \sigma}(x) = \psi_{\Theta, n, \sigma}(D)u(x) := (2\pi)^{-2} \int e^{i(x-y)\xi} \psi_{\Theta, n, \sigma}(\xi) u(y) dy d\xi$$

where $\psi_{\Theta, n, \sigma}(D)$ is the pseudo-differential operator with symbol $a(x, \xi) = \psi_{\Theta, n, \sigma}(\xi)$. Note that $\psi_{\Theta, n, \sigma}(D)$ is the composition $\mathcal{F}^{-1} \circ \Psi_{\Theta, n, \sigma} \circ \mathcal{F}$ where $\mathcal{F}$ is the Fourier transform and $\Psi_{\Theta, n, \sigma}$ is the multiplication operator by $\psi_{\Theta, n, \sigma}$.

For a polarization $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and real numbers p and q , we define the semi-norms $\|\cdot\|_{\Theta,p}^\pm$ on $C^r(R)$ by

$$\|u\|_{\Theta,p}^\sigma = \left(\sum_{n \geq 0} 2^{2pn} \|u_{\Theta,n,\sigma}\|_{L^2}^2 \right)^{1/2} = \left(\sum_{n \geq 0} 2^{2pn} \|\psi_{\Theta,n,\sigma} \cdot \mathcal{F}u\|_{L^2}^2 \right)^{1/2}, \quad \sigma = +, -$$

and the norm $\|\cdot\|_{\Theta,p,q}$ by $\|u\|_{\Theta,p,q} = \left((\|u\|_{\Theta,p}^+)^2 + (\|u\|_{\Theta,q}^-)^2 \right)^{1/2}$. The last norm is called anisotropic Sobolev norm in [2]. Fix $0 < \epsilon < 1$ arbitrarily. Put

$$\|\cdot\|_{\Theta}^+ := \|\cdot\|_{\Theta,1}^+, \quad \|\cdot\|_{\Theta}^- := \|\cdot\|_{\Theta,0}^-, \quad \|\cdot\|_{\Theta} := \|\cdot\|_{\Theta,1,0}$$

and

$$|\cdot|_{\Theta}^+ := \|\cdot\|_{\Theta,1-\epsilon}^+, \quad |\cdot|_{\Theta}^- := \|\cdot\|_{\Theta,-\epsilon}^-, \quad |\cdot|_{\Theta} := \|\cdot\|_{\Theta,1-\epsilon,-\epsilon}.$$

These norms $\|\cdot\|_{\Theta}$ and $|\cdot|_{\Theta}$ are associated to scalar products. The Hilbert spaces $W_*(R; \Theta)$ and $W_{\dagger}(R; \Theta)$ are the completion of $C^\infty(R)$ with respect to the norms $\|\cdot\|_{\Theta}$ and $|\cdot|_{\Theta}$ respectively.

We fix three polarizations

$$\check{\Theta}_0 = (\check{\mathbf{C}}_{0,\pm}, \check{\varphi}_{0,\pm}) < \Theta_0 = (\mathbf{C}_{0,\pm}, \varphi_{0,\pm}) < \hat{\Theta}_0 = (\hat{\mathbf{C}}_{0,\pm}, \hat{\varphi}_{0,\pm})$$

such that

$$(\mathbf{C}(\gamma_0 \theta_f))^* \in \hat{\mathbf{C}}_{0,-} \in (\mathbb{R}^2 \setminus \check{\mathbf{C}}_{0,+}) \in (\mathbf{C}(\theta_f))^*.$$

The product spaces $(W_*(R; \Theta_0))^A$ and $(W_{\dagger}(R; \Theta_0))^A$ are Hilbert spaces respectively with the norms

$$\|\mathbf{u}\| := \left(\sum_{a \in A} \|u_a\|_{\Theta_0}^2 \right)^{1/2} \quad \text{and} \quad |\mathbf{u}| := \left(\sum_{a \in A} |u_a|_{\Theta_0}^2 \right)^{1/2} \quad \text{for } \mathbf{u} = (u_a)_{a \in A}.$$

The operator $\pi_a : L^1(R) \rightarrow L^1(X)$ for $a \in A$ is defined by

$$\pi_a(u)(z) = \begin{cases} u(w)/\det D\kappa_a(w), & \text{if } z = \kappa_a(w) \text{ for some } w \in R; \\ 0, & \text{otherwise.} \end{cases}$$

Take a family of functions $h_a \in C^\infty(R)$, $a \in A$, so that $\sum_{a \in A} \pi(h_a) \equiv 1$ on X .

Let $\Pi : (L^1(R))^A \rightarrow L^1(X)$ be the operator $\Pi((u_a)_{a \in A}) = \sum_{a \in A} \pi_a(u_a)$. We define $W_*(X) = \Pi((W_*(R; \Theta_0))^A)$ and $W_{\dagger}(X) = \Pi((W_{\dagger}(R; \Theta_0))^A)$, which are equipped respectively with the norms

$$\|u\| = \inf\{\|\mathbf{u}\| \mid \Pi(\mathbf{u}) = u\} \quad \text{and} \quad |u| = \inf\{|\mathbf{u}| \mid \Pi(\mathbf{u}) = u\}.$$

These are isomorphic to the orthogonal complement of the kernel of Π in $(W_*(R; \Theta_0))^A$ and $(W_{\dagger}(R; \Theta_0))^A$ respectively and, hence, are Hilbert spaces.

2.3. Transfer operators on local charts. We start to consider the time- t -map T^t for $t \geq \max_{x \in S^1} f(x)$. For $a, b \in A$, we put $Q(a, b, t) = \{z \in Q \mid T^t \circ \kappa_a(z) \in \kappa_b(Q)\}$. The mapping T viewed in the local charts κ_a and κ_b is the C^r map

$$T_{ab}^t : Q(a, b, t) \rightarrow Q, \quad T_{ab}^t(z) = \kappa_b^{-1} \circ T^t \circ \kappa_a(z)$$

provided that $Q(a, b, t) \neq \emptyset$. The operator $\mathcal{P}_{ab} : L^1(R) \rightarrow L^1(R)$, for $a, b \in A$, is defined by

$$\mathcal{P}_{ab}^t u(z) = \sum_{w \in (T_{ab}^t)^{-1}(z)} \frac{h_a(w) h_b(z) u(w)}{\det(DT_{ab}^t)_w}.$$

The operator $\mathbf{P}^t : L^1(R)^A \rightarrow L^1(R)^A$ is defined by

$$\mathbf{P}^t(\mathbf{u}) = \left(\sum_{a \in Z} \mathcal{P}_{ab}^t(u_a) \right)_{b \in Z} \quad \text{for } \mathbf{u} = (u_a)_{a \in A} \in L^1(R)^A.$$

Then the following diagram commutes:

$$(3) \quad \begin{array}{ccc} L^1(R)^A & \xrightarrow{\mathbf{P}^t} & L^1(R)^A \\ \downarrow \Pi & & \downarrow \Pi \\ L^1(X) & \xrightarrow{\mathcal{P}^t} & L^1(X) \end{array}$$

In the following sections, we will prove the following proposition.

Proposition 2.1. *As restrictions of $\mathbf{P}^t$, we have bounded operators*

$$\mathbf{P}^t : W_*(R; \Theta_0)^A \rightarrow W_*(R; \Theta_0)^A \quad \text{and} \quad \mathbf{P}^t : W_\dagger(R; \Theta_0)^A \rightarrow W_\dagger(R; \Theta_0)^A.$$

Further the following Lasota-Yorke type inequality holds:

$$\|\mathbf{P}^t(\mathbf{u})\| \leq C_\sharp \cdot \mathbf{m}(f, t)^{1/2} \|\mathbf{u}\| + C \|\mathbf{u}\| \quad \text{for } \mathbf{u} \in W_*(R)^A$$

where the constant $C_\sharp$ does not depend on t while the constant C may.

Theorem 1.1 follows from this proposition.

Proof of Theorem 1.1. Proposition 2.1 implies that the essential spectral radius of the operator $\mathbf{P}^t : W_*(R; \Theta_0)^A \rightarrow W_*(R; \Theta_0)^A$ is bounded by $C_\sharp \cdot \mathbf{m}(f, t)$. (See [4].) The commutative diagram (4) is restricted to

$$(4) \quad \begin{array}{ccc} W_*(R; \Theta_0)^A & \xrightarrow{\mathbf{P}^t} & W_*(R; \Theta_0)^A \\ \downarrow \Pi & & \downarrow \Pi \\ W_*(X) & \xrightarrow{\mathcal{P}^t} & W_*(X). \end{array}$$

The space $W_*(X)$ is identified with the orthogonal complement of the kernel of Π in $W^*(R)^A$. With this identification, $\mathcal{P}^t$ is identified with the composition of $\mathbf{P}^t$ with the orthogonal projection along the kernel of Π . Thus the essential spectral radius of $\mathcal{P}^t : W^*(X) \rightarrow W^*(X)$ is bounded by that of $\mathbf{P}^t : W^*(R; \Theta_0)^A \rightarrow W^*(R; \Theta_0)^A$ or $C_\sharp \mathbf{m}(f, t)^{1/2}$. Since this holds for any $t > \max_{x \in S^1} f(x)$, the essential spectral radius of $\mathcal{P}^t : W^*(X) \rightarrow W^*(X)$ is actually bounded by $\mathbf{m}(f)^{1/2}$. The inclusion $C_0^1(X) \subset W_*(X) \subset L^2(X)$ follows from Lemma 2.2 in the next subsection. $\square$

2.4. Some properties of anisotropic Sobolev norms. We give some properties of the anisotropic Sobolev spaces. For $s \in \mathbb{R}$, let $W^s(R) = \Delta^{-s/2}(L^2(R))$ be the Sobolev space. The following lemma is easy and proved in [2].

Lemma 2.2. *For any polarizations $\Theta' < \Theta$, we have*

- (a) $W^1(R) \subset W_*(R; \Theta) \subset L^2(R)$ and $W^{1-\epsilon}(R) \subset W_\dagger(R; \Theta) \subset W^{-\epsilon}(R)$.
- (b) $W_*(R; \Theta) \subset W_*(R; \Theta')$ and $W_\dagger(R; \Theta) \subset W_\dagger(R; \Theta')$.
- (c) *The inclusion $W_*(R; \Theta) \subset W_\dagger(R; \Theta)$ is compact.*

Proof. (a) and (b) follow immediately from Parseval's identity. To prove (c), we put $\Psi(\xi) = \sum_{n \geq 0} (\psi_{\Theta, n, -}(\xi) + 2^n \psi_{\Theta, n, +}(\xi))$. Then $\Psi(D) : W_*(R) \rightarrow L^2(R)$ and $\Psi(D) : W_\dagger(R) \rightarrow W^{-\epsilon}(R)$ are isomorphisms. Since the inclusion $\iota : L^2(R) \rightarrow W^{-\epsilon}(R)$ is compact, we obtain (c). $\square$

The two lemmas below are slight modification of the results in [2]. We give the proofs of these lemmas in the appendix. (See Remark 2.5.)

Lemma 2.3. *Let $g_i : \mathbb{R}^2 \rightarrow [0, 1]$, $1 \leq i \leq I$, be a family of C^r functions such that $\sum_{i=1}^k g_i(x) \leq 1$ for $x \in R$ and that $\text{supp}(g_i) \subset Q$. Let Θ and Θ' be polarizations such that $\Theta' < \Theta$. Then we have, $u \in W_*(R; \Theta)$,*

$$(5) \quad \left[\sum_{1 \leq i \leq I} \|g_i u\|_{\Theta'}^2 \right]^{1/2} \leq C_0 \|u\|_{\Theta} + C |u|_{\Theta}$$

and

$$(6) \quad \|u\|_{\Theta'} \leq \nu \left[\sum_{1 \leq i \leq I} \|g_i u\|_{\Theta}^2 \right]^{1/2} + C \sum_{i=1}^I |g_i u|_{\Theta}$$

where ν is the intersection multiplicity of the supports of the functions g_i , $1 \leq i \leq I$ and C_0 is a constant that does not depend on Θ , Θ' nor $\{g_i\}$ while C may.

To state the next lemma, we consider the following situation. Let $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^r diffeomorphism and consider a transfer operator $L : C^{r-1}(R) \rightarrow C^{r-1}(K)$ defined by $Lu(x) = h(x) \cdot u \circ S(x)$ where $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^{r-1} function supported on a closed subset $K \subset R$. Let $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ and $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$ be polarizations. We suppose that $(DS_{\zeta})^{tr}(\mathbb{R}^2 \setminus \mathbf{C}_+) \Subset \mathbf{C}'_-$ for all $\zeta \in K$, where $(DS_{\zeta})^{tr}$ denotes the transpose of DS_{ζ} . We put $\gamma(S) = \min_{\zeta \in K} |\det DS_{\zeta}|$ and

$$\Lambda(S, \Theta', K) = \sup \left\{ \frac{\|(DS_{\zeta})^{tr}(v)\|}{\|v\|} \mid \zeta \in K, (DS_{\zeta})^{tr}(v) \notin \mathbf{C}'_- \right\}.$$

Lemma 2.4. *The operator L extends boundedly to $L : W_*(R; \Theta) \rightarrow W_*(R; \Theta')$ and to $L : W_{\dagger}(R; \Theta) \rightarrow W_{\dagger}(R; \Theta')$. Further we have, for $u \in W_*(R)$,*

$$(7) \quad \|Lu\|_{\Theta'}^- \leq \gamma(S)^{-1/2} \|h\|_{L^\infty} \|u\|_{\Theta}$$

and

$$(8) \quad \|Lu\|_{\Theta'}^+ \leq C_0 \gamma(S)^{-1/2} \Lambda(S, \Theta', K) \|h\|_{L^\infty} \|u\|_{\Theta} + C |u|_{\Theta}$$

where the constant C_0 does not depend on S , h , Θ nor Θ' while C may. In particular, we have, for $u \in W_*(R)$,

$$(9) \quad \|Lu\|_{\Theta'} \leq C_0 \gamma(S)^{-1/2} \max\{1, \Lambda(S, \Theta', K)\} \|h\|_{L^\infty} \|u\|_{\Theta} + C |u|_{\Theta}.$$

Remark 2.5. The inequality (6) in Lemma 2.3 is contained in Lemma 7.1 in [2]. Lemma 2.4 and the inequality (5) of Lemma 2.3 correspond to Proposition 7.2 and 6.1 in [2] respectively. But, since the norms $\|\cdot\|_{\Theta, p, q}$ with $q < 0 < p$ are considered in the corresponding statement in [2], we need to modify the statements and proofs slightly to adapt them to the case $q = 0$. This is one reason why we give the proofs of these claims in the appendix. The other reason is that the proofs become simpler according to our simple setting.

2.5. Decomposition of transfer operators. In the next section, we prove the following lemma, from which Proposition 2.1 follows immediately.

Lemma 2.6. *For $a, b \in A$, we have*

$$\|\mathcal{P}_{ab}^t(u)\|_{\Theta_0}^2 \leq C_{\sharp} \cdot \mathbf{m}(f, t) \|u\|_{\Theta_0}^2 + C|u|_{\Theta_0}^2 \quad \text{for } u \in C^r(R)$$

where the constant $C_{\sharp}$ does not depend on t while C may.

In this section and the next, we fix $a, b \in A$. Let $D(\omega)$, $\omega \in \Omega$, be a family of small closed disks whose interiors cover the closure of R . We assume that the intersection multiplicity is bounded by 4 (or some absolute constant). Note that we can take such family of disks with arbitrarily small diameters. Let $D(\omega, i)$, $1 \leq i \leq I(\omega)$, be the connected components of the preimage $(T_{ab}^t)^{-1}(D(\omega))$ that meet the closure of R . Then $\det T_{ab}^t$ takes constant value on each component $D(\omega, i)$, which is denoted by $e(\omega, i)$. Since $\sum_{y \in (T^t)^{-1}(z)} E(z, t; f)^{-1} \equiv 1$ for any $z \in X$ and $t \geq 0$, we have

$$(10) \quad \sum_{1 \leq i \leq I(\omega)} e(\omega, i)^{-1} \leq 1 \quad \text{for any } \omega \in \Omega.$$

We write $i \curvearrowright_{\omega} j$ for $1 \leq i, j \leq I(\omega)$ if $(DT_{ab}^t)_z(\mathbf{C}_f) \cap (DT_{ab}^t)_w(\mathbf{C}_f) = \{0\}$ for any $z \in D(\omega, i)$ and $w \in D(\omega, j)$. By letting the diameters of the disks $D(\omega)$ be small, we may assume that

$$\sum_{j \curvearrowright_{\omega} i} e(\omega, j)^{-1} \leq \mathbf{m}(f, t) \quad \text{for } 1 \leq i \leq I(\omega) \text{ and } \omega \in \Omega$$

where $\sum_{j \curvearrowright_{\omega} i}$ denote the sum over $1 \leq j \leq I(\omega)$ such that $j \curvearrowright_{\omega} i$. Take polarizations $\Theta(\omega, i) = (\mathbf{C}_{\omega, i, +}, \mathbf{C}_{\omega, i, -}, \varphi_{\omega, i, +}, \varphi_{\omega, i, -})$ for $\omega \in \Omega$ and $1 \leq i \leq I(\omega)$ such that

$$(((DT_{ab}^t)_z)^{-1})^{tr}(\mathbb{R}^2 \setminus \check{\mathbf{C}}_{0, +}) \Subset \mathbf{C}_{\omega, i, -} \Subset (\mathbb{R}^2 \setminus \mathbf{C}_{\omega, i, +}) \Subset \hat{\mathbf{C}}_{0, -} \quad \text{for any } z \in D(\omega, i)$$

and that

$$\overline{(\mathbb{R}^2 \setminus \mathbf{C}_{\omega, i, +})} \cap \overline{(\mathbb{R}^2 \setminus \mathbf{C}_{\omega, j, +})} = \{0\} \quad \text{if } i \curvearrowright_{\omega} j.$$

Also take a family of C^∞ functions $h_\omega : \mathbb{R}^2 \rightarrow [0, 1]$ such that $\text{supp}(h_\omega) \subset D(\omega)$ and that $\sum_{\omega \in \Omega} h_\omega(z) \equiv 1$ for $z \in R$. We define functions $h_{\omega, i} : \mathbb{R}^2 \rightarrow [0, 1]$, $1 \leq i \leq I(\omega)$, by $h_{\omega, i}(z) = h_\omega(T_{ab}^t(z))$ if $z \in D(\omega, i)$ and $h_{\omega, i}(z) = 0$ otherwise.

2.6. The Lasota-Yorke type inequality in local charts. We are going to prove Lemma 2.6. Below $C_{\sharp}$ denotes constants that do not depend on t . We view the operator $\mathcal{P}_{ab}^t$ as the composition of the operations

- (i) breaking a function $u \in C^r(R)$ into $u_{\omega, i} := h_{\omega, i}u$, $\omega \in \Omega$, $1 \leq i \leq I(\omega)$,
- (ii) transforming each $u_{\omega, i}$ to $v_{\omega, i} := \mathcal{P}_{ab}^t(u_{\omega, i})$,
- (iii) summing up $v_{\omega, i}$ for $1 \leq i \leq I(\omega)$ to get $v_\omega := \sum_{1 \leq i \leq I(\omega)} v_{\omega, i} = h_\omega \mathcal{P}_{ab}^t u$,
- (iv) summing up v_ω for $\omega \in \Omega$ to get $\mathcal{P}_{ab}^t u = \sum_{\omega} v_\omega$.

For the operation (i) and (iv), Lemma 2.3 gives

$$\sum_{\omega \in \Omega} \sum_{i=1}^{I(\omega)} \|u_{\omega, i}\|_{\Theta_0}^2 \leq C_{\sharp} \|u\|_{\Theta_0}^2 + C|u|_{\Theta_0}.$$

and

$$\|\mathcal{P}_{ab}^t u\|_{\Theta_0}^2 \leq C_{\sharp} \sum_{\omega \in \Omega} \|v_\omega\|_{\Theta_0}^2 + C \sum_{\omega \in \Omega} |v_\omega|_{\Theta_0}^2$$

Letting S be an extension of the inverse of T_{ab} in Lemma 2.4, we obtain

Lemma 2.7. *We have*

$$(11) \quad \|v_{\omega,i}\|_{\Theta(\omega,i)} \leq C_{\sharp} \cdot e(\omega,i)^{-1/2} \|u_{\omega,i}\|_{\dot{\Theta}_0} + C |u_{\omega,i}|_{\dot{\Theta}_0},$$

$$(12) \quad \|v_{\omega,i}\|_{\dot{\Theta}_0} \leq C_{\sharp} \cdot e(\omega,i)^{-1/2} \|u_{\omega,i}\|_{\dot{\Theta}_0} + C |u_{\omega,i}|_{\dot{\Theta}_0},$$

$$(13) \quad |v_{\omega,i}|_{\Theta(\omega,i)} \leq C |u_{\omega,i}|_{\dot{\Theta}_0} \quad \text{and}$$

$$(14) \quad \|v_{\omega,i}\|_{\dot{\Theta}_0}^+ \leq C_{\sharp} \cdot e(\omega,i)^{-3/2} \|u_{\omega,i}\|_{\dot{\Theta}_0} + C |u_{\omega,i}|_{\dot{\Theta}_0}.$$

The following is the key lemma in the proof.

Lemma 2.8. *If $i \nmid_{\omega} j$, we have*

$$\begin{aligned} & \sum_{n \geq 0} \left| \left(\psi_{\hat{\Theta},n,-}(D)v_{\omega,i}, \psi_{\hat{\Theta},n,-}(D)v_{\omega,j} \right)_{L^2} \right| \\ & \leq \|v_{\omega,i}\|_{\Theta(\omega,i)} |v_{\omega,j}|_{\Theta(\omega,j)} + |v_{\omega,i}|_{\Theta(\omega,i)} \|v_{\omega,j}\|_{\Theta(\omega,j)} + C |v_{\omega,i}|_{\Theta(\omega,i)} |v_{\omega,j}|_{\Theta(\omega,j)}. \end{aligned}$$

Proof. Put $v_{i,n} = \psi_{\hat{\Theta},n,-}(D)v_{\omega,i}$, $\hat{v}_{i,n} = \psi_{\Theta(\omega,i),n,-}(D)v_{\omega,i}$, $\tilde{v}_{i,n} = v_{i,n} - \hat{v}_{i,n}$. From the assumption $i \nmid_{\omega} j$, we have $(\hat{v}_{i,n}, \hat{v}_{j,n})_{L^2} = 0$. Hence

$$(v_{i,n}, v_{j,n})_{L^2} = (\tilde{v}_{i,n}, \hat{v}_{j,n})_{L^2} + (\hat{v}_{i,n}, \tilde{v}_{j,n})_{L^2} + (\tilde{v}_{i,n}, \tilde{v}_{j,n})_{L^2}.$$

Since $\|\tilde{v}_{i,n}\|_{L^2} \leq 2^{-(1-\epsilon)n} |v_{\omega,i}|_{\Theta(\omega,i)}$, we obtain the lemma. $\square$

From Lemma 2.8, it holds

$$\left(\|v_{\omega}\|_{\dot{\Theta}}^- \right)^2 \leq 2 \sum_i \sum_{j \nmid_{\omega} i} \|v_{\omega,i}\|_{\Theta(\omega,i)} \|v_{\omega,j}\|_{\Theta(\omega,j)} + C \sum_i |v_{\omega,i}|_{\Theta(\omega,i)}^2$$

for some constant $C > 0$. Applying Lemma 2.7, we obtain

$$\begin{aligned} \left(\|v_{\omega}\|_{\dot{\Theta}_0}^- \right)^2 & \leq C_{\sharp} \sum_i \sum_{j \nmid_{\omega} i} \frac{e(\omega,i)^{-1} \|u_{\omega,i}\|_{\dot{\Theta}_0}^2 + e(\omega,j)^{-1} \|u_{\omega,j}\|_{\dot{\Theta}_0}^2}{2} + C \sum_i |u_{\omega,i}|_{\dot{\Theta}_0}^2 \\ & \leq C_{\sharp} \mathbf{m}(f, t) \sum_i \|u_{\omega,i}\|_{\dot{\Theta}_0}^2 + C \sum_i |u_{\omega,i}|_{\dot{\Theta}_0}^2. \end{aligned}$$

Lemma 2.7 tells also that

$$\begin{aligned} \left(\|v_{\omega}\|_{\dot{\Theta}_0}^+ \right)^2 & \leq C_{\sharp} \left(\sum_i e(\omega,i)^{-3/2} \|u_{\omega,i}\|_{\dot{\Theta}_0} \right)^2 + C \left(\sum_i |u_{\omega,i}|_{\dot{\Theta}_0} \right)^2 \\ & \leq C_{\sharp} \mathbf{m}(f, t) \sum_i \|u_{\omega,i}\|_{\dot{\Theta}_0}^2 + C \sum_i |u_{\omega,i}|_{\dot{\Theta}_0}^2 \end{aligned}$$

where we used the Schwartz inequality, (10) and the fact $e(\omega,i)^{-1} \leq \mathbf{m}(f, t)$. We have obtained, for the operation (ii) and (iii),

$$\left(\|v_{\omega}\|_{\dot{\Theta}_0} \right)^2 \leq C_{\sharp} \mathbf{m}(f, t) \sum_i \|u_{\omega,i}\|_{\dot{\Theta}_0}^2 + C \sum_i |u_{\omega,i}|_{\dot{\Theta}_0}^2.$$

This together with the estimate on the operation (i) and (iv) gives Lemma 2.6.

3. PROOF OF THEOREM 1.2

The proof of theorem 1.2 below is a modification of the argument in [6].

3.1. Notations. Let $\mathcal{A} = \{1, 2, \dots, \ell\}$ and let $\mathcal{A}^n$ be the space of words of length n on $\mathcal{A}$. For a word $\mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$ and an integer $0 \leq p \leq n$, let $[\mathbf{a}]_p = (a_i)_{i=1}^p$. For $0 \leq p \leq n$, we define equivalence relation $\sim_p$ on $\mathcal{A}^n$ so that $\mathbf{a} \sim_p \mathbf{b}$ if and only if $[\mathbf{a}]_p = [\mathbf{b}]_p$.

Let $\mathcal{P}$ be the partition of S^1 into the intervals $\mathcal{P}(k) = [(k-1)/\ell, k/\ell)$ for $k \in \mathcal{A}$. Then $\mathcal{P} = \bigvee_{i=0}^{n-1} \tau^{-i}(\mathcal{P})$ is the partition into the intervals

$$\mathcal{P}(\mathbf{a}) = \bigcap_{i=0}^{n-1} \tau^{-i}(\mathcal{P}(a_{n-i})), \quad \mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n.$$

For a point $x \in S^1$ and $\mathbf{a} = (a_i)_{i=1}^n \in \mathcal{A}^n$, we denote by $\mathbf{a}(x)$ the unique point $y \in \mathcal{P}(\mathbf{a})$ such that $\tau^n(y) = x$. We denote by $x_{\mathbf{a}}$ the left end point of the interval $\mathcal{P}(\mathbf{a})$. For a C^r function $f \in C^r(S^1)$, $x \in S^1$ and $\mathbf{b} \in \mathcal{A}^n$, we put

$$s(x, \mathbf{b}; f) := f^{(n)}(\mathbf{b}(x)) = \sum_{i=1}^n f([\mathbf{b}]_i(x)).$$

Differentiating the both sides with respect to x , we obtain

$$\frac{ds}{dx}(x, \mathbf{b}; f) = \sum_{i=1}^n \ell^{-i} \frac{df}{dx}([\mathbf{b}]_i(x)).$$

We will identify the unit circle S^1 with the subset $S^1 \times \{0\} \subset X_f$. Then, if $f^{(n)}(\mathbf{b}(x)) \leq t < f^{(n+1)}(\mathbf{b}(x))$, the image of the horizontal tangent vector $(1, 0)$ at $\mathbf{b}(x) \in S^1 \times \{0\}$ by the mapping T_f^t has slope $\frac{d}{dx}s(x, \mathbf{b}; f)$ and so

$$(DT_f^t)_{\mathbf{b}(x)}(\mathbf{C}_f) = \left\{ (\xi, \eta) \in \mathbb{R}^2 \mid \left| \eta - \frac{ds}{dx}(x, \mathbf{b}; f) \cdot \xi \right| \leq \ell^{-n} \theta_f |\xi| \right\}.$$

For $K > 1$, let $C_+^r(S^1; K)$ be the set $f \in C_+^r(S^1)$ such that $K^{-1} \leq f(x) \leq K$ for $x \in S^1$ and $\|f\|_{C^r} \leq K$. In order to prove theorem 1.2, it is enough to show that, for each $\rho > 1$ and $K > 0$, the condition " $\mathbf{m}(f) \leq \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ and $\mathbf{T}$ is weakly mixing" holds for functions f in an open and dense subset in $C_+^r(S^1; K)$. Thereby we henceforth consider fixed $\rho > 1$ and $K > 0$. Note that $\theta_f \leq \theta_K := K/(\gamma_0 \ell - 1)$ and $\ell^{1/K} \leq \lambda_{\min}(\mathbf{T}_f) \leq \ell^K$ for $f \in C_+^r(S^1; K)$.

3.2. Reduction of the conditions. Fix $\gamma > 1$ such that $\gamma^K < \rho$.

Proposition 3.1. *If either $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ or $\mathbf{T}$ is not weakly mixing, then, for any $n \geq 1$, there exist $\mathbf{c} \in \mathcal{A}^n$ and $B \subset \mathcal{A}^n$ with $\#B \geq \gamma^n$ such that*

$$\left| \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{a}; f) - \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}; f) \right| \leq 8\theta_K \cdot \ell^{-n} \quad \text{for } \mathbf{a}, \mathbf{b} \in B.$$

Proof. First we assume $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ and derive the conclusion of the proposition. Take $\gamma < \bar{\gamma} < 1$ such that $\bar{\gamma}^K < \rho$ and then take $1 < \lambda < \lambda_{\min}(\mathbf{T}_f)$ such that $\rho\lambda > \bar{\gamma}^K \lambda_{\min}(\mathbf{T}_f)$. From the assumption, we can take an arbitrarily large $t \geq 0$, $z \in X_f$, $w \in T^{-t}(z)$ and $Z \subset T^{-t}(z)$ such that

$$(15) \quad (DT^t)_{\zeta}(\mathbf{C}_0) \cap (DT^t)_w(\mathbf{C}_0) \neq \{0\} \quad \text{for } \zeta \in Z$$

and

$$\sum_{\zeta \in Z} \frac{1}{E(\zeta, t; f)} \geq \rho^t \cdot \lambda_{\min}(\mathbf{T}_f)^{-t}.$$

We may assume that $z \in S^1 \times \{0\}$. Let $m = \min \{n(x, s+t; f) \mid (x, s) \in Z\}$. Then $\ell^m > \lambda^t$ provided that t is sufficiently large. For each point $\zeta = (x, s) \in Z$, take $I(\zeta) \in \mathcal{A}^{n(x, s+t; f)}$ so that $\zeta \in \mathcal{P}(I(\zeta))$. We put $A = \{[I(y)]_m \mid y \in Z\} \subset \mathcal{A}^m$. Since $\sum_{\xi \in (T_f^s)^{-1}(\omega)} E(\xi, s; f)^{-1} = 1$ for any $\omega \in X_f$ and $s > 0$, we have

$$\#A \cdot \ell^{-m} \geq \sum_{\zeta \in Z} \frac{1}{E(\zeta, t; f)} \geq \rho^t \cdot \lambda_{\min}(\mathbf{T}_f)^{-t}$$

and hence $\#A \geq \ell^m \rho^t \lambda_{\min}(\mathbf{T}_f)^{-t} > \bar{\gamma}^{Kt} \geq \bar{\gamma}^m$. The definition of A and (15) imply

$$\left| \frac{ds}{dx}(z, \mathbf{b}; f) - \frac{ds}{dx}(z, \mathbf{b}'; f) \right| \leq 4\theta_K \cdot \ell^{-m} \quad \text{for } \mathbf{b}, \mathbf{b}' \in A.$$

Take an integer $n \geq 1$. For $0 \leq k \leq m$, let $A_k \subset A$ be an equivalence class in A with respect to the equivalence relation $\sim_k$ with maximum cardinality, and let $q(k) = \#A_k$. Since $q(0) \geq \bar{\gamma}^m$ and $q(k) \leq \ell^{m-k}$ for $0 \leq k \leq m$, we can choose $0 \leq m' \leq m - n$ such that $q(m' + n) \leq \gamma^{-n} q(m')$, provided that we take sufficiently large t in the beginning. We define $A' \subset \mathcal{A}^{m-m'}$ as the set of words that is obtained by removing the first common m' letters (say $\mathbf{c}$) from the words in $A_{m'}$. Put $x = \mathbf{c}(z)$. Then we have

$$\left| \frac{ds}{dx}(x, \mathbf{b}; f) - \frac{ds}{dx}(x, \mathbf{b}'; f) \right| \leq 4\theta_K \cdot \ell^{-(m-m')} \leq 4\theta_K \cdot \ell^{-n} \quad \text{for } \mathbf{b}, \mathbf{b}' \in A'.$$

Put $B = \{[\mathbf{a}]_n \mid \mathbf{a} \in A'\}$. From the choice of m' , we have $\#B \geq \gamma^n$. Since

$$\left| \frac{ds}{dx}(x, [\mathbf{a}]_n; f) - \frac{ds}{dx}(x, \mathbf{a}; f) \right| \leq \theta_K \cdot \ell^{-n} \quad \text{for } \mathbf{a} \in \mathcal{A}^{m-m'},$$

we have

$$\left| \frac{ds}{dx}(z, \mathbf{b}; f) - \frac{ds}{dx}(z, \mathbf{b}'; f) \right| \leq 6\theta_K \cdot \ell^{-n} \quad \text{for } \mathbf{b}, \mathbf{b}' \in B.$$

For $\mathbf{b}, \mathbf{c} \in \mathcal{A}^n$ and $f \in C_+^r(S^1; K)$, the variation of the function $s(\cdot, \mathbf{b}; f)$ on the interval $\mathcal{P}(\mathbf{c})$ is bounded by $\theta_K \ell^{-n}$. Translating the point x to the point $x_{\mathbf{c}}$ with $\mathbf{c} \in \mathcal{A}^n$ such that $z \in \mathcal{P}(\mathbf{c})$, we obtain the conclusion of the theorem.

Next we assume that the semi-flow $\mathbf{T}$ is not weakly mixing. Then there exist a C^1 -function $g : S^1 \rightarrow \mathbb{R}$ and a real number c such that $f = g \circ \tau - g + c$. (See [3].) Thus, for $n \geq 1$, we have

$$\left| \frac{ds}{dx}(z, \mathbf{b}; f) - \frac{ds}{dx}(z, \mathbf{b}'; f) \right| \leq \frac{2 \cdot \max_{x \in S^1} |g'(x)|}{\ell^n}$$

for any $\mathbf{b}, \mathbf{b}' \in \mathcal{A}^n$ and $z \in S^1$. This implies the conclusion of the theorem. $\square$

We fix the constant γ in proposition 3.1. Take real numbers $1 < \beta < \alpha < \gamma$ and then take integers p and ν such that $\beta^{-p} \ell^2 < 1$ and $(\nu + 1)(p + 1)\alpha^{-\nu} < 1$. Put $\delta = 1 - \log(\ell/\gamma)/\log(\ell/\alpha)$. We choose and fix a large integer $N > \nu$ such that $\ell^\nu \alpha^n < \gamma^n$ for $n \geq N$ and $\ell^{-\nu}(\gamma/\beta)^{n'}(1 - (\nu + 1)(p + 1)\alpha^{-\nu}) \geq 1$ for $n' \geq \delta N$.

Proposition 3.2. *If either $\mathbf{m}(f) > \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ or $\mathbf{T}$ is not weakly mixing, then, for $n \geq N$, there exist $\delta n \leq n' \leq n$, $\mathbf{d} \in \mathcal{A}^{n'}$ and disjoint subsets $B_i \subset \mathcal{A}^{n'}$ for $1 \leq i \leq (\nu + 1)(p + 1)$ such that*

- (a) $\left| \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}; f) - \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}'; f) \right| \leq 10\theta_K \cdot \ell^{-n'}$ for $\mathbf{b}, \mathbf{b}' \in \bigcup_{i=0}^{(\nu+1)(p+1)} B_i$,
- (b) $\#B_i \geq \beta^{n'}$ for $0 \leq i \leq (\nu + 1)(p + 1)$, and

(c) for $\mathbf{a} \in B_i$ and $\mathbf{b} \in B_j$, we have $[\mathbf{a}]_\nu = [\mathbf{b}]_\nu$ if and only if $i = j$.

Proof. Let $n \geq N$ and let $B \subset \mathcal{A}^n$ be the subset in the conclusion of proposition 3.1. For $0 \leq k \leq \lfloor n/\nu \rfloor$, let $B_k \subset B$ be an equivalence class in B with respect to the equivalence relation $\sim_k$ with maximum cardinality, and let $q(k) = \#B_k$. We have $q(0) \geq \gamma^n$ and $q(\lfloor n/\nu \rfloor) \leq \ell^\nu$. From the first condition in the choice of N , there exists an integer $1 \leq k \leq \lfloor n/\nu \rfloor$ such that $q(k) < \gamma^n \alpha^{-k\nu}$. Let k_0 be the smallest of such k , so that $q(k_0) < \alpha^{-\nu} q(k_0 - 1)$ and $q(k_0 - 1) \geq \gamma^n \alpha^{-(k_0-1)\nu}$. Put $n' = n - (k_0 - 1)\nu$. Since $q(k) \leq \ell^{n-\nu k}$ obviously, we have $\delta n \leq n' \leq n$. Let $B'_i \subset B_{k_0-1}$, $1 \leq i \leq i_0$, be the equivalence classes in B_{k_0-1} w.r.t. $\sim_{k_0}$. We suppose that they are arranged in decreasing order of cardinality. Then we have

$$\begin{aligned} \min_{1 \leq i \leq (\nu+1)(p+1)} \#B'_i &\geq \frac{q(k_0 - 1) - (\nu + 1)(p + 1)q(k_0)}{\ell^\nu} \\ &\geq \ell^{-\nu} \gamma^n \alpha^{-(k_0-1)\nu} (1 - (\nu + 1)(p + 1)\alpha^{-\nu}) \geq \beta^{n'} \end{aligned}$$

where the last inequality follows from the second condition in the choice of N . For $1 \leq i \leq (\nu + 1)(p + 1)$, let $B_i \subset \mathcal{A}^{n'}$ be the subset of words that are obtained by removing the first common $(k_0 - 1)\nu$ letters (say $\mathbf{c}'$) from the words in B'_i . Then the conditions (b) and (c) hold obviously. From the condition on the subset B in Proposition 3.1, we have

$$\left| \frac{ds}{dx}(x_{\mathbf{c}\mathbf{c}'}, \mathbf{b}; f) - \frac{ds}{dx}(x_{\mathbf{c}\mathbf{c}'}, \mathbf{b}'; f) \right| \leq 8\theta_K \cdot \ell^{-n'} \quad \text{for } \mathbf{b}, \mathbf{b}' \in \bigcup_{i=0}^{(\nu+1)(p+1)} B_i.$$

The condition (a) holds for $\mathbf{d} \in \mathcal{A}^{n'}$ such that $x_{\mathbf{c}\mathbf{c}'} \in \mathcal{P}(\mathbf{d})$, because the variation of the functions $s(\cdot, \mathbf{a}; f)$ for $\mathbf{a} \in \mathcal{A}^{n'}$ on $\mathcal{P}(\mathbf{d})$ is bounded by $\theta_K \ell^{-n'}$. $\square$

3.3. Generic perturbations. We next consider about perturbations of the ceiling function. For $f \in C_+^r(S^1; K)$ and $\varphi_i \in C^\infty(S^1)$, $1 \leq i \leq m$, we consider the family

$$(16) \quad f_{\mathbf{t}}(x) = f(x) + \sum_{i=1}^m t_i \cdot \varphi_i(x)$$

with parameter $\mathbf{t} = (t_i)_{i=1}^m \in \mathbb{R}^m$. For $x \in S^1$ and $\sigma = \{\mathbf{b}_i\}_{0 \leq i \leq q} \in (\mathcal{A}^n)^{p+1}$, we define the affine map $G_{x,\sigma} : \mathbb{R}^m \rightarrow \mathbb{R}^p$ by

$$G_{x,\sigma}(\mathbf{t}) = \left(\frac{ds}{dx}(x, \mathbf{b}_i; f_{\mathbf{t}}) - \frac{ds}{dx}(x, \mathbf{b}_0; f_{\mathbf{t}}) \right)_{i=1}^p.$$

Note that $G_{x,\sigma}(\mathbf{t})$ is independent of f in (16). For an affine map $A : \mathbb{R}^m \rightarrow \mathbb{R}^p$, let $Jac(A)$ be the Jacobian of $DA|_{\ker(DA)^\perp}$, the restriction of the linear part DA of A to the orthogonal complement of its kernel when A is surjective, and put $Jac(A) = 0$ otherwise. In other words, $Jac(A)$ is the maximum among the Jacobians of the restrictions of DA to p -dimensional subspaces in $\mathbb{R}^m$. The following is a variant of [6, Proposition 16].

Proposition 3.3. *There exist functions $\varphi_i \in C^\infty(S^1)$, $1 \leq i \leq m$, such that, for any point $x \in S^1$ and any subsets $A = \{\mathbf{a}_i\}_{1 \leq i \leq (\nu+1)(p+1)}$ of $\mathcal{A}^\nu$, we can choose a subset $A' = \{\mathbf{a}'_i\}_{0 \leq i \leq p}$ of A so that $Jac(G_{x,\sigma}) \geq 1$ holds whenever a subset $\sigma = \{\mathbf{b}_i\}_{0 \leq i \leq p} \in (\mathcal{A}^n)^{p+1}$ with $n \geq \nu$ satisfies $[\mathbf{b}_i]_\nu = \mathbf{a}'_i$ for $0 \leq i \leq p$.*

The proof of proposition 3.3 is similar to that of [6, Proposition 16]. For completeness, we give a proof in the last subsection.

3.4. The end of the proof. Fix the family of functions $\varphi_i \in C^\infty(S^1)$, $1 \leq i \leq m$, in proposition 3.3. For $n \geq \nu$, $\mathbf{c} \in \mathcal{A}^n$ and $\sigma = (\mathbf{b}_i)_{i=0}^p \in (\mathcal{A}^n)^{p+1}$ such that $Jac(G_{x_{\mathbf{c},\sigma}}) \geq 1$, let $Y(n, \mathbf{c}, \sigma)$ be the set of functions $f \in C_+^r(S^1; K)$ such that

$$\left| \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}_i; f) - \frac{ds}{dx}(x_{\mathbf{c}}, \mathbf{b}_0; f) \right| \leq 10\theta_K \cdot \ell^{-n} \quad \text{for all } 1 \leq i \leq p.$$

Note that $Y(n, \mathbf{c}, \sigma)$ is a closed subset in $C_+^r(S^1; K)$. For $n \geq \nu$, let $Y(n)$ be the set of $f \in C_+^r(S^1; K)$ that belongs to $Y(n, \mathbf{c}, \sigma)$ for more than $[\beta^{n(p+1)}]$ combinations of $(\mathbf{c}, \sigma) \subset \mathcal{A}^n \times (\mathcal{A}^n)^{p+1}$ satisfying $Jac(G_{x_{\mathbf{c},\sigma}}) \geq 1$. Let $Y_*(n) = \bigcup_{n'=[\delta n]}^n Y(n')$. Then $Y(n)$ and $Y_*(n)$ are also closed subsets in $C_+^r(S^1; K)$. Proposition 3.2 tells that, if either $\mathbf{m}(f) \geq \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ or $\mathbf{T}_f$ is not weakly mixing for $f \in C_+^r(S^1; K)$, then f belongs to $\bigcap_{n \geq N} Y_*(n)$. To finish the proof of the theorem, we show that the complement of $\bigcap_{n \geq N} Y_*(n)$ is dense in $C_+^r(S^1; K)$.

Take a function $f \in C_+^r(S^1; K)$ arbitrarily and consider the family (16). Take $\epsilon > 0$ so small that $f_{\mathbf{t}} \in C_+^r(S^1; K)$ for all $\mathbf{t} \in [-\epsilon, \epsilon]^m$. Let $X(n, \mathbf{c}, \sigma)$, $X(n)$ and $X_*(n)$ be the set of parameters $\mathbf{t} \in [-\epsilon, \epsilon]^m$ such that $f_{\mathbf{t}} \in Y(n, \mathbf{c}, \sigma)$, that $f_{\mathbf{t}} \in Y(n)$ and that $f_{\mathbf{t}} \in Y_*(n)$, respectively. From the definition of Jacobian, we have $Leb(X(n, \mathbf{c}, \sigma)) \leq C\ell^{-np}$ for some constant $C > 0$ that depends only on m and ϵ . Counting the number of combinations of $(\mathbf{c}, \sigma)$, we can see

$$Leb(X(n)) \leq \frac{C\ell^{-np} \times \ell^n \times \ell^{(p+1)n}}{\beta^{(p+1)n}} < C(\beta^{-p}\ell^2)^n.$$

Recalling that $\beta^{-p}\ell^2 < 1$, we obtain $Leb(\bigcap_{n \geq N} X_*(n)) = 0$. This implies that the complements of $\bigcap_{n \geq N} Y_*(n)$ in $C_+^r(S^1; K)$ is dense.

Remark 3.4. The proof above shows also that $\mathbf{m}(f) \leq \rho \cdot \lambda_{\min}(\mathbf{T}_f)^{-1}$ and $\mathbf{T}$ is weakly mixing for a prevalent subset of $f \in C_+^r(S^1)$ in a measure-theoretical sense.

3.5. The proof of proposition 3.3. It is enough to show

Proposition 3.5. *For each $y \in S^1$, there exist $\varphi_{y,i} \in C^\infty(S^1)$, $1 \leq i \leq \ell^\nu$, and a neighborhood U_y of y such that, for any point $x \in U_y$ and any subsets $A = \{\mathbf{a}_i\}_{1 \leq i \leq (\nu+1)(p+1)}$ of $\mathcal{A}^\nu$, there exists a subset $A' = \{\mathbf{a}'_i\}_{0 \leq i \leq p}$ of A so that $Jac(G_{x,\sigma}) \geq 1$ holds whenever $\sigma = \{\mathbf{b}_i\}_{0 \leq i \leq p} \in (\mathcal{A}^n)^{p+1}$ with $n \geq \nu$ satisfies $[\mathbf{b}_i]_\nu = \mathbf{a}'_i$ for $0 \leq i \leq p$.*

In fact, we can take a finite subset $\{y(j)\}_{j=1}^J$ in S^1 so that the neighborhoods $U_{y(j)}$ in proposition 3.5 cover S^1 and, then, the union $\{\varphi_i\}_{j=1}^m$ of $\{\varphi_{y(j),i}\}_{i=1}^{\ell^\nu}$ for $1 \leq j \leq J$ in the corresponding conclusions of proposition 3.5 satisfies the condition required in proposition 3.3.

Proof of proposition 3.5. Fix $y \in S^1$ arbitrarily. For $\mathbf{a}, \mathbf{b} \in \mathcal{A}^\nu$, we write $\mathbf{a} \prec \mathbf{b}$ if $\tau^q(\mathbf{b}(y)) = \mathbf{a}(y)$ for some $q \geq 0$. By simple combinatorial argument, we can show that this is a partial order on $\mathcal{A}^\nu$ and that, for each $\mathbf{a} \in \mathcal{A}^\nu$, there exists at most $(\nu+1)$ elements $\mathbf{b} \in \mathcal{A}^\nu$ such that $\mathbf{b} \prec \mathbf{a}$. (See the proof of [6, Proposition 16].)

For $0 < \epsilon < 1/2$, let $U(\epsilon)$ be the ϵ -neighborhood of y . For $\mathbf{a} \in \mathcal{A}^\nu$, let $U_{\mathbf{a}}(\epsilon)$ be the connected component of $\tau^{-\nu}(U(\epsilon))$ that contains $\mathbf{a}(y)$. Let $\mu > \nu$ be an integer that we will specify later. We choose $\epsilon_0 > 0$ so small that $\tau^i(U_{\mathbf{b}}(\epsilon_0)) \cap U_{\mathbf{a}}(\epsilon_0) \neq \emptyset$ for some $1 \leq i \leq \mu$ only if $\mathbf{a} \prec \mathbf{b}$. We take C^∞ functions $\varphi_{\mathbf{a}}$ on S^1 for $\mathbf{a} \in \mathcal{A}^\nu$ so that $\text{supp}(\varphi_{\mathbf{a}}) \subset U_{\mathbf{a}}(\epsilon_0)$, $\frac{d}{dx}\varphi_{\mathbf{a}}(y) = \ell^\nu$ on $U_{\mathbf{a}}(\epsilon_0/3)$ and $|\frac{d}{dx}\varphi_{\mathbf{a}}(y)| < 2\ell^\nu$ on S^1 . Let $\varphi_{y,i}$, $1 \leq i \leq \ell^\nu$ be a rearrangement of $\varphi_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{A}^\nu$ and let $U_y = U(\epsilon_0/3)$.

Consider the family (16) for $\varphi_i = \varphi_{y,i}$ and $m = \ell^\nu$. Suppose that a subset $A = \{\mathbf{a}_i\}_{1 \leq i \leq (\nu+1)(p+1)}$ of $\mathcal{A}^\nu$ is given. From the property of the partial order $\prec$ on $\mathcal{A}^\nu$ mentioned above, we can choose a subset $A' = \{\mathbf{a}'_i\}_{0 \leq i \leq p}$ of A that consists of maximal elements in A with respect to $\prec$. Let $\sigma = \{\mathbf{b}_i\}_{0 \leq i \leq p}$ be a subset of $\mathcal{A}^n$ with $n \geq \nu$ such that $[\mathbf{b}_i]_\nu = \mathbf{a}'_i$ for $0 \leq i \leq p$. For $\mathbf{b} \in \mathcal{A}^n$ and $x \in U_y$, we define

$$h_1(x, \mathbf{b}; \mathbf{t}) = \sum_{j=1}^{\min\{n, \mu\}} \ell^{-j} \frac{d}{dx} f_{\mathbf{t}}([\mathbf{b}]_j(x))$$

and

$$h_2(x, \mathbf{b}; \mathbf{t}) = \sum_{j=\min\{n, \mu\}+1}^n \ell^{-j} \frac{d}{dx} f_{\mathbf{t}}([\mathbf{b}]_j(x)),$$

so that

$$\frac{ds}{dx}(x, \mathbf{b}; f_{\mathbf{t}}) = h_1(x, \mathbf{b}; \mathbf{t}) + h_2(x, \mathbf{b}; \mathbf{t}).$$

Accordingly we decompose the affine map $G_{x,\sigma}$ into

$$G_{x,\sigma}^{(1)}(\mathbf{t}) = (h_1(x, \mathbf{b}_i; \mathbf{t}) - h_1(x, \mathbf{b}_0; \mathbf{t}))_{i=1,2,\dots,p} : \mathbb{R}^m \rightarrow \mathbb{R}^p$$

and

$$G_{y,\sigma}^{(2)}(\mathbf{t}) = (h_2(y, \mathbf{b}_i; \mathbf{t}) - h_2(y, \mathbf{b}_0; \mathbf{t}))_{i=1,2,\dots,p} : \mathbb{R}^m \rightarrow \mathbb{R}^p.$$

Let $\xi : \{1, 2, \dots, p\} \rightarrow \{1, 2, \dots, \ell^\nu\}$ be the correspondence such that $\mathbf{a}'_i(y) \in \text{supp}(\varphi_{y,\xi(i)})$ for $1 \leq i \leq p$, and let

$$E = \{\mathbf{t} = (t_j)_{j=1}^{\ell^\nu} \in \mathbb{R}^{\ell^\nu} \mid t_j \neq 0 \text{ only if } j = \xi(i) \text{ for some } 1 \leq i \leq p\} \subset \mathbb{R}^{\ell^\nu}.$$

We naturally identify E with $\mathbb{R}^p$. Consider any point $x \in U_y$ and let $L^{(1)}$ and $L^{(2)}$ be the matrices that represent the linear part of the affine mappings $G_{y,\sigma}^{(1)} : E \rightarrow \mathbb{R}^p$ and $G_{y,\sigma}^{(2)} : E \rightarrow \mathbb{R}^p$ respectively. As a consequence of the choice of $\mathbf{a}'_i$, we can see that $L^{(1)}$ is the identity matrix of size p while all the entries $L^{(2)}$ are bounded by $2\ell^{-\mu+\nu}(1 - \ell^{-1})^{-1}$. Therefore, if we take sufficiently large μ , it holds

$$\text{Jac}(DG_{x,\sigma}) \geq \text{Jac}(DG_{x,\sigma}|_E) \geq 1/2$$

for $x \in U_y$. Multiplying each $\varphi_{y,i}$ by 2, we can replace $1/2$ by 1 on the right hand side. We finished the proof of proposition 3.5. $\square$

APPENDIX A. THE PROOF OF LEMMA 2.3 AND 2.4

We prove Lemma 2.4 first. Put $\Gamma = \mathbb{Z}_+ \times \{+, -\}$, $c(+) = 1$ and $c(-) = 0$. Below we use C_0 for constants that does not depend on S, h, Θ nor Θ' , and C for constants that may depend on these. Take an integer $\mu = \mu(S)$ such that

$$2^{-\mu+6}\|\xi\| \leq \|(DS_x)^{tr}(\xi)\| \leq 2^{\mu-6}\|\xi\| \quad \text{for any } x \in K \text{ and any } \xi \in \mathbb{R}^2.$$

Take another integer $\nu \leq \mu - 6$ such that $2^{\nu-1} < \Lambda(S, \Theta', K) \leq 2^\nu$. So $\|DS_x^{tr}(\xi)\| \leq 2^\nu \|\xi\|$ for $x \in K$ if $(DS_x)^{tr}(\xi) \notin \mathbf{C}'_-$. We write $(m, \tau) \hookrightarrow (n, \sigma)$ if either

- $(\tau, \sigma) = (+, +)$ and $m - \mu \leq n \leq \max\{0, m + \nu + 6\}$, or
- $(\tau, \sigma) \in \{(-, -), (+, -)\}$ and $m - \mu \leq n \leq m + \mu$.

We write $(m, \tau) \not\rightarrow (n, \sigma)$ otherwise.

Consider a function $u \in C^r(R)$ and put $v := Lu$. For $(n, \sigma), (m, \tau) \in \Gamma$, we put $v_{n, \sigma}^{m, \tau} := \psi_{\Theta', n, \sigma}(D)L(u_{\Theta, m, \tau})$ so that $v_{\Theta', n, \sigma} = \sum_{(m, \tau) \in \Gamma} v_{n, \sigma}^{m, \tau}$. By Parseval's identity, it holds

$$(17) \quad \sum_{(n, \sigma) \in \Gamma} \|v_{n, \sigma}^{m, \tau}\|_{L^2}^2 \leq \|L(u_{\Theta, m, \tau})\|_{L^2}^2 \leq C_0 \gamma(S)^{-1} \|h\|_{L^\infty}^2 \|u_{\Theta, m, \tau}\|_{L^2}^2.$$

Also we have the following estimate, whose proof is postponed for a while.

Lemma A.1. *If $(m, \tau) \not\rightarrow (n, \sigma)$, we have*

$$(18) \quad \|v_{n, \sigma}^{m, \tau}\|_{L^2} \leq C 2^{-(r-1) \max\{m, n\}} \|u_{\Theta, m, \tau}\|_{L^2}.$$

We first show that $|v|_{\Theta'} \leq C|u|_{\Theta}$ for some constant C , which implies that L extends boundedly to $L : W_{\dagger}(R; \Theta) \rightarrow W_{\dagger}(R; \Theta)$. We have

$$|v|_{\Theta'}^2 \leq 2 \sum_{(n, \sigma) \in \Gamma} 2^{2(c(\sigma) - \epsilon)n} \left(\left\| \sum_{(m, \tau) \hookrightarrow (n, \sigma)} v_{n, \sigma}^{m, \tau} \right\|_{L^2}^2 + \left\| \sum_{(m, \tau) \not\rightarrow (n, \sigma)} v_{n, \sigma}^{m, \tau} \right\|_{L^2}^2 \right)$$

where $\sum_{(m, \tau) \hookrightarrow (n, \sigma)}$ denotes the sum over $(m, \tau) \in \Gamma$ such that $(m, \tau) \hookrightarrow (n, \sigma)$. Since the relation $(m, \tau) \hookrightarrow (n, \sigma)$ holds only if $c(\sigma) \leq c(\tau)$ and $|m - n| < \mu$,

$$\begin{aligned} \sum_{(n, \sigma) \in \Gamma} \left\| \sum_{(m, \tau) \hookrightarrow (n, \sigma)} 2^{(c(\sigma) - \epsilon)n} v_{n, \sigma}^{m, \tau} \right\|_{L^2}^2 &\leq C \sum_{(n, \sigma) \in \Gamma} \sum_{(m, \tau) \hookrightarrow (n, \sigma)} 2^{2(c(\tau) - \epsilon)m} \|v_{n, \sigma}^{m, \tau}\|_{L^2}^2 \\ &\leq C \sum_{(m, \tau) \in \Gamma} 2^{2(c(\tau) - \epsilon)m} \|u_{\Theta, m, \tau}\|_{L^2}^2 \leq C|u|_{\Theta}^2 \end{aligned}$$

where we used (17) in the second inequality. From (18) and Schwarz inequality,

$$\begin{aligned} &\sum_{(n, \sigma) \in \Gamma} \left\| \sum_{(m, \tau) \not\rightarrow (n, \sigma)} 2^{(c(\sigma) - \epsilon)n} v_{n, \sigma}^{m, \tau} \right\|_{L^2}^2 \\ &\leq \sum_{(n, \sigma) \in \Gamma} \left(\sum_{(m, \tau) \in \Gamma} 2^{2c(\sigma)n - 2c(\tau)m - 2(2 - \epsilon) \max\{n, m\}} \right) \left(\sum_{(m, \tau) \in \Gamma} 2^{2(c(\tau) - \epsilon)m} \|u_{\Theta, m, \tau}\|_{L^2}^2 \right) \\ &\leq C|u|_{\Theta}^2. \end{aligned}$$

Thus we obtain $|v|_{\Theta'} \leq C|u|_{\Theta}$ for $u \in W_{\dagger}(R; \Theta)$.

We next prove (7) and (8). The inequality (7) is easy to prove:

$$(\|v\|_{\Theta'}^-)^2 \leq \|v\|_{L^2}^2 \leq \gamma(S)^{-1} \|h\|_{L^\infty}^2 \|u\|_{L^2}^2 \leq \gamma(S)^{-1} \|h\|_{L^\infty}^2 \|u\|_{\Theta}^2.$$

Note that $(\|v\|_{\Theta'}^+)^2$ is bounded by the double of

$$\sum_{n \geq 0} \left\| \sum_{(m, \tau) \hookrightarrow (n, +)} 2^n v_{n, +}^{m, \tau} \right\|_{L^2}^2 + \sum_{n \geq 0} \left\| \sum_{(m, \tau) \not\rightarrow (n, +)} 2^n v_{n, +}^{m, \tau} \right\|_{L^2}^2.$$

By Schwarz inequality,

$$\left\| \sum_{(m, +) \hookrightarrow (n, +)} 2^n v_{n, +}^{m, \tau} \right\|_{L^2}^2 \leq \left(\sum_{(m, +) \hookrightarrow (n, +)} 2^{2(n-m)} \right) \left(\sum_{(m, +) \hookrightarrow (n, +)} 2^{2m} \|v_{n, +}^{m, \tau}\|_{L^2}^2 \right).$$

Since $(m, \tau) \hookrightarrow (n, +)$ implies $\tau = +$ and $n \leq \max\{0, m + \nu + 4\}$, we have, by (17),

$$\sum_{n \geq 0} \left\| \sum_{(m, \tau) \hookrightarrow (n, +)} 2^n v_{n, +}^{m, \tau} \right\|_{L^2}^2 \leq C_0 \cdot 2^{2\nu} \gamma(S)^{-1} \|h\|_{L^\infty}^2 \|u\|_{\Theta}^2.$$

On the other hand, using Lemma A.1, we can see

$$\sum_{n \geq 0} \left\| \sum_{(m, \tau) \not\hookrightarrow (n, +)} 2^n v_{n, +}^{m, \tau} \right\|_{L^2}^2 < C |u|_{\Theta}^2.$$

These implies (8). Obviously (7) and (8) imply (9) and hence L extends boundedly to $L : W_*(R; \Theta) \rightarrow W_*(R; \Theta')$. We complete the proof by proving Lemma A.1.

Poof of Lemma A.1. Take closed cones $\tilde{\mathbf{C}}_+ \Subset \mathbf{C}_+$ and $\tilde{\mathbf{C}}_- \Subset \mathbf{C}_-$ such that it holds $(DS_\zeta)^{tr}(\mathbb{R}^d \setminus \tilde{\mathbf{C}}_+) \Subset \mathbf{C}'_-$ for $\zeta \in K$. Let $\tilde{\varphi}_+, \tilde{\varphi}_- : S^1 \rightarrow [0, 1]$ be C^∞ functions satisfying

$$\tilde{\varphi}_+(\xi) = \begin{cases} 1, & \text{if } \xi \notin S^1 \cap \mathbf{C}_-; \\ 0, & \text{if } \xi \in S^1 \cap \tilde{\mathbf{C}}_-, \end{cases} \quad \tilde{\varphi}_-(\xi) = \begin{cases} 1, & \text{if } \xi \notin S^1 \cap \mathbf{C}_+; \\ 0, & \text{if } \xi \in S^1 \cap \tilde{\mathbf{C}}_+. \end{cases}$$

Recall the function χ and put $\tilde{\psi}_n(\xi) = \chi(2^{-n-1}|\xi|) - \chi(2^{-n+2}|\xi|)$ for $n \geq 1$ and

$$\tilde{\psi}_{\Theta, n, \sigma}(\xi) = \begin{cases} \tilde{\psi}_n(\xi) \tilde{\varphi}_\sigma(\xi/|\xi|), & \text{if } n \geq 1; \\ \chi(2^{-1}|\xi|), & \text{if } n = 0 \end{cases}$$

for $(n, \sigma) \in \Gamma$. Then we have $\tilde{\psi}_{\Theta, n, \sigma}(\xi) = 1$ if $\xi \in \text{supp}(\psi_{\Theta, n, \sigma})$. From the definition of the relation $\hookrightarrow$, there exists a constant $L > 1$, which may depend on S , such that, if $(m, \tau) \not\hookrightarrow (n, \sigma)$ and $\max\{m, n\} \geq L$, it holds

$$(19) \quad d(\text{supp}(\psi_{\Theta', n, \sigma}), (DS_z)^{tr}(\text{supp}(\tilde{\psi}_{\Theta, m, \tau}))) \geq L^{-1} \cdot 2^{\max\{n, m\}} \quad \text{for } z \in K.$$

In the case $\max\{m, n\} < L$, it is easy to see that the claim of the lemma holds. So we assume $\max\{m, n\} \geq L$ below.

Define the operator $S_{n, \sigma}^{m, \tau}$ by $S_{n, \sigma}^{m, \tau} = \psi_{\Theta', n, \sigma}(D) \circ L \circ \tilde{\psi}_{\Theta, m, \tau}(D)$. Then $v_{n, \sigma}^{m, \tau} = S_{n, \sigma}^{m, \tau} u_{\Theta, m, \tau}$ because $\tilde{\psi}_{\Theta, m, \tau}(D)(u_{\Theta, m, \tau}) = u_{\Theta, m, \tau}$. We rewrite $S_{n, \sigma}^{m, \tau}$ as

$$(S_{n, \sigma}^{m, \tau} u)(x) = (2\pi)^{-2d} \int V_{n, \sigma}^{m, \tau}(x, y) \cdot u \circ S(y) \cdot |\det DS(y)| dy,$$

where

$$(20) \quad V_{n, \sigma}^{m, \tau}(x, y) = \int e^{i(x-w)\xi + i(S(w)-S(y))\eta} h(w) \psi_{\Theta', n, \sigma}(\xi) \tilde{\psi}_{\Theta, m, \tau}(\eta) dw d\xi d\eta.$$

Since $\|u \circ S(y) \cdot |\det DS(y)|\|_{L^2} \leq C \|u\|_{L^2}$, (18) follows if the operator norm of the integral operator $H_{n, \sigma}^{m, \tau} : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$, $H_{n, \sigma}^{m, \tau} v(x) = \int V_{n, \sigma}^{m, \tau}(x, y) v(y) dy$ is bounded by $C \cdot 2^{-(r-1)\max\{n, m\}}$.

Applying the following formula of integration by parts for $(r-1)$ times in (20):

$$\int e^{if(w)} g(w) dw = i \cdot \int e^{if(w)} \cdot \sum_{k=1}^2 \partial_{w_k} \left(\frac{\partial_{w_k} f(w) \cdot g(w)}{\sum_{j=1}^2 (\partial_{w_j} f(w))^2} \right) dw$$

where $w = (w_k)_{k=1}^2 \in \mathbb{R}^2$, we obtain the expression

$$V_{n, \sigma}^{m, \tau}(x, y) = \int e^{i(x-w)\xi + i(S(w)-S(y))\eta} F(\xi, \eta, w) \psi_{\Theta', n, \sigma}(\xi) \tilde{\psi}_{\Theta, m, \tau}(\eta) dw d\xi d\eta$$

where $F(\xi, \eta, w)$ is continuous in w and C^∞ in ξ and η . Note that $F(\xi, \eta, w) = 0$ if $w \notin K$. From (19), there is a constant $C_{\alpha\beta}$ for multi-indices α and β , such that

$$(21) \quad |\partial_\xi^\alpha \partial_\eta^\beta F(\xi, \eta, w)| \leq C_{\alpha\beta} \cdot 2^{-n|\alpha| - m|\beta| - (r-1)\max\{n, m\}}$$

for $w \in \mathbb{R}^2$, $\xi \in \text{supp}(\psi_{\Theta', n, \sigma})$ and $\eta \in \text{supp}(\tilde{\psi}_{\Theta, m, \tau})$. For $n \geq 0$ and $m \geq 0$, we put

$$G_{nm}(\xi, \eta, w) = F(2^n \xi, 2^m \eta, w) \psi_{\Theta', n, \sigma}(2^n \xi) \tilde{\psi}_{\Theta, m, \tau}(2^m \eta).$$

Then we can rewrite $V_{n, \sigma}^{m, \tau}(x, y)$ as

$$(22) \quad \int 2^{nd+md} (\mathcal{F}_{\xi\eta}^{-1} G_{nm})(2^n(x-w), 2^m(S(w) - S(y)), w) dw$$

where $\mathcal{F}_{\xi\eta}^{-1}$ is the inverse Fourier transform with respect to the variables ξ and η . From (21), there exists a constant $C_{\alpha\beta}$ for any multi-indices α and β such that

$$|\partial_\xi^\alpha \partial_\eta^\beta G_{nm}|_{L^\infty} \leq C_{\alpha\beta} 2^{-(r-1)\max\{n, m\}}.$$

Therefore there exists a constant C such that

$$|\mathcal{F}_{\xi\eta}^{-1} G_{nm}(x, y, w)| \leq C 2^{-(r-1)\max\{n, m\}} (1 + |x|^2)^{-2} (1 + |y|^2)^{-2}.$$

This, (22) and Young's inequality give the required estimate on $H_{n, \sigma}^{m, \tau}$. $\square$

Proof of lemma 2.3. We recall the argument in the proof of Lemma 2.4 above, setting $S = id$. Notice the the assumptions of Lemma 2.4 holds since we are assuming $\Theta' < \Theta$. Put $v_i = g_i \cdot u$. Then we have

$$(23) \quad \sum_i (\|v_i\|_{\Theta'}^-)^2 \leq \sum_i \|v_i\|_{L^2}^2 \leq \|u\|_{L^2}^2 \leq \|u\|_{\Theta}^2.$$

In the proof of Lemma 2.4, we have proved that

$$\sum_{n \geq 0} 2^{2n} \left\| \sum_{(m, \tau) \not\rightarrow (n, +)} \psi_{\Theta', n, +}(D)(g_i u_{\Theta, m, \tau}) \right\|_{L^2}^2 \leq C |u|_{\Theta}$$

Since we can put $\mu = 6$ in our specific setting $S = id$, we have

$$\begin{aligned} & \sum_i \sum_{n \geq 0} 2^{2n} \left\| \sum_{(m, \tau) \hookrightarrow (n, +)} \psi_{\Theta', n, +}(D)(g_i u_{\Theta, m, \tau}) \right\|_{L^2}^2 \\ & \leq 13 \cdot \sum_i \sum_{n \geq 0} \sum_{|m-n| \leq 6} 2^{2n} \|\psi_{\Theta', n, +}(D)(g_i u_{\Theta, m, +})\|_{L^2}^2 \\ & \leq 13 \cdot 2^{24} \cdot \sum_{m \geq 0} \sum_i 2^{2m} \|g_i u_{\Theta, m, +}\|_{L^2}^2 \leq C_0 \|u\|_{\Theta}^2. \end{aligned}$$

Thus we obtain $\sum_i (\|v_i\|_{\Theta'}^+)^2 \leq C_0 \|u\|_{\Theta}^2 + C |u|_{\Theta}$. This and (23) give the lemma. $\square$

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