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EQUIVALENCE BETWEEN THE BOUNDARY HARNACK PRINCIPLE AND THE CARLESON ESTIMATE

HIROAKI AIKAWA

ABSTRACT. Both the boundary Harnack principle and the Carleson estimate describe the boundary behavior of positive harmonic functions vanishing on a portion of the boundary. These notions are inextricably related and have been obtained simultaneously for domains with specific geometrical conditions. The main aim of this paper is to show that the boundary Harnack principle and the Carleson estimate are equivalent for arbitrary domains.

1. INTRODUCTION

The purpose of this note is to investigate the relationship between the boundary Harnack principle and the Carleson estimate. Roughly speaking, the boundary Harnack principle is a principle asserting that two positive harmonic functions vanishing on a portion of the boundary decay at the same speed toward a smaller portion, while the Carleson estimate is an estimate asserting that a positive harmonic function vanishing on a portion of the boundary is bounded up to a smaller portion by the value at a fixed point in the domain with a multiplicative constant independent of the function. These notions have many variants inextricably related from the very beginning. In fact, when Kemper [15] formulated the notions for the first time, he referred to the global Carleson estimate (Definition 2 below) and the global boundary Harnack principle (Definition 1 below) as the boundary Harnack principle and Property III, respectively.

For a Lipschitz domain, Kemper observed that the global Carleson estimate follows from the global boundary Harnack principle and tried to verify the global boundary Harnack principle, though his argument had a gap ([15, page 253]). After Kemper's pioneering work, the global boundary Harnack principle was legitimately proved for a Lipschitz domain by Ancona [5], Dahlberg [12] and Wu [18] independently. Since then the terminology, "the boundary Harnack principle", has been mainly used for the global boundary Harnack principle in this note. Caffarelli-Fabes-Mortola-Salsa [11], Jerison-Kenig [14] and Bass-Burdzy-Bañuelos [9, 8] gave significant extensions. The boundary Harnack principle and the Carleson estimate have been obtained for domains with specific geometrical conditions. As far as we know, they were proved simultaneously. This is not a coincidence. The main aim of this note is to show that the global boundary Harnack principle and the global Carleson estimate are equivalent for *arbitrary domains*.

To this end, the precise formulations of the boundary Harnack principle and the Carleson estimate are crucial. Not only the dependencies of constants but also the domains for harmonic functions are very important. We shall need to distinguish the global notions and the local notions. Throughout the paper we let D be a bounded domain in $\mathbb{R}^d$ with $d \geq 2$ and let $\delta_D(x) = \operatorname{dist}(x, \partial D)$. We write $B(x, r)$ and $S(x, r)$ for the open ball and the sphere of center at x and

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radius r, respectively. By the symbol A we denote an absolute positive constant whose value is unimportant and may change from one occurrence and the next. If necessary, we use $A_0, A_1, \dots$, to specify them.

Let us begin with the definitions of the global boundary Harnack principle and the global Carleson estimate. We consider a pair (V, K) of a bounded open set $V \subset \mathbb{R}^d$ and a compact set $K \subset \mathbb{R}^d$ such that

$$K \subset V,\ K \cap D \neq \emptyset \text{ and } K \cap \partial D \neq \emptyset. \tag{1}$$

Definition 1. We say that a domain D enjoys the *global boundary Harnack principle* if for each pair (V, K) with (1), there exists a constant A_1 depending only on D, V and K with the following property: If u and v are positive superharmonic functions on D such that

(i) u and v are bounded, positive and harmonic in $V \cap D$,
(ii) u and v vanish on $V \cap \partial D$ except for a polar set,

then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A_1 \quad \text{for } x, y \in K \cap D. \tag{2}$$

Definition 2. We say that a domain D enjoys the *global Carleson estimate* if for each pair (V, K) with (1) and a point $x_0 \in K \cap D$, there exists a constant A_2 depending only on D, V, K and x_0 with the following property: If u is a positive superharmonic function on D such that

(i) u is bounded, positive and harmonic in $V \cap D$,
(ii) u vanishes on $V \cap \partial D$ except for a polar set,

then

$$u(x) \le A_2 u(x_0) \quad \text{for } x \in K \cap D. \tag{3}$$

Remark 1. Since $K \cap D$ may be disconnected, the superharmonicity of u and v over the whole D is needed. Jerison-Kenig [14] and Bass-Burdzy-Bañuelos [9, 8] assume that u and v are positive and harmonic over the whole D for their global boundary Harnack principle. Our boundary Harnack principle is slightly stronger.

The main result of this note is the following theorem.

Theorem 1. *The global boundary Harnack principle and the global Carleson estimate are equivalent.*

Let us give an application of Theorem 1. Let $D \subset \mathbb{R}^d$ be a bounded domain. We define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds(z)}{\delta_D(z)},$$

where the infimum is taken over all rectifiable curves γ connecting x to y in D and $ds(z)$ stands for the line element on γ. We say that D satisfies a quasihyperbolic boundary condition if

$$k_D(x, x_0) \le A \log \frac{\delta_D(x_0)}{\delta_D(x)} + A' \quad \text{for all } x \in D \tag{4}$$

with some positive constants A and A'. A domain satisfying the quasihyperbolic boundary condition is called a Hölder domain by Smith-Stegenga [16, 17]. Bañuelos [7] said that such a domain is a Hölder domain of order 0. As a corollary to Theorem 1 we prove that the global

boundary Harnack principle holds for a domain satisfying the quasihyperbolic boundary condition. This provides a new class of domains satisfying the global boundary Harnack principle. See Remark 6 in the next section for further remarks.

Corollary 1. *The global Carleson estimate holds for a domain satisfying the quasihyperbolic boundary condition. Consequently, the global boundary Harnack principle holds.*

In [1] we have established the *local boundary Harnack principle* or the *scale invariant boundary Harnack principle* for a uniform domain. Actually, the local boundary Harnack principle characterizes a uniform domain ([2]). Let us study the relationship between the local boundary Harnack principle and the local Carleson estimate.

Definition 3. We say that a domain D enjoys the *local boundary Harnack principle* if there exist constants $A_3,\ A_4 > 1$ and $r_0 > 0$ depending only on D with the following property: If $\xi \in \partial D$, $0 < r < r_0$ and

(i) u and v are bounded, positive and harmonic in $B(\xi, A_3 r) \cap D$,
(ii) u and v vanish on $B(\xi, A_3 r) \cap \partial D$ except for a polar set,

then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A_4 \quad \text{for } x, y \in B(\xi, r) \cap D. \tag{5}$$

Definition 4. We say that a domain D enjoys the *local Carleson estimate* if there is a constant $A_5 > 1$ with the following property: If $\xi \in \partial D$, $0 < r < r_0$ and

(i) u is bounded, positive and harmonic in $B(\xi, A_5 r) \cap D$,
(ii) u vanishes on $B(\xi, A_5 r) \cap \partial D$ except for a polar set,

then

$$u(x) \le A_\varepsilon u(y) \quad \text{for } x \in B(\xi, r) \cap D, \tag{6}$$

whenever $y \in B(\xi, r) \cap D$ and $\delta_D(y) \ge \varepsilon r$ with $0 < \varepsilon < 1$. Here A_ε depends only on D and ε.

Remark 2. The local boundary Harnack principle implies the following connectivity of D near the boundary: If $\xi \in \partial D$ and $0 < r < r_0$, then $B(\xi, r) \cap D$ is included in one connected component of $B(\xi, A_3 r) \cap D$, though $B(\xi, r) \cap D$ itself may be disconnected. Similarly, the local Carleson estimate implies that if $\xi \in \partial D$ and $0 < r < r_0$, then $B(\xi, r) \cap D$ is included in one connected component of $B(\xi, A_5 r) \cap D$.

The existence of a point $y \in B(\xi, r) \cap D$ with $\delta_D(y) \ge \varepsilon r$ is crucial, as the statement of the local Carleson estimate would be vacuous if there were no such points. The existence is guaranteed by the *corkscrew condition*: There exists $\varepsilon > 0$ such that

$$B(\xi, r) \cap D \text{ includes a ball of radius } \varepsilon r, \text{ whenever } \xi \in \partial D \text{ and } 0 < r < r_0. \tag{7}$$

See [14, p. 93]. John domains satisfy the corkscrew condition; the converse is not necessarily true. We shall prove the equivalence between the local boundary Harnack principle and the local Carleson estimate, provided the domain D satisfies the corkscrew condition.

Theorem 2. *Assume that D satisfies the corkscrew condition. Then the local boundary Harnack principle and the local Carleson estimate are equivalent.*

This theorem immediately gives another characterization of a uniform domain. See [2].

Definition 5. By Cap we denote the logarithmic capacity if $d = 2$, and the Newtonian capacity if $d \geq 3$. We say that the capacity density condition holds if there exist constants $A > 1$ and $r_0 > 0$ such that

$$\mathrm{Cap}(B(\xi, r) \setminus D) \geq \begin{cases} A^{-1} r & \text{if } d = 2, \\ A^{-1} r^{d-2} & \text{if } d \geq 3, \end{cases}$$

whenever $\xi \in \partial D$ and $0 < r < r_0$.

Corollary 2. *Let D be a John domain satisfying the capacity density condition. Then the following are equivalent:*

(i) *D is a uniform domain.*
(ii) *D enjoys the local boundary Harnack principle.*
(iii) *D enjoys the local Carleson estimate.*

Remark 3. The Carleson estimate can be proved rather easily by the Domar argument (see Proof of Corollary 1 below). The Domar argument extends to even p-harmonic functions in a metric measure space and the Carleson estimate holds in this general settings ([4]). Our equivalence between the boundary Harnack principle and the Carleson estimate relies on the relationship between the Green function and the harmonic measure (see Lemma 1 below). So, if the counterpart for p-harmonic functions is established, we might be able to obtain the boundary Harnack principle for p-harmonic functions. The relationship between the p-Green function and the p-harmonic measure is an open problem.

2. Proof of Theorem 1 and Corollary 1

Let Ω be an open set. We write $\omega(E, \Omega)$ for the harmonic measure over the open set Ω of $E \subset \partial\Omega$, i.e., $\omega(E, \Omega)$ is the Dirichlet solution in Ω of the boundary function χ_E. The value of $\omega(E, \Omega)$ at $x \in \Omega$ is denoted by $\omega^x(E; \Omega)$. Let G_Ω be the Green function for Ω, i.e., for each fixed $y \in \Omega$, the minus of the distributional Laplacian $-\Delta G_\Omega(\cdot, y)$ is the point measure at y and $G_\Omega(\cdot, y)$ vanishes on $\partial\Omega$ except for a polar set. The harmonic measure $\omega(E, \Omega)$ and the Green function G_Ω are related as follows.

Lemma 1. *Let $x \in \Omega$. If $\varphi \in C_0^\infty(\mathbb{R}^d)$, then*

$$\int_{\partial\Omega} \varphi(y) \omega^x(dy; \Omega) = \varphi(x) + \int_\Omega G_\Omega(x, y) \Delta\varphi(y) dy.$$

Proof. We give a proof for the reader's convenience, though the proof is not so difficult (see e.g. Armitage-Gardiner [6, p. 264] and Jerison-Kenig [14, (4.6)]). For simplicity we treat only the case $d \geq 3$. Let $a_d = \sigma_d(d-2)$ with σ_d being the surface area of a unit ball in $\mathbb{R}^d$. Then

$$G_\Omega(x, z) = \begin{cases} \dfrac{1}{a_d} \left\{ |x - z|^{2-d} - \displaystyle\int_{\partial\Omega} |y - z|^{2-d} \omega^x(dy; \Omega) \right\} & \text{for } z \in \Omega, \\ 0 & \text{for q.e. } z \in \mathbb{R}^d \setminus \Omega, \end{cases}$$

where "q.e." stands for "quasieverywhere" and means that "outside a polar set". Since $\Delta_z(|x-z|^{2-d}) = -a_d\delta_x$ in the distribution sense, it follows from Fubini's theorem that

$$\begin{aligned}\int_\Omega G_\Omega(x,z)\Delta\varphi(z)dz &= \int_{\mathbb{R}^d}\frac{1}{a_d}\left\{|x-z|^{2-d} - \int_{\partial\Omega}|y-z|^{2-d}\omega^x(dy;\Omega)\right\}\Delta\varphi(z)dz\\ &= -\varphi(x) - \int_{\partial\Omega}\omega^x(dy;\Omega)\int_{\mathbb{R}^d}\frac{1}{a_d}|y-z|^{2-d}\Delta\varphi(z)dz\\ &= -\varphi(x) + \int_{\partial\Omega}\varphi(y)\omega^x(dy;\Omega).\end{aligned}$$

□

Remark 4. The identity of the lemma may be written as

$$\omega^x(dy;\Omega) = \delta_x + \Delta_y G_\Omega(x,y)$$

in the distribution sense. Such an identity for a general elliptic operator was used by Ancona [5] to show the boundary Harnack principle in a Lipschitz domain. His idea goes back to Brelot's remark [10]. Ancona applied his identity to the *top of a cylinder*. Since we have no geometric assumptions, we shall use the identity in a different fashion. See Remark 5 after the proof of Theorem 1. The author thanks Alano Ancona for informing him of the background of the identity.

Proof of Theorem 1. First suppose that D satisfies the global boundary Harnack principle. Take a pair (V,K) with (1) and a point $x_0 \in K\cap D$. Let u be a positive superharmonic function in D satisfying (i) and (ii) of Definition 2. Observe that the lower regularization of

$$v = \begin{cases}\omega(\partial V\cap D, V\cap D) & \text{on } V\cap D,\\ 1 & \text{on } D\setminus V\end{cases}$$

is a positive bounded superharmonic function on D satisfying (i) and (ii) of Definition 2. The regularization and v differ only on a polar set of $D\cap\partial V$. The global boundary Harnack principle (2) for u and v gives

$$\frac{u(x)/u(x_0)}{v(x)/v(x_0)} \le A_1 \quad \text{for } x\in K\cap D$$

with A_1 depending only on V and K. Since $0\le v(x)\le 1$, it follows that

$$u(x) \le \frac{A_1}{v(x_0)}u(x_0) \quad \text{for } x\in K\cap D.$$

Thus (3) holds with $A_2 = A_1/v(x_0)$, which depends only on D, V, K and x_0.

Conversely, suppose that D satisfies the global Carleson estimate. Take a pair (V,K) with (1). Find open sets U_0, U_1 and U_2 such that

$$K\subset U_0\subset\overline{U}_0\subset U_1\subset\overline{U}_1\subset U_2\subset\overline{U}_2\subset V$$

and such that the pairs $(U_2\setminus\overline{U}_0, \partial U_1)$ and $(V,\overline{U}_2)$ satisfy (1). For simplicity let $E_j = \partial U_j\cap D$ for $j=1,2$. See Figure 1. Let us apply Lemma 1 to $\Omega = U_2\cap D$. Take $\varphi\in C_0^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on E_2 and $\operatorname{supp}\varphi\cap\overline{U}_1 = \emptyset$. Then Lemma 1 gives

$$\omega^x(E_2;\Omega) \le \int_{\Omega\cap\operatorname{supp}\varphi} G_\Omega(x,y)|\Delta\varphi(y)|dy \le \int_{\Omega\cap\operatorname{supp}\varphi} G_D(x,y)|\Delta\varphi(y)|dy \quad \text{for } x\in K\cap D. \tag{8}$$

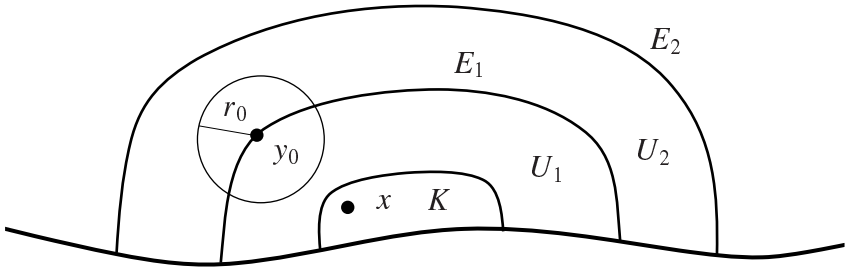

FIGURE 1. Global boundary Harnack principle and Carleson estimate.

For a moment fix $x \in K \cap D$. By the global Carleson estimate for the pair $(U_2 \setminus \overline{U}_0, \partial U_1)$ and the positive superharmonic function $G_D(x,\cdot)$ in D, which is bounded and harmonic on $U_2 \setminus \overline{U}_0$, we have

$$G_D(x,y) \le A_2 G_D(x,y_0) \quad \text{for } y \in E_1,$$

where $y_0 \in E_1$ is fixed and A_2 depends only on D, U_0, U_1, U_2 and y_0. The maximum principle gives $G_D(x,y) \le A_2 G_D(x,y_0)$ for $y \in D \setminus U_1$, so that (8) yields

$$\omega^x(E_2;\Omega) \le A_6 G_D(x,y_0) \quad \text{for } x \in K \cap D \tag{9}$$

with $A_6 = A_2 \int_{\Omega \cap \operatorname{supp}\varphi} |\Delta\varphi(y)|dy$.

Let u and v be positive superharmonic functions in D satisfying (i) and (ii) of Definition 1. Then the Carleson estimate for the pair $(V, \overline{U}_2)$ gives

$$u(x) \le A_2' u(y_0) \quad \text{for } x \in \overline{U}_2 \cap D \supset E_2,$$

where A_2' depends only on D, U_2, V and y_0. Hence the maximum principle and (9) imply

$$u(x) \le A_2' u(y_0)\omega^x(E_2;\Omega) \le A_2' u(y_0) A_6 G_D(x,y_0) \quad \text{for } x \in K \cap D.$$

Let $0 < r_0 \le \delta_D(y_0)/2$ be such that $B(y_0,r_0) \cap K = \emptyset$. Then $G_D(x,y_0) \le A r_0^{2-d} \le A v(x)/v(y_0)$ for $x \in S(y_0,r_0)$ with A independent of v. By the maximum principle we have $G_D(x,y_0) \le A v(x)/v(y_0)$ for $x \in D \setminus B(y_0,r_0) \supset K \cap D$. Hence we obtain

$$u(x) \le A_7 u(y_0) v(x)/v(y_0) \quad \text{for } x \in K \cap D$$

with A_7 independent of u and v. This gives (2) with $A_1 = A_7^2$. □

Remark 5. The intersection of the boundary ∂D and a ball with center on the boundary is called a *surface ball*. The estimate of the harmonic measure of a surface ball plays a very important role if D is a Lipschitz domain. In fact, Dahlberg [12] recognized the following relationship between the Green function and the harmonic measure:

$$\omega^x(B(\xi,r) \cap \partial D; D) \approx r^{d-2} G_D(A_r(\xi), x) \quad \text{for } \xi \in \partial D,\ x \in D \setminus B(\xi, Ar) \text{ and } 0 < r < r_0, \tag{10}$$

where $A > 1$ depends on the Lipschitz nature of D and $A_r(\xi)$ is a "nontangential point", i.e., $|A_r(\xi) - \xi| \approx \delta_D(A_r(\xi)) \approx r$. Caffarelli-Fabes-Mortola-Salsa [11, Lemma 2.2] extended this comparison to the Green function and the harmonic measure for general elliptic operators in divergence form. They employed the counterpart of Lemma 1. The same technique was used by Jerison-Kenig [14, (4.6)] to prove (10) for NTA domains.

The comparison (10) holds only for domains with *the capacity density condition*, whereas the boundary Harnack principle holds even for irregular domains for which a surface ball may be a polar set ([9, 8] and [1]). The point of the above proof is the usage of Lemma 1. We estimate

the harmonic measure $\omega(\partial U_2 \cap D, U_2 \cap D)$ instead of the harmonic measure of the surface ball $B(\xi, r) \cap \partial D$, which may vanish for a general domain D.

We shall prove Corollary 1 with the aid of Domar's argument and the exponential integrability of the quasihyperbolic metric of a domain satisfying the quasihyperbolic boundary condition. For $u \geq 0$ we write

$$\log^+ u = \begin{cases} \log u & \text{if } u \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma A (Domar [13] and [3]). *Let u be a nonnegative subharmonic function on a bounded domain Ω in $\mathbb{R}^d$. Suppose there is $\varepsilon > 0$ such that*

$$I = \int_\Omega (\log^+ u)^{d-1+\varepsilon} dx < \infty.$$

Then

$$u(x) \leq \exp(2 + AI^{1/\varepsilon} \delta_\Omega(x)^{-d/\varepsilon}), \tag{11}$$

where A is a positive constant depending only on ε and the dimension d.

Lemma B (Smith-Stegenga [17]). *Let D satisfy the quasihyperbolic boundary condition and let $x_0 \in D$. Then there exists a positive number τ such that*

$$\int_D \exp(\tau k_D(x, x_0)) dx < \infty.$$

Proof of Corollary 1. In view of Theorem 1, it is sufficient to establish the global Carleson estimate. Let (V, K) be a pair satisfying (1) and let $x_0 \in K \cap D$. Let u be a positive superharmonic function in D satisfying (i) and (ii) of Definition 2.

First suppose u is harmonic on D. We extend u to $V \setminus D$ by 0 and take the upper regularization to obtain a nonnegative subharmonic function on $V \cup D$ coinciding with u in D. With a slight abuse of the notation, we denote by u this extended subharmonic function as well. By the Harnack inequality we obtain

$$\frac{u(x)}{u(x_0)} \leq A \exp(Ak_D(x, x_0)) \quad \text{for } x \in D.$$

Let $\varepsilon > 0$. Lemma B yields

$$I := \int_{V \cup D} \left[\log^+ \frac{u(x)}{u(x_0)} \right]^{d-1+\varepsilon} dx \leq A \int_D [k_D(x, x_0)]^{d-1+\varepsilon} dx \leq A \int_D \exp(\tau k_D(x, x_0)) dx < \infty,$$

where A depends only on d, ε and τ. By Lemma A

$$\frac{u(x)}{u(x_0)} \leq \exp(2 + AI^{1/\varepsilon} \delta_{V \cup D}(x)^{-d/\varepsilon}) \quad \text{for } x \in V \cup D,$$

where A is independent of u. Since $\delta_{V \cup D}(x) \geq \operatorname{dist}(K, V^c)$ for $x \in K$, we obtain (3).

Next consider the general case. By the Riesz decomposition we can write

$$u(x) = \int_{D \setminus V} G_D(x, y) d\mu(y) + h(x) \quad \text{for } x \in D$$

where μ is a measure on $D \setminus V$ and h is a positive harmonic function on D. Let U be an open set such that $K \subset U \subset \overline{U} \subset V$. By the Harnack inequality we obtain

$$\frac{G_D(x, y)}{G_D(x_0, y)} \leq A \exp(Ak_{D \setminus \{y\}}(x, x_0)) \quad \text{for } x \in D.$$

Let $y \in D \setminus V$. Then [3, Lemma 7.2] gives

$$k_{D\setminus\{y\}}(x, x_0) \le Ak_D(x, x_0) + A \quad \text{for } x \in U \cap D,$$

where A is independent of y. In the same way as above, we obtain from Lemma B

$$\int_{U\cup D}\left[\log^+ \frac{G_D(x,y)}{G_D(x_0,y)}\right]^{d-1+\varepsilon} dx \le A\int_D [k_D(x,x_0)]^{d-1+\varepsilon}dx \le A\int_D \exp(\tau k_D(x,x_0))dx < \infty,$$

so that

$$G_D(x, y) \le AG_D(x_0, y) \quad \text{for } x \in U \cap D,$$

where A is independent of $y \in D \setminus V$. Integrating both sides of the inequality, we obtain

$$\int_{D\setminus V} G_D(x,y)d\mu(y) \le A\int_{D\setminus V} G_D(x_0,y)d\mu(y) \quad \text{for } x \in U \cap D.$$

By the first part $h(x) \le Ah(x_0)$ for $x \in K\cap D$. We obtain (3) by adding the above two inequalities. Thus the global Carleson estimate holds, and hence the global boundary Harnack principle follows from Theorem 1. □

Remark 6. We say that D is a Hölder domain of order α ($0 < \alpha \le 1$) if the boundary is locally represented as the graph of a α-Hölder continuous function. Bass-Burdzy-Bañuelos [9, 8] established the global boundary Harnack principle for a Hölder domain of order α ($0 < \alpha \le 1$). As a further generalization, Bass-Burdzy [9] gave the global boundary Harnack principle for a twisted Hölder domain of order α ($1/2 < \alpha \le 1$), one of which conditions is the capacity density condition. Their crucial estimate is obtained by the so-called *box argument*. Our proof of Corollary 1 is completely different. It is based on Theorem 1 and the Domar argument ([3, 4]), which requires neither the graph representation nor the capacity density condition. See also [1, 2].

3. Proof of Theorem 2

In this section we shall prove the equivalence between the local boundary Harnack principle and the local Carleson estimate. To this end we observe that the local boundary Harnack principle has the following extended form ([2, Lemma 4.3]). For the convenience sake of the reader we provide a proof.

Lemma 2. *Let the local boundary Harnack principle hold. Then for each $0 < \varepsilon < 1/2$ there is a positive constant A_ε depending only on ε, A_3, A_4 and d with the following property: Let $\xi \in \partial D$ and $0 < r < r_0$. Suppose that there is $x^* \in B(\xi, r) \cap D$ with $\delta_D(x^*) \ge 2\varepsilon r$ and let U be the connected component of $B(\xi, r) \cap D$ containing x^*. If u and v are positive harmonic functions on $B(\xi, 2r) \cap D \setminus B(x^*, \varepsilon r/2)$ vanishing on $\partial D \cap B(\xi, 2r)$ and bounded apart from x^*, then*

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A_\varepsilon \tag{12}$$

whenever $x, y \in U \setminus B(x^*, \varepsilon r)$. See Figure 2.

Proof. We claim that if $\zeta \in \overline{U \setminus B(x^*, \varepsilon r)}$, then (12) holds for $x, y \in B(\zeta, (6A_3)^{-1}\varepsilon r) \cap D$. If $B(\zeta, (3A_3)^{-1}\varepsilon r) \subset D$, then this follows from the usual Harnack inequality. Otherwise, we find $\zeta' \in \partial D$ such that $|\zeta' - \zeta| < (3A_3)^{-1}\varepsilon r$. Observe that

$$B(\zeta', \frac{\varepsilon}{2}r) \subset B(\xi, (1 + \frac{\varepsilon}{3A_3} + \frac{\varepsilon}{2})r) \subset B(\xi, 2r)$$

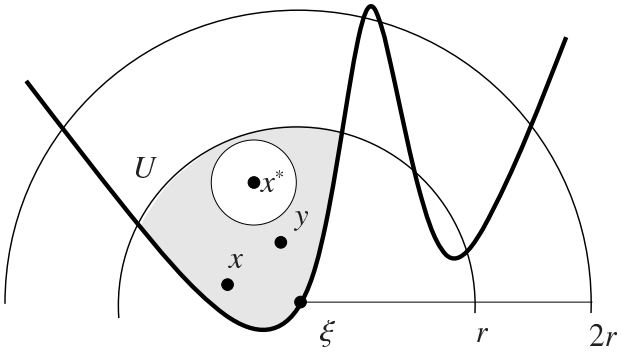

FIGURE 2. Extended boundary Harnack principle.

and

$$|\zeta' - x^*| \geq \varepsilon r - \frac{\varepsilon}{3A_3} r > \frac{\varepsilon}{2} r.$$

Hence, u and v are positive harmonic functions on $B(\zeta', \varepsilon r/2) \cap D$ vanishing on $\partial D \cap B(\zeta', \varepsilon r/2)$, so that the local boundary Harnack principle implies (12) for $x, y \in B(\zeta', (2A_3)^{-1}\varepsilon r) \cap D$, which includes $B(\zeta, (6A_3)^{-1}\varepsilon r) \cap D$. Thus the claim follows. Now we find points $\zeta_1, \ldots, \zeta_N \in \overline{U \setminus B(x^*, \varepsilon r)}$ such that $\cup_{j=1}^N B(\zeta_j, (7A_3)^{-1}\varepsilon r)$ is connected and covers $U \setminus B(x^*, \varepsilon r)$, where N depends only on ε, A_3 and d. Applying the claim to ζ_j repeatedly, we obtain (12) for $x, y \in U \setminus B(x^*, \varepsilon r)$. The lemma is proved. □

Proof of Theorem 2. Assume that D satisfies the corkscrew condition. First suppose that D satisfies the local boundary Harnack principle. Let $\xi \in \partial D$ and $0 < r < r_0/A_3$. In view of Remark 2 and the scaling that $B(\xi, r) \cap D$ is contained in a connected component U of $B(\xi, A_3 r) \cap D$. Suppose u is a bounded positive harmonic function in $B(\xi, 2A_3 r) \cap D$ vanishing on $\partial D \cap B(\xi, 2A_3 r)$. Let $y \in B(\xi, r) \cap D$ with $\delta_D(y) \geq \varepsilon r$. Apply Lemma 2 to u and $v = G_{B(\xi, 2A_3 r) \cap D}(\cdot, y)$ with $x^* = y$, $A_3 r$ in place of r, and $\varepsilon/(2A_3)$ in place of ε. Since $v(y') \approx r^{2-d}$ for $y' \in S(y, \varepsilon r/2)$ and $v(x) \leq A r^{2-d}$ for $x \in B(\xi, 2A_3 r) \cap D \setminus B(y, \varepsilon r/2)$, it follows from (12) that

$$\frac{u(x)}{u(y')} \leq A_\varepsilon \frac{v(x)}{v(y')} \leq A_\varepsilon \quad \text{for } x \in U \cap \setminus B(y, \varepsilon r/2) \supset B(\xi, r) \cap D \setminus B(y, \varepsilon r/2).$$

Hence the Harnack inequality implies that

$$u(x) \leq A_\varepsilon u(y) \quad \text{for } x \in B(\xi, r) \cap D.$$

Thus the local Carleson estimate holds with $A_5 = 2A_3$.

Conversely, suppose that D satisfies the local Carleson estimate. Without loss of generality, we may assume that $A_5 \geq 2$. Let $\xi \in \partial D$ and $0 < r < r_0$. Let $\Omega = B(\xi, A_5^2 r) \cap D$, $\Omega^* = B(\xi, A_5^4 r) \cap D$ and $E = \partial\Omega \cap D$. We find $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ on $S(\xi, A_5^2 r)$, $\operatorname{supp}\varphi \subset B(\xi, (A_5^2 + 1)r) \setminus B(\xi, (A_5^2 - 1)r)$ and $|\Delta\varphi| \leq A r^{-2}$, where A depends only on the dimension d. Lemma 1 yields

$$\omega^x(E; \Omega) \leq \int_{\Omega \cap \operatorname{supp}\varphi} G_\Omega(x, y) |\Delta\varphi(y)| dy \leq A r^{-2} \int_{\Omega \cap \operatorname{supp}\varphi} G_{\Omega^*}(x, y) dy \quad \text{for } x \in B(\xi, r) \cap D. \tag{13}$$

For a moment fix $x \in B(\xi, r) \cap D$ and apply the local Carleson estimate to the harmonic function $G_{\Omega^*}(x, \cdot)$ on $\Omega^* \setminus B(\xi, r)$. In view of (7) and the Harnack inequality, we find $y_1, \ldots, y_N \in B(\xi, (A_5^2 +$

$1)r) \setminus B(\xi, (A_5^2 - 1)r) \subset B(\xi, A_5^3 r)$ such that $\delta_D(y_j) \geq \varepsilon r$ and

$$G_{\Omega^*}(x, y) \leq A \max_{j=1,\dots,N} G_{\Omega^*}(x, y_j) \quad \text{for } y \in \operatorname{supp} \varphi \cap D,$$

where N depends only on ε and the dimension d. See Figure 3.

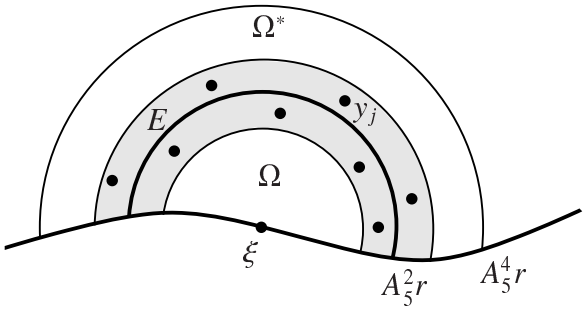

FIGURE 3. $G_{\Omega^*}(x, y) \leq A \max_{j=1,\dots,N} G_{\Omega^*}(x, y_j)$ for $y \in \operatorname{supp} \varphi \cap D$.

This, together with (13), gives

$$\omega^x(E; \Omega) \leq A r^{d-2} \max_{j=1,\dots,N} G_{\Omega^*}(x, y_j) \quad \text{for } x \in B(\xi, r) \cap D. \tag{14}$$

Now suppose that u and v are bounded harmonic functions in $B(\xi, A_5^4 r) \cap D$ vanishing on $B(\xi, A_5^4 r) \cap \partial D$ except for a polar set. Apply the Carleson estimate to u to obtain

$$u(x) \leq A_\varepsilon \min_{j=1,\dots,N} u(y_j) \quad \text{for } x \in B(\xi, A_5^3 r) \cap D.$$

The maximum principle gives

$$u(x) \leq A_\varepsilon \min_{j=1,\dots,N} u(y_j) \omega^x(E; \Omega) \quad \text{for } x \in B(\xi, A_5^2 r) \cap D.$$

Hence (14) implies

$$u(x) \leq A_\varepsilon r^{d-2} \min_{j=1,\dots,N} u(y_j) \cdot \max_{j=1,\dots,N} G_{\Omega^*}(x, y_j) \quad \text{for } x \in B(\xi, r) \cap D. \tag{15}$$

On the other hand

$$G_{\Omega^*}(\cdot, y_j) \approx \delta_D(y_j)^{2-d} \approx r^{2-d} \quad \text{on } S(y_j, \delta_D(y_j)/2),$$

where the constant of comparison depends only on the dimension d and ε. Sine $v/v(y_j) \approx 1$ on $S(y_j, \delta_D(y_j)/2)$ by the Harnack inequality, it follows that $G_{\Omega^*}(\cdot, y_j) \approx r^{2-d} v(y_j)^{-1} v$ on $S(y_j, \delta_D(y_j)/2)$, and hence

$$G_{\Omega^*}(\cdot, y_j) \leq A r^{2-d} \frac{v}{v(y_j)} \quad \text{on } \Omega^* \setminus B(y_j, \delta_D(y_j)/2)$$

by the maximum principle. Since $B(\xi, r) \cap D \subset \Omega^* \setminus B(y_j, \delta_D(y_j)/2)$, this, together with (15), gives

$$u(x) \leq A \min_{j=1,\dots,N} u(y_j) \cdot \max_{j=1,\dots,N} \frac{v(x)}{v(y_j)} \quad \text{for } x \in B(\xi, r) \cap D.$$

In other words,

$$\frac{u(x)}{v(x)} \leq A \frac{\min_{j=1,\dots,N} u(y_j)}{\min_{j=1,\dots,N} v(y_j)}.$$

Replacing the roles of u and v and changing x and y, we obtain

$$\frac{u(x)}{v(x)} \cdot \frac{v(y)}{u(y)} \le A^2 \frac{\min_{j=1,\dots,N} u(y_j)}{\min_{j=1,\dots,N} v(y_j)} \cdot \frac{\min_{j=1,\dots,N} v(y_j)}{\min_{j=1,\dots,N} u(y_j)} = A^2 \quad \text{for } x \in B(\xi, r) \cap D.$$

Thus the local boundary Harnack principle holds.

□

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Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

E-mail address: aik@math.sci.hokudai.ac.jp