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NONLINEAR SCHRÖDINGER EQUATION WITH A POINT DEFECT

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ABSTRACT. We study nonlinear Schrödinger equation with a delta-function impurity in one space dimension. Global well-posedness is proved for the Cauchy problem in $L^2(\mathbb{R})$ under subcritical nonlinearity, as well as under critical nonlinearity with smallness assumption on the data. In the attractive case, orbital stability and instability of the ground state is proved in $H^1(\mathbb{R})$.

1. INTRODUCTION

In this paper we study nonlinear Schrödinger equations of the form

$$i\partial_t u + \frac{1}{2}D^2 u + Z\delta u = f(u), \quad (1.1)$$

where u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}$, $\partial_t = \partial/\partial t$, $D = \partial/\partial x$, δ is the Dirac measure at the origin, $Z \in \mathbb{R}$, and f is a complex valued function of $z \in \mathbb{C}$. A typical example of f is a double power nonlinearity of the form

$$f(z) = \lambda_1 |z|^{p_1-1} z + \lambda_2 |z|^{p_2-1} z,$$

where $\lambda_j \in \mathbb{R}$ and $1 \leq p_1 \leq p_2 < \infty$. For $Z \neq 0$, the equations of the form (1.1) arise in a wide variety of physical models with a point defect on the line [15] and references therein. In spite of a large literature on (1.1) with $Z = 0$, there seems only one mathematical study in (1.1) with $Z \neq 0$ [15] available so far.

To be more specific, it was shown in [15] that the Cauchy problem is globally well-posed in $H^1(\mathbb{R}) = (1 - D^2)^{-1/2} L^2(\mathbb{R})$ in the case where $Z > 0$ and $f(u) = -|u|^2 u$. Conserved quantities are the energy E and the charge Q :

$$E(v) = \frac{1}{4} \|Dv\|_{L^2}^2 - \frac{Z}{2} \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx + \int_{\mathbb{R}} F(v) dx, \quad Q(v) = \frac{1}{2} \|v\|_{L^2}^2, \quad v \in H^1(\mathbb{R}),$$

where $F(z) = \int_0^{|z|} f(t) dt$. Moreover, the authors in [15] studied the stability of nonlinear bound states u_{Def} given by

$$u_{\text{Def}}(x) = \sqrt{2\omega} e^{i\omega t} \text{sech} \left(\sqrt{2\omega} |x| + \tanh^{-1} \frac{Z}{\sqrt{2\omega}} \right) \quad (1.2)$$

with $\sqrt{2\omega} > Z$ and $\omega > 0$ in the orbitally Lyapunov sense in H^1 in the case where $Z > 0$ and $f(u) = -|u|^2 u$.

The purpose in this paper is to generalize those results in various directions and to compare our results with the available results with $Z = 0$.

To state our results precisely, we introduce the following notation. We define $D^j \equiv \frac{d^j}{dx^j}$ for $j \in \mathbb{N} \cup \{0\}$. Let H be the self-adjoint operator in $L^2(\mathbb{R})$ associated with $-(1/2)D^2 - Z\delta$ [1]. Then the equation (1.1) is converted to the integral equation

$$u(t) = U(t)u_0 - i \int_0^t U(t-t')f(u(t'))dt', \quad (1.3)$$

where $U(t) = \exp(-itH)$ and u_0 is the prescribed Cauchy data at $t = 0$. We consider the following assumptions on f .

(H1) $f \in C(\mathbb{C}, \mathbb{C})$ and there exist p_1 and p_2 with $1 \leq p_1 \leq p_2 < \infty$ such that f satisfies the estimate

$$\begin{aligned} |f(z)| &\leq C(|z|^{p_1} + |z|^{p_2}), \\ |f(z) - f(z')| &\leq C(|z|^{p_1-1} + |z'|^{p_2-1})|z - z'| \end{aligned}$$

for all $z, z' \in \mathbb{C}$.

(H2) $\text{Im}(\bar{z}f(z)) = 0$ for all $z \in \mathbb{C}$.

Concerning the well-posedness of the equation (1.3) in $L^2(\mathbb{R})$, we prove:

Theorem 1. *Let f satisfy (H1) and (H2) with $1 \leq p_1 \leq p_2 < 5$. Then for any $u_0 \in L^2$, the equation (1.3) has a unique solution $u \in C(\mathbb{R}; L^2) \cap L_{\text{loc}}^4(\mathbb{R}; L^\infty)$. Moreover, u satisfies the conservation of charge:*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad (1.4)$$

for all $t \in \mathbb{R}$. For any $T > 0$ the map $u_0 \mapsto u$ is continuous from L^2 to $L^\infty(I; L^2) \cap L^4(I; L^\infty)$, where $I = [-T, T]$.

Theorem 2. *Let f satisfy (H1) and (H2) with $p_1 = p_2 = 5$. Then there exists $R > 0$ such that for any $u_0 \in B_R(0) = \{\phi \in L^2 : \|\phi\|_{L^2} \leq R\}$ the equation (1.3) has a unique solution $u \in (C \cap L^\infty)(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$. Moreover, u satisfies (1.4) and has a unique pair $u_\pm \in L^2$ such that*

$$\|u(t) - U(t)u_\pm\|_{L^2} \rightarrow 0 \quad (1.5)$$

as $t \rightarrow \pm\infty$. The map $u_0 \mapsto u$ is continuous from $B_R(0)$ to $L^\infty(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$.

Remark 1.1. Theorem 1 implies the global well-posedness of (1.1) with subcritical power at the level of L^2 and so does Theorem 2 with critical power under the smallness assumption above. Those results are reminiscent of the standard theory for NLS, namely, $Z = 0$, at the level of L^2 [5, 29] and references therein, where the notion of critical and subcritical powers for NLS comes from the associated scaling argument. We should remark that the

corresponding scaling argument breaks down for (1.1) with $Z \neq 0$, while the notion of critical and subcritical powers still remains.

Remark 1.2. We recall the definition of the self-adjoint operator H as the precise formulation of a formal expression $-(1/2)D^2 - Z\delta$.

$$Hu = -\frac{1}{2}D^2u, \quad u \in \text{Dom}(H),$$

where

$$\begin{aligned} \text{Dom}(H) &= \{u \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\}) : Du(0+) - Du(0-) = -2Zu(0)\}, \\ H^m(I) &= \{u \in L^2(I) : D^j u \in L^2 \text{ for all } j \text{ with } 0 \leq j \leq m\}, \quad I \subset \mathbb{R}. \end{aligned}$$

All self-adjoint extensions of $\dot{H} \equiv -(1/2)D^2$ with domain

$$\text{Dom}(\dot{H}) = \{u \in H^2(\mathbb{R}) : u(0) = 0\}$$

are parametrized by H with $Z \in [-\infty, +\infty)$ (see [1]).

Nonlinear bound states mean the solutions to (1.1) having the form $u_\omega(t, x) = e^{i\omega t}\phi_\omega(x)$, where $\omega > 0$ is the frequency and ϕ_ω should satisfy the following semilinear elliptic equations:

$$-\frac{1}{2}D^2\phi + \omega\phi - Z\delta\phi = f(\phi), \quad x \in \mathbb{R}, \quad Z \in \mathbb{R}. \quad (1.6)$$

In the case that $f(u) = -|u|^{p-1}u$ with $p > 1$, there exists a unique positive symmetric solution of (1.6) which is explicitly described as:

$$\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \text{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{\sqrt{2}}|x| + \tanh^{-1} \left(\frac{Z}{\sqrt{2\omega}} \right) \right) \right\}^{\frac{1}{p-1}} \quad (1.7)$$

if $\sqrt{2\omega} > |Z|$. Precisely, this solution is constructed from the solution with $Z = 0$ on each side of the defect pasted together at $x = 0$ to satisfy the conditions of continuity and symmetry at $x = 0$ and the jump condition in the first derivative at $x = 0$, $Du(0+) - Du(0-) = -2Zu(0)$.

In case of $Z = 0$ it is unique for $\omega > 0$ up to translations, which, we denote by $\psi_\omega(x)$. Orbital stability for the case of $Z = 0$ has been well studied (see [2, 5, 6, 8, 17, 18, 30, 31]). For the case where $f(u) = -|u|^{p-1}u$ with $p > 1$, Cazenave and Lions [6] proved that $e^{i\omega t}\psi_\omega(x)$ is stable for any $\omega > 0$ if $p < 5$. On the other hand, it was shown that $e^{i\omega t}\psi_\omega(x)$ is unstable for any $\omega > 0$ if $p \geq 5$ (see Berestycki and Cazenave [2] for $p > 5$, and Weinstein [30] for $p = 5$).

As we have mentioned, Goodman, Holmes and Weinstein [15] claimed in the case where $Z > 0$ and $f(u) = -|u|^2u$ that (1.2) are nonlinearly orbitally Lyapunov stable by the same method as that of Rose and Weinstein [25], Weinstein [30]. More exactly, the authors established a variational characterization for (1.2) and proved the stability using the bifurcation from the linear mode, i.e., the corresponding eigenfunction to the eigenvalue $\lambda = -Z^2/2$.

They also remark that as $\omega \rightarrow \infty$, (1.2) looks more and more like the solitary standing wave of (1.1) with $Z = 0$ and $\omega = 1$. However, we address in this article a different point from the case $Z = 0$.

The notion of the stability and instability in this paper is formulated as follows.

Definition 1. For $\eta > 0$, we put

$$U_\eta(\phi_\omega) := \left\{ v \in H^1(\mathbb{R}) : \inf_{\theta \in \mathbb{R}} \|v - e^{i\theta} \phi_\omega\|_{H^1} < \eta \right\}.$$

We say that a standing wave solution $e^{i\omega t} \phi_\omega(x)$ of (1.1) is stable in $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\eta > 0$ such that for any $u_0 \in U_\eta(\phi_\omega)$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in U_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{i\omega t} \phi_\omega(x)$ is said to be unstable in $H^1(\mathbb{R})$.

Before we mention our result, we should remark a variational characterization of ϕ_ω for the discussion below. From now on, we will consider the nonlinearity $f(u) = -|u|^{p-1}u$ only and the case of $Z > 0$. The global well-posedness of the Cauchy problem holds in $H^1(\mathbb{R})$ for any p with $1 < p < 5$ by the same method as in [15].

Definition 2. For $Z > 0$ and $\omega > Z^2/2$, we define two C^1 functionals on $H^1(\mathbb{R})$:

$$\begin{aligned} S_\omega(v) &:= E(v) + \omega Q(v), \\ I_\omega(v) &:= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx - \|v\|_{L^{p+1}}^{p+1} \\ &= \frac{1}{2} \|Dv\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - Z |v(0)|^2 - \|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Let $\mathcal{G}_\omega$ be the set of all nonnegative minimizers for the minimization problem

$$d(\omega) = \inf \{ S_\omega(v) : v \in H^1(\mathbb{R}) \setminus \{0\}, I_\omega(v) = 0 \}. \quad (1.8)$$

The existence of non-negative minimizers for (1.8) is proved by the standard variational argument. We will briefly show the following proposition in Section 3 for the sake of completeness.

Proposition 1. *Let $Z > 0$. For any $\omega > Z^2/2$, the minimization problem (1.8) is attained by a symmetric nonincreasing function vanishing at infinity.*

Remark 1.3. (i) For $Z > 0$, let

$$\lambda = \inf \left\{ \frac{1}{2} \|Dv\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |v(x)|^2 dx : \|v\|_{L^2} = 1, v \in H^1(\mathbb{R}) \right\}.$$

Then we have $\lambda = -Z^2/2$ and the corresponding eigenfunction is $\Phi(x) = Ze^{-Z|x|}$.

(ii) We note that

$$I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v \rangle$$

for $\lambda > 0$.

(iii) Let $v_\omega \in \mathcal{G}_\omega$. Then, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S'_\omega(v_\omega) = \Lambda I'_\omega(v_\omega)$. Thus, we have $\langle S'_\omega(v_\omega), v_\omega \rangle = \Lambda \langle I'_\omega(v_\omega), v_\omega \rangle$. Since $\langle S'_\omega(v_\omega), v_\omega \rangle = I_\omega(v_\omega) = 0$ and $\langle I'_\omega(v_\omega), v_\omega \rangle = -(p-1)\|v_\omega\|_{p+1}^{p+1} < 0$, we have $\Lambda = 0$. Namely, v_ω satisfies (1.6). Moreover, for any $v \in H^1(\mathbb{R}) \setminus \{0\}$ satisfying $S'_\omega(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of $\mathcal{G}_\omega$, we have $S_\omega(v_\omega) \leq S_\omega(v)$. Namely, $v_\omega \in \mathcal{G}_\omega$ is a ground state (minimal action solution) of (1.6) in $H^1(\mathbb{R})$. It is easy to see that a ground state of (1.6) in $H^1(\mathbb{R})$ is a minimizer of (1.8).

(iii) The minimizer $v \in \mathcal{G}_\omega$ obtained above, which is nonnegative, symmetric and nonincreasing, satisfies the initial boundary value problem (see Lemma 3.1 below):

$$\begin{aligned} v &\in C^2(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad v(x) > 0, \quad x \in \mathbb{R}, \\ -\frac{1}{2}D^2v + \omega v - v^p &= 0, \quad x \neq 0, \\ Dv(0+) - Dv(0-) &= -2Zv(0), \\ Dv(x), v(x) &\rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \end{aligned}$$

Remark 1.4. The minimizer obtained in Proposition 1 is precisely the same as ϕ_ω defined by (1.7) since the positive symmetric solution with above initial boundary value problem (3.4)–(3.7) is uniquely determined.

To prove stability and instability, we use the following sufficient condition originally obtained by Shatah [26] and Shatah and Strauss [27] (see also [10, 12] for the proof).

Proposition 2. *Let $p > 1$, $Z > 0$ and $\omega > Z^2/2$. Let $v_\omega \in \mathcal{G}_\omega$. Assume that $\omega \mapsto v_\omega$ is a C^1 mapping.*

- (i) *If $\partial_\omega \|v_\omega\|_{L^2}^2 > 0$ at $\omega = \omega_0$, then $e^{i\omega_0 t} v_{\omega_0}(x)$ is stable in $H^1(\mathbb{R})$.*
- (ii) *If $\partial_\omega \|v_\omega\|_{L^2}^2 < 0$ at $\omega = \omega_0$, then $e^{i\omega_0 t} v_{\omega_0}(x)$ is unstable in $H^1(\mathbb{R})$.*

Remark 1.5. In case of $Z = 0$, it is easy to verify this condition since we have $\|\psi_\omega\|_{L^2}^2 = \omega^{2/(p-1)-1/2} \|\psi_1\|_{L^2}^2$ by the scaling invariance even in the higher dimensional case. Due to the potential term we lost the scaling invariance in general (see [7, 12, 13, 14, 20, 22, 32, 33] for example). However, in the present one-dimensional case where the potential is a Dirac-delta, we can compute exactly increase and decrease of L^2 norm of (1.7).

Theorem 3. *Let $Z > 0$ and $\omega > Z^2/2$.*

- (i) *Let $1 < p \leq 5$. Then $e^{i\omega t} \phi_\omega(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/2, \infty)$.*
- (ii) *Let $p > 5$. Then there exists unique $\omega_1 > 0$ such that $e^{i\omega t} \phi_\omega(x)$ is stable in $H^1(\mathbb{R})$ for any $\omega \in (Z^2/2, \omega_1)$, and that it is unstable in $H^1(\mathbb{R})$ for any $\omega \in (\omega_1, \infty)$, where*

ω_1 is exactly defined as follows:

$$\begin{aligned} \frac{p-5}{p-1}J(\omega_1) &= \frac{Z}{\sqrt{2\omega_1}} \left(1 - \frac{Z^2}{2\omega_1}\right), \\ J(\omega_1) &= \int_{A(\omega_1)}^{\infty} \operatorname{sech}^{4/(p-1)} y dy, \quad A(\omega_1) = \tanh^{-1} \left(\frac{Z}{\sqrt{2\omega_1}} \right). \end{aligned}$$

Remark 1.6. Concerning the critical case $\partial_{\omega} \|\phi_{\omega}\|_{L^2}^2 = 0$, we conjecture that $e^{i\omega_1 t} \phi_{\omega_1}(x)$ would be unstable in view of the result of Comech and Pelinovsky [8]. For that purpose, the variational characterizations of ϕ_{ω} above would be useful to investigate the number of nonpositive eigenvalues of the linearized operators around $e^{i\omega t} \phi_{\omega}(x)$ (see [19, 9]).

Remark 1.7. A similar result is known for the case where $Z = 0$ and $f(z) = \lambda_1 |z|^{p_1-1} z + \lambda_2 |z|^{p_2-1} z$, where $\lambda_j \in \mathbb{R}$ and $1 \leq p_1 \leq p_2 < \infty$ (see Ohta [23]).

In Section 2, we give a proof of Theorems 1 and 2. In Section 3, we prove Proposition 1 and we complete the proof of Theorem 3 by checking the increase and the decrease of L^2 norm of ϕ_{ω} as a function of ω . Also, we give the outline of the proof of Proposition 2.

2. PROOF OF THEOREMS 1 AND 2

We recall that $U(t) = \exp(-itH)$ is represented as

$$(U(t)\phi)(x) = \int K(t, x, y) \phi(y) dy \quad (2.1)$$

for $\phi \in L^1 \cap L^2$ and $t \neq 0$ [4, 16], where $K = K_0 + K_1$ with

$$\begin{aligned} K_0(t, x, y) &= \frac{1}{\sqrt{2\pi it}} \exp\left(\frac{i(x-y)^2}{2t}\right), \\ K_1(t, x, y) &= \frac{Z}{2} \exp(-Z\rho + \frac{it}{2}Z^2) \operatorname{erfc}(\xi) \\ &= \frac{Z}{2} \exp\left(\frac{i\rho^2}{2t}\right) \exp(\xi^2) \operatorname{erfc}(\xi), \\ \xi &= \rho/\sqrt{2it} - Z\sqrt{it/2}, \quad \rho \equiv |x| + |y|, \\ \operatorname{erfc}(\xi) &= \frac{2}{\sqrt{\pi}} \int_{\xi}^{\infty} e^{-r^2} dr, \quad \xi \in \mathbb{C}. \end{aligned}$$

Here the last integral is taken along any path from the origin to ξ in the complex plane.

The main tool for the proof of Theorems 1 and 2 is the Strichartz estimates.

Proposition 3. *Let q_j, r_j satisfy $0 \leq 2/q_j = 1/2 - 1/r_j \leq 1/2$, $j = 0, 1, 2$. Then the following estimates hold:*

$$\begin{aligned} \|U(\cdot)\phi\|_{L^{q_0}(\mathbb{R}; L^{r_0})} &\leq C \|\phi\|_{L^2}, \\ \|Gf\|_{L^{q_1}(I; L^{r_1})} &\leq C \|f\|_{L^{q'_2}(I; L^{r'_2})}, \end{aligned}$$

where

$$(Gf)(t) = \int_0^t U(t-s)f(s)ds,$$

I is an interval with $0 \in \bar{I}$, C is a constant independent of I , and q' is a conjugate exponent to q defined by $1/q + 1/q' = 1$.

Proof. By the standard argument for the Strichartz estimates [5, 29], it suffices to prove that

$$\sup_{t,x,y \in \mathbb{R}} \sqrt{|t|} |K(t, x, y)| = C < \infty, \quad (2.2)$$

which follows from (2.2) with K replaced by K_1 . By the definition of ξ , we have

$$|\xi| = \sqrt{\frac{\rho^2}{2|t|} + \frac{Z^2|t|}{2}} \geq \sqrt{\frac{|t|}{2}} Z. \quad (2.3)$$

If $|\xi| \geq 1$, then

$$\begin{aligned} |K_1(t, x, y)| &\leq \frac{|Z|}{2} |\exp(\xi^2) \operatorname{erfc}(\xi)| \\ &\leq \frac{C|Z|}{|\xi|} \leq \frac{C}{\sqrt{|t|}}, \end{aligned} \quad (2.4)$$

where we have used an inequality for the error function [28] and (2.3). If $|\xi| \leq 1$, then $|Z|\sqrt{\frac{|t|}{2}} \leq 1$ and therefore

$$\begin{aligned} |K_1(t, x, y)| &\leq \frac{|Z|}{2} |\operatorname{erfc}(\xi)| \\ &\leq C|Z| \leq \frac{C}{\sqrt{|t|}}, \end{aligned} \quad (2.5)$$

where we have used another inequality for the error function [28]. The required estimate (2.2) then follows from (2.4) and (2.5).

Once the Strichartz estimates are established, the local well-posedness follows by the standard contraction argument [5, 29]. We only give a sketch of the argument. For $T > 0$ we define

$$X_T = L^\infty(-T, T; L^2) \cap L^4(-T, T; L^\infty)$$

with norm $\|u\| = \|u\|_{L_t^\infty(L^2)} \vee \|u\|_{L_t^4(L^\infty)}$, where $a \vee b = \max(a, b)$. For $u \in X_T$ and $u_0 \in L^2$ we define

$$(\Phi(u))(t) = U(t)u_0 - i \int_0^t U(t-s)f(u(s))ds.$$

For simplicity we assume that $p_1 = 1$, $p_2 = p < 5$ for Theorem 1 and that $p_1 = p_2 = p = 5$ for Theorem 2. By Proposition 3 and Hölder inequalities in space and time, we have

$$\begin{aligned} |||\Phi(u)||| &\leq C\|u_0\|_{L^2} + C\|f(u)\|_{L_t^1(L^2)} \\ &\leq C\|u_0\|_{L^2} + CT\|u\|_{L_t^\infty(L^2)} + CT^{1-\theta}\|u\|_{L_t^4(L^\infty)}^{p-1}\|u\|_{L_t^\infty(L^2)}, \end{aligned} \quad (2.6)$$

where $\theta = (p-1)/4$. Similarly, for $u, v \in X_T$, we have

$$\begin{aligned} |||\Phi(u) - \Phi(v)||| &\leq C\|f(u) - f(v)\|_{L_t^1(L^2)} \\ &\leq CT\|u - v\|_{L_t^\infty(L^2)} + CT^{1-\theta}(\|u\|_{L_t^4(L^\infty)} \vee \|v\|_{L_t^4(L^\infty)})^{p-1}\|u - v\|_{L_t^\infty(L^2)}. \end{aligned} \quad (2.7)$$

If $p < 5$, then $\theta < 1$ and therefore (2.6) and (2.7) show that Φ is a contraction on a closed ball in X_T with $T > 0$ sufficiently small. If $p = 5$, then $\theta = 0$ and therefore (2.6) and (2.7) show that Φ is a contraction on $L^\infty(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$ if the size of L^2 norm of the Cauchy data is sufficiently small. The conservation law (1.4) follows in the same way as in [24]. This leads to the existence of global solutions [5, 29]. Uniqueness and continuous dependence of solutions follows by the standard method [5, 29], so that it suffices to show (1.5). Let $u \in (C \cap L^\infty)(\mathbb{R}; L^2) \cap L^4(\mathbb{R}; L^\infty)$ be a solution of (1.3) with $u(0) = u_0 \in L^2$. Then by Proposition 3 and (1.4), we have for $t > s$

$$\begin{aligned} \|U(-t)u(t) - U(-s)u(s)\|_{L^2} &= \left\| \int_s^t U(-t')f(u(t'))dt' \right\|_{L^2} \\ &= \left\| \int_s^t U(t-t')f(u(t'))dt' \right\|_{L^2} \\ &\leq C\|u\|_{L^4(s,t;L^\infty)}^4\|u_0\|_{L^2} \\ &\rightarrow 0 \end{aligned}$$

as $t > s \rightarrow +\infty$, since $u \in L^4(\mathbb{R}; L^\infty)$. This implies the existence of $u_+ \in L^2$ such that

$$\|u(t) - U(t)u_+\|_{L^2} \rightarrow 0$$

as $t \rightarrow +\infty$. The case $t \rightarrow -\infty$ follows in the same way.

3. EXISTENCE OF GROUND STATES AND PROOF OF THEOREM 3

First, we remark that the following variational problem is equivalent to $d(\omega)$:

$$d_1(\omega) = \inf \left\{ \frac{p-1}{2(p+1)}\|v\|_{L^{p+1}}^{p+1} : v \in H^1(\mathbb{R}) \setminus \{0\}, I_\omega(v) \leq 0 \right\}. \quad (3.1)$$

Proof of Proposition 1. Let $\{v_j\}$ be a minimizing sequence for $d_1(\omega)$ and v_j^* be a rearrangement of $|v_j|$. Therefore, $\{v_j^*\}$ are nonnegative, symmetric and nonincreasing functions. For

the main properties of rearrangement functions, see Lieb and Loss [21], Folland [11] and Appendix of Berestycki and Lions [3]. In particular,

$$|v_j(0)|^2 = \int_{\mathbb{R}} \delta(x) |v_j(x)|^2 dx \leq \int_{\mathbb{R}} \delta(x) [|v_j(x)|^*]^2 dx = [|v_j(0)|^*]^2 = |v_j^*(0)|^2. \quad (3.2)$$

Indeed, it is known (see [21, Theorem 3.4]) that for any nonnegative functions f, g vanishing at infinity,

$$\int_{\mathbb{R}} f(x)g(x)dx \leq \int_{\mathbb{R}} f^*(x)g^*(x)dx, \quad (3.3)$$

where f^* and g^* are the symmetric-decreasing rearrangements of f and g . Here we employ

$$\rho_\varepsilon(x) = \frac{1}{2\sqrt{\varepsilon\pi}} e^{-|x|^2/4\varepsilon}, \quad \varepsilon > 0,$$

as f in (3.3), and we have the desired inequality in (3.2) letting $\varepsilon \rightarrow 0$. Therefore, $\{v_j^*\}$ satisfies $I_\omega(v_j^*) \leq 0$ and is also a minimizing sequence for $d_1(\omega)$. Here, we rewrite $w_j = v_j^*$ for simplicity and we have

$$\frac{1}{2} \|Dw_j\|_{L^2}^2 + \omega \|w_j\|_{L^2}^2 - Z \int_{\mathbb{R}} \delta(x) |w_j(x)|^2 dx \leq C.$$

By (i) of Remark 1.3, $(\omega + \lambda) \|w_j\|_{L^2}^2 \leq C$. Also, by Sobolev embedding,

$$\begin{aligned} \frac{1}{2} \|Dw_j\|_{L^2}^2 &\leq C + Z |w_j(0)|^2 \\ &\leq C + C \|w_j\|_{L^2} \|Dw_j\|_{L^2} \\ &\leq C + C \left(\frac{1}{2\varepsilon} \|w_j\|_{L^2}^2 + \frac{\varepsilon}{2} \|Dw_j\|_{L^2}^2 \right). \end{aligned}$$

Taking $\varepsilon > 0$ sufficiently small, we have that $\|Dw_j\|_{L^2}^2$ is bounded and so that $\|w_j\|_{H^1}^2$ is bounded. From Strauss' radial lemma for one dimensional version (see Cazenave [5, Proposition 1.7.1]), there exists a subsequence still denoted by $\{w_j\}$ and $w_0 \in H^1(\mathbb{R})$ such that w_j converges weakly to w_0 in $H^1(\mathbb{R})$ and that w_j converges strongly to w_0 in $L^r(\mathbb{R})$ for any r with $2 < r \leq \infty$. Accordingly, $w_j \rightarrow w_0$ a.e. x in $\mathbb{R}$ and we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{\mathbb{R}} \delta(x) |w_j(x)|^2 dx &= \lim_{j \rightarrow \infty} |w_j(0)|^2 = |w_0(0)|^2 = \int_{\mathbb{R}} \delta(x) |w_0(x)|^2 dx, \\ \lim_{j \rightarrow \infty} \|w_j\|_{L^{p+1}}^{p+1} &= \|w_0\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Thus, $I_\omega(w_0) \leq \liminf_{j \rightarrow \infty} I_\omega(w_j) \leq 0$ by the weak limit. By definition of $d_1(\omega)$, we have

$$d_1(\omega) \leq \frac{p-1}{2(p+1)} \|w_0\|_{L^{p+1}}^{p+1} = \lim_{j \rightarrow \infty} \frac{p-1}{2(p+1)} \|w_j\|_{L^{p+1}}^{p+1} = d_1(\omega).$$

This concludes that w_0 is a minimizer of $d_1(\omega)$. $\square$

After the verification of existence of a minimizer, it is easy to conclude Proposition 1 since the minimizer is already nonnegative, symmetric and nonincreasing by the process of

rearrangement. To assure that this minimizer satisfies the boundary condition and decays at infinity, we prove the following lemma.

Lemma 3.1. *Let $p > 1$, $Z > 0$ and $\omega > Z^2/2$. Assume that $v \in \mathcal{G}_\omega$. Then v is symmetric, positive and satisfies the following.*

$$v \in C^j(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R}), \quad j = 1, 2. \quad (3.4)$$

$$-\frac{1}{2}D^2v + \omega v - v^p = 0, \quad x \neq 0, \quad (3.5)$$

$$Dv(0+) - Dv(0-) = -2Zv(0), \quad (3.6)$$

$$Dv(x), v(x) \rightarrow 0, \quad \text{as } |x| \rightarrow \infty. \quad (3.7)$$

Proof. Since v satisfies $S'_\omega(v) = 0$, v satisfies (1.6). To check (3.4) and (3.7), we take an appropriate test function $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Then ξv satisfies

$$-\frac{1}{2}D^2(\xi v) + \omega \xi v = -\frac{1}{2}(D^2\xi)v - (D\xi)(Dv) + \xi v^p,$$

in the sense of distributions. We employ the standard bootstrap argument for this equation (see Section 8 of [5] for details). The right hand side is in $L^2(\mathbb{R})$ and so $\xi v \in H^2(\mathbb{R})$, that is, $v \in H^2(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R} \setminus \{0\})$. The case of $j = 2$ is similar. The equation (3.5) follows from the fact that $C_0^\infty(\mathbb{R} \setminus \{0\})$ is dense in $L^2(\mathbb{R})$. Concerning (3.6), we integrate $S'_\omega(v) = 0$ from $-\varepsilon$ to ε .

$$-\frac{1}{2} \int_{-\varepsilon}^{\varepsilon} D^2v dx + \omega \int_{-\varepsilon}^{\varepsilon} v dx - Z \int_{-\varepsilon}^{\varepsilon} \delta(x)v dx = \int_{-\varepsilon}^{\varepsilon} v^p dx.$$

Then we have the initial boundary condition

$$Dv(0+) - Dv(0-) = -2Zv(0)$$

letting $\varepsilon \rightarrow 0$. Multiplying the equation (3.5) by Dv and integrating resulting terms in $x > 0$ and in $x < 0$, we have

$$-\frac{1}{4}(Dv)^2 = F(v(x)), \quad x \neq 0. \quad (3.8)$$

We note that $v(x) > 0$ for $x \in \mathbb{R}$. If not, there exists x_0 such that $v(x_0) = 0$. From (3.8), we have $Dv(x_0) = 0$. It implies $v \equiv 0$, which is impossible. $\square$

To show Theorem 3, we check the sufficient condition for stability and instability in Proposition 2.

Proof of Theorem 3. We put $\alpha = \omega^{-1/2}$ and then it follows from (1.7) that

$$\begin{aligned}\frac{\partial}{\partial \omega} \|\phi_\omega\|_{L^2}^2 &= -\frac{\partial}{\partial \alpha} \|\phi_\alpha\|_{L^2}^2 = C_p \alpha^{-4/(p-1)} g(\alpha), \\ g(\alpha) &= \frac{p-5}{p-1} J(\alpha) - \alpha Z (1 - C_\alpha^2)^{-(p-3)/(p-1)}, \\ J(\alpha) &= \int_{A(\alpha)}^\infty \operatorname{sech}^{4/(p-1)} y dy, \quad A(\alpha) = \tanh^{-1}(C_\alpha),\end{aligned}$$

where $C_\alpha = Z\alpha$ and C_p is a constant depending only on p . It suffices to check the sign of $g(\alpha)$. In the case where $Z > 0$ and $p \leq 5$, we have $g(\alpha) < 0$ for any $\alpha \in (0, 1/Z)$. In the case where $Z > 0$ and $p > 5$, we see that $g'(\alpha) > 0$ in a neighborhood of 0, $g'(\alpha) < 0$ in a neighborhood of $1/Z$ and that $g''(\alpha) < 0$ for any $\alpha \in (0, 1/Z)$. Therefore, there exists a unique $\alpha^* \in (0, 1/Z)$ such that $g(\alpha^*) = 0$, $g(\alpha) > 0$ for any $\alpha \in (0, \alpha^*)$ and that $g(\alpha) < 0$ for any $\alpha \in (\alpha^*, 1/Z)$ since $g(0) > 0$. $\square$

For the sake of completeness, we give a remark on the proof of Proposition 2. First, we consider the stability. We explain briefly because the proof is similar to that of [12, Proposition 1] (see also Fibich and Wang[10]). We remark that $d'(\omega) = Q(\phi_\omega)$ and it follows from the explicit form of (1.7) that the mapping $\omega \mapsto \phi_\omega$ is C^1 .

We introduce the C^1 map $\omega(\cdot) : U_\eta(\phi_\omega) \rightarrow \mathbb{R}$ defined by

$$\omega(u) = d^{-1} \left(\frac{p-1}{2(p+1)} \|u\|_{L^{p+1}}^{p+1} \right). \quad (3.9)$$

Here, let us denote ϕ_{ω_0} by ϕ_0 for simplicity. The following lemma is important to have the stability. We omit the proof since it is the same as that of Lemma 4.2 in [12].

Lemma 3.2. *Let $p > 1$, $Z > 0$ and $\omega > Z^2/2$. Assume $d''(\omega) > 0$ at $\omega = \omega_0$ for some $\omega_0 \in (Z^2/2, \infty)$. Then there exists $\eta = \eta(\omega_0) > 0$ such that for all $u \in U_\eta(\phi_0)$,*

$$E(u) - E(\phi_0) + \omega(u) \{Q(u) - Q(\phi_0)\} \geq \frac{1}{4} d''(\omega_0) (\omega(u) - \omega_0)^2.$$

We verify the statement of Proposition 2 (i) by contradiction. Assume that $e^{i\omega_0 t} \phi_0(x)$ is unstable in $H^1(\mathbb{R})$. Then we have $\varepsilon_0 > 0$ and initial data $u_k(0) \in U_{1/k}(\phi_0)$ such that

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u_k(t) - e^{i\theta} \phi_0\|_{H^1} \geq \varepsilon_0,$$

where $u_k(t)$ is the solution of (1.1) with initial data $u_k(0)$. Let t_k be the first time at which

$$\inf_{\theta \in \mathbb{R}} \|u_k(t_k) - e^{i\theta} \phi_0\|_{H^1} = \frac{\varepsilon_0}{2}. \quad (3.10)$$

We put $v_k = u_k(t_k)$. Since E and Q are conserved in t , we have

$$|E(v_k) - E(\phi_0)| = |E(u_k(0)) - E(\phi_0)| \rightarrow 0, \quad (3.11)$$

$$|Q(v_k) - Q(\phi_0)| = |Q(u_k(0)) - Q(\phi_0)| \rightarrow 0 \quad (3.12)$$

as $k \rightarrow \infty$. From (3.10), we have $\|v_k\|_{H^1} \leq C$ uniformly in k . Also we note that $\omega_k = \omega(v_k)$ is uniformly bounded in k since $\omega(u)$ is a continuous map. Here, we take η small enough so that Lemma 3.2 may be applied. Then we have

$$E(v_k) - E(\phi_0) + \omega_k\{Q(v_k) - Q(\phi_0)\} \geq \frac{1}{4}d''(\omega_0)(\omega_k - \omega_0)^2. \quad (3.13)$$

Since $d''(\omega_0) > 0$, this implies that $\omega_k \rightarrow \omega_0$ as $k \rightarrow \infty$. By using $I_{\omega_0}(\phi_0) = 0$ and the fact that $d(\cdot)$ is continuous, it follows that

$$\lim_{k \rightarrow \infty} \frac{p-1}{2(p+1)} \|v_k\|_{L^{p+1}}^{p+1} = \lim_{k \rightarrow \infty} d(\omega_k) = d(\omega_0) = \frac{p-1}{2(p+1)} \|\phi_0\|_{L^{p+1}}^{p+1}. \quad (3.14)$$

From (3.11) and (3.12), we have

$$\begin{aligned} S_{\omega_0}(v_k) &= S_{\omega_0}(v_k) - S_{\omega_0}(\phi_0) + S_{\omega_0}(\phi_0) \\ &= E(v_k) - E(\phi_0) + \omega_0(Q(v_k) - Q(\phi_0)) + d(\omega_0) \\ &\rightarrow d(\omega_0), \end{aligned} \quad (3.15)$$

as $k \rightarrow \infty$. Let $w_k = (\|\phi_0\|_{L^{p+1}}/\|v_k\|_{L^{p+1}})v_k$. Then, w_k satisfies $\|w_k\|_{L^{p+1}} = \|\phi_0\|_{L^{p+1}}$ and $\|w_k - v_k\|_{H^1} \rightarrow 0$ as $k \rightarrow \infty$. Furthermore, by (3.14) and (3.15), $S_{\omega_0}(w_k) \rightarrow d(\omega_0)$ as $k \rightarrow \infty$. Therefore, $\{w_k\}$ is a minimizing sequence for $d(\omega_0)$. By the compactness of rearrangement functions (see the proof of Proposition 1) and the uniqueness of minimizers of $d(\omega_0)$, there exists a sequence $\{\theta_k\} \subset \mathbb{R}$ such that $\|w_k - e^{i\theta_k}\phi_0\|_{H^1} \rightarrow 0$ as $k \rightarrow \infty$. Namely, we get

$$\|v_k - e^{i\theta_k}\phi_0\|_{H^1} \rightarrow 0$$

as $k \rightarrow \infty$, which is a contradiction to (3.10). $\square$

Next, concerning the sufficient condition for instability, i.e., Proposition 2 (ii), we can apply a similar method by Shatah and Strauss [27] to our present case. Indeed, we modify the function $\psi(\omega)$, which was defined in the proof of Theorem 5 in [27] and used to determine an unstable direction in solving the ordinary differential equation of Lemma 11 in [27], as the following: For any fixed $\omega_0 \in (Z^2/2, \infty)$, we put for any $v_\omega \in \mathcal{G}_\omega$,

$$\psi(\omega) = \lambda(\omega)v_\omega(x), \quad \lambda(\omega) = \frac{\|v_{\omega_0}\|_{L^2}}{\|v_\omega\|_{L^2}}$$

for any $\omega \in (Z^2/2, \infty)$ which is close to ω_0 . We also remark that our variational characterization is different from theirs only in the point that $d(\omega) = \frac{p-1}{2(p+1)}\|v_\omega\|_{L^{p+1}}^{p+1}$. Accordingly, we change the norm in the statement of Lemma 11 iv) in [27] to L^{p+1} norm. In consequence, their method works and Theorem 17 in [27] holds for the present case.

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