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THE INVERSE SCATTERING PROBLEM FOR SCHRÖDINGER AND KLEIN-GORDON EQUATIONS WITH A NONLOCAL NONLINEARITY

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ABSTRACT. We study the inverse scattering problem for the nonlinear Schrödinger equation and for the nonlinear Klein-Gordon equation with the generalized Hartree type nonlinearity. We reconstruct the nonlinearity from knowledge of the scattering operator, which improves the known results.

1. INTRODUCTION

We consider the inverse scattering problem for the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = f(u) \tag{NLS}$$

and for the nonlinear Klein-Gordon equation

$$\partial_t^2 w - \Delta w + w = f(w) \tag{NLKG}$$

in space-time $\mathbb{R} \times \mathbb{R}^n$. Here u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, w is either a complex-valued or real-valued function, $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}^n$. The nonlocal nonlinear term $f(v)$ has the form

$$f(v) = \int_{\mathbb{R}^n} \mu(x, y) |v(x - y)|^2 v(x) dy. \tag{1.1}$$

In order to state the condition of μ , we define A_σ^l for $\sigma \in (0, n)$ and $l = 1, 2, \dots$ as the set of all functions $\nu : [\mathbb{R}^n \setminus \{0\}]^2 \mapsto \mathbb{R}$ satisfying the following conditions:

- (1) For $y \in \mathbb{R}^n \setminus \{0\}$, $\nu(\cdot, y) \in C^l(\mathbb{R}_x^n \setminus \{0\})$ and for $x \in \mathbb{R}^n \setminus \{0\}$, $\nu(x, \cdot)$ is measurable on $\mathbb{R}_y^n \setminus \{0\}$.
- (2) For $(x, y) \in [\mathbb{R}^n \setminus \{0\}]^2$, $|\partial_x^\alpha \nu(x, y)| \leq C_0 |y|^{-\sigma}$, where C_0 is independent of x, y and $0 \leq |\alpha| \leq l$.
- (3) There exists $\lambda_0 \in C([\mathbb{R}^n \setminus \{0\}]^2)$ such that

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- λ_0 is bounded on $[\mathbb{R}^n \setminus \{0\}]^2$
- λ_0 is not 0-function.
- λ_0 satisfies that either $\lambda_0 \geq 0$ or $\lambda_0 \leq 0$.
- $\lim_{\alpha \downarrow 0} \nu(\alpha x, \alpha y) |\alpha y|^\sigma = \lambda_0(x, y)$ a.e.

Suppose that μ belongs to A_σ^1 with unknown σ .

The nonlinear term $f(v)$ is the generalization of the Hartree term

$$G_0^\sigma(v) = \lambda(|\cdot|^{-\sigma} * |v|^2)v, \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

The term G_0^σ is an approximative expression of the nonlocal interaction of specific elementary particles. The equations (NLS) and (NLKG) with $f = G_0^\sigma$ are initially studied by Chadam-Glassey [3] and Menzala-Strauss [8], respectively. There is a substantial literature on the scattering theory for Hartree equations (see for instance [11] and references therein).

The inverse scattering problem for the nonlinear equation is recovering the nonlinearity from the knowledge of the scattering operator. For the definition of the scattering operator for the nonlinear equation, see, e.g., Section 2 in [14]. As we will show later, under suitable conditions, the scattering operator is well-defined for (NLS) and for (NLKG).

The inverse scattering problem for the nonlinear Schrödinger equation with the Hartree term is initially studied by [17]. To introduce the other results, we define the following two terms:

$$\begin{aligned} G_1(v) &= (\lambda_1(\cdot)|\cdot|^{-\sigma} * |v|^2)v, \\ G_2(v) &= \lambda_2(x)(|\cdot|^{-\sigma} * |v|^2)v, \end{aligned}$$

where $\lambda_j \in C^1(\mathbb{R}^n) \cap W_\infty^1(\mathbb{R}^n)$, $\lambda_j(0) \neq 0$, $j = 1, 2$. We remark that the term G_1 and G_2 satisfy (1.1) with $\mu = \lambda_1(y)|y|^{-\sigma} \in A_\sigma^1$ and $\mu = \lambda_2(x)|y|^{-\sigma} \in A_\sigma^1$, respectively. Watanabe [19] determined σ of the term G_1 if λ_1 is a constant. However, the method of [19] is not applicable in the case where λ_1 is not constant. It was suggested in [19] that we can easily reconstruct the term λ_2 if σ is a given number. Thus, if we can make a formula for determining σ of G_2 , then we can reconstruct G_2 . However, the method to determine σ of G_2 is not known in the case where λ_2 is not a constant.

The inverse scattering problem for the nonlinear Klein-Gordon equation with the Hartree term is initially studied by [14] which proved the uniqueness on identifying $\mu = \mu(y)$. We remark that for $\lambda > 0$ with $\lambda \neq 1$, and for any nontrivial solution of a free Klein-Gordon equation $\phi(t, x)$, a rescaled function $\phi(\lambda t, \lambda x)$ does not solve the equation. As a result, the method to make the reconstruction formula for G_0^σ in (NLS) is not applicable to the same term in (NLKG).

Our aim in this paper is to give an almost complete answer of the above problem which have not been shown. More precisely, for (NLS) and (NLKG), and for $j = 1, 2$, we shall determine σ of G_j even if λ_j is not constant. Since f with $\mu \in A_\sigma^1$ is a generalization of

G_j , it is sufficient to make the formula for determining σ of $\mu \in A_\sigma^1$.

To state our results, we give some notation. Let $\langle x \rangle = (1 + |x|^2)^{1/2}$. For $s, k \in \mathbb{R}$, let H_p^s and $H^{s,k}$ be the Sobolev space $(1 - \Delta)^{-s/2} L_p(\mathbb{R}^n)$ and the weighted Sobolev space $(1 - \Delta)^{-s/2} \langle x \rangle^{-k} L_2(\mathbb{R}^n)$, respectively. Especially, H^s denotes H_2^s . For $1 \leq r \leq \infty$, let r' be the Hölder conjugate of r . For $\alpha > 0$ and $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$, we denote $\phi(\alpha^{-1}x)$ by $\phi_\alpha(x)$. For (NLS), we set

$$T[\phi] = \lim_{\varepsilon \downarrow 0} \frac{i}{\varepsilon^3} \langle (S - I)(\varepsilon \phi), \phi \rangle_{L^2(\mathbb{R}^n)},$$

where S is the scattering operator for (NLS). For (NLKG), we put

$$K[\phi] = \lim_{\varepsilon \downarrow 0} \frac{i}{\varepsilon^3} \langle (S - I)(\varepsilon {}^t(\phi, 0)), {}^t(0, \phi) \rangle_{H^1(\mathbb{R}^n) \oplus L_2(\mathbb{R}^n)},$$

where S is the scattering operator for (NLKG).

Again we assume that the nonlinearity f has the form (1.1), and μ belongs to A_σ^1 with unknown σ . We are in a position to state our main Theorems.

Theorem 1.1. *Let $n \geq 2$, $1 < \sigma \leq 4$, $\sigma < n$. Assume that $\phi \in H^1 \cap H^{0,1}$ and $\phi \neq 0$. For (NLS), we have the formula for determining σ*

$$\sigma = 2n + 2 - \lim_{\alpha \downarrow 0} \ln \frac{|T[\phi_{e\alpha}]|}{|T[\phi_\alpha]| + \alpha^{2n+2}}. \quad (1.2)$$

Theorem 1.2. *Let $I = (6(n-1)/(3n-5), 2n/(n-2)]$. Suppose that $n \geq 3$, $\max\{n/(n-1), 4/3\} < \sigma \leq 4$, $\sigma < n$,*

$$\phi \in H^{1,1/3} \cap \bigcap_{r \in I} H_r^{(n+1)(1/2-1/r)}$$

and $\phi \neq 0$. For (NLKG), we have the formula for determining σ

$$\sigma = 2n + 1 - \lim_{\alpha \downarrow 0} \ln \frac{|K[\phi_{e\alpha}]|}{|K[\phi_\alpha]| + \alpha^{2n+1}}. \quad (1.3)$$

Remark 1. From Theorem 1.1 or 1.2, we can determine σ of G_1 and of G_2 . Moreover, by using the determined σ and the method of [15] (see also (1.6) in [19]), we have the reconstruction formula for λ_1 of (NLS)

$$\lambda_1(x_0) = \frac{\lim_{\alpha \rightarrow 0} \alpha^{-(2n+2-\sigma)} T[\phi_{\alpha, x_0}]}{\int |y|^{-\sigma} |u_0(t, x-y)|^2 |u_0(t, x)|^2 d(t, x, y)}. \quad (1.4)$$

Here, $\phi_{\alpha, x_0}(x)$ denotes $\phi(\alpha^{-1}(x - x_0))$, and $u_0 = e^{it\Delta} \phi$. For (NLKG), the reconstruction formula for λ_1 can be also given by

$$\lambda_1(x_0) = \frac{\lim_{\alpha \rightarrow 0} \alpha^{-(2n+1-\sigma)} K[\phi_{\alpha, x_0}]}{\int |y|^{-\sigma} |w_0(t, x-y)|^2 |w_0(t, x)|^2 d(t, x, y)}, \quad (1.5)$$

where $w_0 = \cos(t\sqrt{-\Delta})\phi$. For the proof of (1.5), see Section 3.2.

Remark 2. In [19], The condition for σ of the equation (NLS) is $2 \leq \sigma \leq 4$ and $\sigma < n$. We extend this condition to $1 < \sigma \leq 4$ and $\sigma < n$.

The contents of this paper are as follows. In Section 2, we prove that we can treat $\mu \in A_\sigma^1$ as $|y|^{-\sigma}$ whenever we consider only the scattering problems for (NLS) and for (NLKG). For this purpose, we first give an estimate for the nonlinearity by using the Hardy-Littlewood-Sobolev inequality and the complex interpolation method for the Sobolev spaces. From the estimate, we see the existence of the scattering operators for (NLS) and for (NLKG) under suitable condition of σ .

In Section 3, we first determine σ of (NLS). As the first step, by using the method of [15] and the proposition in Section 2, we show that $|T[\phi_{e\alpha}]|/(|T[\phi_\alpha]| + \alpha^{2n+2})$ is nearly equal to

$$e^{2n+2-\sigma} \frac{\int \mu(e\alpha x, e\alpha y) |e\alpha y|^\sigma R(\phi)(t, x, y) d(t, x, y)}{\int \mu(\alpha x, \alpha y) |\alpha y|^\sigma R(\phi)(t, x, y) d(t, x, y)},$$

where a function $R(\phi)(t, x, y)$ is integrable on $\mathbb{R}_{(t,x,y)}^{1+n+n}$. The condition for μ and the Lebesgue dominate theorem enable us to show (1.2). We next determine σ of (NLKG). As in the (NLS) case, we first prove that $|K[\phi_{e\alpha}]|/(|K[\phi_\alpha]| + \alpha^{2n+1})$ is nearly equal to

$$e^{2n+1-\sigma} \frac{\int \mu(e\alpha x, e\alpha y) |e\alpha y|^\sigma Q^\alpha(\phi)(t, x, y) d(t, x, y)}{\int \mu(\alpha x, \alpha y) |\alpha y|^\sigma Q^\alpha(\phi)(t, x, y) d(t, x, y)},$$

where

$$Q^m(t, x, y) = |y|^\sigma |\Psi^m(t, x - y)|^2 |\Psi^m(t, x)|^2,$$

and

$$\Psi^m(t) = \cos(t\sqrt{m^2 - \Delta})\phi.$$

In order to complete the theorem, we have only to prove that $\int Q^\alpha \rightarrow \int Q^0$ as $\alpha \rightarrow 0$. To show the convergence, we use the $L^p - L^q$ estimates for the linear Klein-Gordon and for the linear wave equation. Finally, we show the reconstruction formula (1.5).

2. SCATTERING

Before considering the inverse scattering problem, we have to solve the direct problem. It has been proved that there exist the scattering operators for (NLS) and for (NLKG) with $f = G_0^\sigma$ (see, e.g., [5, 6, 7, 13, 16]).

The following proposition is helpful to consider the direct problem:

Proposition 2.1. *Let $l = 1, 2, \dots$, $0 \leq s \leq l$, $1 < r_1 < r < \infty$, $0 < \sigma < n$ and $1 + 1/r = \sigma/n + 1/r_1$. If ν satisfies the conditions (1) and (2) in the definition of A_σ^l ,*

then we have for any $g \in H_{r_1}^s$

$$\left\| \int_{\mathbb{R}^n} \nu(\cdot, y) g(\cdot - y) dy \right\|_{H_r^s} \leq C \|g\|_{H_{r_1}^s}. \quad (2.1)$$

Proof. Let $0 \leq |\alpha| \leq l$. Since $|\partial_x^\alpha \nu(x, y)| \leq C_0 |y|^{-\sigma}$, it follows from the Hardy-Littlewood-Sobolev inequality and the Hölder inequality that

$$\begin{aligned} & \left\| \partial_x^\alpha \int_{\mathbb{R}^n} \nu(\cdot, y) g(\cdot - y) dy \right\|_{L_r} \\ & \leq C \sum_{\alpha=\beta+\gamma} \left\| \int_{\mathbb{R}^n} (\partial_x^\beta \nu(\cdot, y) |y|^\sigma) |y|^{-\sigma} \partial_x^\gamma g(\cdot - y) dy \right\|_{L_r} \\ & \leq C \sum_{\alpha=\beta+\gamma} \left\| \int_{\mathbb{R}^n} |y|^{-\sigma} \partial_x^\gamma g(\cdot - y) dy \right\|_{L_r} \\ & \leq C \sum_{\gamma \leq \alpha} \left\| \partial_x^\gamma g \right\|_{L_{r_1}}. \end{aligned}$$

In particular, we have

$$\begin{aligned} & \left\| \int_{\mathbb{R}^n} \nu(\cdot, y) g(\cdot - y) dy \right\|_{L_r} \leq C \|g\|_{L_{r_1}}, \\ & \left\| \int_{\mathbb{R}^n} \nu(\cdot, y) g(\cdot - y) dy \right\|_{H_r^l} \leq C \|g\|_{H_{r_1}^l}. \end{aligned}$$

Accordingly, by using the complex interpolation method (see, e.g., [2]) for the linear operator $L_{r_1} \ni g \mapsto \int \nu(\cdot, y) g(\cdot - y) dy \in L_r$, we obtain (2.1). $\square$

From (2.1), and the methods of [5, 6, 13], we can see the existence of the scattering operator for some σ . In order to mention the scattering states in detail, we list some notation. For a Banach space A , and for $\delta > 0$, let $B(\delta; A)$ be the set $\{a \in A; \|a|A\| \leq \delta\}$. Put $\omega = \sqrt{1 - \Delta}$

$$U(t) = \begin{pmatrix} \cos(t\omega) & \omega^{-1} \sin(t\omega) \\ -\omega \sin(t\omega) & \cos(t\omega) \end{pmatrix}$$

and

$$f(v_1, v_2, v_3) = \int_{\mathbb{R}^n} \mu(x, y) v_1(x - y) \overline{v_2(x - y)} v_3(x) dy.$$

We set

$$Y_\sigma^1 = \begin{cases} H^1 \cap H^{0,1} & \text{for } 1 < \sigma < 2 \\ H^1 & \text{for } 2 \leq \sigma \leq 4 \end{cases}$$

and

$$Y_\sigma^2 = \begin{cases} H^{1, \frac{1}{3}} \oplus H^{0, \frac{1}{3}} & \text{for } 4/3 < \sigma < 2 \\ H^1 \oplus L_2 & \text{for } 2 \leq \sigma \leq 4. \end{cases}$$

For a Banach space A , we denote $L^p(\mathbb{R}; A)$ by $L^p A$. Let

$$Z_\sigma^1 = \begin{cases} B_1 & \text{for } 1 < \sigma \leq 4/3 \\ B_2 & \text{for } 4/3 < \sigma < 2 \\ L^3 H_{q_1}^1 & \text{for } 2 \leq \sigma \leq 4, \end{cases}$$

where

$$\begin{aligned} B_1 &= \{e^{-it\Delta}u(t) \in C(\mathbb{R}; Y_\sigma^1); \sup_{t \in \mathbb{R}} \|e^{-it\Delta}u(t)\|_{Y_\sigma^1} < \infty\}, \\ B_2 &= \{e^{-it\Delta}u(t) \in C(\mathbb{R}; Y_\sigma^1), \omega u \in L^{8/\sigma} L^{\tilde{q}}, e^{it\Delta}x^\beta e^{-it\Delta}u \in L^{8/\sigma} L^{\tilde{q}}; |\beta| \leq 1, \\ &\quad \sup_{t \in \mathbb{R}} \|e^{-it\Delta}u(t)\|_{Y_\sigma^1} + \|\omega u\|_{L^{8/\sigma} L^{\tilde{q}}} + \sum_{|\beta| \leq 1} \|e^{it\Delta}x^\beta e^{-it\Delta}u\|_{L^{8/\sigma} L^{\tilde{q}}} < \infty\}, \end{aligned}$$

$1/q_1 = 1/2 - 2/3n$, $1/\tilde{q} = 1/2 - \sigma/8$. Further, we define

$$Z_\sigma^2 = \begin{cases} L^{p_2} H_{q_2}^{1-\rho_2} & \text{for } 4/3 < \sigma < 2 \\ L^3 H_{q_3}^{1-\rho_3} & \text{for } 2 \leq \sigma \leq 4, \end{cases}$$

where

$$\rho_2 = \max\left\{\frac{n+2}{n}\left(1 - \frac{1}{p_2}\right), \frac{2-\sigma}{4}\right\}, \quad \rho_3 = \frac{n+1+\theta}{3(n-1+\theta)},$$

$1/q_3 = 1/2 - 2/3(n-1+\theta)$, $\theta = 2 - \sigma/2$. Here, p_2 and q_2 satisfy that $3 < p_2 < \infty$, $2 < q_2 < \infty$,

$$\frac{n}{2}\left(\frac{1}{2} - \frac{1}{q_2}\right) < \frac{1}{p_2} < n\left(\frac{1}{2} - \frac{1}{q_2}\right), \quad \sigma = 2 - 2\left\{\frac{2}{p_2} - n\left(\frac{1}{2} - \frac{1}{q_2}\right)\right\}.$$

The scattering states are as follows:

Theorem 2.2. *Assume that μ belongs to A_σ^1 with $0 < \sigma < n$. Then we have the following properties:*

- (1) Let $n \geq 2$, $1 < \sigma \leq 4$ and $\sigma < n$. There exists some $\rho_1 > 0$ such that for any $\phi_- \in B_{\rho_1}(Y_\sigma^1)$, we uniquely have $u \in Z_\sigma^1$ and $\phi_+ \in H^1$ such that

$$u \in C(\mathbb{R}; H^1), \quad (2.2)$$

$$u(t) = e^{it\Delta}\phi_- + \frac{1}{i} \int_{-\infty}^t e^{i(t-\tau)\Delta} f(u(\tau)) d\tau, \quad (2.3)$$

$$\|u(t) - e^{it\Delta}\phi_\pm\|_{H^1} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty, \quad (2.4)$$

$$\|u\|_{Z_\sigma^1} \leq C\|\phi_-\|_{Y_\sigma^1}, \quad (2.5)$$

$$\|u - e^{it\Delta}\phi_-\|_{Z_\sigma^1} \leq C\|\phi_-\|_{Y_\sigma^1}^3, \quad (2.6)$$

$$\left\| \int_{\mathbb{R}} e^{-it\Delta} f(u_1, u_2, u_3) dt \right\|_{L_2} \leq C \Pi_{j=1}^3 \|u_j\|_{Z_\sigma^1} \quad \text{for } u_j \in Z_\sigma^1, j = 1, 2, 3. \quad (2.7)$$

Moreover, we can define the scattering operator for (NLS)

$$S : B_{\rho_1}(Y_\sigma^1) \ni \phi_- \mapsto \phi_+ = \phi_- + \frac{1}{i} \int_{\mathbb{R}} e^{-it\Delta} f(u(t)) dt \in H^1.$$

- (2) Let $n \geq 2$, $4/3 < \sigma \leq 4$ and $\sigma < n$. There exists some $\rho_2 > 0$ such that for any $\psi_- = {}^t(\psi_-^1, \psi_-^2) \in B_{\rho_2}(Y_\sigma^2)$, we uniquely have $w \in Z_\sigma^2$ and $\psi_+ \in H^1 \oplus L_2$ such that

$$W = {}^t(w, \partial_t w) \in C^1(\mathbb{R}; H^1) \oplus C(\mathbb{R}; L_2), \quad (2.8)$$

$$W(t) = U(t)\phi_- + \frac{1}{i} \int_{-\infty}^t U(t-\tau) \begin{pmatrix} 0 \\ if(w(\tau)) \end{pmatrix} d\tau, \quad (2.9)$$

$$\|W(t) - U(t)\psi_\pm\|_{H^1 \oplus L_2} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty, \quad (2.10)$$

$$\|w\|_{Z_\sigma^2} \leq C\|\psi_-\|_{Y_\sigma^2}, \quad (2.11)$$

$$\left\| w - \left(\cos(t\omega)\psi_-^1 + \omega^{-1} \sin(t\omega)\psi_-^2 \right) \right\|_{Z_\sigma^2} \leq C\|\psi_-\|_{Y_\sigma^2}^3, \quad (2.12)$$

$$\left\| \int_{\mathbb{R}} e^{it\omega} f(w_1, w_2, w_3) dt \right\|_{L_2} \leq C \Pi_{j=1}^3 \|w_j\|_{Z_\sigma^2}. \quad \text{for } w_j \in Z_\sigma^1, j = 1, 2, 3. \quad (2.13)$$

Moreover, we can define the scattering operator for (NLKG)

$$S : B_{\rho_2}(Y_\sigma^2) \ni \psi_- \mapsto \psi_+ = \psi_- + \frac{1}{i} \int_{\mathbb{R}} U(-t) \begin{pmatrix} 0 \\ if(w(t)) \end{pmatrix} dt \in H^1 \oplus L_2.$$

Proof. In the case of both (NLS) and (NLKG) with $2 \leq \sigma \leq 4$, the estimate (2.1) enables us to see that (3.6) and (3.7) in [6] hold.

In the case of (NLS) with $1 < \sigma < 2$, the estimates (2.5)–(2.8) in [5] are immediately shown by (2.1).

In the case of (NLKG) with $4/3 < \sigma < 2$, Lemma 2.3 in [13] holds from (2.1).

Accordingly, we can treat the estimates for f as the estimates for the Hartree type G_0^σ . For the rest of the proof, we have only to apply to methods of [5, 6, 13]. This completes the proof. $\square$

3. INVERSE SCATTERING

In this section, we consider the inverse scattering problem for (NLS) and for (NLKG).

3.1. Nonlinear Schrödinger equation. Inverse scattering problem for the nonlinear Schrödinger equation was initially studied by [15]. Weder [21, 22, 24] considered the nonlinear Schrödinger equation with a power type nonlinearity.

It was mentioned by [14] that the method for the power type nonlinearity is not applicable to our problem of determining $\mu(x, y)$ in (NLS).

Now we prove Theorem 1.1. Let $1 < \sigma \leq 4$ and $\sigma < n$. We here assume $\mu \in A_\sigma^1$. From Theorem 2.2,(1), if $\phi \in H^1 \cap H^{0,1}$, then $T[\phi]$ is well-defined.

By [15] and [18], it follows from (2.5)–(2.7) that

$$T[\phi] = \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \mu(x, y) |e^{it\Delta}\phi(x-y)|^2 |e^{it\Delta}\phi(x)|^2 d(t, x, y). \quad (3.1)$$

Having in mind that

$$e^{it\Delta}\phi_\alpha = (e^{it\alpha^{-2}\Delta}\phi)_\alpha,$$

we see that

$$T[\phi_\alpha] = \alpha^{2n+2-\sigma} \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \mu(\alpha x, \alpha y) |\alpha y|^\sigma R(\phi)(t, x, y) d(t, x, y),$$

where

$$R(\phi)(t, x, y) = |y|^{-\sigma} |e^{it\Delta}\phi(x-y)|^2 |e^{it\Delta}\phi(x)|^2.$$

Substituting $\mu(x, y) = |y|^{-\sigma}$ into (3.1), we see that $R(\phi)$ is integrable. Hence it follows from the assumption $\mu \in A_\sigma^1$ that

$$\begin{aligned} & \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \mu(\beta x, \beta y) |\beta y|^\sigma R(\phi)(t, x, y) d(t, x, y) \\ & \rightarrow \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda_0(x, y) R(\phi)(t, x, y) d(t, x, y) \neq 0 \quad \text{as } \beta \rightarrow 0 \end{aligned}$$

if $\phi \neq 0$. Hence we obtain

$$\frac{|T[\phi_{e\alpha}]|}{|T[\phi_\alpha]| + \alpha^{2n+2}} = \frac{e^{2n+2-\sigma} |\int \mu(e\alpha x, e\alpha y) |e\alpha y|^\sigma R(\phi)(t, x, y)|}{|\int \mu(\alpha x, \alpha y) |\alpha y|^\sigma R(\phi)(t, x, y)| + \alpha^\sigma} \rightarrow e^{2n+2-\sigma} \quad \text{as } \alpha \rightarrow 0.$$

Thus, we have (1.2). This completes the proof.

3.2. Nonlinear Klein-Gordon equation. Inverse scattering problem for the nonlinear Klein-Gordon equation was initially studied by [9]. Weder [20, 23] (see also [1]) considered the nonlinear Klein-Gordon equation with a power type nonlinearity

$$\frac{\partial^2 w}{\partial t^2}(x, t) - \Delta w(x, t) + w(x, t) = V_0(x)w(x, t) + \sum_{j=1}^{\infty} V_j(x)|w|^{2(j_0+j)}w(x, t).$$

It was proved that the small amplitude limit of the scattering operator determines uniquely all the $V_j(x)$, $j = 0, 1, \dots$.

It was mentioned by [14] that the method for the power type nonlinearity is not applicable to our problem of determining $\mu(x, y)$ in (NLKG).

Now we prove Theorem 1.2, and the reconstruction formula (1.5). To derive (1.3), we follow the line of the proof of Theorem 1.1. Let $4/3 < \sigma \leq 4$, $n/(n-1) < \sigma < n$ and let S be the scattering operator for (NLKG).

We here assume that $\mu \in A_\sigma^1$,

$$\phi \in \Lambda = H^{1,1/3} \cap \bigcap_{r \in I} H_r^{(n+1)(1/2-1/r)}$$

and $\phi \neq 0$, where $I = (6(n-1)/(3n-5), 2n/(n-2)]$. By Theorem 2.2, (2), and $\Lambda \subset H^{1,1/3}$, $K[\phi]$ is well-defined. By (2.11)–(2.13), it follows from the method of [15] and [18] that

$$K[\phi] = \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \mu(x, y) |\Psi^1(t, x-y)|^2 |\Psi^1(t, x)|^2 d(t, x, y),$$

where $\Psi^m(t) = \cos(t\sqrt{m^2 - \Delta})\phi$.

Having in mind that

$$e^{it\sqrt{1-\Delta}}\phi_\alpha = (e^{it\alpha^{-1}\sqrt{\alpha^2-\Delta}}\phi)_\alpha,$$

we also have

$$K[\phi_\alpha] = \alpha^{2n+1-\sigma} \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \mu(\alpha x, \alpha y) |\alpha y|^\sigma Q^\alpha(\phi)(t, x, y) d(t, x, y),$$

where

$$Q^m(\phi)(t, x, y) = |y|^{-\sigma} |\Psi^m(t, x-y)|^2 |\Psi^m(t, x)|^2.$$

In order to see the convergence of $|K[\phi_{e\alpha}]|/(|K[\phi_\alpha]| + \alpha^{2n+1})$, we give the following lemma:

Lemma 3.1. *Let $I = (6(n-1)/(3n-5), 2n/(n-2)]$ and $0 \leq \beta \leq 1$. For $n/(n-1) < \sigma \leq 4$, $\sigma < n$, and*

$$\phi \in H^1 \cap \bigcap_{r \in I} H_r^{(n+1)(1/2-1/r)},$$

then $Q^\beta(\phi)(t, x, y)$ is integrable on $\mathbb{R}_{(t,x,y)}^{1+n+n}$. Moreover, we have

$$\int Q^\beta(\phi) d(t, x, y) \rightarrow \int Q^0(\phi) d(t, x, y) \quad \text{as } \beta \rightarrow 0. \quad (3.2)$$

Proof. We first state $L^p - L^{\dot{p}}$ estimates (see, e.g., [10, 12])

$$\|e^{it\sqrt{-\Delta}}\phi\|_{L^r} \leq C|t|^{-(n-1)(1/2-1/r)} \|\phi\|_{\dot{H}_r^{(n+1)(1/2-1/r)}} \quad (3.3)$$

and

$$\|e^{it\sqrt{1-\Delta}}\phi\|_{L^r} \leq C|t|^{-(n-1)(1/2-1/r)} \|\phi\|_{H_r^{(n+1)(1/2-1/r)}}, \quad (3.4)$$

where $2 < r < \infty$ and $\dot{H}_p^s$ is the homogeneous Sobolev space (for the definition, see, e.g., [2, 4]).

By the embedding $H_r^s \hookrightarrow \dot{H}_r^s$ and $H^{n(1/2-1/r)} \hookrightarrow L^r$, it follows from (3.3) that

$$\|e^{it\sqrt{-\Delta}}\phi\|_{L^r} \leq C \langle t \rangle^{-(n-1)(1/2-1/r)} N_r(\phi), \quad (3.5)$$

where

$$N_r(\phi) = \max\{\|\phi\|_{H^{n(1/2-1/r)}}, \|\phi\|_{H_r^{(n+1)(1/2-1/r)}}\}.$$

On the other hand, Using the identity

$$e^{it\sqrt{m^2-\Delta}}\phi = (e^{itm\sqrt{1-\Delta}}\phi_m)_{m^{-1}},$$

and (3.4), we obtain

$$\begin{aligned} \|e^{it\sqrt{m^2-\Delta}}\phi\|_{L^r} &= m^{-n/r} \|e^{itm\sqrt{1-\Delta}}\phi_m\|_{L^r} \\ &\leq C m^{-n/r} |tm|^{-(n-1)(1/2-1/r)} \|\phi_m\|_{H_r^{(n+1)(1/2-1/r)}} \\ &\leq C m^{(n+1)(1/2-1/r)} (1 + m^{-(n+1)(1/2-1/r)}) \\ &\quad \times |t|^{-(n-1)(1/2-1/r)} \|\phi\|_{H_r^{(n+1)(1/2-1/r)}} \\ &\leq C |t|^{-(n-1)(1/2-1/r)} \|\phi\|_{H_r^{(n+1)(1/2-1/r)}} \end{aligned}$$

if $0 < m \leq 1$. By the embedding $H^{n(1/2-1/r)} \hookrightarrow L^r$, it follows from (3.5) that

$$\|e^{it\sqrt{\beta^2-\Delta}}\phi\|_{L^r} \leq C \langle t \rangle^{-(n-1)(1/2-1/r)} N_r(\phi). \quad (3.6)$$

Let $s \in [0, 1]$. By the Hardy-Littlewood-Sobolev inequality, the Hölder inequality and the embedding $L^{(1/2+s/n)^{-1}} \hookrightarrow H^{-s}$, we have

$$\begin{aligned}
& \|Q^m(\phi) - Q^0(\phi)\|_{L^1(\mathbb{R}_{(x,y)}^{2n})} \\
& \leq \|(| \cdot |^{-\sigma} * (\Psi^m - \Psi^0) \overline{\Psi^m}) \Psi^m \overline{\Psi^m}\|_{L^1(\mathbb{R}^n)} \\
& \quad + \|(| \cdot |^{-\sigma} * \Psi^0 \overline{(\Psi^m - \Psi^0)}) \Psi^m \overline{\Psi^m}\|_{L^1(\mathbb{R}^n)} \\
& \quad + \|(| \cdot |^{-\sigma} * \Psi^0 \overline{\Psi^0}) (\Psi^m - \Psi^0) \overline{\Psi^m}\|_{L^1(\mathbb{R}^n)} \\
& \quad + \|(| \cdot |^{-\sigma} * \Psi^0 \overline{\Psi^0}) \Psi^0 \overline{(\Psi^m - \Psi^0)}\|_{L^1(\mathbb{R}^n)} \\
& \leq \sum_{\beta_1, \beta_2 \in \{0,1\}} \|(| \cdot |^{-\sigma} * |\Psi^{\beta_1}|^2) |\Psi^{\beta_2}| |\Psi^m - \Psi^0|\|_{L^1(\mathbb{R}^n)} \\
& \leq C \|\Psi^m - \Psi^0\|_{H^s} \sum_{\beta_1, \beta_2 \in \{0,1\}} \|(| \cdot |^{-\sigma} * |\Psi^{\beta_1}|^2) |\Psi^{\beta_2}|\|_{H^{-s}} \\
& \leq C \|\Psi^m - \Psi^0\|_{H^1} \sum_{\beta_1, \beta_2 \in \{0,1\}} \|(| \cdot |^{-\sigma} * |\Psi^{\beta_1}|^2) |\Psi^{\beta_2}|\|_{L^{(\frac{1}{2} + \frac{s}{n})^{-1}}} \\
& \leq C \|\Psi^m - \Psi^0\|_{H^1} \{\|\Psi^m\|_{L^r}^3 + \|\Psi^0\|_{L^r}^3\},
\end{aligned}$$

where we have used the equality

$$\int (| \cdot |^{-\sigma} * v_1) v_2 = \int (| \cdot |^{-\sigma} * v_2) v_1$$

for the second inequality, and r satisfies

$$\frac{3}{2} + \frac{s}{n} = \frac{\sigma}{n} + \frac{3}{r}, \quad 2 < r < \infty.$$

Using (3.6), we see that

$$\|Q^m(\phi) - Q^0(\phi)\|_{L^1(\mathbb{R}_{(x,y)}^{2n})} \leq C \langle t \rangle^{-3(n-1)(1/2-1/r)} (N_r(\phi))^3 \|\Psi^m - \Psi^0\|_{L^2}. \quad (3.7)$$

Since $n/(n-1) < \sigma \leq 4$ and $\sigma < n$, we can put $r \in I$. Thus, we have $n(1/2 - 1/r) \leq 1$ and $-3(n-1)(1/2 - 1/r) < -1$. Hence, $Q^\beta(\phi)$ is integrable, and the right hand side of (3.7) is bounded by some integrable function which is independent of m . For all $t \in \mathbb{R}$, we can easily show that

$$\|\Psi^m(t) - \Psi^0(t)\|_{H^s} \rightarrow 0 \quad \text{as } m \rightarrow 0.$$

Thus, applying the Lebesgue dominate theorem on $\mathbb{R}_t$, we have (3.2). This completes the proof. $\square$

Let us go back to the proof of Theorem 1.2. We again assume $\phi \in \Lambda$. From Lemma 3.1, we see that

$$\begin{aligned} & \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \mu(\alpha x, \alpha y) |\alpha y|^\sigma Q^\alpha(\phi)(t, x, y) d(t, x, y). \\ & \rightarrow \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda_0(x, y) Q^0(\phi)(t, x, y) d(t, x, y) \neq 0 \quad \text{as } \alpha \rightarrow 0. \end{aligned}$$

So we obtain

$$\begin{aligned} \frac{|K[\phi_{e\alpha}]|}{|K[\phi_\alpha]| + \alpha^{2n+1}} &= \frac{e^{2n+1-\sigma} \int \mu(e\alpha x, e\alpha y) |e\alpha y|^\sigma Q^0(\phi)(t, x, y) d(t, x, y)}{|\int \mu(\alpha x, \alpha y) |\alpha y|^\sigma Q^0(\phi)(t, x, y) d(t, x, y)| + \alpha^\sigma} \\ &\rightarrow e^{2n+1-\sigma} \quad \text{as } \alpha \rightarrow 0. \end{aligned}$$

Thus, we have (1.3). This completes the proof of Theorem 1.2.

It remains to show the reconstruction formula (1.5). We put $\phi \in \Lambda$ and $\phi_{\alpha, x_0} = \phi(\alpha^{-1}(x - x_0))$. Then we have

$$K[\phi_{\alpha, x_0}] = \alpha^{2n+1-\sigma} \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda_1(\alpha x + x_0) Q^\alpha(\phi)(t, x, y) d(t, x, y).$$

Here, σ is a known number which is determined by Theorem 1.2. Since λ_1 is bounded and continuous, it follows from Lemma 3.1 that

$$\lim_{\alpha \rightarrow 0} \alpha^{-(2n+1-\sigma)} K[\phi_{\alpha, x_0}] = \int_{\mathbb{R}^{1+n+n}_{(t,x,y)}} \lambda_1(x_0) Q^0(\phi)(t, x, y) d(t, x, y).$$

Thus, we have (1.5).

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