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Title	On the semi-relativistic Hartree type equation
Author(s)	Cho, Yonggeun; Ozawa, Tohru
Citation	Hokkaido University Preprint Series in Mathematics, 773, 1-16
Issue Date	2006
DOI	https://doi.org/10.14943/83923
Doc URL	https://hdl.handle.net/2115/69581
Type	departmental bulletin paper
File Information	re773.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

ON THE SEMI-RELATIVISTIC HARTREE TYPE EQUATION

YONGGEUN CHO AND TOHRU OZAWA

Abstract. We study the global Cauchy problem and scattering problem for the semi-relativistic equation in $\mathbb{R}^n, n \geq 1$ with nonlocal nonlinearity $F(u) = \lambda(|x|^{-\gamma} * |u|^2)u, 0 < \gamma < n$. We prove the existence and uniqueness of global solutions for $0 < \gamma < \frac{2n}{n+1}, n \geq 2$ or $\gamma > 2, n \geq 3$ and the non-existence of asymptotically free solutions for $0 < \gamma \leq 1, n \geq 3$. We also specify asymptotic behavior of solutions as the mass tends to zero and infinity.

1. Introduction

In this paper we consider the following Cauchy problem:

$$\begin{cases} i\partial_t u = \sqrt{m^2 - \Delta}u + F(u), \ \text{ in } \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\ u(x,0) = \varphi(x) \ \text{ in } \ \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $m > 0$ denotes the mass of bosons in units $\hbar = c = 1$, $F(u)$ is nonlinear functional of Hartree type such that $F(u) = (V_\gamma * |u|^2)u$, where $*$ denotes the convolution in $\mathbb{R}^n$, $V_\gamma(x) = \lambda|x|^{-\gamma}$ for some fixed constant $\lambda \in \mathbb{R}$, and $0 < \gamma < n$.

The equation (1.1) is called a semi-relativistic Hartree equation which was used to describe Boson stars. See [7, 8, 17] and the references therein.

The purpose of this paper is to establish the local and global existence theory to the equation (1.1) and the scattering theory of the global solutions. In this paper we study the Cauchy problem (1.1) in the form of the integral equation:

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \tag{1.2}$$

where

$$U(t)\varphi(x) = (e^{-it\sqrt{m^2-\Delta}}\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t\sqrt{m^2+|\xi|^2})}\widehat{\varphi}(\xi)\,d\xi.$$

Here $\widehat{\varphi}$ denotes the Fourier transform of φ such that $\widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$.

One of the key tools for the existence and scattering is the conservation law. If the solution u of (1.1) has sufficient decay at infinity and smoothness, it satisfies two conservation laws:

$$\begin{aligned} \|u(t)\|_{L^2} &= \|\varphi\|_{L^2}, \\ E(u) \equiv K_m(u) + V(u) &= E(\varphi), \end{aligned} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 35Q40, 35Q55, 47J35.

Key words and phrases. semi-relativistic Hartree type equation, global solution, scattering, non-existence of asymptotically free solution .

The first author is JSPS Research Fellow.

where $K_m(u) = \frac{1}{2}\langle\sqrt{m^2-\Delta}\,u, u\rangle$, $V(u) = \frac{1}{4}\langle F(u), u\rangle$ and $\langle,\rangle$ is the complex inner product in L^2. For actual proof of (1.3) a regularizing method is simply applicable as in [18] in the case of $0 < \gamma \le 1$. For local solutions constructed by a contraction argument based on the Strichartz estimate stated below, the case of $1 < \gamma \le 2$ is treated by exactly the same method as in [26] without using approximate or regularizing approach.

In Section 2, a local existence is shown for $0 < \gamma < n$ and $\varphi \in H^s$ with $s \ge \frac{\gamma}{2}$ by the Plancheral theorem and the standard contraction mapping theorem without resort to a Strichartz estimate. Then we use the conservation laws to obtain the global existence for $s \ge \frac{\gamma}{2}$, $0 < \gamma \le 1$, $n \ge 2$ and $0 < \gamma < 1$, $n = 1$. This result is an extension of the work of Lenzmann [18] in which global well-posedness is considered for a Coulomb type potential in 3 space dimensions. From the energy conservation, we get uniform bound on the mass m on any finite time interval, if m is bounded from above, and then get a strong convergence of solutions of (1.1) to a solution of the equation without mass. However if m is large, then the kinetic energy $K_m(u)$ is not bounded globally in time any more. Instead, we can get a uniform bound of local solutions in H^s, provided $s \ge \frac{\gamma}{2}$. Then after a phase modulation, we prove the modulated solution is closely approximated by a solution of a Schrödinger equation of Hartree type if m is sufficiently large. This phenomenon can be interpreted as a kind of non-relativistic limit and eventually as a semi-classical or vanishing dispersion limit. See Proposition 2.5 below.

The second tool is the Strichartz estimate. We consider the following Strichartz estimate for the unitary group $U(t)$ (see [19, 20]):

$$\begin{gathered} \|U(t)\varphi\|_{L_T^{q_0}H_{r_0}^{s-\sigma_0}} \lesssim \|\varphi\|_{H^{s_0}}, \\ \left\|\int_0^t U(t-t')f(t')\,dt'\right\|_{L_T^{q_1}H_{r_1}^{s_1-\sigma_1}} \lesssim \|f\|_{L_T^1 H^{s_1}}, \end{gathered} \tag{1.4}$$

where $(q_i, r_i), i = 0, 1$, satisf that for any $\theta \in [0, 1]$

$$\begin{gathered} \frac{2}{q_i} = (n-1+\theta)\left(\frac{1}{2}-\frac{1}{r_i}\right), \quad 2\sigma_i = (n+1+\theta)\left(\frac{1}{2}-\frac{1}{r_i}\right), \\ 2 \le q_i, r_i \le \infty, \quad (q_i, r_i) \ne (2, \infty). \end{gathered} \tag{1.5}$$

We call the pair (q, r, σ) satisfying (1.5) *admissible pair*. If $\theta = 0$, it is called wave admissible and if $\theta = 1$, then Schrödinger admissible. Here $H_r^s = (1-\Delta)^{-s/2}L^r$ is the usual Sobolev space and $H^s = H_2^s$. Hereafter, we denote the space $L_T^q(B)$ by $L^q(0, T; B)$ and its norm by $\|\cdot\|_{L_T^q B}$ for some Banach space B, and also $L^q(B)$ with norm $\|\cdot\|_{L^q B}$ by $L^q(0, \infty; B)$, $1 \le q \le \infty$.

In Section 3, we consider the global existence and scattering in case where $0 < \gamma < n$. We first show the local existence for $0 < \gamma < n$, $n \ge 1$ and s slightly less than $\frac{\gamma}{2}$ by the Strichartz estimate of non-endpoint wave admissible pairs. Then we extend the local solution to the global one for $0 < \gamma < \frac{2n}{n+1}$ by the energy conservation and continuation procedure. The gain of upper bound $\frac{2n}{n+1}$ follows

from the fact that the Sobolev exponent s can be made smaller than $\frac{\gamma}{2}$, which enables us to use the continuation procedure. Secondly, we get a small data global existence results and scattering for the case $2 < \gamma < n$ and $n \geq 3$ by using the endpoint Strichartz estimate for Schrödinger admissible pair.

In the last section, as the usual case of nonlinearity with long range potential, non-existence of nontrivial asymptotically free solutions is shown for the case $0 < \gamma \leq 1, n \geq 3$ and $0 < \gamma < \frac{n}{2}, n = 1, 2$ by a similar method applied to a large class of dispersive equations. See [2, 5, 11, 12, 21, 32].

Until now, it remains open to show the global existence for $\frac{2n}{n+1} \leq \gamma \leq 2$ as well as the scattering for $1 < \gamma \leq 2$.

There is a large literature on partial differential equations with Hartree type nonlinearity. We refer the reader to [4, 9, 10, 13, 14, 15, 25, 23, 24] for Schrödinger related equations, to [1, 22, 27, 28, 31, 30, 33] for Klein-Gordon related equations in both massive and massless cases.

If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \leq CB$ and $A \geq C^{-1}B$, respectively. Different positive constants possibly depending on n, m, λ and γ might be denoted by the same letter C. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

Acknowledgments. The authors would like to thank Professor Kenji Nakanishi for enlightening discussions.

2. Global existence I

In this section, we study the global existence and the limiting problem as $m \to 0$ or as $m \to \infty$ with $0 < \gamma \leq 1$.

Let us first introduce the following local existence result.

Proposition 2.1. *Let $0 < \gamma < n$ and $n \geq 1$. Suppose $\varphi \in H^s(\mathbb{R}^n)$ with $s \geq \frac{\gamma}{2}$. Then there exists a positive time T independent of m such that (1.2) has a unique solution $u \in C([0,T]; H^s)$ with $\|u\|_{L^\infty_T H^s} \leq C\|\varphi\|_{H^s}$, where C does not depend on m.*

Proof. Let $(X^s_{T,\rho}, d)$ be a complete metric space with metric d defined by

$$X^s_{T,\rho} = \{u \in L^\infty_T(H^s(\mathbb{R}^n)) : \ \|u\|_{L^\infty_T H^s} \leq \rho\}, \ \ d(u,v) = \|u - v\|_{L^\infty_T L^2}.$$

Now we define a mapping $N : u \mapsto N(u)$ on $X^s_{T,\rho}$ by

$$N(u)(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')\,dt'. \tag{2.1}$$

Our strategy is to use the standard contraction mapping argument. To do so, let us introduce a generalized Leibniz rule (see Lemma A1 $\sim$ Lemma A4 in Appendix of [16]).

Lemma 2.2. *For any $s \geq 0$ we have*

$$\|D^s(uv)\|_{L^r} \lesssim \|D^s u\|_{L^{r_1}} \|v\|_{L^{q_2}} + \|u\|_{L^{q_1}} \|D^s v\|_{L^{r_2}},$$

where $D^s = (-\Delta)^{s/2}$

and $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{q_2} = \dfrac{1}{q_1} + \dfrac{1}{r_2}, \quad r_i \in (1,\infty),\ q_i \in (1,\infty], \quad i = 1,2.$

Then for all $u \in X(T,\rho)$ we have

$$\begin{aligned}
\|N(u)\|_{L^\infty_T H^s} &\leq \|\varphi\|_{H^s} + T\|F(u)\|_{L^\infty_T H^s} \\
&\lesssim \|\varphi\|_{H^s} + T\left(\|I_{n-\gamma}(|u|^2)\|_{L^\infty_T L^\infty} \|u\|_{L^\infty_T H^s} \right. \\
&\qquad\qquad \left. + \|I_{n-\gamma}(|u|^2)\|_{L^\infty_T H^s_{\frac{2n}{\gamma}}} \|u\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\right) \qquad (2.2)\\
&\lesssim \|\varphi\|_{H^s} + T\left(\|u\|^2_{L^\infty_T H^{\frac{\gamma}{2}}} \|u\|_{L^\infty_T H^s} + \|u\|^2_{L^\infty_T L^{\frac{2n}{n-\gamma}}} \|u\|_{L^\infty_T H^s}\right) \\
&\lesssim \|\varphi\|_{H^s} + T\|u\|^2_{L^\infty_t H^{\frac{\gamma}{2}}} \|u\|_{L^\infty_t H^s} \lesssim \|\varphi\|_{H^s} + T\rho^3,
\end{aligned}$$

where I_α is the fractional integral operator given by $I_\alpha(v)(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} v(y)\, dy$. It is well-known that I_α satisfies the inequality (see [29] for instance)

$$\|I_\alpha(\psi)\|_{L^q} \lesssim \|\psi\|_{L^p},\ \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, 1 < p < q < \infty.$$

For the third inequality we used the fractional integral inequality, generalized Leibniz rule (Lemma 2.2) and the fact that

$$\sup_{x\in\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{|u(x-y)|^2}{|y|^\gamma}\, dy \right| \lesssim \|u\|^2_{\dot{H}^{\frac{\gamma}{2}}}. \qquad (2.3)$$

For the last one, we used the Sobolev embedding $H^{\frac{\gamma}{2}} \hookrightarrow L^{\frac{2n}{n-\gamma}}$.

If we choose ρ and T such as $\|\varphi\|_{H^s} \leq \rho/2$ and $CT\rho^3 \leq \rho/2$, then N maps $X^s_{T,\rho}$ to itself.

Now we have only to show that N is a Lipschitz map for sufficiently small T. Let $u, v \in X^s_{T,\rho}$. Then we have

$$\begin{aligned}
&d(N(u), N(v)) \\
&\quad\lesssim T\left\|I_{n-\gamma}(|u|^2)u - I_{n-\gamma}(|v|^2)v\right\|_{L^\infty_T L^2} \\
&\quad\lesssim T\left(\left\|I_{n-\gamma}(|u|^2)(u-v)\right\|_{L^\infty_T L^2} + \left\|I_{n-\gamma}(|u|^2 - |v|^2)v\right\|_{L^\infty_T L^2}\right) \\
&\quad\lesssim T\left(\|u\|^2_{L^\infty_T H^{\frac{\gamma}{2}}} d(u,v) + \|I_{n-\gamma}(|u|^2 - |v|^2)\|_{L^\infty_T L^{\frac{2n}{\gamma}}} \|v\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}\right) \\
&\quad\lesssim T(\rho^2 d(u,v) + \rho\||u|^2 - |v|^2\|_{L^\infty_T L^{\frac{2n}{2n-\gamma}}}) \\
&\quad\lesssim T(\rho^2 + \rho(\|u\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}} + \|v\|_{L^\infty_T L^{\frac{2n}{n-\gamma}}}))d(u,v) \\
&\quad\lesssim T\rho^2 d(u,v).
\end{aligned}$$

The above estimate implies that the mapping N is a contraction, if T is sufficiently small.

The uniqueness and time continuity follows easily from the equation (1.2) and contraction argument. This completes the proof of proposition. □

From the conservation laws (1.3), we get the following global well-posedness.

Theorem 2.3. *Let $0 < \gamma \le 1$ for $n \ge 2$, $0 < \gamma < 1$ for $n = 1$ and $s \ge \frac{1}{2}$. Let T^* be the maximal existence time of the solution u as in Theorem 2.1. Then if $\lambda \ge 0$, or if $\lambda < 0$ and $\|\varphi\|_{L^2}$ is sufficiently small, then $T^* = \infty$. Moreover $\|u(t)\|_{H^s} \le C\|\varphi\|_{H^s} e^{C(E(\varphi)+\|\varphi\|_{L^2})t}$, where C does not depend on m.*

Proof. From the estimate (2.3) and L^2 conservation, we have

$$|V(u)| \lesssim \|u\|^2_{\dot{H}^{\frac{\gamma}{2}}} \|u\|_{L^2} \lesssim \|u\|^{2\gamma}_{\dot{H}^{\frac{1}{2}}} \|\varphi\|^{3-2\gamma}_{L^2}.$$

Hence if $\lambda \ge 0$ or if $\lambda < 0$ and $\|\varphi\|_{L^2}$ is sufficiently small, then for some $\theta > 0$

$$\|u(t)\|^2_{\dot{H}^{\frac{\gamma}{2}}} \le C(E(u) + \|\varphi\|^\theta_{L^2}) = C(E(\varphi) + \|\varphi\|^\theta_{L^2}). \tag{2.4}$$

From (2.4) and a similar estimate to (2.2), we have

$$\begin{aligned}\|u(t)\|_{H^s} &\lesssim \|\varphi\|_{H^s} + \int_0^t \|u\|^2_{H^{\frac{\gamma}{2}}} \|u\|_{H^s}\, dt' \\ &\lesssim \|\varphi\|_{H^s} + (E(\varphi) + \|\varphi\|^\theta_{L^2}) \int_0^t \|u\|_{H^s}\, dt'.\end{aligned}$$

Gronwall's inequality shows that

$$\|u(t)\|_{H^s} \le C\|\varphi\|_{H^s} \exp(C(E(\varphi) + \|\varphi\|^\theta_{L^2})t).$$

This completes the proof. □

If m is bounded above, then the energy $E(\varphi)$ is also bounded and hence the H^s norm of solution u is bounded in a finite time interval uniformly on small m. This enables us to treat a limit problem as $m \to 0$. We have the following. See [1] for related second order equations.

Proposition 2.4. *If $u_m \in (C \cap L^\infty)(H^s)$ be the solution of (1.2) as in Theorem 2.3, then for any finite time T, $u_m \to u_0$ in $L^\infty_T(H^s)$ with $s \ge \frac{1}{2}$ as $m \to 0$, where u_0 is the global solution to the Cauchy problem*

$$i\partial_t u_0 = \sqrt{-\Delta} u_0 + F(u_0), \quad u_0(x,0) = \varphi(x). \tag{2.5}$$

Proof. One can easily show the global existence of (2.5) by the same argument as in the proof of Theorem 2.3. The solution u_0 can be written as

$$u_0(t) = U_0(t)\varphi - i\int_0^t U_0(t-t')F(u_0)(t')\, dt',$$

where $U_0(t) = e^{-it\sqrt{-\Delta}}$.

For any $T>0$ there exists M such that $\sup_{0<m\le 1}(\|u_m\|_{L_T^\infty H^s}+\|u_0\|_{L_T^\infty H^s})\le M$. Then we first observe that for any $t\in[0,T]$

$$\begin{aligned}
&\|u_m(t)-u_0(t)\|_{H^s}\\
&\le \|(U(t)-U_0(t))\varphi\|_{H^s}\\
&\qquad+\int_0^t (\|F(u_m)-F(u_0)\|_{H^s}+\|(U(t-t')-U_0(t-t'))F(u_0)\|_{H^s})\,dt'\\
&\lesssim Tm\|\varphi\|_{H^s}+\int_0^t \|I_{n-\gamma}(|u_m|^2-|u_0|^2)u_m\|_{H^s}\,dt'\\
&\qquad+\int_0^t \|I_{n-\gamma}(|u_0|^2)(u_m-u_0)\|_{H^s})\,dt'+mT\int_0^t\|F(u_0)\|_{H^s}\,dt'.
\end{aligned}\tag{2.6}$$

From Lemma 2.2, fractional integration and the estimate (2.3), it follows that

$$\begin{aligned}
&\|I_{n-\gamma}(|u_m|^2-|u_0|^2)u_m\|_{H^s}\\
&\lesssim \|I_{n-\gamma}(|u_m|^2-|u_0|^2)\|_{L^\infty}\|u_m\|_{H^s}+\|I_{n-\gamma}(|u_m|^2-|u_0|^2)\|_{H^s_{\frac{2n}{\gamma}}}\|u_m\|_{L^{\frac{2n}{n-\gamma}}}\\
&\lesssim \|u_m-u_0\|_{H^{\frac{\gamma}{2}}}(\|u_m\|_{H^{\frac{\gamma}{2}}}+\|u_0\|_{H^{\frac{\gamma}{2}}})\|u_m\|_{H^s}\\
&\qquad+\||u_m|^2-|u_0|^2\|_{L^{\frac{2n}{2n-\gamma}}}\|u_m\|_{H^{\frac{\gamma}{2}}}\\
&\lesssim M^2\|u_m-u_0\|_{H^s}
\end{aligned}$$

and similarly that

$$\begin{aligned}
&\|I_{n-\gamma}(|u_0|^2)(u_m-u_0)\|_{H^s}\\
&\lesssim \|u_0\|^2_{H^{\frac{\gamma}{2}}}\|u_m-u_0\|_{H^s}+\|I_{n-\gamma}(|u_0|^2)\|_{H^s_{\frac{2n}{\gamma}}}\|u_m-u_0\|_{L^{\frac{2n}{n-\gamma}}}\\
&\lesssim M^2\|u_m-u_0\|_{H^s}.
\end{aligned}$$

Substituting these into (2.6), we have for any $t\in[0,T]$ that

$$\|u_m(t)-u_0(t)\|_{H^s}\lesssim MTm+M^3mT^2+M^2\int_0^t\|u_m(t')-u_0(t')\|_{H^s}\,dt'.$$

Then Gronwall's inequality implies the strong convergence $u_m\to u_0$ in $L^\infty_T(H^s)$. □

In the case of large mass, the situation is different. Since $E(u)=E(\varphi)=\frac{1}{2}\langle\sqrt{m^2-\Delta}\varphi,\varphi\rangle+V(\varphi)$ diverges as $m\to\infty$, it is difficult to obtain the uniform bound for $\|u\|_{H^{\frac{1}{2}}}$ from the energy conservation law. However, from the Proposition 2.1 we see that the local existence time T and the constant C can be chosen independently of the mass m, if $s\ge\frac{\gamma}{2}$. To be more specific, we have $\|u_m(t)\|_{H^s}\le C\|\varphi\|_{H^s}$, where u_m is the solution of the equation with mass m. Now using the phase modulation $v_m=e^{imt}u_m$, the function v_m satisfies the equation

$$i\partial_t v_m=(\sqrt{m^2-\Delta}-m)v_m+F(v_m),\ \ v_m(0)=\varphi,$$

and equivalently

$$v_m(t)=U_m(t)\varphi-i\int_0^t U_m(t-t')F(v_m)(t')\,dt',\tag{2.7}$$

where $U_m(t) = e^{-it(\sqrt{m^2-\Delta}-m)}$. Let $\widetilde{U}_m$ be the unitary group $e^{-it\frac{1}{2m}\Delta}$. As was first observed by Segal [28] at a formal level, we expect that the linear solutions $U_m\varphi$ and $\widetilde{U}_m\varphi$ become very close in $L^\infty_T(H^s)$ norm, if T is finite and $\varphi \in H^s$. That observation is in fact justified by

$$\begin{aligned}
&\|(U_m(t) - \widetilde{U}_m(t))\varphi\|^2_{L^\infty_T H^s} \\
&\quad\le \sup_{0<t<T} \int_{|\xi|\le\sqrt{m}} \left|1 - e^{-it(\sqrt{m^2+|\xi|^2}-m-\frac{|\xi|^2}{2m})}\right|^2 (1+|\xi|)^{2s}|\widehat{\varphi}|^2\,d\xi \\
&\qquad + 2\int_{|\xi|\ge\sqrt{m}} (1+|\xi|)^{2s}|\widehat{\varphi}|^2\,d\xi \\
&\quad\le T\int \left|2m/(\sqrt{m^2+|\xi|^2}+m) - 1\right|^2 (1+|\xi|)^{2s}|\widehat{\varphi}|^2\,d\xi \\
&\qquad + 2\int_{|\xi|\ge\sqrt{m}} (1+|\xi|)^{2s}|\widehat{\varphi}|^2\,d\xi \\
&\quad\to 0 \ \text{ as } \ m\to\infty.
\end{aligned}$$

Hence we can expect that v_m is very close to a function w_m in $L^\infty_T(H^s)$, where w_m is a solution of the nonlinear Schrödinger equation:

$$i\partial_t w_m = \frac{1}{2m}\Delta w_m + F(w_m), \ \ w_m(0) = \varphi. \tag{2.8}$$

Of course, by the same argument as the proof of Proposition 2.1, we find T and C independent of m and a unique solution $w_m \in C([0,T];H^s)$ of the equation (2.8) for $s \ge \frac{\gamma}{2}$ such that $\|w_m\|_{L^\infty_T H^s} \le C\|\varphi\|_{H^s}$.

Now let $T^*_{v_m}$ and $T^*_{w_m}$ be the maximal existence time of the solutions u_m and w_m, respectively. Then from the local existence result (Proposition 2.1) we deduce that $T^* \equiv \inf_{m>1}\min(T^*_{v_m}, T^*_{w_m})$ is strictly positive and have the following.

Proposition 2.5. *If $s \ge \frac{\gamma}{2}$ and $T < T^*$, then $v_m - w_m \to 0$ in $L^\infty_T(H^s)$ as $m\to\infty$.*

Proof. First we consider the integral equation

$$u_\infty = \varphi - i\int_0^t F(u_\infty)\,dt',$$

which is equivalent to the ordinary differential equation $i\partial_t u_\infty = F(u_\infty),\ u_\infty(x,0) = \varphi$. This equation has an exact solution $u_\infty(x,t) = \varphi(x)e^{-i\lambda t(|\cdot|^{-\gamma} * |\varphi|^2)(x)}$ for any $t \ge 0$. If $s \ge \frac{\gamma}{2}$, then the uniqueness of u_∞ is guaranteed.

To prove $v_m - w_m \to 0$ in $L^\infty_T(H^s)$, we have only to prove that $v_m - u_\infty \to 0$ in $L^\infty_T(H^s)$ and $w_m - u_\infty \to 0$ in $L^\infty_T(H^s)$. At first we have

$$\begin{aligned}
&\|v_m(t) - u_\infty(t)\|_{H^s} \\
&\le \|\,(U_m(t)-1)\,\varphi\|_{H^s} + \int_0^t \|(U_m(t-t')-1)F(u_\infty)\|_{H^s}\,dt' \\
&\qquad\qquad + \int_0^t \|F(v_m) - F(u_\infty)\|_{L^\infty}\,dt'.
\end{aligned}$$

and

$$
\begin{aligned}
&\|\,(U_m(t)-1)\,\varphi\|_{H^s}^2\\
&=\int\left|e^{-it(\sqrt{m^2+|\xi|^2}-m)}-1\right|^2|\widehat{\varphi}(\xi)|^2\,d\xi\\
&=\int_{|\xi|\le m^{\frac14}}+\int_{|\xi|>m^{\frac14}}\\
&\le\int_{|\xi|\le m^{\frac14}}\frac{t^2|\xi|^4}{(\sqrt{m^2+|\xi|^2}+m)^2}(1+|\xi|)^{2s}|\widehat{\varphi}(\xi)|^2\,d\xi+4\int_{|\xi|>m^{\frac14}}(1+|\xi|)^{2s}|\widehat{\varphi}(\xi)|^2\,d\xi\\
&=\frac{T^2}{4m}\|\varphi\|_{H^s}^2+4\int_{|\xi|>m^{\frac14}}(1+|\xi|)^{2s}|\widehat{\varphi}(\xi)|^2\,d\xi\\
&\to 0\ \text{ as }\ m\to\infty.
\end{aligned}
$$

We take $M=M(T)$ such that $\sup_{m\ge1}(\|v_m\|_{L_T^\infty H^s}+\|w_m\|_{L_T^\infty H^s}+\|u_\infty\|_{L_T^\infty H^s})\le M$. Then since $F(u_\infty)\in L_T^\infty(H^s)$, we have

$$\int_0^T\|(U_m(t-t')-1)F(u_\infty)\|_{H^s}\,dt'\to 0\ \text{ as }\ m\to\infty.$$

We also have

$$\|F(v_m)-F(u_\infty)\|_{H^s}\le CM^2\|v_m-u_\infty\|_{H^s}.$$

Thus

$$\|v_m(t)-u_\infty(t)\|_{H^s}\le o(1)+CM^2\int_0^t\|v_m-u_\infty\|_{H^s}\,dt' \tag{2.9}$$

and as for w_m by the same argument as that of v_m

$$\|w_m(t)-u_\infty(t)\|_{H^s}\le o(1)+CM^2\int_0^t\|w_m-u_\infty\|_{H^s}\,dt'. \tag{2.10}$$

Therefore Gronwall's inequality yields the claim. □

3. Global existence II

In this section, we reexamine the existence result and get a slightly low regularity by using Strichartz estimate. The first result is the following local existence for $0<\gamma<n$.

Proposition 3.1. *Let $0<\gamma<n$ and $n\ge2$. Then there is a number α with $0<\alpha<\min(\gamma,\frac{2n}{n-1})$ satisfying that given $s>\frac{\gamma}{2}-\frac{(n-1)\alpha}{4n}$ and $\varphi\in H^s$ there exists a positive time T such that (1.2) has a unique solution $u\in C([0,T];H^s)\cap L_T^q(H_r^{s-\sigma})$, where $q=\frac{4n}{(n-1)\alpha}$, $r=\frac{2n}{n-\alpha}$ and $\sigma=\frac{(n+1)\alpha}{4n}$.*

Proof. Given n and γ, choose a number α with $0<\alpha<\min(\gamma,\frac{2n}{n-1})$ and fix $s>\frac{\gamma}{2}-\frac{(n-1)\alpha}{4n}$. Then for some positive number T to be chosen later, let us define a complete metric space $(Y_{T,\rho}^s,d_T)$ with metric d_T by

$$
\begin{aligned}
&Y_{T,\rho}^s=\left\{v\in L_T^\infty(H^s)\cap L_T^q(H_r^{s-\sigma}):\|v\|_{L_T^\infty H^s}+\|v\|_{L_T^qH_r^{s-\sigma}}\le\rho\right\},\\
&d_T(u,v)=\|u-v\|_{L_T^\infty H^s\cap L_T^qH_r^{s-\sigma}},
\end{aligned}
$$

where q, r, σ are the same indices as in Proposition 3.1.

From now on, we will prove that the nonlinear mapping N defined as (2.1) is a contraction on $Y^s_{T,\rho}$, provided T is sufficiently small. We will use the following lemma instead of (2.3), which follows by estimating the (fractional) integral inside and outside of the ball with radius $R > 0$ separately by Hölder's inequality and by minimizing the resulting estimates with respect to R.

Lemma 3.2. *Let $0 < \gamma < n$. Then for any $0 < \varepsilon < n - \gamma$ we have*

$$\left\| I_{n-\gamma}(|u|^2) \right\|_{L^\infty} \lesssim \|u\|_{L^{\frac{2n}{n-\gamma-\varepsilon}}} \|u\|_{L^{\frac{2n}{n-\gamma+\varepsilon}}}.$$

If we take $\theta = 0$ in the Strichartz estimate (1.4), then the pair $(q, r, \sigma) = \left(\frac{(n-1)\alpha}{4n}, \frac{2n}{n-\alpha}, \frac{(n+1)\alpha}{4n}\right)$ becomes an admissible one. Hence the Strichartz estimate together with Plancheral theorem, Lemma 3.2 and generalized Leibniz rules (Lemma 2.2), enables us to deduce that for sufficiently small ε

$$\begin{aligned}
&\|N(u)\|_{L^\infty_T H^s \cap L^q_T H^{s-\sigma}_r} \\
&\quad\lesssim \|\varphi\|_{H^s} + \|F(u)\|_{L^1_T H^s} \\
&\quad\lesssim \|\varphi\|_{H^s} + \|I_{n-\gamma}(|u|^2)\|_{L^1_T L^\infty} \|u\|_{L^\infty_T H^s} \\
&\qquad\qquad + \int_0^T \|I_{n-\gamma}(|u|^2)\|_{H^s_{\frac{2n}{\gamma+\varepsilon}}} \|u\|_{L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \, dt \\
&\quad\lesssim \|\varphi\|_{H^s} + \|u\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{L^\infty_T H^s} \\
&\qquad\qquad + \int_0^T \||u|^2\|_{H^s_{\frac{2n}{2n-(\gamma-\varepsilon)}}} \|u\|_{L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \, dt \\
&\quad\lesssim \|\varphi\|_{H^s} + \|u\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{L^\infty_T H^s}.
\end{aligned} \tag{3.1}$$

Using Hölder's inequality for time integral, we have

$$\begin{aligned}
&\|N(u)\|_{L^\infty_T H^s \cap L^q_T H^{s-\sigma}_r} \\
&\quad\lesssim \|\varphi\|_{H^s} + T^{1-\frac{2}{q}} \|u\|_{L^q_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \|u\|_{L^q_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u\|_{L^\infty_T H^s}.
\end{aligned} \tag{3.2}$$

Now if we choose $\varepsilon > 0$ so small that $\varepsilon < \min\left(\gamma - \alpha, 2(s + \frac{(n-1)\alpha}{4n}) - \gamma\right)$, then since

$$\frac{2n}{n-\alpha} \le \frac{2n}{n-(\gamma-\varepsilon)} < \frac{2n}{n-(\gamma+\varepsilon)} \le \frac{2n}{n-\alpha-2(s-\sigma)},$$

we have from (3.2) and Sobolev embedding $H^{s-\sigma}_r \hookrightarrow L^r \cap L^{\frac{2n}{n-\alpha-2(s-\sigma)}}$ that

$$\begin{aligned}
\|N(u)\|_{L^\infty_T H^s \cap L^q_T H^{s-\sigma}_r} &\le C(\|\varphi\|_{H^s} + T^{1-\frac{2}{q}} \|u\|_{L^\infty_T H^s} \|u\|^2_{L^q_T H^{s-\sigma}_r}) \\
&\le C(\|\varphi\|_{H^s} + T^{1-\frac{2}{q}} \rho^3)
\end{aligned}$$

for some constant C. Here we used the conventional embedding that if $2(s-\sigma) \ge n - \alpha$ then $H^{s-\sigma}_r \hookrightarrow L^{r_1}$ for any $r_1 \ge r$. Thus if we choose ρ and T so that $C\|\varphi\|_{H^s} \le \frac{\rho}{2}$ and $CT^{1-\frac{2}{q}}\rho^3 \le \frac{\rho}{2}$, then we conclude that N maps from $Y^s_{T,\rho}$ to itself.

For any $u, v \in Y^s_{T,\rho}$, we have

$$\begin{aligned} d_T(N(u), N(v)) &\lesssim \|F(u) - F(v)\|_{L^1_T H^s} \\ &\lesssim \|I_{n-\gamma}(|u|^2 - |v|^2)u\|_{L^1_T H^s} + \|I_{n-\gamma}(|v|^2)(u-v)\|_{L^1_T H^s}. \end{aligned} \tag{3.3}$$

By Lemma 3.2 and Hölder's inequality, we have for sufficiently small $\varepsilon > 0$

$$\begin{aligned} &\|I_{n-\gamma}(|u|^2 - |v|^2)u\|_{L^1_T H^s} \\ &\quad\lesssim \|I_{n-\gamma}(|u|^2 - |v|^2)\|_{L^2_T L^\infty} \|u\|_{L^\infty_T H^s} \\ &\qquad\quad + \|I_{n-\gamma}(|u|^2 - |v|^2)\|_{L^2_T H^s_{\frac{2n}{\gamma+\varepsilon}}} \|u\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \\ &\quad\lesssim \rho \||u|^2 - |v|^2\|^{\frac12}_{L^1_T L^{\frac{n}{n-(\gamma+\varepsilon)}}} \||u|^2 - |v|^2\|^{\frac12}_{L^1_T L^{\frac{n}{n-(\gamma-\varepsilon)}}} \\ &\qquad + \rho \|u-v\|_{L^\infty_T H^s} (\|u\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} + \|v\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}}) \\ &\qquad + \rho \|u-v\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} (\|u\|_{L^\infty_T H^s} + \|v\|_{L^\infty_T H^s}). \end{aligned} \tag{3.4}$$

Now by another Hölder's inequality with respect to the time variable, we have

$$\|I_{n-\gamma}(|u|^2 - |v|^2)u\|_{L^1_T H^s} \lesssim T^{1-\frac{2}{q}} \rho^2 d_T(u,v).$$

Similarly,

$$\begin{aligned} &\|I_{n-\gamma}(|v|^2)(u-v)\|_{L^1_T H^s} \\ &\lesssim \|I_{n-\gamma}(|v|^2)\|_{L^1_T L^\infty} \|u-v\|_{L^\infty_T H^s} + \|I_{n-\gamma}(|v|^2)\|_{L^2_T L^{\frac{2n}{\gamma+\varepsilon}}} \|u-v\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} \\ &\lesssim \|v\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|v\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}} d_T(u,v) \\ &\quad + \|v\|_{L^\infty_T H^s} \|v\|_{L^2_T L^{\frac{2n}{n-(\gamma-\varepsilon)}}} \|u-v\|_{L^2_T L^{\frac{2n}{n-(\gamma+\varepsilon)}}}. \end{aligned} \tag{3.5}$$

Hence we get

$$\|I_{n-\gamma}(|v|^2)(u-v)\|_{L^1_T H^s} \lesssim T^{1-\frac{2}{q}} \rho^2 d_T(u,v).$$

Substituting these two estimates into (3.3) and then using the fact $CT^{1-\frac{2}{q}}\rho^2 \le \frac12$ for small T, we conclude that N is a contraction mapping. □

Remark 1. If we follow the proof above with the Schrödinger admissible pairs, we conclude that Proposition 3.1 holds for $n \ge 3$, $0 < \alpha < \gamma$, $\alpha \le 2$, $s > \frac{\gamma}{2} - \frac{n-2}{4n}\alpha$, $q = \frac{4}{\alpha}$, $r = \frac{2n}{n-2}$ and $\sigma = \frac{(n+2)\alpha}{4n}$.

Remark 2. In general, the Strichartz estimate (1.4) is not uniform on m. However, using Lemma 4 of [20] where a non-relativistic limit problem of Dirac equation is treated, we can find a solution u_m with uniform norms on m stated in Proposition 3.1 above and Theorem 3.3 below, provided $n = 3$ and m is large. Thus it is naturally expected that $v_m = e^{imt}u_m$ is very close to w_m the solution of Schrödinger equation (2.8). However, the Strichartz estimates of the solution w_m are not be uniform on m even on a finite time interval. This causes a trouble in the limit problem concerning a low regularity than $H^{\frac{\gamma}{2}}$. To overcome this difficulty, we need

a much more subtle estimate. But we will not pursue this topic here, which will be treated somewhere.

Now we show the local solutions can be extended globally in time by using the energy conservation law.

Theorem 3.3. *Let $0 < \gamma < \frac{2n}{n+1}$, $n \geq 2$. Then there exists an α with $0 < \alpha < \gamma$ such that if $\varphi \in H^{\frac{1}{2}}$ and if $\lambda > 0$, or $\lambda < 0$ but $\|\varphi\|_{L^2}$ is sufficiently small, then (1.2) has a unique solution $u \in C([0,\infty); H^{\frac{1}{2}}) \cap L^q_{loc}(H_r^{\frac{1}{2}-\sigma})$, where $q = \frac{4n}{(n-1)\alpha}$, $r = \frac{2n}{n-\alpha}$ and $\sigma = \frac{(n+1)\alpha}{4n}$.*

Proof. Let T^* be the maximal existence time and it be finite. The local existence theory shows that $\|u\|_{L^q_{T^*} H_r^{\frac{1}{2}-\sigma}} = \infty$. Since $\gamma < 2$, from the local existence Lemma 3.1, we see that the energy conservation law (1.3) holds. Thus at any $t < T^*$, the solution u satisfies that

$$\begin{aligned}
\frac{1}{2}\|u(t)\|^2_{H^{\frac{1}{2}}} &\leq E(u) + |V(u)| \\
&\leq E(\varphi) + C\|u\|^2_{L^{\frac{2n}{n-\gamma+1}}} \|u\|_{H^{\frac{1}{2}}} \\
&\leq E(\varphi) + C\|u\|^{2-\gamma}_{L^2} \|u\|^{\gamma}_{H^{\frac{1}{2}}} \\
&= E(\varphi) + C\|\varphi\|^{2-\gamma}_{L^2} \|u\|^{\gamma}_{H^{\frac{1}{2}}}.
\end{aligned}$$

and hence by Young's inequality

$$\|u(t)\|^2_{H^{\frac{1}{2}}} \leq CE(\varphi). \tag{3.6}$$

The smallness of $\|\varphi\|_{L^2}$ is used to guarantee the positivity of $E(\varphi)$ when $\lambda < 0$.

From the estimate (3.2) and (3.6), we have

$$\|u\|_{L^q_T H_r^{\frac{1}{2}-\sigma}} \lesssim CE(\varphi) + T^{1-\frac{2}{q}} E(\varphi)^{\frac{1}{2}} \|u\|^2_{L^q_T H_r^{\frac{1}{2}-\sigma}}.$$

Thus for sufficiently small T depending on $E(\varphi)$,

$$\|u\|_{L^q(T_{j-1},T_j;H_r^{\frac{1}{2}-\sigma})} \leq CE(\varphi),$$

where $T_j - T_{j-1} = T$ for $j \leq k-1$ and $T_k = T^*$ This means that

$$\|u\|^q_{L^q(0,T^*;H_r^{\frac{1}{2}-\sigma})} \leq \sum_{1\leq j\leq k} \|u\|^q_{L^q(T_{j-1},T_j;H_r^{\frac{1}{2}-\sigma})} \leq (kCE(\varphi))^q < \infty$$

and hence that $T^* = \infty$.

The condition $\gamma < \frac{2n}{n+1}$ is necessary for the existence of α satisfying $s = \frac{1}{2} > \frac{\gamma}{2} - \frac{(n+1)\alpha}{4n}$ and $\alpha < \gamma$. This completes the proof. □

Remark 3. If we choose $\theta = 1$, then we deduce the same result as in Theorem 3.3 with $0 < \gamma < \frac{2n}{n+2}$, $q = \frac{4}{\alpha}$, $r = \frac{2n}{n-\alpha}$ and $\sigma = \frac{(n+2)\alpha}{4n}$.

Now we consider the small data global existence and scattering for $2 < \gamma < n$.

Theorem 3.4. *Let $2 < \gamma < n$, $n \geq 3$ and $s > \frac{\gamma}{2} - \frac{n-2}{2n}$. Then there exists $\rho > 0$ such that for any $\varphi \in H^s$ with $\|\varphi\|_{H^s} \leq \rho$, (1.2) has a unique solution $u \in (C \cap L^\infty)(H^s) \cap L^2(H^{s-\frac{n+2}{2n}}_{\frac{2n}{n-2}})$, if $\|\varphi\|_{H^s}$ is sufficiently small. Moreover there is $\varphi^+ \in H^s$ such that*

$$\|u(t) - U(t)\varphi^+\|_{H^s} \to 0 \text{ as } t \to \infty.$$

Proof. We will use the Strichartz estimate (1.4) with $\theta = 1$ and endpoint admissible pair $(q, r, \sigma) = \left(2, \frac{2n}{n-2}, \frac{n+2}{2n}\right)$ (See Remark 1).

Let us define a complete metric space (Y^s_ρ, d) with metric d by

$$Y^s_\rho = \left\{ v \in L^\infty(H^s) \cap L^2(H^{s-\sigma}_r) : \|v\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r} \leq \rho \right\},$$
$$d(u, v) = \|u - v\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r},$$

Then from the estimate (3.2), we have

$$\|N(u)\|_{L^\infty H^s \cap L^2 H^{s-\sigma}_r} \leq C\|\varphi\|_{H^s} + C\|u\|^2_{L^2 H^{s-\sigma}_r}\|u\|_{L^\infty H^s}.$$

If we choose sufficiently small ρ such that $C\|\varphi\|_{H^s} \leq \frac{\rho}{2}$ and $C\rho^3 \leq \frac{\rho}{2}$, then N maps Y^s_ρ to itself. Similarly, from (3.3)–(3.5), one can show that $d(N(u), N(v)) \leq \frac{1}{2}d(u, v)$. This proves the existence part.

To prove the scattering, let us define a function φ^+ by

$$\varphi^+ = \varphi - i\int_0^\infty U(-t')F(u)(t')\,dt'.$$

Then since the solution u is in Y^s_ρ, $\varphi^+ \in H^s$, and therefore there holds

$$\begin{aligned}\|u(t) - u^+(t)\|_{H^s} &\lesssim \int_t^\infty \|F(u)\|_{H^s}\,dt' \\ &\lesssim \|u\|_{L^\infty H^s}\int_t^\infty \|u\|^2_{H^{s-\sigma}_r}\,dt' \to 0 \text{ as } t \to \infty.\end{aligned}$$

□

4. Non-existence of scattering

We prove the non-existence of non-trivial asymptotically free solution.

Theorem 4.1. *Assume that $0 < \gamma \leq 1$ for $n \geq 3$ and $0 < \gamma < \frac{n}{2}$ for $n = 1, 2$. Suppose that u is a smooth global solution to (1.1) and there exists a smooth function φ^+ such that*

$$\|u(t) - u^+(t)\|_{L^2} \to 0 \quad as \quad t \to \infty,$$

where $u^+(t) = U(t)\varphi^+$. Then $u = u^+ = 0$.

Proof. Let us define a function of $H(t)$ by

$$H(t) = sgn(\lambda)\mathrm{Re}\langle u(t), u^+(t)\rangle.$$

Then from the condition of u and u^+, $H(t)$ is uniformly bounded on t and by the regularization

$$\frac{d}{dt}H(t) = |\lambda|\mathrm{Im}\langle I_{n-\gamma}(|u|^2)u, u^+\rangle. \tag{4.1}$$

Suppose $\varphi^+ \neq 0$. Then we derive a contradiction to the uniform boundedness of H on t.

The integration in (4.1) is rewritten as

$$\langle I_{n-\gamma}(|u|^2)u, u^+\rangle = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= \langle I_{n-\gamma}(|u^+|^2)u^+, u^+\rangle, \\ J_2 &= \langle I_{n-\gamma}(|u|^2 - |u^+|^2)u^+, u^+\rangle, \\ J_3 &= \langle I_{n-\gamma}(|u|^2)(u - u^+), u^+\rangle. \end{aligned}$$

To estimate each J_i, we need the following time decay estimate.

Lemma 4.2. *If φ^+ is sufficiently smooth, then*

$$\|U(t)\varphi^+\|_{L^\infty} \lesssim t^{-\frac{n}{2}}.$$

As for J_2, from Lemma 3.2, we have

$$\begin{aligned} &|J_2(t)| \\ &\quad = |\langle(|u|^2 - |u^+|^2), I_{n-\gamma}(|u^+|^2)\rangle| \\ &\quad \le \|u - u^+\|_{L^2}(\|u\|_{L^2} + \|u^+\|_{L^2})\|I_{n-\gamma}(|u^+|^2)\|_{L^\infty} \\ &\quad \lesssim \|u - u^+\|_{L^2}(\|u\|_{L^2} + \|u^+\|_{L^2})\|u^+\|_{L^{\frac{2n}{n-\gamma-\varepsilon}}}\|u^+\|_{L^{\frac{2n}{n-\gamma+\varepsilon}}} \\ &\quad \lesssim \|u - u^+\|_{L^2}(\|u\|_{L^2} + \|u^+\|_{L^2})\|u^+\|_{L^2}^{\frac{n-\gamma-\varepsilon}{n}+\frac{n-\gamma+\varepsilon}{n}}\|u^+\|_{L^\infty}^{2-\frac{n-\gamma-\varepsilon}{n}+\frac{n-\gamma+\varepsilon}{n}} \\ &\quad \lesssim \|u - u^+\|_{L^2}(\|u\|_{L^2} + \|u^+\|_{L^2})\|u^+\|_{L^2}^{\frac{2(n-\gamma)}{n}}\|u^+\|_{L^\infty}^{\frac{2\gamma}{n}}. \end{aligned} \tag{4.2}$$

For the fourth inequality we used Hölder's inequality

$$\|u\|_{L^r} \le \|u\|_{L^2}^{\frac{2}{r}}\|u\|_{L^\infty}^{1-\frac{2}{r}}. \tag{4.3}$$

Now from Lemma 4.2 we get

$$|J_2(t)| = o(t^{-\gamma}). \tag{4.4}$$

Since $\gamma \le 1$ for $n \ge 3$ and $\gamma < \frac{n}{2}$ for $n = 1, 2$, we can take $\varepsilon > 0$ such that $\gamma + \varepsilon < \frac{n}{2}$. Hence by the same argument for J_2 we have for J_3 that

$$\begin{aligned}
|J_3(t)| &= |\langle |u|^2, I_{n-\gamma}((\overline{u} - \overline{u^+})u^+)\rangle| \\
&\le \|u\|_{L^2}^2 \|I_{n-\gamma}((\overline{u} - \overline{u^+})u^+)\|_{L^\infty} \\
&\lesssim \|u\|_{L^2}^2 \left\| |(\overline{u} - \overline{u^+})u^+|^{\frac{1}{2}} \right\|_{L^{\frac{2n}{n-\gamma-\varepsilon}}} \left\| |(\overline{u} - \overline{u^+})u^+|^{\frac{1}{2}} \right\|_{L^{\frac{2n}{n-\gamma+\varepsilon}}} \\
&\lesssim \|u\|_{L^2}^2 \left\| (\overline{u} - \overline{u^+})u^+ \right\|^{\frac{1}{2}}_{L^{\frac{n}{n-\gamma-\varepsilon}}} \left\| (\overline{u} - \overline{u^+})u^+ \right\|^{\frac{1}{2}}_{L^{\frac{n}{n-\gamma+\varepsilon}}} \qquad (4.5) \\
&\lesssim \|u\|_{L^2}^2 \|u - u^+\|_{L^2} \|u^+\|^{\frac{1}{2}}_{L^{\frac{2n}{n-2(\gamma+\varepsilon)}}} \|u^+\|^{\frac{1}{2}}_{L^{\frac{2n}{n-2(\gamma-\varepsilon)}}} \\
&\lesssim \|u\|_{L^2}^2 \|u - u^+\|_{L^2} \|u^+\|^{\frac{n-2\gamma}{n}}_{L^2} \|u^+\|^{1-\frac{n-2\gamma}{n}}_{L^\infty} \\
&= o(t^{-\gamma}).
\end{aligned}$$

As for J_1, if $|x| \le At$ for some $A > 1$, then for any $t > 0$

$$\begin{aligned}
I_{n-\gamma}(|u^+|^2)(x) &\ge \int_{|y| \le At} |x - y|^{-\gamma} |u^+(y)|^2 \, dy \\
&\ge \frac{1}{(2At)^\gamma} \int_{|y| \le At} |u^+(y)|^2 \, dy.
\end{aligned}$$

Now we prove

$$\int_{|y| \le At} |u^+(y)|^2 \, dy \gtrsim \|\varphi^+\|_{L^2}^2 \qquad (4.6)$$

for large t, provided φ^+ is sufficiently smooth. Choose a large R such that $\|\eta_R \varphi^+\|_{L^2}^2 \ge \frac{2}{3}\|\varphi^+\|_{L^2}^2$, where η_R is smooth cut-off function supported in the ball of radius $2R$ with center at the origin. Then

$$\|u^+\|_{L^2(|x| \le At)}^2 \ge \left| \|U(t)(\eta_R \varphi^+)\|_{L^2(|x| \le At)}^2 - \|\varphi^+\|_{L^2(|x| > R)}^2 \right|.$$

Since the linear solution u^+ has the finite propagation propagation speed (actually speed 1), one can easily show that $|U(t)(\eta_R \varphi^+)(x)| \lesssim |x|^{-N} \|\varphi^+\|_{L^2}$ for any N, provided $|x| > 1 + 2R + t$. Hence we deduce that if $N > \frac{n}{2}$ and t is large enough so that $At > 1 + 3R + t$, then

$$\begin{aligned}
\|U(t)(\eta_R \varphi^+)\|_{L^2(|x| \le At)}^2 &= \|U(t)(\eta_R \varphi^+)\|_{L^2}^2 - \int_{|x| > At} |U(t)\eta_R \varphi^+|^2 \, dx \\
&\ge \|\eta_R \varphi^+\|_{L^2} - C \int_{|x| > At} |x|^{-2N} \, dx \|\varphi^+\|_{L^2}^2 \\
&\ge \frac{2}{3}\|\varphi^+\|_{L^2}^2 - C(At)^{n-2N} \|\varphi^+\|_{L^2}^2.
\end{aligned}$$

Therefore for t large enough,

$$\|u^+\|_{L^2(|x| \le At)}^2 \ge \frac{1}{3}\|\varphi^+\|_{L^2}^2$$

and hence

$$J_1(t) \gtrsim t^{-\gamma}. \qquad (4.7)$$

Now combining (4.7) with (4.4) and (4.5), we deduce that for t sufficiently large

$$\frac{d}{dt}H(t) \gtrsim t^{-\gamma} \geq t^{-1}.$$

This is a contradiction to the uniform boundedness of $H(t)$ on t. □

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Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: ygcho@math.sci.hokudai.ac.jp

E-mail address: ozawa@math.sci.hokudai.ac.jp