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Scaling limit for a generalization of the Nelson model and its application to nuclear physics

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Abstract

We study a mathematically rigorous derivation of a quantum mechanical Hamiltonian in a general framework. We derive such a Hamiltonian by taking a scaling limit for a generalization of the Nelson model, which is an abstract interaction model between particles and a Bose field with some internal degrees of freedom.

Applying it to a model for the field of the nuclear force with an isospin, we obtain a Schrödinger Hamiltonian with a matrix valued potential, describing an effective interaction between nucleons.

Keywords: scaling limit, weak coupling limit, GSB model, Nelson model, effective potential, isospin, nuclear force

AMS Subject Classification: 81Q10

1 Introduction

We consider a scaling limit of an abstract quantum field theoretical Hamiltonian of particles coupled to a Bose field with some internal degrees of freedom such as a spin or an isospin. The purpose of this paper is to derive a quantum mechanical Hamiltonian in the scaling limit of such a quantum field theoretical Hamiltonian in a general framework.

A typical example of scaling limits is the weak coupling limit for a model of norelativistic quantum particles coupled to a Bose field whose Hamiltonian is given by

$$H = -\frac{1}{2M}\Delta \otimes I + I \otimes H_{\rm b} + gH_{\rm I},$$

where $M > 0$ denotes the mass of the particles, Δ the generalized Laplacian, $H_{\rm b}$ the free Hamiltonian of the Bose field, $H_{\rm I}$ an interaction between the particles and the Bose field, $g \in \mathbb{R}$ a coupling constant which represents the strength of the interaction. A scaled Hamiltonian of H is introduced by

$$H(\Lambda) = -\frac{1}{2M}\Delta \otimes I + \Lambda^2 I \otimes H_{\rm b} + g\Lambda H_{\rm I},\ \Lambda > 0.$$

One can show that, under suitable conditions, there exists a symmetric operator $V_{\rm eff}$, called an *effective potential*, such that

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda) - z)^{-1} = \left(-\frac{1}{2M}\Delta + V_{\rm eff} - z\right)^{-1} \otimes P_0, \tag{1.1}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$, where P_0 denotes the orthogonal projection onto $\ker H_{\rm b}$. Physically, a vector belonging to the subspace $\ker H_{\rm b}$ represents the vacuum of the free Bose field. Therefore one obtains a quantum mechanical Hamiltonian, called a *Schrödinger Hamiltonian*, in the vacuum of the free Bose field in the resolvent sense. According to Davies [5], the limit (1.1) is the weak coupling limit at the same time as the mass of the particles becomes infinity, since we can write

$$H(\Lambda) = \Lambda^2\left(-\frac{1}{2M\Lambda^2}\Delta \otimes I + I \otimes H_{\rm b} + \frac{g}{\Lambda}H_{\rm I}\right),$$

where the factor Λ^2 on the whole Hamiltonian is interpreted as a time scaling. However, Davies dealt with spinless particles and bosons. Hiroshima [6, 7] investigated the weak coupling limit and removing ultraviolet cutoff for the *Nelson model*, which describes spinless particles interacting with spinless bosons.

On the other hand, Arai [1] studied scaling limits for a spin-boson interaction model, called the *spin boson model*, and a model in nonrelativistic quantum electrodynamics, called the *Pauli-Fierz model*, in the dipole approximation without the self-interaction of photons. The methods in [1] have been extended to the *generalized spin boson (GSB) model* [3] and the Pauli-Fierz model with the self-interaction of photons ([8] and the refernces therein).

In this paper, we study a scaling limit for a generalization of the Nelson model, which describes particles coupled to a Bose field with some internal

degrees of freedom. Various branches of physics, such as nuclear physics and condensed matter physics, have many examples of the model, and the interaction $H_{\rm I}$ depends on models [3, 13]. From this point of view, it seems natural to consider scaling limits of these general models under conditions as weak as possible. As a result, we are now able to derive an effective potential that is an operator-valued potential in the weak coupling limit. Note that, since the Nelson model has no internal degree of freedom, the effective potential is a scalar potential. However, in nuclear physics, matrix valued potentials appear as effective potentials. A new feature of our work is in that a quantum mechanical Hamiltonian with such a potential is derived.

This paper is organized as follows. Sec. 2 is devoted to a proof of an abstract scaling limit theorem. We introduce some notations, and modify a scaling limit theorem established in [1, 6, 13] for later use. In Sec. 3, we define our model and state the main theorem of this paper. We prove the main results in Sec. 4. In the last section, we discuss an application to a model in nuclear physics.

2 Preliminaries

In this section, we describe an abstract scaling limit theorem ([1, 6, 13]) in a convenient form to establish the scaling limit for our model. We denote the inner product and the associated norm of a Hilbert space $\mathcal{L}$ by $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}}$, respectively. If there is no danger of confusion, we omit the subscript $\mathcal{L}$ in $\langle\cdot,\cdot\rangle_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}}$. Moreover, the domain and range of an operator T is denoted by $D(T)$ and $\mathrm{Ran}(T)$, respectively.

2.1 Uniform relative boundedness

To begin with, we introduce the following notions which are useful for describing a condition of a scaling limit theorem.

Definition 2.1 *Let $\mathcal{L}$ be a Hilbert space, t_0 a point in an interval $I_0 \subset [-\infty,+\infty]$, and $L(t)$ and $M(t)$ $(t \in I_0)$ operators on $\mathcal{L}$ satisfying*

$$\bigcap_{t\in I_0} D(L(t)) \neq \emptyset.$$

(1) We say that $M(t)$ is $L(t)$-bounded uniformly near t_0 if there exist a neighborhood $I \subset I_0$ of t_0 and constants $a, b \geq 0$ such that for any $t \in I \setminus \{t_0\}$, $D(M(t)) \supset D(L(t))$ and

$$\|M(t)\Psi\| \leq a\|L(t)\Psi\| + b\|\Psi\|, \qquad \Psi \in D(L(t)).$$

(2) We say that $M(t)$ is $L(t)$-infinitesimally small uniformly near t_0 if for any $\varepsilon > 0$, there exist an interval $I(\varepsilon) \subset I_0$ and a constant $b(\varepsilon)$ such that for any $t \in I(\varepsilon) \setminus \{t_0\}$, $D(M(t)) \supset D(L(t))$ and

$$\|M(t)\Psi\| \leq \varepsilon \|L(t)\Psi\| + b(\varepsilon)\|\Psi\|, \qquad \Psi \in D(L(t)).$$

Note that, from the Kato-Rellich Theorem, if $M(t)$ is $L(t)$-infinitesimally small uniformly near t_0, then $L(t) + M(t)$ is self-adjoint on $D(L(t))$ for all $t \in I \setminus \{t_0\}$ with some neighborhood I of t_0, and moreover, if $L(t)$ is bounded from below, then so is $L(t) + M(t)$.

2.2 Abstract scaling limit

Let A be a non-negative self-adjoint operator on a Hilbert space $\mathcal{H}$ and B a non-negative self-adjoint operator on a Hilbert space $\mathcal{K}$ with

$$\ker B \neq \{0\}.$$

We denote by P_{B} the orthogonal projection onto $\ker B$ from $\mathcal{K}$. Let $\{C_\Lambda\}_{\Lambda>0}$ be symmetric operators on $\mathcal{X} := \mathcal{H} \otimes \mathcal{K}$. Put

$$K(\Lambda) := K_0(\Lambda) + C_\Lambda,$$

where

$$K_0(\Lambda) := A \otimes I + \Lambda I \otimes B.$$

We consider the scaling limit of $K(\Lambda)$.

Put

$$K_0 := A \otimes I + I \otimes B.$$

The following lemma on unifom relative boundedness is useful for proving scaling limits.

Lemma 2.1 *If C_Λ is $I \otimes B$-bounded uniformly near ∞, then C_Λ is $K_0(\Lambda)$-infinitesimally small uniformly near ∞. Also, if C_Λ is K_0-infinitesimally small uniformly near ∞, then C_Λ is $K_0(\Lambda)$-infinitesimally small uniformly near ∞.*

Note that the followimg lemma holds.

Lemma 2.2 *Let C_Λ be $K_0(\Lambda)$-infinitesimally small uniformly near ∞. Then for $z \in \mathbb{C} \setminus [0, \infty)$*

$$\text{s-}\lim_{\Lambda\to\infty} C_\Lambda \left(K_0(\Lambda) - z\right)^{-1} I \otimes (I - P_{\mathrm{B}}) = 0. \tag{2.1}$$

Proof. It is easy to see that for any $\varepsilon > 0$ there exists a constant $\Lambda_0(\varepsilon)$ such that for all $\Lambda > \Lambda_0(\varepsilon)$

$$\begin{aligned}&\|C_\Lambda(K_0(\Lambda) - z)^{-1} I \otimes (I - P_{\rm B})\Psi\| \\ &\quad \leq \varepsilon \|I \otimes (I - P_{\rm B})\Psi\| + (\varepsilon|z| + b(\varepsilon))\|(K_0(\Lambda) - z)^{-1} I \otimes (I - P_{\rm B})\Psi\|, \qquad \Psi \in \mathcal{X}.\end{aligned}$$

Since the second term on the right hand side converges to zero as $\Lambda \to \infty$, we obtain the desired result. □

Now, we can prove the following scaling limit theorem.

Theorem 2.3 *(scaling limit [1, 6, 13]) Suppose that C_Λ is $K_0(\Lambda)$-infinitesimally small uniformly near ∞ and there exists a symmetric operator C on $\mathcal{X}$ such that $D(C) \supset D(A) \otimes \ker B$ and for all $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\text{s-}\lim_{\Lambda\to\infty} C_\Lambda (A - z)^{-1} \otimes P_{\rm B} = C(A - z)^{-1} \otimes P_{\rm B}. \tag{2.2}$$

Then, the following (1)-(3) hold.
(1)For any $\Lambda > \Lambda_0$ with some Λ_0, $K(\Lambda)$ is self-adjoint on $D(K_0)$ and bounded from below uniformly in $\Lambda > \Lambda_0$. Moreover, it is essentially self-adjoint on any core for K_0.
(2)The operator

$$K_{\rm eff} := A \otimes I + (I \otimes P_{\rm B})C(I \otimes P_{\rm B})$$

is self-adjoint on $D(A \otimes I)$ and bounded from below. Moreover, it is essentially self-adjoint on any core for $A \otimes I$.
(3)For any $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large

$$\text{s-}\lim_{\Lambda\to\infty} (K(\Lambda) - z)^{-1} = (K_{\rm eff} - z)^{-1} I \otimes P_{\rm B}. \tag{2.3}$$

Proof. It is easy to prove (1) and (2). (See [13].) By Lemma 2.2 and (2.2), one can easily show that

$$\text{s-}\lim_{\Lambda\to\infty} C_\Lambda (K_0(\Lambda) - z)^{-1} = C(A - z)^{-1} \otimes P_{\rm B}. \tag{2.4}$$

There exists a constant $\epsilon_0 < 0$ such that $K_\Lambda \geq \epsilon_0$, for all $\Lambda > \Lambda_0$. Let $|z| > 0$ be sufficiently large so that $z < \epsilon_0 < 0$. Iterating the second resolvent formula, we have

$$(K(\Lambda) - z)^{-1} = \sum_{n=0}^{N} (-1)^n (K_0(\Lambda) - z)^{-1} {T_\Lambda}^n + R_N(\Lambda), \tag{2.5}$$

where

$$T_\Lambda := C_\Lambda (K_0(\Lambda) - z)^{-1},$$

and

$$R_N(\Lambda) := (-1)^{N+1} (K_\Lambda - z)^{-1} {T_\Lambda}^{N+1}.$$

Since C_Λ is $K_0(\Lambda)$-infinitesimally small uniformly near ∞,

$$\|T_\Lambda\| \leq 2\varepsilon + \frac{b(\varepsilon)}{|z|} \tag{2.6}$$

for all $\Lambda > \Lambda_0$ with some Λ_0. Hence, taking $\varepsilon < 1/2$,

$$\|T_\Lambda\| < 1$$

uniformly in $\Lambda > \Lambda_0$, if $|z|$ is surfficiently large. Since one can easily prove

$$\|R_N(\Lambda)\| \leq \frac{\|T_\Lambda\|^{N+1}}{|\epsilon_0 - z|},$$

it follows that

$$(K(\Lambda) - z)^{-1} = \sum_{n=0}^{\infty} (-1)^n (K_0(\Lambda) - z)^{-1} {T_\Lambda}^n \tag{2.7}$$

in operator norm uniformly in $\Lambda > \Lambda_0$. Taking $\Lambda \to \infty$ and using (2.4), we get the desired result with $z < 0$, $|z|$ being sufficiently large. (For the details, see [1, Theorem 2.2].) The proof in the case where z is not real is similar to that in [1]. □

We denote by $\otimes_{\rm alg}$ the algebraic tensor product. We can state the scaling limit theorem under easier conditions:

Corollary 2.4 *Suppose that C_Λ is $K_0(\Lambda)$-infinitesimally small uniformly near ∞ and there exists a closed symmetric operator C on $\mathcal{X}$ satisfying the following condition:*
There exists a core $\mathcal{D}$ for $A \otimes I$ such that $D(C) \supset \mathcal{D} \otimes_{\rm alg} \ker B$ and for all $\Psi \in \mathcal{D} \otimes_{\rm alg} \ker B$

$$\text{s-}\lim_{\Lambda \to \infty} C_\Lambda \Psi = C\Psi. \tag{2.8}$$

Then, (1)-(3) in Theorem 2.3 hold.
Proof. We need only to prove (2.2). It is easy to see that for any $\Lambda \geq \Lambda_0$ with some Λ_0

$$\|C_\Lambda (A-z)^{-1} \otimes P_{\rm B} \Psi\| \leq \left[\varepsilon + (\varepsilon|z| + b(\varepsilon)) \, \|(A-z)^{-1}\|\right] \|\Psi\|, \quad \Psi \in \mathcal{X}. \tag{2.9}$$

Hence, $C_\Lambda(A-z)^{-1}\otimes P_{\rm B}$ is a bounded operator with

$$\sup_{\Lambda\geq\Lambda_0}\|C_\Lambda(A-z)^{-1}\otimes P_{\rm B}\|<\infty. \tag{2.10}$$

Put

$$\mathcal{L}:=(A-z)\mathcal{D}.$$

By the injectivity of $(A-z)^{-1}$,

$$(A-z)\mathcal{D}=\{u\in\mathcal{H}\mid (A-z)^{-1}u\in\mathcal{D}\}.$$

Since $\mathcal{D}$ is core for A, for every $\Psi\in D(A)$ there exists a sequence $\{\Psi_n\}\subset\mathcal{D}$ such that

$$\Psi_n\longrightarrow\Psi,\quad A\Psi_n\longrightarrow A\Psi.$$

Thus we obtain that

$$\overline{(A-z)\mathcal{D}}=\mathrm{Ran}(A-z)=\mathcal{H}.$$

Hence, $\mathcal{L}$ is dense in $\mathcal{H}$. For all $\Psi\in\mathcal{L}\otimes_{\rm alg}\mathcal{K}$,

$$(A-z)^{-1}\otimes P_{\rm B}\Psi\in\mathcal{D}\otimes_{\rm alg}\ker B$$

and

$$\text{s-}\lim_{\Lambda\to\infty}C_\Lambda(A-z)^{-1}\otimes P_{\rm B}\Psi=C(A-z)^{-1}\otimes P_{\rm B}\Psi.$$

Therefore, taking $\Lambda\to\infty$ in (2.9),

$$\|C(A-z)^{-1}\otimes P_{\rm B}\Psi\|\leq\left[\varepsilon+(\varepsilon|z|+b(\varepsilon))\,\|(A-z)^{-1}\|\right]\|\Psi\|,\quad \Psi\in\mathcal{L}\otimes_{\rm alg}\mathcal{K}. \tag{2.11}$$

By closedness of C, $C(A-z)^{-1}\otimes P_{\rm B}$ is a closed operator. Since $\mathcal{L}\otimes_{\rm alg}\mathcal{K}$ is dense in $\mathcal{X}$, $C(A-z)^{-1}\otimes P_{\rm B}$ is bounded on $\mathcal{H}$. By (2.10) and the density of $\mathcal{L}\otimes_{\rm alg}\mathcal{K}$ again, we obtain (2.2). □

3 Definitions and the main theorem

3.1 Boson Fock space

To describe a Bose field, one uses the Boson Fock space over a complex Hilbert space $\mathcal{K}$:

$$\begin{aligned}\mathcal{F}_{\rm b}(\mathcal{K}):&=\bigoplus_{n=0}^{\infty}\bigotimes_{\rm s}^{n}\mathcal{K}\\&=\left\{\psi=\{\Psi^{(n)}\}_{n=0}^{\infty}\;\middle|\;n\geq0,\quad\Psi^{(n)}\in\bigotimes_{\rm s}^{n}\mathcal{K},\quad\sum_{n=0}^{\infty}\|\psi^{(n)}\|^2<\infty\right\},\end{aligned}$$

where $\bigotimes_{\rm s}^n \mathcal{K}$ denotes the n-fold symmetric tensor product of $\mathcal{K}$ with $\bigotimes_{\rm s}^0 \mathcal{K} := \mathbb{C}$.

The *annihilation operator* $a(f)(f \in \mathcal{K})$ is a densely defined closed linear operator on $\mathcal{F}_{\rm b}(\mathcal{K})$ such that, for all $\psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

where S_n is the symmetrization operator on $\bigotimes^n \mathcal{K}$ ($S_n{}^* = S_n$, $S_n{}^2 = S_n$, $\bigotimes_s^n \mathcal{K} = S_n(\bigotimes^n \mathcal{K})$). The adjoint $a(f)^*$, called the *creation operator*, and the annihilation operator $a(g)(g \in \mathcal{K})$ obey the *canonical commutation relations*

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0 \tag{3.1}$$

for all $f, g \in \mathcal{K}$ on the dense subspace

$$\mathcal{F}_0(\mathcal{K}) := \{\psi \in \mathcal{F}_{\rm b}(\mathcal{K}) \mid \text{there exists a number } n_0 \text{ such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X, Y] := XY - YX$.

The *Segal field operator* $\phi(f)$ $(f \in \mathcal{K})$ is defined by

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}.$$

It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{K})$ [11, §X.7]. We denote its closure by the same symbol $\phi(f)$.

It follows from (3.1) that, for all $f, g \in \mathcal{K}$,

$$[\phi(f), \phi(g)] = i\Im\langle f, g \rangle \tag{3.2}$$

on $\mathcal{F}_0(\mathcal{K})$. Moreover, we have

$$e^{i\phi(f)} e^{i\phi(g)} = e^{-i\Im\langle f,g\rangle} e^{i\phi(g)} e^{i\phi(f)}, \quad f, g \in \mathcal{K}, \tag{3.3}$$

which are called the *Weyl relations* of $\{\phi(f) | f \in \mathcal{K}\}$ [11, §X.7].

For every self-adjoint operator T on $\mathcal{K}$, one can define a self-adjoint operator $d\Gamma(T)$, called the *second quantization* of T [10, p.302], by

$$d\Gamma(T) := \bigoplus_{n=0}^{\infty} T^{(n)},$$

with $T^{(0)} = 0$ and $T^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\overbrace{T}} \otimes \cdots \otimes I \right) \Bigg| \bigotimes_{\rm alg}^{n} D(T).$$

If T is non-negative, then so is $d\Gamma(T)$.

3.2 Definition of a generalization of the Nelson model

We consider a model of a quantum system $\mathcal{S}$ coupled to a Bose field with some internal degrees of freedom. The Hilbert space of the system $\mathcal{S}$ is taken to be $L^2(\mathbb{R}^d;\mathcal{H})$. Here $d \in \mathbb{N}$ and $\mathcal{H}$ is taken to be an arbitary separable complex Hilbert space. In concrete realizations, $\mathcal{S}$ may be a system of quantum particles with some internal degrees of freedom such as spin and isospin. The Hilbert space of the coupled system of $\mathcal{S}$ and the Bose field is given by the tensor product

$$\mathcal{F} := L^2(\mathbb{R}^d;\mathcal{H}) \otimes \mathcal{F}_{\rm b}(\mathcal{K}) \simeq \mathcal{H} \otimes L^2(\mathbb{R}^d;\mathcal{F}_{\rm b}(\mathcal{K})). \tag{3.4}$$

We assume that T is a non-negative, injective and self-adjoint operator on $\mathcal{K}$. Then, the free Hamiltonian of the Bose field is defined by

$$H_{\rm b} := d\Gamma(T)$$

acting on $\mathcal{F}_{\rm b}(\mathcal{K})$.

We define the quantized scalar field by

$$\Phi(g) := \int_{\mathbb{R}^d}^{\oplus} \phi(g(x))dx$$

on $L^2(\mathbb{R}^d;\mathcal{F}_{\rm b}(\mathcal{K}))$, where $g : x \in \mathbb{R}^d \mapsto g(x) \in \mathcal{K}$. If g is a strongly continuous function, then $\Phi(g)$ is self-adjoint (see [12, Theorem XIII.85 (b)]).

Now we define a total Hamiltonian H acting on $\mathcal{F}$ by

$$H := -\Delta \otimes I + I \otimes H_{\rm b} + gH_{\rm I}, \tag{3.5}$$

where $g \in \mathbb{R}$ denotes a coupling constant and

$$H_{\rm I} := \sum_{j=1}^{J} B_j \otimes \Phi(g_j). \tag{3.6}$$

Here, $B_j (j = 1, \cdots, J)$ are bounded self-adjont operators on $\mathcal{H}$. The functions $g_j (j = 1, \cdots, J, J \in \mathbb{N})$ from $\mathbb{R}^d$ to $\mathcal{K}$ are strongly continuous.

Example 3.1 *(the Nelson model)* If $\dim \mathcal{H} = 1$ and $B_j = 1$, then the Hamiltonian H is written as

$$H_{\rm Nelson} := -\Delta \otimes I + I \otimes H_{\rm b} + g\sum_{j=1}^{J} \Phi(g_j)$$

on $L^2(\mathbb{R}^d;\mathcal{F}_{\rm b}(\mathcal{K}))$, which is called the *Nelson Hamiltonian*. The weak coupling limit of this Hamiltonian is studied by Hiroshima [6, 7].

3.3 Scaling limit for the model

To begin with, we introduce a scaled Hamiltonian $H(\Lambda)$ $(\Lambda > 0)$ by

$$H(\Lambda) := -\Delta \otimes I + \Lambda^2 I \otimes H_{\rm b} + g\Lambda H_{\rm I}. \tag{3.7}$$

In order to describe our result, we introduce some notations, and formulate our assumption.

We denote by $L^\infty(\mathbb{R}^d;\mathcal{K})$ the set of measurable functions $f : \mathbb{R}^d \mapsto \mathcal{K}$ for which

$$\|f\|_\infty := \text{ess. sup}_{x\in\mathbb{R}^d} \|f(x)\|_{\mathcal{K}} < \infty.$$

For $\alpha \in \mathbb{R}$, we define a $\mathcal{K}$-valued function $T^\alpha f$ on $\mathbb{R}^d$ as follows: if $f(x) \in D(T^\alpha)$ a.e.$x \in \mathbb{R}^d$ with respect to Lebesuge mesure,

$$(T^\alpha f)(x) := T^\alpha f(x).$$

Definition 3.1 *Let $\alpha \in \mathbb{R}$. $L^\infty_\alpha(\mathbb{R}^d;\mathcal{K})$ denotes the set of $\mathcal{K}$-valued functions f on $\mathbb{R}^d$ satisfing the following conditions:*
(i) f is strongly continuous with $f \in L^\infty(\mathbb{R}^d;\mathcal{K})$.
(ii) $f(x) \in D(T^\alpha)$ $(x \in \mathbb{R})$ and $T^\alpha f \in L^\infty(\mathbb{R}^d;\mathcal{K})$.

A $\mathcal{K}$-valued function f on $\mathbb{R}^d$ is said to be differentiable with respect to x_μ if the net

$$\frac{f(x_1,\cdots,x_\mu+\varepsilon,\cdots,x_d) - f(x)}{\varepsilon} \tag{3.8}$$

converges as $\varepsilon \to 0$ for any $x = (x_1,\cdots,x_d) \in \mathbb{R}^d$. Then, we denote the limit of (3.8) by $\partial_\mu f$. One can define the n times diferentiability $(n \in \mathbb{N})$, inductively:

$$\partial_\mu^n f := \partial_\mu(\partial_\mu^{(n-1)} f),\ n \geq 1.$$

Hypothesis I *The functions $g_j (j = 1,\cdots,J)$ are twice diferensiable and satisfy the following conditions:*
(i) $g_j \in L^\infty_{-3/2}(\mathbb{R}^d;\mathcal{K})$.
(ii) $\partial_\mu(T^{-1}g_j) \in L^\infty_{-1/2}(\mathbb{R}^d;\mathcal{K}) \cap L^\infty_{1/2}(\mathbb{R}^d;\mathcal{K})$ for $\mu = 1,\cdots,d$.
(iii) $\partial^2_\mu(T^{-1}g_j) \in L^\infty_{-1/2}(\mathbb{R}^d;\mathcal{K})$ for $\mu = 1,\cdots,d$.
Moreover, we assume that for any $j,k = 1,\cdots,J$ and $x \in \mathbb{R}^d$

$$\langle g_j(x), g_k(x)\rangle,\quad \langle g_j(x), T^{-1}g_k(x)\rangle,\quad \langle T^{-1}g_j(x), T^{-1}g_k(x)\rangle \in \mathbb{R}. \tag{3.9}$$

We are now ready to describe our result. Let

$$V_{\rm eff} = -\frac{g^2}{2} \sum_{1 \le j,k \le J} B_k B_j V_{j,k}, \tag{3.10}$$

on $L^2(\mathbb{R}^d; \mathcal{H})$, where

$$V_{j,k}(x) = \left\langle g_j(x), T^{-1} g_k(x) \right\rangle, \quad \text{a.e.} x \in \mathbb{R}. \tag{3.11}$$

Let $\Omega := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\rm b}(\mathcal{K})$, the Fock vacuum, and P_0 be the orthogonal projection onto the subspace $\{\alpha\Omega \mid \alpha \in \mathbb{C}\} = \ker H_{\rm b}$.

Theorem 3.1 *Assume Hypothesis I. Let $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large. Then,*

$$\text{s-}\lim_{\Lambda \to \infty} (H(\Lambda) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_0, \tag{3.12}$$

where

$$H_{\rm eff} = -\Delta + V_{\rm eff} \tag{3.13}$$

on $L^2(\mathbb{R}^d; \mathcal{H})$.

4 Proof of Theorem 3.1

Note that the following facts hold: Let $f_1 \in L^\infty_{-1/2}(\mathbb{R}^d; \mathcal{K})$. Then, $D(H_{\rm b}^{1/2}) \subset D(\Phi(f_1))$ and for some constans $c(f_1), d(f_1) \ge 0$

$$\|\Phi(f_1)\Psi\| \le c(f_1) \|I \otimes H_{\rm b}^{1/2} \Psi\| + d(f_1)\|\Psi\|, \quad \Psi \in D(I \otimes H_{\rm b}^{1/2}). \tag{4.1}$$

If $f_2 \in L^\infty_{-1/2}(\mathbb{R}^d; \mathcal{K}) \cap L^\infty_{1/2}(\mathbb{R}^d; \mathcal{K})$, then $D(I \otimes H_{\rm b}) \subset D(\Phi(f_1)\Phi(f_2))$ and for some constants $c(f_1, f_2), d(f_1, f_2) \ge 0$

$$\|\Phi(f_1)\Phi(f_2)\Psi\| \le c(f_1, f_2) \|I \otimes H_{\rm b} \Psi\| + d(f_1, f_2)\|\Psi\|, \quad \Psi \in D(I \otimes H_{\rm b}). \tag{4.2}$$

These facts imply that $\Phi(f_1)$ is $I \otimes H_{\rm b}^{1/2}$-bounded and $\Phi(f_1)\Phi(f_2)$ is $I \otimes H_{\rm b}$-bounded. Hence, using the method in [13], we can prove Theorem 3.1. Throughout this section, it is taken for granted that Hypothesis I hold.

4.1 Unitary transformation

Let

$$\mathcal{F}_{\rm fin}(\mathcal{M}) := \mathscr{L}\{\Omega, a(f_1)^* \cdots a(f_n)^*\Omega \mid n \geq 1, f_j \in \mathcal{M}, j = 1, \cdots, n\},$$

for a subset $\mathcal{M}$ of $\mathcal{K}$. For every $D \subset L^2(\mathbb{R}^d)$ and $\mathcal{E} \subset \mathcal{F}_{\rm b}(\mathcal{K})$, let

$$D \cdot \mathcal{E} := \mathscr{L}\{f \times \psi \mid f \in D, \psi \in \mathcal{E}\} \subset L^2(\mathbb{R}^d; \mathcal{F}_{\rm b}(\mathcal{K})),$$

where

$$(f \times \psi)(x) := f(x)\psi \in \mathcal{F}_{\rm b}(\mathcal{K}), \quad \text{a.e.} x \in \mathbb{R}^d.$$

Now, we introduce a unitary transformation as follows. Let

$$S = \sum_{j=1}^{J} B_j \otimes \Phi(iT^{-1}g_j)$$

acting on $\mathcal{H} \otimes L^2(\mathbb{R}^d; \mathcal{F}_{\rm b}(\mathcal{K}))$.

Lemma 4.1 *The operator S is essentially self-adjoint.*

Proof. Use Nelson's analytic vector theorem. It is easy to see from Hypothesis I (i) that $\mathcal{H} \otimes_{\rm alg} \left(C_0^\infty(\mathbb{R}^d) \cdot \mathcal{F}_{\rm fin}(\mathcal{K})\right)$ is a dense set of analytic vectors of S, where where $C_0^\infty(\mathbb{R}^d)$ denotes the set of infinitely differentiable functions with compact support. □

We denote the closure of S by the same symbol S. Then, we can define the unitary operator $U(t)$ $(t \in \mathbb{R})$ generated by S:

$$U(t) := e^{-itS}, \quad t \in \mathbb{R}.$$

The subspace

$$\mathcal{F}_T := \mathcal{H} \otimes_{\rm alg} \left(C_0^\infty(\mathbb{R}^d) \cdot \mathcal{F}_{\rm fin}(D(T)\right).$$

is a core for $-\Delta \otimes I$, $I \otimes H_{\rm b}$, and therefore for $-\Delta \otimes I + I \otimes H_{\rm b}$. It is easy to prove that for $\Psi \in \mathcal{F}_T$

$$\sum_{m,n} \frac{|s|^n |t|^m}{n!m!} \|S^n I \otimes H_{\rm b} S^m \Psi\| < \infty, \quad s, t \in \mathbb{R}.$$

(See [13, Lemma 4.1].)

Let

$$E(t) = U(t)\left[\mathrm{ad}^2(S)I \otimes H_{\mathrm{b}}\right] U(t)^{-1}, \quad t \in \mathbb{R}, \tag{4.3}$$

where, for each X in an algebra $\mathcal{A}$, we inductively define $\mathrm{ad}^N(X)$ by

$$\mathrm{ad}^0(X)Y = Y, \quad \mathrm{ad}^N(X)Y = [X, \mathrm{ad}^{N-1}(X)Y], \quad Y \in \mathcal{A}, \ N \geq 1.$$

One can prove the following lemma in the same way as in [13, Proposition 4.2].

Lemma 4.2 *For each $\Psi \in \mathcal{F}_T$, there exist $\xi(t),\ \eta(t) \in [-|t|, |t|]$ such that*

$$U(t)I \otimes H_{\mathrm{b}}U(t)^{-1}\Psi = \left[I \otimes H_{\mathrm{b}} + tH_{\mathrm{I}} - \frac{t^2}{2}E(\xi(t))\right]\Psi, \tag{4.4}$$

and

$$U(t)H_{\mathrm{I}}U(t)^{-1}\Psi = \left[H_{\mathrm{I}} - tE(\eta(t))\right]\Psi. \tag{4.5}$$

Applying (4.4) and (4.5) with $t = g/\Lambda$, we obtain the following equality:

$$\begin{aligned} &U(g/\Lambda)\left[\Lambda^2 I \otimes H_{\mathrm{b}} + g\Lambda H_{\mathrm{I}}\right] U(g/\Lambda)^{-1}\Psi \\ &\qquad = \left[\Lambda^2 I \otimes H_{\mathrm{b}} - \frac{g^2}{2}E(\xi(g/\Lambda)) + g^2 E(\eta(g/\Lambda))\right]\Psi \end{aligned} \tag{4.6}$$

for $\Psi \in \mathcal{F}$.

In what follows, we consider the unitary transformation of $-\Delta \otimes I$ by $U(t)$. Under the condition Hypothesis I, it is easy to show that

$$\sum_{n,m} \frac{|s|^n|t|^m}{n!m!}\,\|S^n[-\Delta \otimes I]S^m\Psi\| < \infty$$

for any $s, t \in \mathbb{R}$ and $\Psi \in \mathcal{F}_T$. Thus, in particular, we can prove

$$U(t)\mathcal{F}_T \subset D(-\Delta \otimes I) \quad (t \in \mathbb{R}).$$

We set

$$\delta A(t) := U(t)[-\Delta \otimes I]U(t)^{-1} - [-\Delta \otimes I], \tag{4.7}$$

Then, we can write

$$\begin{aligned} &U(g/\Lambda)^{-1}H(\Lambda)U(g/\Lambda)\Psi \\ &\qquad = \left[-\Delta \otimes I + \Lambda^2 I H_{\mathrm{b}} + C_\Lambda\right]\Psi, \end{aligned} \tag{4.8}$$

for $\Psi \in \mathcal{F}_T$, where we have set

$$C_\Lambda := \delta A(g/\Lambda) - \frac{g^2}{2} E(\xi(g/\Lambda)) + g^2 E(\eta(g/\Lambda)). \tag{4.9}$$

We denote the closure of C_Λ by the same symbol. Indeed, we can prove the following theorem:

Theorem 4.3 *Let*

$$\widehat{H}(\Lambda) := -\Delta \otimes I \ + \Lambda^2 I \otimes H_{\rm b} + C_\Lambda. \tag{4.10}$$

Then, $\widehat{H}(\Lambda)$ is essentilly self-adjoint on $\mathcal{F}_T$ and the following operator equality holds:

$$U(g/\Lambda) H(\Lambda) U(g/\Lambda)^{-1} = \widehat{H}(\Lambda). \tag{4.11}$$

We prove Theorem 4.3 in the following subsection.

4.2 Uniform relative boundedness of C_Λ

Let

$$H_0(\Lambda) := -\Delta \otimes I + \Lambda^2 I \otimes H_{\rm b}. \tag{4.12}$$

In this section, we prove that C_Λ is $H_0(\Lambda)$-infinitesimally small uniformly near ∞. As a result, we obtain Theorem 4.3.

To begin with, we state a uniform relative boundedness of the operator $E(t)$:

Lemma 4.4 *The operator $E(t)$ defined by* (4.3) *is $I \otimes H_{\rm b}$-bounded uniformly near* 0.

Proof. It is easy to see that

$$\begin{aligned} \|E(\xi(t)\Psi\| \leq & \left(\sum_{j,k} \|B_j B_k\| \left\|T^{-1} g_j\right\|_\infty \|g_k\|_\infty \right) \|\Psi\| \\ & + \sum_{j,k} \|[B_j, B_k]\| \left\| I \otimes \Phi(g_k) \Phi(iT^{-1} g_j) U(\xi(t))^{-1} \Psi \right\| \end{aligned}$$

for all $\Psi \in \mathcal{F}_T$. Hence, we need only to prove that $I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)U(\xi(t))^{-1}$ is $I \otimes H_{\rm b}$-bounded uniformly near 0.

For each $\Psi \in \mathcal{F}_T$ and $t \in \mathbb{R}$, there exists a constant $\delta(t) \in [-|t|, |t|]$ such that

$$\begin{aligned} &U(t)I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)U(t)^{-1}\Psi \\ &\quad = I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)\Psi \\ &\qquad\qquad - itU(\delta(t))\mathrm{ad}^1(S)\left[I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)\right] U(\delta(t))^{-1}\Psi \end{aligned} \tag{4.13}$$

(See [13, Lemma 4.5].) By (4.2), the first term on the right hand side in (4.13) is H_{b}-bounded, since $\mathcal{F}_T$ is core for $I \otimes H_{\mathrm{b}}$. Therefore, we need only to estimate the second term.

It is easy to see, from (3.9), that

$$\begin{aligned} &\mathrm{ad}^1(S)\left[I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)\right]\Psi \\ &\qquad = -i\sum_l B_l \otimes \left\langle T^{-1}g_l, g_k\right\rangle \Phi(iT^{-1}g_j)\Psi. \end{aligned}$$

Hence, we obtain the following:

$$\begin{aligned} &\left\|I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)U(t)^{-1}\Psi\right\| \\ &\quad = \left\|I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)\Psi\right\| \\ &\qquad + |t|\sum_l \|B_l\| \left\|T^{-1}g_l\right\|_\infty \|g_k\|_\infty \left\|I \otimes \Phi(iT^{-1}g_j)\Psi\right\|, \end{aligned}$$

where we have used the commutativity of $I \otimes \Phi(iT^{-1}g_j)$ and $U(\delta(t))^{-1}$. One can assume that $|t| \leq 1$ without loss of generality. Note that $|\xi(\delta(t))| \leq |t|$. Thus, by (4.1) and (4.2), $I \otimes \Phi(g_k)\Phi(iT^{-1}g_j)U(\xi(\delta(t)))^{-1}\Psi$ is $I \otimes H_{\mathrm{b}}$-bounded uniformly near 0. Hence, we obtain the desired result. □

By (4.9), we need only to prove the uniform relative boundedness of the operator $\delta A(t)$. The following lemma is a fundamental fact to do this.

Lemma 4.5 *The operator $U(t)I \otimes H_{\mathrm{b}}U(t)^{-1}$ is $I \otimes H_{\mathrm{b}}$-bounded uniformly near 0.*

Proof. By Lemma 4.2,

$$\left\|U(t)I \otimes H_{\mathrm{b}}U(t)^{-1}\Psi\right\| \leq \|I \otimes H_{\mathrm{b}}\Psi\| + |t|\,\|H_{\mathrm{I}}\Psi\| + \frac{|t|^2}{2}\,\|E(\xi(t))\Psi\|$$

for all $\Psi \in \mathcal{F}_T$. From Lemma 4.4, $E(\xi(t))$ is $I \otimes H_{\mathrm{b}}$-bounded unifomly near 0. Therefore, we need only to prove that H_{I} is $(I \otimes H_{\mathrm{b}})$-bounded unifomly near 0, since we can assume that $|t| \leq 1$ without loss of generality.

It is easy to see that

$$\|H_I\Psi\| \leq \sum_{j=1}^{J} \|B_j\| \, \|I \otimes \Phi(g_j)\Psi\|$$

for all $\Psi \in \mathcal{F}_T$. Hence, we obtain the desired result by (4.1). $\square$

The following lemma follows from (4.1), (4.2) and Lemma 4.5.

Lemma 4.6 *Let $f_1 \in L^\infty_{-1/2}(\mathbb{R}^d;\mathcal{K})$ and $f_2 \in L^\infty_{-1/2}(\mathbb{R}^d;\mathcal{K}) \cap L^\infty_{1/2}(\mathbb{R}^d;\mathcal{K})$. Then, the operator $U(t)[I \otimes \Phi(f_1)]U(t)^{-1}$ is $I \otimes H_{\rm b}$-infinitesimally small uniformly near 0 and $U(t)[I \otimes \Phi(f_1)\Phi(f_2)]U(t)^{-1}$ is $I \otimes H_{\rm b}$-bounded uniformly near 0.*

In what follows, we describe a uniform relative boundedness of the operator $\delta A(t)$.

Lemma 4.7 *The operator $\delta A(t)$ is $-\Delta \otimes I + I \otimes H_{\rm b}$-infinitesimally small uniformly near 0 and for all $\Psi \in \mathcal{F}_T$*

$$\text{s-}\lim_{t\to\infty} \delta A(t)\Psi = 0. \tag{4.14}$$

Proof. It is easy to see that $U(t)\Psi \in D(\Delta \otimes I)$ for all $t \in \mathbb{R}$ and that

$$\delta A(t)\Psi = -\left[U(t)(\Delta \otimes I)U(t)^{-1} - \Delta \otimes I\right]\Psi$$

for all $\Psi \in \mathcal{F}_T$. In the same way as in Lemma 4.2, one can prove that there exists $\omega(t) \in [-|t|, |t|]$ $(t \in \mathbb{R})$ such that

$$\left[U(t)(\Delta \otimes I)U(t)^{-1} - \Delta \otimes I\right]\Psi = -itU(\omega(t))\mathrm{ad}^1(S)[\Delta \otimes I]U(\omega(t))^{-1}\Psi.$$

Thus, under the identification (3.4), we have

$$\|\delta A(t)\Psi\| \leq |t| \sum_{j=1}^{J} \|B_j\| \left\|I \otimes \left[\Phi(iT^{-1}g_j), \Delta\right] U(\omega(t))^{-1}\Psi\right\| \tag{4.15}$$

for all $\Psi \in \mathcal{F}_T$. This inequality implies that (4.14) holds.

We need only to prove a uniform relative boundedness of the operator

$$I \otimes \left[\Phi(iT^{-1}g_j), \Delta\right] U(\omega(t))^{-1}.$$

Let D_μ ($\mu = 1, \cdots, d$) be the partial differential operators on $L^2(\mathbb{R}^d; \mathcal{F}_{\rm b}(\mathcal{K}))$. It is is easy to see that

$$\left[\Phi(iT^{-1}g_j), \Delta\right] = \sum_{\mu=1}^{d} \left(D_\mu \left[\Phi(iT^{-1}g_j), D_\mu\right] + \left[\Phi(iT^{-1}g_j), D_\mu\right] D_\mu\right)$$

on $C_0^\infty(\mathbb{R}^d) \cdot \mathcal{F}_{\rm fin}(\mathcal{K})$. One can easily prove that

$$\left[\Phi(iT^{-1}g_j), D_\mu\right] \psi = -\Phi(i\partial_\mu T^{-1}g_j)\psi$$

for all $\psi \in C_0^\infty(\mathbb{R}^d) \cdot \mathcal{F}_{\rm fin}(\mathcal{K})$. Hence, we obtain the following:

$$\begin{aligned}\left[\Phi(iT^{-1}g_j), \Delta\right] \psi &= -\sum_{\mu=1}^{d} \left(D_\mu \Phi(i\partial_\mu T^{-1}g_j) + \Phi(i\partial_\mu T^{-1}g_j) D_\mu\right) \psi \\ &= -\sum_{\mu=1}^{d} \left(\Phi(i\partial_\mu^2 T^{-1}g_j) + 2\Phi(i\partial_\mu T^{-1}g_j) D_\mu\right) \psi.\end{aligned}$$

Thus, we get the following inequality.

$$\left\| \left[\Phi(iT^{-1}g_j), \Delta\right] \psi \right\| \leq \sum_{\mu=1}^{d} \left\| \left[\Phi(i\partial_\mu^2 T^{-1}g_j) + 2\Phi(i\partial_\mu T^{-1}g_j) D_\mu\right] \psi \right\|$$

for all $\psi \in C_0^\infty(\mathbb{R}^d) \cdot \mathcal{F}_{\rm fin}(\mathcal{K})$. By (4.15), we get

$$\begin{aligned}\|\delta A(t)\Psi\| \leq |t| \sum_{j=1}^{J} \|B_j\| \left\| I \otimes \Phi(i\partial_\mu^2 T^{-1}g_j) U(\omega(t))^{-1}\Psi \right\| \\ + 2|t| \sum_{j=1}^{J} \|B_j\| \left\| I \otimes \Phi(i\partial_\mu T^{-1}g_j) D_\mu U(\omega(t))^{-1}\Psi \right\| \quad (4.16)\end{aligned}$$

for all $\Psi \in \mathcal{F}_T$. By (4.6), the first term on the right hand side is $I \otimes H_{\rm b}$-infinitesimally small uniformly near 0. Hence we need only to prove that the operator $I \otimes \Phi(i\partial_\mu T^{-1}g_j) D_\mu U(\omega(t))^{-1}$ is $-\Delta \otimes I + I \otimes H_{\rm b}$-bounded uniformly near 0.

Using the same way as in (4.2), one can show that

$$\begin{aligned}&U(t) I \otimes \Phi(i\partial_\mu T^{-1}g_j) D_\mu U(t)^{-1}\Psi \\ &\quad = I \otimes \Phi(i\partial_\mu T^{-1}g_j) D_\mu \Psi + itU(\theta(t)) \mathrm{ad}^1(S) I \otimes \Phi(i\partial_\mu T^{-1}g_j) D_\mu U(\theta(t))^{-1}\Psi\end{aligned}$$

for all $\Psi \in \mathcal{F}_T$ with some $\theta(t) \in [-|t|, |t|]$. Thus, we obtain the following:

$$\begin{aligned}&\left\|I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu U(t)^{-1} \Psi\right\| \\ &\quad \leq \left\|I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu \Psi\right\| + |t| \left\|\mathrm{ad}^1(S)\left[I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu\right] U(\theta(t))^{-1} \Psi\right\|. \end{aligned} \tag{4.17}$$

The first term on the right hand side above is estimated as follows:

$$\left\|I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu \Psi\right\| \leq c \left\|I \otimes H_{\mathrm{b}}^{1/2} D_\mu \Psi\right\| + d \left\|I \otimes D_\mu \Psi\right\|$$

Note that $H_{\mathrm{b}}^{1/2} D_\mu$ is $-\Delta + H_{\mathrm{b}}$-bounded. We can now estimate the second term on the right hand side of (4.17). It is easy to see that

$$\begin{aligned}&\mathrm{ad}^1(S)\left[I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu\right] \\ &\quad = \sum_l B_l \otimes \left[\Phi(iT^{-1} g_l), \Phi(i\partial_\mu T^{-1} g_j) D_\mu\right] \\ &\quad = -\sum_l B_l \otimes \Phi(i\partial_\mu T^{-1} g_l) \Phi(i\partial_\mu T^{-1} g_j) \\ &\qquad\quad + \sum_l B_l \otimes i\Im \left\langle T^{-1} g_l(\cdot), \partial_\mu T^{-1} g_j(\cdot)\right\rangle D_\mu.\end{aligned}$$

Thus, one can prove that $U(t)\Psi$ ($t \in \mathbb{R}$) belongs to $D(\mathrm{ad}^1(S)\,[I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu])$ for all $\Psi \in \mathcal{F}$ and that the following inequality holds:

$$\begin{aligned}&\left\|\mathrm{ad}^1(S)\left[I \otimes \Phi(i\partial_\mu T^{-1} g_j) D_\mu\right] U(\theta(t))^{-1} \Psi\right\| \\ &\qquad \leq \sum_l \|B_l\| \left\|I \otimes \Phi(i\partial_\mu T^{-1} g_l) \Phi(i\partial_\mu T^{-1} g_j) U(\theta(t))^{-1} \Psi\right\| \\ &\qquad\quad + \sum_l \|B_l\| \left\|T^{-1} g_l\right\|_\infty \left\|\partial_\mu T^{-1} g_j\right\|_\infty \left\|I \otimes D_\mu U(\theta(t))^{-1} \Psi\right\|.\end{aligned}$$

Note that the operator $U(\theta(t)) I \otimes \Phi(i\partial_\mu T^{-1} g_l) \Phi(i\partial_\mu T^{-1} g_j) U(\theta(t))^{-1}$ is $I \otimes H_{\mathrm{b}}$-bounded uniformly near 0 by Hypothesis I (ii) and Lemma 4.6. Hence, we need only to estimate the second term. Indeed, one can prove that the operator $U(t)\,[I \otimes D_\mu]\,U(t)^{-1} - I \otimes D_\mu$ is $I \otimes H_{\mathrm{b}}$-infinitesimally small uniformly near 0, in the same way as in Lemma 4.2. Hence we obtain the desired result. □

We can now state a uniform relative boundedness of the operator C_Λ defined by (4.9). Note that Lemma 2.1 holds. Hence, Lemmas 4.4 and 4.7 imply the following proposition, so that Theorem 4.3 is proved by the Kato-Rellich theorem.

Proposition 4.8 *The operator C_Λ is $H_0(\Lambda)$-infinitesimally small uniformly near ∞.*

4.3 Proof of Theorem 3.1

We are now ready to prove Theorem 3.1. We need the following proposition:

Proposition 4.9 *For all* $\Psi \in \mathcal{F}_T$

$$\text{s-}\lim_{\Lambda\to\infty} C_\Lambda \Psi = C\Psi \tag{4.18}$$

holds, where

$$C = \frac{g^2}{2}\mathrm{ad}^2(S) I \otimes H_{\mathrm{b}}.$$

Proof. By (4.14), we need only to prove

$$\text{s-}\lim_{t\to 0} E(t)\Psi = E\Psi,$$

where

$$E = \mathrm{ad}^2(S) I \otimes H_{\mathrm{b}}.$$

Note that the following fact holds:

$$E(t)\Psi = \left\{ E - itU(\kappa(t)) \left[\mathrm{ad}^1(S)E\right] U(\kappa(t))^{-1} \right\} \Psi$$

with some $\kappa(t) \in [-|t|, |t|]$ $(t \in \mathbb{R})$. Therefore, we obtain the desired result. □

Now, we apply Corollary 2.4 to $H(\Lambda)$. Let

$$C_\infty := (I \otimes P_0) C (I \otimes P_0), \tag{4.19}$$

where we have denoted the closure of the operator C by the same notation. Note that

$$\left(\widehat{H}(\Lambda) - z\right)^{-1} = U(g/\Lambda)(H(\Lambda) - z)^{-1} U(g/\Lambda)^{-1}$$

for all z contained in the resolvent set of $H(\Lambda)$. It is easy to see that

$$\text{s-}\lim_{t\to\infty} U(t) = I.$$

By Proposition 4.8 and 4.9, we obtain the following: For $z \in \mathbb{C} \setminus \mathbb{R}$

$$\text{s-}\lim_{\Lambda\to\infty} (H(\Lambda) - z)^{-1} = \left(-\Delta \otimes I + C_\infty\right)^{-1} I \otimes P_0 \tag{4.20}$$

holds.

By [1, Theorem 2.12] we need only to prove that the partial expectaion of C with respect to the Fock vacuum Ω is eaqual to $V_{\rm eff}$ defined by (3.10). It is easy to see that

$$\begin{aligned}&\langle u_1 \otimes f_1 \otimes \Omega, C(u_2 \otimes f_2 \otimes \Omega)\rangle \\ &\qquad = -\frac{g^2}{2}\sum_{j,k}\langle f_1, V_{j,k} f_2\rangle \left\langle u_1, \left(\frac{1}{2}[B_j, B_k] + B_k B_j\right) u_2\right\rangle\end{aligned}$$

for all $u_i \in \mathcal{H}$, $f_i \in L^2(\mathbb{R}^d)$ $(i = 1, 2)$. Note that $[B_j, B_k] = -[B_k, B_j]$. It follows that

$$\langle u_1 \otimes f_1 \otimes \Omega, C(u_2 \otimes f_2 \otimes \Omega)\rangle = \langle u_1 \otimes f_1 \otimes \Omega, V_{\rm eff} u_2 \otimes f_2 \otimes \Omega\rangle\,.$$

Hence, we obtain that

$$C_\infty = V_{\rm eff} \otimes P_0,$$

and therefore (3.12) holds.

5 Nucleon-Pion interaction

In this section, we give a concrete realization of the abstract model H, which is an interaction model between nucleons and pions with isospin (see [13, Section 5.1]).

Let $\sigma_j, \tau_j (j = 1, 2, 3)$ be the Pauli matrices:

$$\sigma_1 = \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\begin{aligned}{\sigma_j}^{(i)} &= 1_2 \otimes \cdots \otimes \overset{i\text{th}}{\breve{\sigma_j}} \otimes \cdots \otimes 1_2, \quad j = 1, 2, 3, \\ {\tau_\alpha}^{(i)} &= 1_2 \otimes \cdots \otimes \overset{i\text{th}}{\breve{\tau_\alpha}} \otimes \cdots \otimes 1_2, \quad \alpha = 1, 2, 3,\end{aligned}$$

where 1_2 is the 2×2 identity matrix. Physically, $\sigma^{(i)} = ({\sigma_1}^{(i)}, {\sigma_2}^{(i)}, {\sigma_3}^{(i)})$ and $\tau^{(i)} = ({\tau_1}^{(i)}, {\tau_2}^{(i)}, {\tau_3}^{(i)})$ denote the spin and the isospin of the ith particle, respectively. Set

$$\mathcal{H}_N := \left[\bigotimes^N \mathbb{C}^2\right] \bigotimes \left[\bigotimes^N \mathbb{C}^2\right].$$

If there is no danger of confusion, we denote the operators ${\sigma_j}^{(i)} \otimes (\bigotimes^N 1_2)$ and $(\bigotimes^N 1_2) \otimes {\tau_\alpha}^{(i)}$ acting on $\mathcal{H}_N$ by the same symbol ${\sigma_j}^{(i)}$ and ${\tau_\alpha}^{(i)}$, respectively. We denote by $\hbar$ the Planck constant divided by 2π. Put

$$B_{j,\alpha}^{(i)} := \frac{\hbar}{2} {\sigma_j}^{(i)} {\tau_\alpha}^{(i)}, \quad i = 1, \cdots, N, \ j, \alpha = 1, 2, 3,$$

which act on $\mathcal{H}_N$. It is straightforward to see that

$$\left[B^{(i)}_{j,\alpha}, B^{(l)}_{k,\beta}\right] = 0, \quad i \neq l, \ j,k,\alpha,\beta = 1,2,3. \tag{5.1}$$

By the anticommutativity of the Pauli matrices, it follows that, for $i = 1, \cdots, N$, $j,k,\alpha,\beta = 1,2,3$,

$$\left\{B^{(i)}_{j,\alpha}, B^{(i)}_{k,\beta}\right\} = \frac{\hbar^2}{4}\delta_{jk}\delta_{\alpha\beta}, \tag{5.2}$$

where $\{X,Y\} = XY + YX$ and δ_{ij} is Kronecker's delta.

We denote by m and c the mass of a pion and the speed of light, respectively. Let

$$\omega(k) = \sqrt{\hbar^2 k^2 c^2 + m^2 c^4} \ (k \in \mathbb{R}^3),$$

where ω denotes a dispersion relation of one free pion. Let

$$\mathcal{K} = \bigoplus^3 L^2(\mathbb{R}^3).$$

The function ω defines a multiplication operator on $\mathcal{K}$. We denote it by the same symbol ω:

$$\omega f := (\omega f_1, \omega f_2, \omega f_3), \quad f = (f_1, f_2, f_3) \in \mathcal{K}$$

with $f_i \in D(\omega)$. Let $H_{\rm b} = d\Gamma(\omega)$. Then, $H_{\rm b}$ represents the free Hamiltonian of the pion field.

Let

$$\phi_\alpha(f) := \phi(f_\alpha), \qquad f \in L^2(\mathbb{R}^3),$$

where

$$f_\alpha := (\delta_{\alpha 1} f, \delta_{\alpha 2} f, \delta_{\alpha 3} f).$$

We denote by ρ the density of a nucleon, which is a real distribution satisfying $\widehat{\partial_j \rho}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, where $\widehat{\partial_j \rho}$ denotes the Fourier transform of $\partial_j \rho$. Let

$$\Phi_\alpha(g^{(i)}_j) = \int^{\oplus}_{\mathbb{R}^{3N}} \phi_\alpha(g^{(i)}_j(x))dx,$$

where

$$g^{(i)}_j(x) = -\frac{\sqrt{\hbar}}{\sqrt{(2\pi)^3\omega}}\widehat{\hbar\partial_j \rho}\ e^{-ik\cdot x_i}$$

for $x := (x_1, \cdots, x_N) \in \mathbb{R}^{3N}$. Here $x_i \in \mathbb{R}^3$ indicates the coordinate of the ith nucleon.

A Hamiltonian of spin-nucleons interacting with pions, acting on $\mathcal{H}_N \otimes L^2(\mathbb{R}^{3N}; \mathcal{F}_{\rm b}(\mathcal{K}))$, is defined by

$$H(\hbar, c, M) := -\frac{\hbar^2}{2M}\Delta \otimes I + \frac{\hbar}{2}\sum_{i=1}^{N} {\sigma_3}^{(i)} \otimes I + I \otimes H_{\rm b} + g \sum_{\substack{1\le i\le N \\ 1\le j,\alpha\le 3}} B_{j,\alpha}^{(i)} \otimes \Phi_\alpha({g_j}^{(i)}),$$

where $g \in \mathbb{R}$ is a coupling constant.

Now, we define the scaled Hamiltonian by

$$H(\Lambda) := \frac{1}{\Lambda^2} H(\Lambda^2\hbar, \Lambda^2 c, \Lambda^2 M).$$

Then, we can write

$$H(\Lambda) = -\frac{\hbar^2}{2M}\Delta \otimes I + \frac{\hbar}{2}\sum_{i=1}^{N} {\sigma_3}^{(i)} \otimes I + \Lambda^2 I \otimes H_{\rm b} + g\Lambda H_{\rm I},$$

where

$$H_{\rm I} = \sum_{i=1}^{N} \sum_{1\le j,\alpha\le 3} B_{j,\alpha}^{(i)} \otimes \Phi_\alpha({g_j}^{(i)}).$$

We now ready to derive a quantum mechanical Hamiltonian from $H(\Lambda)$. Let

$$H_{\rm eff} = -\frac{\hbar^2}{2M}\Delta + \frac{\hbar}{2}\sum_{i=1}^{N} {\sigma_3}^{(i)} + \sum_{1\le i<l\le N} E_{i,l} + NE_0,$$

where

$$E_{i,l}(x) = -\frac{g^2\hbar}{(2\pi)^3}\int_{\mathbb{R}^3} \left(\frac{\hbar}{2}\sigma^{(i)}\cdot\hbar k\right)\left(\frac{\hbar}{2}\sigma^{(l)}\cdot\hbar k\right)\frac{|\hat\rho(k)|^2}{\omega(k)^2} e^{-ik\cdot(x_i - x_l)}dk,$$

a.e.$x = (x_1, \cdots, x_N) \in \mathbb{R}^{3N}$ and

$$E_0 = -\frac{3}{2}\frac{g^2\hbar}{(2\pi)^3}\sum_{\alpha=1}^{3} {\tau_\alpha}^{(i)}{\tau_\alpha}^{(l)} \left(\frac{\hbar}{2}\right)^2 \int_{\mathbb{R}^3} |\hbar k|^2 \frac{|\hat\rho(k)|^2}{\omega(k)^2}dk.$$

Theorem 5.1 *Suppose that*

$$\omega^{-3/2}{g_j}^{(i)} \in L^2(\mathbb{R}^3), \quad i = 1, \cdots, N, \quad j = 1, 2, 3.$$

Then, for all $z \in \mathbb{C}\setminus\mathbb{R}$ or $z < 0$ with sufficiently large,

$$\text{s-}\lim_{\Lambda\to\infty} (H(\Lambda) - z)^{-1} = (H_{\rm eff} - z)^{-1} \otimes P_0.$$

Proof. Applying Theorem 3.1, we have

$$\text{s-}\lim_{\Lambda\to\infty}(H(\Lambda)-R\otimes I-z)^{-1}=(H_{\text{eff}}-R-z)^{-1}\otimes P_0,$$

where

$$R=\frac{\hbar}{2}\sum_{i=1}^{N}{\sigma_3}^{(i)}.$$

Since R is bounded, we can prove the desired result in the same way as in Theorem 2.3. □

Remark 5.1 *Physically, E_0 and $E_{i,l}$ above are considered respectively as the self-energy of each nucleon and an effective potential of the force between two nucleons caused by the exchange of pions.*

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