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CONVERGENCE OF SCATTERING OPERATORS FOR THE KLEIN-GORDON EQUATION WITH A NONLOCAL NONLINEARITY

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ABSTRACT. We consider the scattering problems for two types of nonlinear Klein-Gordon equations. One is the equation of the Hartree type, and the other one is the equation with power nonlinearity. We show that the scattering operator for the equation of the Hartree type converges to that for the one with power nonlinearity in some sense. Our proof is based on some inequalities in the Lorentz space, and a strong limit of Riesz potentials.

1. INTRODUCTION

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = \gamma F_\sigma(u)$$

in space-time $\mathbb{R} \times \mathbb{R}^n$, where u is a real or complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in $\mathbb{R}^n$ and γ is a real number. The nonlinearity F_σ has the form

$$F_\sigma(u) = \big(V_\sigma * |u|^{p-1}\big)u \tag{1.1}$$

with $p > 1$ and

$$|V_\sigma(x)| \le C_V |x|^{-\sigma}, \quad 0 < \sigma < n. \tag{1.2}$$

The constant C_V is independent of σ, and $*$ denotes the convolution in space variables. Furthermore, we suppose that there exists some real number λ_0 such that

$$\| |V_\sigma| \cdot |^\sigma - \lambda_0 \|_{L^\infty} \to 0 \qquad \text{as} \quad \sigma \to n. \tag{1.3}$$

The term $F_\sigma(u)$ is the generalization of the Hartree term $(| \cdot |^{-\sigma} * |u|^2)u$ which is an approximative expression of the nonlocal interaction with potential $|x|^{-\sigma}$. Menzala and

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Strauss started to study the Klein-Gordon equation with the Hartree term in [3].

In order to treat the scattering problem, we define the scattering operator for the nonlinear Klein-Gordon equation $\partial_t^2 u - \Delta u + u = f(u)$. First, we list some notation to give the definition. Put $\omega = \sqrt{1-\Delta}$. Let H^s be the Sobolev space $\omega^{-s}L^2(\mathbb{R}^n)$. We denote $H^1 \oplus L^2$ by X. For a positive number δ and a Banach space A, we denote the set $\left\{a \in A; \|a|A\| \le \delta\right\}$ by $B(\delta; A)$.

The the scattering operator is defined as the mapping $S : B(\delta; X) \ni (\phi_-^1, \phi_-^2) \mapsto (\phi_+^1, \phi_+^2) \in X$ if the following condition holds for some $\delta > 0$:

Let Z be a suitable subspace of $C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$, and $u_\pm(t)$ are solutions of linear Klein-Gordon equations whose initial data are $(\phi_\pm^1, \phi_\pm^2)$, respectively. For any $(\phi_-^1, \phi_-^2) \in B(\delta; X)$, there uniquely exist $u \in Z$ and data $(\phi_+^1, \phi_+^2) \in X$ such that

$$u(t) = u_-(t) + \int_{-\infty}^{t} \frac{\sin(t-t')\omega}{\omega} f(u(t'))dt'$$

and

$$\|u(t) - u_\pm(t)\|_{H^1} + \|\partial_t u(t) - \partial_t u_\pm(t)\|_{L^2} \to 0 \qquad as \quad t \to \pm\infty.$$

We call that "(S, X) is well-defined" if we can define the scattering operator $S : B(\delta; X) \to X$ for some $\delta > 0$.

For simplicity, for the present, we assume that F_σ has the form $(|\cdot|^{-\sigma} * |u|^{p-1})u$, which clearly satisfies the conditions (1.2) and (1.3). For $0 < \sigma < n$ and a suitable function v, the term $|\cdot|^{-\sigma} * v$ is expressed by the Riesz potential. In fact, we have

$$(-\Delta)^{(-n+\sigma)/2} v = \gamma(\sigma)^{-1} |\cdot|^{-\sigma} * v, \tag{1.4}$$

where

$$\gamma(\sigma) = \frac{\pi^{n/2} 2^{n-\sigma} \Gamma((n-\sigma)/2)}{\Gamma(\sigma/2)} \tag{1.5}$$

and Γ is the usual Gamma function. Thus, it seems natural to expect that

$$\gamma(\sigma)^{-1} F_\sigma(v) \to |v|^{p-1} v \qquad \text{as} \quad \sigma \to n$$

in some sense. From the above expectation, we shall prove the following properties:

Suppose that F_σ and V_σ satisfy (1.1)–(1.3). Consider the nonlinear Klein-Gordon equations

$$\partial_t^2 u - \Delta u + u = \gamma(\sigma)^{-1} F_\sigma(u) \tag{NLKG$_\sigma$}$$

and

$$\partial_t^2 u - \Delta u + u = \lambda_0 |u|^{p-1} u. \tag{NLKG}$$

Let S_σ and S_n be the scattering operators for (NLKG$_\sigma$) and for (NLKG), respectively. Then, for sufficiently small data $\phi_- = (\phi_-^1, \phi_-^2) \in X$, we have

$$\|S_\sigma(\phi_-) - S_n(\phi_-)\|_X \to 0 \qquad as \quad \sigma \to n. \tag{1.6}$$

Before we state our main result, we remark some results on the scattering problems for (NLKG$_\sigma$) and for (NLKG).

There is a large literature on the scattering problem for (NLKG). Especially, it was proved by Strauss [9] and Pecher [7] that if

$$1 + \frac{4}{n} \le p \begin{cases} < \infty & \text{if } n = 1, 2, \\ \le 1 + \frac{4}{n-2} & \text{if } n \ge 3, \end{cases} \tag{1.7}$$

then (S_n, X) is well-defined. On the other hand, for (NLKG$_\sigma$), we can easily see from Mochizuki [4] that if $p \ge 2$, $0 < \sigma < n$ and

$$2 - \frac{n(p-3)}{2} \le \sigma \le p + 1 - \frac{n(p-3)}{2},$$

then (S_σ, X) is well-defined.

Therefore, if we assume that $p \ge 2$ and

$$2 - \frac{n(p-3)}{2} < n \le p + 1 - \frac{n(p-3)}{2}, \tag{1.8}$$

then it follows that (S_σ, X) is well-defined if $2 - n(p-3)/2 \le \sigma < n$ and $\sigma > 0$. Moreover, (S_n, X) is also well-defined because (1.8) is equivalent to

$$1 + \frac{4}{n} < p \begin{cases} < \infty & \text{if } n = 1, 2, \\ \le 1 + \frac{4}{n-2} & \text{if } n \ge 3, \end{cases} \tag{1.9}$$

which implies (1.7).

Again, suppose that $p \ge 2$ and (1.8) (or equivalently, (1.9)) holds. Then, the dimensional number n is automatically restricted to $1 \le n \le 6$. For $0 < \sigma < n$, we again denote the scattering operator for (NLKG$_\sigma$) by S_σ. Let S_n be the scattering operator for (NLKG). As we stated before, the existence of the scattering operator S_κ has been already known if $\kappa \in (0, n]$ is sufficiently close to n. In this paper, we prove that for sufficiently small data $\phi_- \in X$, (1.6) holds.

In order to state our result precisely, we give some notation which will be used later. For function spaces $A(\mathbb{R})$ and $B(\mathbb{R}^n)$, we denote $A(\mathbb{R}; B(\mathbb{R}^n))$ by AB. For $l = 0, 1$, let $\mathcal{B}^l(\mathbb{R}) = C^l(\mathbb{R}) \cap W^l_\infty(\mathbb{R})$, where $W^l_\infty(\mathbb{R})$ is the Sobolev space. We define $\mathcal{H}$ by $\mathcal{B}H^1 \cap \mathcal{B}^1 L^2$ with the norm $\|v\|_{\mathcal{H}} = \|v\|_{L^\infty H^1} + \|\partial_t v\|_{L^\infty L^2}$. We denote the Fourier transform and its

inverse by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively. Let $\{\widetilde{\psi_j}\}_{j=0}^{\infty}$ be the Littlewood-Paley decomposition on $\mathbb{R}^n$. For $s \in \mathbb{R}$, $1 \le q \le \infty$ and $1 \le r < \infty$, the inhomogeneous Besov space $B^s_{q,r}$ is defined by

$$B^s_{q,r} = \{w \in \mathcal{S}'; \|w\|_{B^s_{q,r}} < \infty\}.$$

Here,

$$\|w\|_{B^s_{q,r}} = \Big\{ \sum_{j=0}^{\infty} 2^{sjr} \|\psi_j * w\|^r_{L^q} \Big\}^{\frac{1}{r}}$$

and $\psi_j = \mathcal{F}^{-1}\widetilde{\psi_j}$. For $s \in \mathbb{R}$ and $1 \le q < \infty$, we put $B^s_q = B^s_{q,2}$. For $n \ge 3$, let ϑ_p be a real number satisfying

$$p = \vartheta_p\big(1 + \frac{4}{n}\big) + (1 - \vartheta_p)\big(1 + \frac{4}{n-2}\big).$$

Set

$$\theta = \begin{cases} 1 & \text{if } n = 1, 2, \\ \vartheta_p & \text{if } 3 \le n \le 6, \end{cases}$$

and

$$\frac{1}{r(p)} = \frac{1}{2} - \frac{2}{p(n-1+\theta)}, \quad \rho(p) = \frac{n+1+\theta}{p(n-1+\theta)}.$$

We denote $L^p B^{1-\rho(p)}_{r(p)} \cap \mathcal{H}$ by $Z(p)$. Put

$$f_\kappa(v) = \begin{cases} \gamma(\kappa)^{-1} F_\kappa(v) & \text{if} \quad 0 < \kappa < n, \\ \lambda_0 |v|^{p-1} v & \text{if} \quad \kappa = n. \end{cases}$$

We are ready to state our main result.

Theorem 1.1. *Let $1 \le n \le 6$. Assume that $p \ge 2$ satisfies (1.9), and that F_σ and V_σ satisfy (1.1)–(1.3). Put $u_\star(t) = \cos(t\omega)\phi^1_\star + \omega^{-1}\sin(t\omega)\phi^2_\star$, where $\star$ denotes either $+$ or $-$. Then there exist some $\delta_0 > 0$ and $\sigma(p) \in (0, n)$ satisfying the following properties:*

If $\kappa \in [\sigma(p), n]$ and $(\phi^1_-, \phi^2_-) \in B(\delta_0; X)$, then there uniquely exist $u^\kappa \in Z(p)$ and $(\phi^1_+, \phi^2_+) \in X$ such that u^κ satisfies

$$u^\kappa(t) = u_-(t) + \int_{-\infty}^{t} \frac{\sin(t-t')\omega}{\omega} f_\kappa(u^\kappa(t'))dt', \tag{1.10}$$

and we have

$$\|u^\kappa(t) - u_\pm(t)\|_{\mathcal{H}} \to 0 \qquad as \quad t \to \pm\infty. \tag{1.11}$$

Furthermore, the scattering operator S_κ is well-defined on $B(\delta_0; X)$ and we have (1.6).

Remark 1. The radius δ_0 depends on p, independent of κ.

The contents of this paper is as follows. In Section 2, we show some inequalities in the Lorentz space, and we give a strong limit of $(-\Delta)^{(-n+\sigma)/2}$ as σ tends to n. In Section 3, we first show an embedding inequality for the Besov spaces. Using the inequality and the Strichartz estimate, we prove that the radius δ_0 is given independently with respect to $\sigma \in [\sigma(p), n)$. Finally, applying a strong limit of $(-\Delta)^{(-n+\sigma)/2}$, we prove (1.6).

2. Preliminaries

In this section, we first state some properties of the Lorentz space. We define the Lorentz space as follows: We denote the n-dimensional Lebesgue measure by μ. For a measurable function $f : \mathbb{R}^n \to \mathbb{C}$, and for real numbers a, y and z, we define $m(f,a) = \mu(\{x; |f(x)| > a\})$, $f^*(y) = \inf\{a; m(f,a) \le y\}$ and $f^{**}(z) = z^{-1}\int_0^z f^*(y)dy$. For $1 \le q, r \le \infty$, Set

$$\|f\|_{q,r} = \begin{cases} \big[\int_0^\infty \big(z^{1/q} f^{**}(z)\big)^r \frac{dz}{z}\big]^{1/r} & \text{if } 1 \le r < \infty, \\ \sup_{z>0} z^{1/q} f^{**}(z) & \text{if } r = \infty. \end{cases}$$

For $1 \le q, r \le \infty$, the Lorentz space $L^{q,r} = L^{q,r}(\mathbb{R}^n)$ is the set of all measurable functions satisfying $\|f\|_{q,r} < \infty$.

Put $\|\cdot\|_p = \|\cdot\|_{L^p}$. For $1 \le q \le \infty$, q' denotes the Hölder conjugate of q. The following proposition is shown by O'Neil [6]:

Proposition 2.1. *Let $1 \le q, r, q_j, r_j \le \infty$, $j = 1, 2$. Then, we have the properties*
(1) If $1 < q < \infty$, then

$$\|f\|_q \le \|f\|_{q,q} \le q'\|f\|_q. \tag{2.1}$$

(2) If $1 < q < \infty$ and $1 \le r_1 < r_2 \le \infty$, then

$$\|f\|_{q,r_2} \le \big(\frac{r_1}{q}\big)^{\frac{1}{r_1}-\frac{1}{r_2}} \|f\|_{q,r_1}. \tag{2.2}$$

(3) If $1/q_1 + 1/q_2 > 1$, $1 + 1/q = 1/q_1 + 1/q_2$, $r \ge 1$ and $1/r_1 + 1/r_2 \ge 1/r$, then

$$\|f * g\|_{q,r} \le 3q\|f\|_{q_1,r_1}\|g\|_{q_2,r_2}. \tag{2.3}$$

The Hardy-Littlewood-Sobolev inequality implies that the operator $(-\Delta)^{(-n+\sigma)/2}$ is a bounded operator from some L^{q_2}-space into some L^{q_1}-space. In general, the operator norm of $(-\Delta)^{(-n+\sigma)/2}$ is dependent on q_1, q_2 and σ. Applying the following proposition, we can see the concrete dependancy.

Proposition 2.2. *Let $0 < \sigma < n$, $1 < q_2 < q_1 < \infty$. If $1 + 1/q_1 = \sigma/n + 1/q_2$, then we have*

$$\||\cdot|^{-\sigma} * f\|_{q_1} \le 3q_1 q_2' n\alpha(n)^{\frac{\sigma}{n}}(n-\sigma)^{-1}\|f\|_{q_2}, \tag{2.4}$$

$$\|(-\Delta)^{\frac{-n+\sigma}{2}} f\|_{q_1} \le C_H(q_1, q_2, \sigma)\|f\|_{q_2}, \tag{2.5}$$

where

$$C_H(q_1,q_2,\sigma)=\frac{3q_1q_2'n\alpha(n)^{\sigma/n}\Gamma(n/2)}{2^{n-\sigma+1}\pi^{n/2}\Gamma(1+(n-\sigma)/2)}$$

and

$$\alpha(n)=\mu(B_1(0))=\pi^{\frac{n}{2}}\big(\frac{n}{2}\Gamma(\frac{n}{2})\big)^{-1}.$$

Proof. For $0<\sigma<n$, we obtain

$$\||\cdot|^{-\sigma}\|_{\frac{n}{\sigma},\infty}=n\alpha(n)^{\frac{\sigma}{n}}(n-\sigma)^{-1}.$$

Thus, it follows from (2.1)–(2.3) that

$$\begin{aligned}\||\cdot|^{-\sigma}*f\|_{q_1}&\le\||\cdot|^{-\sigma}*f\|_{q_1,q_1}\\&\le 3q_1\||\cdot|^{-\sigma}\|_{n/\sigma,\infty}\|f\|_{q_2,q_1}\\&\le 3q_1n\alpha(n)^{\sigma/n}(n-\sigma)^{-1}\|f\|_{q_2,q_2}\\&\le 3q_1q_2'n\alpha(n)^{\sigma/n}(n-\sigma)^{-1}\|f\|_{q_2}.\end{aligned}$$

Hence (2.4) holds. By (1.4), (1.5), (2.4) and the equivalence $y\Gamma(y)=\Gamma(y+1)$, we can easily see (2.5). □

Remark 2. Let $0<\sigma_0<n$. Assume that K is a compact subset of $(1,\infty)$. Then from (2.5), we see that

$$C_H=\sup_{\sigma\in[\sigma_0,n),\ q_1,q_2\in K}C_H(q_1,q_2,\sigma)<\infty.$$

Next, we consider a strong limit of the operator $(-\Delta)^{(-n+\sigma)/2}$. For this purpose, we introduce some function spaces. Set

$$\widehat{\mathcal{D}_0}=\big\{\phi\in\mathcal{S};\mathcal{F}\phi\in C_0^\infty(\mathbb{R}^n\setminus\{0\})\big\},$$

where $C_0^\infty(\mathbb{R}^n\setminus\{0\})$ is the set of all compactly supported, and smooth functions. Let $\{\widetilde{\phi_j}\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley decomposition on $\mathbb{R}^n\setminus\{0\}$. For $s\in\mathbb{R}$, $1\le q<\infty$ and $1\le r<\infty$, the homogeneous Triebel-Lizorkin space $\dot F^s_{q,r}$ is defined by

$$\dot F^s_{q,r}=\big\{w\in\mathcal{S}';\|w\|_{\dot F^s_{q,r}}<\infty\big\}/\mathcal{P}.$$

Here,

$$\|w\|_{\dot F^s_{q,r}}=\Big\|\Big\{\sum_{j=-\infty}^{\infty}2^{sjr}\big|\phi_j*w\big|^r\Big\}^{\frac{1}{r}}\Big\|_q,$$

$\phi_j=\mathcal{F}^{-1}\widehat{\phi_j}$, and $\mathcal{P}$ is the set of all polynomials.

Proposition 2.3. *For $1<q_2<q_1<\infty$, $\widehat{\mathcal{D}_0}$ is dense in $L^{q_1}\cap L^{q_2}$.*

Proof. For $v \in \mathcal{S}$, set

$$v_N = \sum_{j=-N}^{N} \phi_j * v, \qquad N = 1, 2, \cdots .$$

Clearly, we have $v_N \in \widehat{\mathcal{D}_0}$. If $1 < q < \infty$, then L^q is equivalent to $\dot{F}^0_{q,2}$ (see, e.g., Ginibre-Velo [1]). Therefore, we obtain

$$\begin{aligned}
\|v - v_N\|_q &\le C\|v - v_N\|_{\dot{F}^0_{q,2}} \\
&= C\Big\| \Big\{ \sum_{k=-\infty}^{\infty} \big|\phi_k * (v - \sum_{j=-N}^{N} \phi_j * v)\big|^2 \Big\}^{\frac{1}{2}} \Big\|_q \\
&= C\Big\| \Big\{ \sum_{k=-\infty}^{\infty} \big|(\phi_k - \sum_{j=-N}^{N} \phi_k * \phi_j) * v\big|^2 \Big\}^{\frac{1}{2}} \Big\|_q .
\end{aligned}$$

Since

$$\phi_k - \sum_{j=-N}^{N} \phi_k * \phi_j = \begin{cases} 0 & \text{if} \quad |k| \le N-1, \\ \phi_{-N-1} * (\phi_{-N-2} + \phi_{-N-1}) & \text{if} \quad k = -N-1, \\ \phi_{-N} * \phi_{-N-1} & \text{if} \quad k = -N, \\ \phi_N * \phi_{N+1} & \text{if} \quad k = N, \\ \phi_{N+1} * (\phi_{N+1} + \phi_{N+2}) & \text{if} \quad k = N+1, \\ \phi_k & \text{if} \quad |k| > N+1 \end{cases}$$

and

$$\|\phi_k * \phi_j * v\|_q \le \|\phi_0\|_1 \|\phi_k * v\|_q,$$

we see that

$$\begin{aligned}
\|v - v_N\|_q \le C\Big\| \Big\{ &\big|\phi_{-N-1} * (\phi_{-N-2} + \phi_{-N-1}) * v\big|^2 \\
&+ \big|\phi_{N+1} * (\phi_{N+1} + \phi_{N+2}) * v\big|^2 \\
&+ \big|\phi_{-N} * \phi_{-N-1} * v\big|^2 + \big|\phi_N * \phi_{N+1} * v\big|^2 \\
&+ \sum_{|k|>N+1} \big|\phi_k * v\big|^2 \Big\}^{\frac{1}{2}} \Big\|_q \\
\le C\Big\{ &\sum_{|k|=N,N+1} \|\phi_k * v\|_q + \Big\| \Big\{ \sum_{|k|>N+1} |\phi_k * v|^2 \Big\}^{\frac{1}{2}} \Big\|_q \Big\} \\
\le C\Big\| &\Big\{ \sum_{|k|\ge N} |\phi_k * v|^2 \Big\}^{\frac{1}{2}} \Big\|_q .
\end{aligned}$$

It follows from $v \in \dot{F}^0_{q,r}$ and the Lebesgue dominated theorem that

$$\Big\| \Big\{ \sum_{|k|\geq N} |\phi_k * v|^2 \Big\}^{\frac{1}{2}} \Big\|_q \to 0 \qquad \text{as} \quad N \to \infty.$$

Hence, we have $\lim_{N\to\infty} \|v - v_N\|_{q_j} = 0$ for $j = 1, 2$. Therefore, we obtain

$$\|v - v_N\|_{L^{q_1}\cap L^{q_2}} \to 0 \qquad \text{as} \quad N \to \infty.$$

Thus, $\widehat{\mathcal{D}_0}$ is dense in $L^{q_1} \cap L^{q_2}$ because $\mathcal{S}$ is dense in the same space. □

Using the above proposition and Remark 2, we have the following strong limit of $(-\Delta)^{(-n+\sigma)/2}$:

Lemma 2.4. *For $1 < q < \infty$ and $0 < \sigma < n$, define*

$$\frac{1}{q_\sigma} = 1 + \frac{1}{q} - \frac{\sigma}{n}. \tag{2.6}$$

Let $0 < \sigma_0 < n$. If $1 < q_{\sigma_0} < q < \infty$, $q \geq 2$ and $f \in L^{q_{\sigma_0}} \cap L^q$, then we have

$$\big\|(-\Delta)^{\frac{-n+\sigma}{2}} f - f\big\|_q \to 0 \qquad \textit{as} \quad \sigma \to n. \tag{2.7}$$

Proof. From Proposition 2.3, for all $\varepsilon > 0$, there exists some $f_\varepsilon \in \widehat{\mathcal{D}_0}$ such that

$$\|f - f_\varepsilon\|_{L^{q_{\sigma_0}}\cap L^q} \leq \varepsilon. \tag{2.8}$$

We can take some $\eta \in (0, 1)$ satisfying

$$\text{supp } \widehat{f_\varepsilon} \subset \{x \in \mathbb{R}^n; |x| > \eta\}.$$

Using the Hausdorff-Young inequality, we obtain

$$\begin{aligned}\big\|(-\Delta)^{\frac{-n+\sigma}{2}} f_\varepsilon - f_\varepsilon\big\|_q &\leq C\big\|(|\cdot|^{-n+\sigma} - 1)\widehat{f_\varepsilon}\big\|_{q'} \\ &\leq C_\varepsilon|\eta^{-n+\sigma} - 1|,\end{aligned}$$

where C_ε is independent of σ. Therefore, we see from (2.5) and Remark 2 that

$$\begin{aligned}\big\|(-\Delta)^{\frac{-n+\sigma}{2}} f - f\big\|_q \leq& \big\|(-\Delta)^{\frac{-n+\sigma}{2}}(f - f_\varepsilon)\big\|_q + \big\|(-\Delta)^{\frac{-n+\sigma}{2}} f_\varepsilon - f_\varepsilon\big\|_q \\ &+ \|f_\varepsilon - f\|_q \\ \leq& C_H\|f - f_\varepsilon\|_{q_\sigma} + C_\varepsilon|\eta^{-n+\sigma} - 1| + \|f_\varepsilon - f\|_q \\ \leq& C_\varepsilon|\eta^{-n+\sigma} - 1| + (C_H + 1)\|f - f_\varepsilon\|_{L^{q_{\sigma_0}}\cap L^q},\end{aligned}$$

where C_H is independent of σ and ε. Thus, if $n - \sigma$ is sufficiently small, then we see from (2.8) that

$$\big\|(-\Delta)^{\frac{-n+\sigma}{2}} f - f\big\|_q \leq (C_H + 2)\varepsilon.$$

Hence, we have (2.7). □

3. Proof of theorem

In this section, we prove Theorem 1.1. Throughout this section, we assume that $1 \le n \le 6$, $p \ge 2$ and (1.8) (or equivalently, (1.9)). We set $r = r(p)$, $\rho = \rho(p)$ and $Z = Z(p)$.

For $0 \le \kappa \le n$, let p_κ be a real number which satisfies

$$\frac{3}{2} = \frac{\kappa}{n} + \frac{p-1}{2p_\kappa} + \frac{1}{2p}. \tag{3.1}$$

We can easily see that

$$p_\kappa < p \qquad \text{if} \quad 0 < \kappa < n, \tag{3.2}$$

$$\lim_{\sigma \to n} p_\sigma = p. \tag{3.3}$$

The space L^{2p_σ} has the following property:

Proposition 3.1. *For p, there exists some $\sigma(p) \in (0, n)$ such that*

$$\|f\|_{2p_\kappa} \le C_E \|f\|_{B_r^{1-\rho}} \qquad \text{if} \quad \sigma(p) \le \kappa \le n, \tag{3.4}$$

where C_E is independent of f and κ.

Proof. If

$$\frac{1}{r} \ge \frac{1}{2p_\kappa} \ge \frac{1}{r} - \frac{1-\rho}{n}, \tag{3.5}$$

then we have $B_r^{1-\rho} \hookrightarrow L^{2p_\kappa}$ (see, e.g., Triebel [10]). The condition (3.5) is equivalent to

$$p \ge \frac{p}{p_\kappa} + \frac{4}{n-1+\theta} \tag{3.6}$$

and

$$\frac{p}{p_\kappa} \ge \frac{p(n-2)}{n} + \frac{2(-n+1+\theta)}{n(n-1+\theta)}. \tag{3.7}$$

For $n \le 2$, (3.6) and (3.7) obviously holds if κ is sufficiently close to n. Suppose that $3 \le n \le 6$. Since the function $1 + 4/(n-s)$ is convex with respect to $s \in (-\infty, n)$, it follows from $\vartheta_p \in [0, 1)$ that

$$\begin{aligned} p &= \vartheta_p\big(1 + \frac{4}{n}\big) + (1-\vartheta_p)\big(1 + \frac{4}{n-2}\big) \\ &\ge 1 + \frac{4}{n - 2(1-\vartheta_p)} > 1 + \frac{4}{n-1+\vartheta_p}. \end{aligned}$$

Therefore, we see from (3.2) and (3.3) that (3.6) holds if κ is sufficiently close to n. By elementary calculations, we have (3.7).

For all $\kappa \in [\sigma(p), n]$, p_κ belongs to some compact set of $(1, \infty)$. Then we can take C_E which is independent f and κ (see [1]).

□

Let us give a proof of Theorem 1.1 dividing into three steps.
(Step I.) Put $\phi_- = (\phi_-^1, \phi_-^2) \in X$. Let $\sigma(p)$ be a real number which satisfies (3.4) and that

$$\frac{2p_{\sigma(p)}}{p-1} > 1. \tag{3.8}$$

For $\kappa \in [\sigma(p), n]$, we define a mapping

$$\Phi^\kappa : u \mapsto u_- + \int_{-\infty}^{t} \frac{\sin(t-t')\omega}{\omega} f_\kappa(u(t'))dt'.$$

We prove that Φ^κ is a contraction mapping on a suitable complete metric space. By using the Strichartz estimate shown in Machihara-Nakanishi-Ozawa [2], we have

$$\begin{aligned} \|u_-\|_Z &\le C_S \|\phi_-\|_X, \\ \|\Phi^\kappa u - u_-\|_Z &\le C_S \|f_\kappa(u)\|_{L^1 L^2}. \end{aligned}$$

Here, the constant C_S is dependent only on n and p.

Let q be a real number which satisfies

$$\frac{1}{2} = \frac{1}{q} + \frac{1}{2p}. \tag{3.9}$$

We immediately see from (3.1) that

$$1 + \frac{1}{q} = \frac{\sigma}{n} + \frac{p-1}{2p_\sigma}. \tag{3.10}$$

By (1.2), (3.9), the Hölder inequality and (1.4), we obtain

$$\|f_\kappa(u(t))\|_{L^2} \le \max\{C_V, \lambda_0\} \|(-\Delta)^{(-n+\kappa)/2} |u(t)|^{p-1}\|_q \|u(t)\|_{2p}.$$

From (3.10), (2.5) and Remark 2, we see that

$$\begin{aligned} \|(-\Delta)^{(-n+\kappa)/2} |u(t)|^{p-1}\|_q &\le C_H \||u(t)|^{p-1}\|_{2p_\kappa/(p-1)} \\ &\le C_H \|u(t)\|_{2p\kappa}^{p-1}, \end{aligned}$$

where the constant C_H is dependent only on n and p. Accordingly, it follows from (3.4) that there exists a positive constant C_0 depending only on n and p, such that

$$\begin{aligned} \|\Phi^\kappa u - u_-\|_Z &\le C_0 \|u\|_{L^p B_r^{1-\rho}}^p, \\ \|\Phi^\kappa u\|_Z &\le C_0 \big(\|\phi_-\|_X + \|u\|_{L^p B_r^{1-\rho}}^p \big). \end{aligned}$$

We can choose C_0 satisfying that

$$\|\Phi^\kappa u - \Phi^\kappa v\|_Z \le C_0\|u-v\|_{L^pB^{1-\rho}_r}\big(\|u\|^{p-1}_{L^pB^{1-\rho}_r} + \|v\|^{p-1}_{L^pB^{1-\rho}_r}\big).$$

Hence we obtain

$$\|\Phi^\kappa u - u_-\|_Z \le C_0\|u\|^p_Z, \tag{3.11}$$

$$\|\Phi^\kappa u\|_Z \le C_0\big(\|\phi_-\|_X + \|u\|^p_Z\big), \tag{3.12}$$

$$\|\Phi^\kappa u - \Phi^\kappa v\|_Z \le C_0\|u-v\|_Z\big(\|u\|^{p-1}_Z + \|v\|^{p-1}_Z\big). \tag{3.13}$$

(Step II.) Set $Z_0 = B(2C_0\|\phi_-\|_X, Z)$ and $d(u,v) = \|u-v\|_Z$. Clearly, (Z_0, d) is a complete metric space.

From (3.11) and elementary calculations, it follows that $\Phi^\kappa u \in \mathcal{H}$ if $u \in Z$. Hence, by (3.12) and (3.13), we see that Φ^κ is a contraction mapping from Z_0 into itself if $\|\phi_-\|_X \le \delta$ and $\delta \le (4C_0)^{-1/(p-1)}$. Consequently, we obtain a unique fixed point u^κ of Φ^κ. Since the fixed point is also unique in Z (see Nakamura [5]), we have the time global solution $u^\kappa \in Z$ of (1.10).

Put

$$\phi_+ = \phi_- + \int_{\mathbb{R}} \frac{\sin(-t)\omega}{\omega} f_\kappa(u^\kappa(t))dt.$$

By the analogous argument on (3.12), we see that (1.11) holds. We remark that we can easily show the uniqueness of $u_\pm(t)$ satisfying (1.11) (see Sasaki [8]). Thus, the scattering operator S_κ is well-defined.

(Step III.) Since u^κ satisfies $\Phi^\kappa u^\kappa = u^\kappa$, we see that for $\sigma \in [\sigma(p), n)$,

$$\begin{aligned} u^\sigma - u^n &= \int_{-\infty}^t \frac{\sin(t-t')\omega}{\omega}\big\{f_\sigma(u^\sigma(t')) - f_n(u^n(t'))\big\}dt', \\ f_\sigma(u^\sigma) - f_n(u^n) &= \gamma(\sigma)^{-1}\big(V_\sigma * |u^\sigma|^{p-1}\big)(u^\sigma - u^n) \\ &\quad + \big(\gamma(\sigma)^{-1}V_\sigma * |u^\sigma|^{p-1} - \lambda_0|u^n|^{p-1}\big)u^n. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} &\gamma(\sigma)^{-1}V_\sigma * |u^\sigma|^{p-1} - \lambda_0|u^n|^{p-1} \\ &\quad = \gamma(\sigma)^{-1}V_\sigma * |u^\sigma|^{p-1} - \gamma(\sigma)^{-1}\lambda_0|\cdot|^{-\sigma} * |u^\sigma|^{p-1} \\ &\qquad + \gamma(\sigma)^{-1}\lambda_0|\cdot|^{-\sigma} * |u^\sigma|^{p-1} - \gamma(\sigma)^{-1}\lambda_0|\cdot|^{-\sigma} * |u^n|^{p-1} \\ &\qquad + \gamma(\sigma)^{-1}\lambda_0|\cdot|^{-\sigma} * |u^n|^{p-1} - \lambda_0|u^n|^{p-1}. \end{aligned}$$

Therefore, following the line of Step I, we obtain

$$\begin{aligned}
&\|u^\sigma - u^n\|_{L^pB_r^{1-\rho}} \\
&\le C_0\|u^\sigma - u^n\|_{L^pB_r^{1-\rho}}\big(\|u^\sigma\|^{p-1}_{L^pB_r^{1-\rho}} + \|u^n\|^{p-1}_{L^pB_r^{1-\rho}}\big) \\
&\quad + C_0\||V_\sigma|\cdot|^\sigma - \lambda_0\|_\infty\|u^\sigma\|^p_{L^pB_r^{1-\rho}} \\
&\quad + C_0\big\|(-\Delta)^{(-n+\sigma)/2}|u^n|^{p-1} - |u^n|^{p-1}\big\|_{L^{p/(p-1)}L^{2p/(p-1)}}\|u^n\|_{L^pB_r^{1-\rho}}. \\
&\le \frac{1}{4}\|u^\sigma - u^n\|_{L^pB_r^{1-\rho}} + \||V_\sigma|\cdot|^\sigma - \lambda_0\|_\infty \\
&\quad + C_0\big\|(-\Delta)^{(-n+\sigma)/2}|u^n|^{p-1} - |u^n|^{p-1}\big\|_{L^{p/(p-1)}L^{2p/(p-1)}},
\end{aligned} \tag{3.14}$$

where we have used

$$\|u^\kappa\|_Z \le 2C_0\|\phi_-\|_X \le 2C_0\delta.$$

By $u^n \in Z$ and (3.4), we see that

$$\begin{aligned}
&|u^n|^{p-1}(t) \in L^{2p_{\sigma(p)}/(p-1)}(\mathbb{R}^n) \cap L^{2p/(p-1)}(\mathbb{R}^n) \quad \text{for a.e. } t, \\
&1 + \frac{p-1}{2p} = \frac{\sigma}{n} + \frac{p-1}{2p_{\sigma(p)}}.
\end{aligned}$$

Thus, it follows from (3.8) and Lemma 2.4 that

$$\lim_{\sigma\to n}\big\|(-\Delta)^{(-n+\sigma)/2}|u^n|^{p-1}(t) - |u^n|^{p-1}(t)\big\|_{2p/(p-1)} = 0 \quad \text{for a.e. } t.$$

From (2.5) and (3.4), we have

$$\big\|(-\Delta)^{(-n+\sigma)/2}|u^n|^{p-1}(t) - |u^n|^{p-1}(t)\big\|_{2p/(p-1)} \le 2C_H\|u^n(t)\|^{p-1}_{B_r^{1-\rho}}.$$

Hence, by $\|u^n(\cdot)\|^{p-1}_{B_r^{1-\rho}} \in L^{p/(p-1)}(\mathbb{R})$ and the Lebesgue dominated theorem with respect to t, it follows that

$$\lim_{\sigma\to n}\big\|(-\Delta)^{(-n+\sigma)/2}|u^n|^{p-1} - |u^n|^{p-1}\big\|_{L^{p/(p-1)}L^{2p/(p-1)}} = 0. \tag{3.15}$$

From (1.3), (3.14) and (3.15), we have

$$\|u^\sigma - u^n\|_{L^pB_r^{1-\rho}} \to 0 \qquad \text{as} \quad \sigma \to n. \tag{3.16}$$

Similarly, we obtain

$$\begin{aligned}
\|S_\sigma(\phi_-) - S_n(\phi_-)\|_X &\le \frac{1}{2}\|u^\sigma - u^n\|_{L^pB_r^{1-\rho}} + 2\||V_\sigma|\cdot|^\sigma - \lambda_0\|_\infty \\
&\quad + 2C_0\big\|(-\Delta)^{(-n+\sigma)/2}|u^n|^{p-1} - |u^n|^{p-1}\big\|_{L^{p/(p-1)}L^{2p/(p-1)}}.
\end{aligned}$$

By (1.3), (3.15) and (3.16), We obtain (1.6). This completes the proof.

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