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Author(s)	Nakazi, Takahiko; Seto, Michio
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Double commuting compressed shifts and generalized interpolation in the Hardy space over the bidisk *

Takahiko Nakazi and Michio Seto

Abstract

This paper deals with an operator theory of compressed shifts on the Hardy space over the bidisk. We give commutant lifting type theorems and some interpolation theorems in two variables.

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1 Introduction

In [8], D. Sarason gave an operator theory approach to classical interpolation problems of complex function theory, Carathéodory interpolation problem and Nevanlinna-Pick interpolation problem, in the Hardy space over the unit disk. It shows a deep connection between operator theory and complex function theory. Sz.-Nagy and Foiaş generalized Sarason's technique as their commutant lifting theorem for a contraction on an abstract Hilbert space in [9] (cf. [1] and [7]). It is one of the most useful tools in operator theory, so that one would attempt to find a commutant lifting theorem for tuples of contractions. However, for tuples of contractions, there are examples in which the commutant lifting theorem fails.

Clark [4] and Amar-Menini [2] gave interesting counter-examples of the commutant lifting theorem in the Hardy space over the polydisk. Their examples are tuples of compressed shifts of coordinate functions. In particular, Amar-Menini dealt with the bidisk. The authors have been motivated to study commutant lifting type theorems in the Hardy space over the bidisk by their example. Our interest is the following question:

- *Why does the commutant lifting theorem fail for compressed shifts in the Hardy space over the bidisk ?*

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It seems to be one of the first steps toward answering the above question that we study non-trivial cases where commutant lifting type theorems hold for compressed shifts.

In this paper, we attempt to find non-trivial commutant lifting type theorems for compressed shifts as a sequel to [5]. Section 2 introduces notions used in this paper. In Section 3, we shall prove commutant lifting type theorems in two variables. In Section 4, we deal with norm estimates of compressed operators. In Section 5, a question related to commutant lifting is posed. In Sections 6 and 7, we deal with Carathéodory type interpolation and Nevanlinna-Pick type interpolation problems in two variables, respectively.

2 Preliminaries

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, the unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T} = \{e^{i\theta} \in \mathbb{C} : 0 \leq \theta < 2\pi\}$, the unit circle in $\mathbb{C}$. $L^2 = L^2(\mathbb{T}^2)$ denotes the Hilbert space of all square integrable functions with respect to the normalized Lebesgue measure $d\sigma$ on $\mathbb{T}^2$. For a closed subspace $\mathcal{M}$ of L^2, $P_{\mathcal{M}}$ denotes the orthogonal projection from L^2 onto $\mathcal{M}$. $L^\infty = L^\infty(\mathbb{T}^2)$ denotes the commutative Banach algebra of all essentially bounded measurable functions with respect to the measure $d\sigma$ on $\mathbb{T}^2$. $H^2 = H^2(\mathbb{D}^2)$ will denote the Hardy space over the bidisk $\mathbb{D}^2$ in $\mathbb{C}^2$, H^2 consists of all analytic functions on $\mathbb{D}^2$ satisfying the following condition:

$$\sup_{0\leq r<1} \int_{\mathbb{T}} \int_{\mathbb{T}} |f(rz, rw)|^2 \, d\sigma < +\infty,$$

where z and w are the usual coordinate functions on $\mathbb{C}^2$. Let $H^2(z)$ (resp. $H^2(w)$) denote the usual one-variable Hardy space over $\mathbb{D}$ of the variable z (resp. w) with respect to the normalized Lebesgue measure $d\sigma_1$ (resp. $d\sigma_2$) on $\mathbb{T}$. Then it is well known that $H^2 = H^2(z) \otimes H^2(w)$, the Hilbert space tensor product of $H^2(z)$ and $H^2(w)$. $H^\infty = H^\infty(\mathbb{D}^2)$ will denote the commutative Banach algebra consisting of all bounded analytic functions on $\mathbb{D}^2$ with the norm $\|f\|_\infty = \sup_{\lambda,\mu\in\mathbb{D}} |f(\lambda, \mu)|$. The Toeplitz operator T_φ on H^2 of a bounded analytic function φ is the restriction of the multiplication operator of φ on H^2.

Definition 2.1 A closed subspace $\mathcal{N}$ of H^2 is called a backward shift invariant subspace if $\mathcal{N}$ is invariant under T_z^* and T_w^*. For a bounded analytic function φ, S_φ denotes the compression of T_φ in a backward shift invariant subspace $\mathcal{N}$, that is, $S_\varphi f = P_{\mathcal{N}}\varphi f$ for any f in $\mathcal{N}$. In particular, S_z and S_w are called compressed shifts.

$\mathbb{C}[z, w]$ will denote the polynomial ring over $\mathbb{C}$ with variables z and w. Every backward shift invariant subspace $\mathcal{N}$ has a module structure over $\mathbb{C}[z, w]$ by the action defined as follows: $f \cdot h = S_f h = P_{\mathcal{N}} f h$ for any f in $\mathbb{C}[z, w]$ and h in $\mathcal{N}$.

In [5], Izuchi and the authors showed the following theorem:

Theorem 2.1 ([5]) *If a backward shift invariant subspace $\mathcal{N}$ satisfies the condition $[S_z^*, S_w] = S_z^* S_w - S_w S_z^* = 0$, then one and only one of the following occurs*:

(i) $\mathcal{N} = H^2 \ominus q_1(z)H^2 = (H^2(z) \ominus q_1(z)H^2(z)) \otimes H^2(w)$,

(ii) $\mathcal{N} = H^2 \ominus q_2(w)H^2 = H^2(z) \otimes (H^2(w) \ominus q_2(w)H^2(w))$,

(iii) $\mathcal{N} = (H^2(z) \ominus q_1(z)H^2(z)) \otimes (H^2(w) \ominus q_2(w)H^2(w))$.

where $q_1(z)$ and $q_2(w)$ are one-variable inner functions.

In this paper, we deal mainly with operators on the tensor product Hilbert space $\mathcal{N}$ obtained in Theorem 2.1, and use the notation $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$, for short.

3 Commutant lifting type theorems for S_z and S_w

Let $\mathcal{A}$ be the weak closed commutative Banach algebra generated by S_z, S_w and the identity operator $I_{\mathcal{N}}$ on a backward shift invariant subspace $\mathcal{N}$, and let $\mathcal{A}'$ denote the commutant of $\mathcal{A}$ in the Banach algebra of all bounded linear operators on $\mathcal{N}$.

Regarding the algebra $\mathcal{A}$ on a backward shift invariant subspace $\mathcal{N}$, we observe the following statements:

(A) $\mathcal{A}' = \{S_\varphi : \varphi \in H^\infty\}$,

(B) $\mathcal{A} = \mathcal{A}'$,

(C) $\mathcal{A}' = \mathcal{A}''$, where $\mathcal{A}''$ is the double commutant of $\mathcal{A}$.

First, we note that $\mathcal{A}$ is equal to the weak closure of the set $\{S_\varphi : \varphi \in H^\infty\}$ and the statement (B) is equivalent to the following (B′):

(B′) for any operator A in $\mathcal{A}'$, there exists a sequence $\{\varphi_n\}$ in H^∞ such that S_{φ_n} converges to A in the weak operator topology.

Indeed, for any function φ in H^∞, there exists a sequence $\{p_n\}$ in $\mathbb{C}[z, w]$ such that p_n converges to φ in the w*-topology. Then $\langle S_{p_n} f, g\rangle$ converges to $\langle S_\varphi f, g\rangle$ for any functions f and g in $\mathcal{N}$. This implies that $\mathcal{A}$ is equal to the weak closure of the set $\{S_\varphi : \varphi \in H^\infty\}$. Hence (B) is equivalent to (B′). Further, it is easy to see that (A) $\Rightarrow$ (B) $\Rightarrow$ (C) and (A) is the straightforward version of the Sarason theorem in the bidisk. However, Amar and Menini gave a counter-example of (A) in [2].

In this section, first, we prove lemmas needed later. Next, we will prove (C) on $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$ in Theorem 3.1, and we will give examples where (B) and (A) hold in Theorems 3.2 and 3.3, respectively.

The following Lemma 3.1 is known in [3]. For the sake of reader's convenience, we prove this in our setting.

Lemma 3.1 *If A is in $\mathcal{A}'$ on $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$, then there exists a function Φ in $\mathcal{N}$ such that $A = S_\Phi$, where S_Φ is the bounded operator on $\mathcal{N}$ defined by the following identity: $S_\Phi f = P_\mathcal{N} \Phi f$ for any f in $\mathcal{N} \cap H^\infty$.*

Proof Let $AP_\mathcal{N}1 = \Phi$, and let $f(z)$ and $g(w)$ be bounded analytic functions in $\mathcal{N}_z$ and $\mathcal{N}_w$, respectively. Then we have

$$\begin{aligned} Af(z)g(w) &= AS_{f(z)}S_{g(w)}P_\mathcal{N}1 \\ &= S_{f(z)}S_{g(w)}AP_\mathcal{N}1 \\ &= S_{f(z)}S_{g(w)}\Phi \\ &= P_\mathcal{N} f(z)g(w)\Phi \\ &= S_\Phi f(z)g(w). \end{aligned}$$

Since $\mathcal{N} \cap H^\infty$ is dense in $\mathcal{N}$, we have $A = S_\Phi$.

Note that Φ may be an unbounded function in Lemma 3.1. However S_Φ is a bounded operator on $\mathcal{N}_z \otimes \mathcal{N}_w$. We set $\Phi_r(z,w) = \Phi(rz, rw)$ for $0 \leq r < 1$. Then it is known that Φ_r is a bounded analytic function converging to Φ as r tends to 1 in the L^2-norm topology. Setting

$$\mathcal{D} = \mathcal{D}(T) = \{f \in \mathcal{N} : \sup_{0 \leq r < 1} \|S_{\Phi_r} f\| < \infty\},$$

a densely defined linear operator T can be defined as follows:

$$Th = \lim_{r \to 1-0} S_{\Phi_r} h$$

in the weak topology, where h is any function in $\mathcal{D}$. Trivially, S_Φ is the closed extension of T.

Lemma 3.2 *We set $\xi = P_\mathcal{N}1$. Then ξ is a cyclic separating vector for $\mathcal{A}'$ on $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$. Moreover ξ is a bounded analytic function.*

Proof It is easy to show that $\xi = P_\mathcal{N}1 = (1 - \overline{q_1(0)}q_1(z))(1 - \overline{q_2(0)}q_2(w))$, which is a bounded analytic function. First, we show ξ is a cyclic vector for $\mathcal{A}'$. If there exists a function f in $\mathcal{N}$ such that $\langle S_z^k S_w^l \xi, f\rangle = 0$ for any non-negative integers k and l, then we have $f = 0$. Indeed, we have

$$\langle z^k w^l, f\rangle = \langle S_z^k S_w^l \xi, f\rangle = 0,$$

which implies that $f = 0$. Hence ξ is a cyclic vector for $\mathcal{A}$. Since $\mathcal{A}$ is contained in $\mathcal{A}'$, ξ is cyclic for $\mathcal{A}'$. Further, if $A\xi = 0$ for an operator A in $\mathcal{A}'$ then $AS_z^k S_w^l \xi = S_z^k S_w^l A\xi = 0$. Since ξ is cyclic for $\mathcal{A}'$, we have $A = 0$. This completes the proof.

Let $K_{\lambda,\mu}$ be the Cauchy kernel of H^2 at (λ,μ) in $\mathbb{D}^2$. We set

$$\xi_{\lambda,\mu} = P_{\mathcal{N}} K_{\lambda,\mu} = \frac{\left(1-\overline{q_1(\lambda)}q_1(z)\right)\left(1-\overline{q_2(\mu)}q_2(w)\right)}{\left(1-\overline{\lambda}z\right)\left(1-\overline{\mu}w\right)},$$

where $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$. Trivially, every $\xi_{\lambda,\mu}$ is a bounded analytic function. Further, it is easy to check that $\xi_{0,0} = \xi$ and $\xi_{\lambda,\mu} = S_{K_{\lambda,\mu}}\xi$. It follows that every $\xi_{\lambda,\mu}$ is a cyclic separating vector for $\mathcal{A}'$ by Lemma 3.2.

Lemma 3.3 *Let $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$. If A is in $\mathcal{A}'$ such that $A = S_\Phi$ for some function Φ in $\mathcal{N}$, then $S^*_{\Phi_r}\xi_{\lambda,\mu}$ converges to $A^*\xi_{\lambda,\mu}$ as r tends to 1 in the L^2-norm topology for any (λ,μ) in $\mathbb{D}^2$.*

Proof First we show that $A^*\xi_{\lambda,\mu} = P_{\mathcal{N}}\overline{\Phi}\xi_{\lambda,\mu}$. For any bounded function f in $\mathcal{N}$, we have

$$\langle A^*\xi_{\lambda,\mu}, f\rangle = \langle \xi_{\lambda,\mu}, Af\rangle = \langle \xi_{\lambda,\mu}, P_{\mathcal{N}}\Phi f\rangle = \int \xi_{\lambda,\mu}\overline{\Phi f}\, d\sigma = \langle P_{\mathcal{N}}\overline{\Phi}\xi_{\lambda,\mu}, f\rangle.$$

Since $\mathcal{N}\cap H^\infty$ is dense in $\mathcal{N}$, we have that $A^*\xi_{\lambda,\mu} = P_{\mathcal{N}}\overline{\Phi}\xi_{\lambda,\mu}$. Using this, we obtain that

$$\begin{aligned}
\|S^*_{\Phi_r}\xi_{\lambda,\mu} - A^*\xi_{\lambda,\mu}\| &= \|P_{\mathcal{N}}\overline{\Phi_r}\xi_{\lambda,\mu} - P_{\mathcal{N}}\overline{\Phi}\xi_{\lambda,\mu}\| \\
&\le \|\overline{\Phi_r}\xi_{\lambda,\mu} - \overline{\Phi}\xi_{\lambda,\mu}\| \\
&\le \|\overline{\Phi_r} - \overline{\Phi}\|\|\xi_{\lambda,\mu}\|_\infty \to 0
\end{aligned}$$

as r tends to 1. This concludes the proof.

Theorem 3.1 *If $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$, then $\mathcal{A}'$ is commutative, that is, $\mathcal{A}' = \mathcal{A}''$.*

Proof If A and B are any elements in $\mathcal{A}'$, then it suffices to show $AB\xi = BA\xi$ by Lemma 3.2. Then there exists a function Φ in $\mathcal{N}$ such that $A = S_\Phi$ by Lemma 3.1. Further, it is easy to check that

$$BA\xi = \lim_{r\to 1-0} BS_{\Phi_r}\xi = \lim_{r\to 1-0} S_{\Phi_r}B\xi.$$

Hence we have

$$\begin{aligned}
\langle BA\xi, \xi_{\lambda,\mu}\rangle &= \lim_{r\to 1-0}\langle S_{\Phi_r}B\xi, \xi_{\lambda,\mu}\rangle \\
&= \lim_{r\to 1-0}\langle B\xi, S^*_{\Phi_r}\xi_{\lambda,\mu}\rangle \\
&= \langle B\xi, A^*\xi_{\lambda,\mu}\rangle \\
&= \langle AB\xi, \xi_{\lambda,\mu}\rangle,
\end{aligned}$$

by Lemma 3.3. Since every $\xi_{\lambda,\mu}$ is the reproducing kernel of $\mathcal{N}$ at (λ,μ), we have that $AB\xi = BA\xi$. This concludes the proof.

Theorem 3.2 *If A is in $\mathcal{A}'$ on $\mathcal{N} = H^2 \ominus q(z)H^2$, then there exists a family of bounded analytic functions $\{\varphi_j(z)\}$ depending only on the variable z such that*

$$A = \sum_{j \geq 0} S_{\varphi_j(z)} T_w^j$$

in the weak operator topology.

Proof Since A commutes with T_w, A is an analytic function which takes values in operators on $\mathcal{N}_z$. Therefore, we have the following expansion of A:

$$A = \sum_{j=0}^{\infty} A_j \otimes T_w^j.$$

Let Q_k denote the orthogonal projection from $\mathcal{N} = \mathcal{N}_z \otimes H^2(w)$ onto $\mathcal{N}_z \otimes \mathbb{C}w^k$. Then $A = \sum_{k,l \geq 0} Q_k A Q_l$ in the weak operator topology. Further, we have $Q_k A Q_l = Q_k(A_{k-l} \otimes T_w^{k-l})Q_l$ for $k \geq l$. Indeed, it is easy to check the following:

$$Q_k A Q_l = \sum Q_k(A_j \otimes T_w^j)Q_l = Q_k(A_{k-l} \otimes T_w^{k-l})Q_l.$$

Since S_z commutes with every Q_k, for any function $f(z)$ in $\mathcal{N}_z$, we have the following:

$$\begin{aligned} S_z A_{k-l} f(z) w^k &= S_z Q_k (A_{k-l} \otimes T_w^{k-l}) Q_l f(z) w^l \\ &= Q_k (A_{k-l} \otimes T_w^{k-l}) Q_l S_z f(z) w^l = A_{k-l} S_z f(z) w^k. \end{aligned}$$

It follows that S_z commutes with every A_j on $\mathcal{N}_z$. Hence there exists a bounded analytic function $\varphi_j(z)$ such that $A_j = S_{\varphi_j(z)}$ by Thoerem 1 in [8]. This concludes the proof.

Corollary 3.1 *$\mathcal{A} = \mathcal{A}'$ on $\mathcal{N} = H^2 \ominus q(z)H^2$.*

If we assume that the dimension of $\mathcal{N}$ is finite in Lemma 3.1, then every element in $\mathcal{N}$ is a rational function. Hence it is an immediate consequence that there exists a bounded analytic function φ such that $A = S_\varphi$ when $\mathcal{N}$ is of finite dimension. Then, trivially $\mathcal{A} = \mathcal{A}'$. We show further details of this fact in the next theorem.

For λ in $\mathbb{D}$, we define two functions as follows:

$$b_\lambda(z) = (z - \lambda)/(1 - \overline{\lambda}z), \quad k_\lambda(z) = \left(1 - \overline{\lambda}z\right)^{-1}.$$

Lemma 3.4 *Let $q_1(z)$ be a finite Blaschke product, each zero λ_i having multiplicity k_i, that is,*

$$q_1(z) = \prod_{i=0}^{l} \left(\frac{z - \lambda_i}{1 - \overline{\lambda_i} z} \right)^{k_i} = \prod_{i=0}^{l} b_{\lambda_i}^{k_i}(z).$$

Then

$$\left\{ e_i^{(j)}(z) = k_{\lambda_i}(z) b_{\lambda_0}^{k_0}(z) \cdots b_{\lambda_{i-1}}^{k_{i-1}}(z) b_{\lambda_i}^{j}(z) : 0 \leq i \leq l,\ 0 \leq j \leq k_i - 1 \right\}$$

is an orthogonal basis of $H^2(z) \ominus q_1(z)H^2(z)$.

Proof It is easy to verify from the following:

$$\begin{aligned} & H^2(z) \ominus q_1(z)H^2(z) \\ = \; & \left\{H^2(z) \ominus b_{\lambda_0}^{k_0}(z)H^2(z)\right\} \oplus \left\{b_{\lambda_0}^{k_0}(z)\left(H^2(z) \ominus b_{\lambda_1}^{k_1}(z)H^2\right)\right\} \\ & \oplus \cdots \oplus \left\{\left(b_{\lambda_0}^{k_0}(z)\cdots b_{\lambda_{l-1}}^{k_{l-1}}(z)\right)\left(H^2(z) \ominus b_{\lambda_l}^{k_l}(z)H^2(z)\right)\right\}. \end{aligned}$$

Theorem 3.3 Let $q_1(z)$ be a finite Blaschke product having the following form:

$$q_1(z) = \prod_{i=0}^{l}\left(\frac{z-\lambda_i}{1-\overline{\lambda}_i z}\right)^{k_i} = \prod_{i=0}^{l} b_{\lambda_i}^{k_i}(z),$$

and $\{e_i^{(j)}\}$ be the basis obtained in Lemma 3.4. Suppose that $\mathcal{N} = H^2 \ominus q_1(z)H^2$ or $\mathcal{N} = (H^2(z) \ominus q_1(z)H^2(z)) \otimes (H^2(w) \ominus q_2(w)H^2(w))$ as in Theorem 2.1. Then, A is in $\mathcal{A}'$ if and only if there exists a finite set of bounded analytic functions $\{\varphi_i^{(j)}(w)\}$ such that $A = S_\varphi$ where $\varphi(z,w) = \sum_{i,j} e_i^{(j)}(z)\varphi_i^{(j)}(w)$.

Proof It suffices to consider the case

$$\mathcal{N} = \left(H^2(z) \ominus q_1(z)H^2(z)\right) \otimes \left(H^2(w) \ominus q_2(w)H^2(w)\right).$$

By Lemma 3.1, there is a function Φ in $\mathcal{N}$ such that $A = S_\Phi$. Since Φ is in $\mathcal{N}$, we have the following finite expansion of Φ:

$$\Phi = \sum_{i,j} e_i^{(j)}(z)\phi_i^{(j)}(w),$$

where $\{e_i^{(j)}\}$ is the basis of $H^2(z) \ominus q_1(z)H^2(z)$ obtained in Lemma 3.4, and every $\phi_i^{(j)}(w)$ is in $H^2(w) \ominus q_2(w)H^2(w)$. To prove our assertion, we use an induction.

First, for any bounded functions $f(w)$ and $g(w)$ in $H^2(w) \ominus q_2(w)H^2(w)$, $k_{\lambda_0}(z)f(w)$ and $k_{\lambda_0}(z)g(w)$ are in $\mathcal{N}$, and we have

$$\begin{aligned} & \langle Ak_{\lambda_0}(z)f(w), k_{\lambda_0}(z)g(w)\rangle \\ = \; & \langle S_\Phi k_{\lambda_0}(z)f(w), k_{\lambda_0}(z)g(w)\rangle \\ = \; & \sum_{i,j}\left\langle e_i^{(j)}(z)\phi_i^{(j)}(w)k_{\lambda_0}(z)f(w), k_{\lambda_0}(z)g(w)\right\rangle \\ = \; & \sum_{i,j}\left\langle e_i^{(j)}(z)k_{\lambda_0}(z), k_{\lambda_0}(z)\right\rangle\left\langle \phi_i^{(j)}(w)f(w), g(w)\right\rangle \\ = \; & \left\langle e_0^{(0)}(z)k_{\lambda_0}(z), k_{\lambda_0}(z)\right\rangle\left\langle \phi_0^{(0)}(w)f(w), g(w)\right\rangle \\ = \; & \frac{1}{(1-|\lambda_0|^2)^2}\left\langle \phi_0^{(0)}(w)f(w), g(w)\right\rangle. \end{aligned}$$

Hence we have

$$\left|\frac{1}{(1-|\lambda_0|^2)^2}\left\langle \phi_0^{(0)}(w)f(w), g(w)\right\rangle\right| \le \|A\|\|k_{\lambda_0}(z)\|^2\|f(w)\|\|g(w)\|,$$

furthermore

$$\left|\left\langle \phi_0^{(0)}(w)f(w), g(w)\right\rangle\right| \leq (1-|\lambda_0|^2)\|A\|\|f(w)\|\|g(w)\|.$$

By Theorem 1 in [8], there exists a bounded analytic function $\varphi_0^{(0)}$, we can substitute $\varphi_0^{(0)}(w)$ for $\phi_0^{(0)}(w)$, that is, $S_{\phi_0^{(0)}(w)} = S_{\varphi_0^{(0)}(w)}$.

Next, for $0 \leq i \leq m,\ 0 \leq j \leq k_i - 1$ $(i < m)$ and $0 \leq j \leq n \leq k_m - 1$ $(i = m)$, we assume that each $\varphi_i^{(j)}(w)$ is a bounded analytic function. Then

$$A_{m,n} := S_\Phi - \sum{}' S_{e_i^{(j)}\varphi_{i(j)}}$$

is a bounded operator on $\mathcal{N}$, where the sum $\sum'$ is taken only over $0 \leq i \leq m,\ 0 \leq j \leq k_i - 1$ $(i < m)$ and $0 \leq j \leq n$ $(i = m)$.

If $n < k_m - 1$, then

$$\begin{aligned}
&\left\langle A_{m,n}k_{\lambda_m}(z)f(w), k_{\lambda_m}(z)b_{\lambda_m}^{n+1}(z)g(w)\right\rangle \\
= &\sum{}'' \left\langle e_i^{(j)}(z)\phi_i^{(j)}(w)k_{\lambda_m}(z)f(w), k_{\lambda_m}(z)b_{\lambda_m}^{n+1}(z)g(w)\right\rangle \\
= &\sum{}'' \left\langle e_i^{(j)}(z)k_{\lambda_m}(z), k_{\lambda_m}(z)b_{\lambda_m}^{n+1}(z)\right\rangle \left\langle \phi_i^{(j)}(w)f(w), g(w)\right\rangle \\
= &\left\langle e_m^{(n+1)}(z)k_{\lambda_m}(z), k_{\lambda_m}(z)b_{\lambda_m}^{n+1}(z)\right\rangle \left\langle \phi_m^{(n+1)}(w)f(w), g(w)\right\rangle \\
= &\frac{1}{(1-|\lambda_m|^2)^2} b_{\lambda_0}^{k_0}(\lambda_m)\cdots b_{\lambda_{m-1}}^{k_{m-1}}(\lambda_m)\left\langle \phi_m^{(n+1)}(w)f(w), g(w)\right\rangle,
\end{aligned}$$

where the sum $\sum''$ is taken over $m \leq i \leq l,\ n+1 \leq j \leq k_m - 1$ $(i = m)$ and $0 \leq j \leq k_i - 1$ $(i > m)$. Hence we have

$$\left|\left\langle \phi_m^{(n+1)}(w)f(w), g(w)\right\rangle\right| \leq K_{m,n}\|A_{m,n}\|\|f(w)\|\|g(w)\|,$$

for some constant $K_{m,n}$. Therefore, by Theorem 1 in [8], there exists a bounded analytic function $\varphi_m^{(n+1)}$, we can substitute $S_{\varphi_m^{(n+1)}(w)}$ for $S_{\phi_m^{(n+1)}(w)}$.

If $n = k_m - 1$, then

$$\begin{aligned}
&\left\langle A_{m,n}k_{\lambda_{m+1}}(z)f(w), k_{\lambda_{m+1}}(z)g(w)\right\rangle \\
= &\sum{}''' \left\langle e_i^{(j)}(z)\phi_i^{(j)}(w)k_{\lambda_{m+1}}(z)f(w), k_{\lambda_{m+1}}(z)g(w)\right\rangle \\
= &\sum{}''' \left\langle e_i^{(j)}(z)k_{\lambda_{m+1}}(z), k_{\lambda_{m+1}}(z)\right\rangle \left\langle \phi_i^{(j)}(w)f(w), g(w)\right\rangle \\
= &\left\langle e_{m+1}^{(0)}(z)k_{\lambda_{m+1}}(z), k_{\lambda_{m+1}}(z)\right\rangle \left\langle \phi_{m+1}^{(0)}(w)f(w), g(w)\right\rangle \\
= &\frac{1}{(1-|\lambda_{m+1}|^2)^2} b_{\lambda_0}^{k_0}(\lambda_{m+1})\cdots b_{\lambda_{m-1}}^{k_m}(\lambda_{m+1})\left\langle \phi_{m+1}^{(0)}(w)f(w), g(w)\right\rangle,
\end{aligned}$$

where the sum $\sum'''$ is taken over $m+1 \leq i \leq l$ and $0 \leq j \leq k_i - 1$. Hence we have

$$\left|\left\langle \phi_{m+1}^{(0)}(w)f(w), g(w)\right\rangle\right| \leq K'_{m,n}\|A_{m,n}\|\|f(w)\|\|g(w)\|,$$

for some constant $K'_{m,n}$. Therefore, by Theorem 1 in [8], there exists a bounded analytic function $\varphi^{(0)}_{m+1}(w)$, we can substitute $S_{\varphi^{(0)}_{m+1}(w)}$ for $S_{\phi^{(0)}_{m+1}(w)}$. Therefore $\varphi(z,w) = \sum_{i,j} e^{(j)}_i(z)\varphi^{(j)}_i(w)$ is a bounded analytic function and $S_\Phi = S_\varphi$.

Conversely, it is easy to check that if φ is in H^∞ then S_φ commutes with S_z and S_w.

Corollary 3.2 *If $\mathcal{N}_z$ is of finite dimension, then $\mathcal{A} = \mathcal{A}'$ on $\mathcal{N} = \mathcal{N}_z \otimes \mathcal{N}_w$.*

4 Norm estimates

In this section, we deal with norm estimates of S_φ. To begin with, we should recall the one-variable case, in which the following norm equality was obtained by Sarason in Proposition 2.1 in [8]:

$$\|S_{\varphi(z)}\| = \|\varphi(z) + q(z)H^\infty(z)\|_\infty \ \text{ on } \ \mathcal{N}_z = H^2(z) \ominus q(z)H^2(z),$$

where $H^\infty(z)$ is the Banach algebra of all bounded analytic functions with the variable z on $\mathbb{D}$, $\varphi(z)$ is in $H^\infty(z)$ and $q(z)$ is an inner function in $H^\infty(z)$.

Let H_f denote the Hankel operator of a bounded function f, that is, $H_f = (I_{L^2} - P_{H^2})M_f|_{H^2}$, where M_f is the multiplication operator by f on L^2.

The following lemma is known for the one-variable case.

Lemma 4.1 *Suppose that $\mathcal{N} = H^2 \ominus qH^2$ for an inner function q. Then $M_q H_{\overline{q}\varphi} P_\mathcal{N} = S_\varphi P_\mathcal{N}$ for any φ in H^∞ and $\|H_{\overline{q}\varphi}\| = \|S_\varphi\|$.*

Proof Since

$$\begin{aligned} M_q H_{\overline{q}\varphi} P_\mathcal{N} f = q(I_{L^2} - P_{H^2})\overline{q}\varphi P_\mathcal{N} f &= (I_{L^2} - qP_{H^2}\overline{q})\varphi P_\mathcal{N} f \\ &= (P_{H^2} - qP_{H^2}\overline{q})\varphi P_\mathcal{N} f \\ &= P_\mathcal{N}\varphi P_\mathcal{N} f \\ &= S_\varphi P_\mathcal{N} f, \end{aligned}$$

we have the first assertion. Next, we show the second one. It is easy to check that $qH^2 \subseteq \ker H_{\overline{q}\varphi}$, we have $\|S_\varphi\| = \|S_\varphi P_\mathcal{N}\| = \|M_q H_{\overline{q}\varphi} P_\mathcal{N}\| = \|H_{\overline{q}\varphi} P_\mathcal{N}\| = \|H_{\overline{q}\varphi}\|$.

The next lemma is elementary.

Lemma 4.2 *Let X be a w*-closed subspace of L^∞. Then, for any f in L^∞, there exists a function F in $f + X$ such that $\|F\|_\infty \le \|f + X\|_\infty = \inf_{g\in X} \|f + g\|_\infty$.*

Proof We set $\alpha = \|f+X\|_\infty$. For any n in $\mathbb{N}$, there exists a function f_n in $f+X$ such that $\|f_n\|_\infty \le \alpha+1/n$. Since $\|f_n\|_\infty \le \alpha+1$, $\{f_n\}_{n\in\mathbb{N}}$ is w*-compact in L^∞. Hence there exists a subsequence $\{f_{n_k}\}_k$ such that f_{n_k} converges to some function F in the w*-topology of L^∞ as k tends to $+\infty$. Then $\|F\|_\infty \le \liminf_{k\in\mathbb{N}} \|f_{n_k}\|_\infty \le \liminf_{k\in\mathbb{N}}(\alpha + 1/n_k) = \alpha$ and F is in $f + X$.

Let $\mathbb{H}^\infty_z$ (resp. $\mathbb{H}^\infty_w$) denote the w*-closed subalgebra of L^∞ generated by z, w and $\overline{w}$ (resp. z, $\overline{z}$ and w).

Theorem 4.1 *Let $q_1(z)$ be an inner function depending only on the variable z and let $\mathfrak{Z}(f(z))$ be the set of all zero points of a one-variable function $f(z)$ in $\mathbb{D}$, that is, $\mathfrak{Z}(f(z)) = \{\lambda \in \mathbb{D} : f(\lambda) = 0\}$. Suppose that φ is in H^∞. Then the following norm estimates hold:*

(i) *Suppose that $\mathcal{N} = H^2 \ominus q_1(z)H^2$ and let I_1 denote*

$$\max\left\{\sup_{\lambda\in\mathbb{D}} \|\varphi(z,\lambda) + q_1(z)H^\infty(z)\|_\infty, \sup_{\lambda\in\mathfrak{Z}(q_1)} \|\varphi(\lambda,w)\|_\infty\right\}.$$

Then

$$I_1 \le \|S_\varphi\| = \|\varphi + q_1(z)\mathbb{H}^\infty_z\|_\infty.$$

(ii) *Suppose that $\mathcal{N} = (H^2(z) \ominus q_1(z)H^2(z)) \otimes (H^2(w) \ominus q_2(w)H^2(w))$ for an inner function $q_2(w)$ depending only on the variable w and let I_2 denote*

$$\max\left\{\sup_{\lambda\in\mathfrak{Z}(q_1)} \|\varphi(\lambda,w) + q_2(w)H^\infty(w)\|_\infty, \sup_{\lambda\in\mathfrak{Z}(q_2)} \|\varphi(z,\lambda) + q_1(z)H^\infty(z)\|_\infty\right\}.$$

Then

$$I_2 \le \|S_\varphi\| \le \|\varphi + (q_1(z)H^2 + q_2(w)H^2) \cap H^\infty\|_\infty.$$

Proof We show the assertion of (i). By Lemma 4.1, $\|S_\varphi\| = \|H_{\overline{q_1}\varphi}\|$. Since $\|H_{\overline{q_1}\varphi}\| = \left\|\overline{q_1(z)}\varphi + \mathbb{H}^\infty_z\right\|$ by Corollary 1 in [6], we have $\|S_\varphi\| = \left\|\overline{q_1(z)}\varphi + \mathbb{H}^\infty_z\right\|$. Hence we have the equality in (i).

Next, for any λ in $\mathbb{D}$, any f_1 and g_1 in $H^2(z) \ominus q_1(z)H^2(z)$,

$$\begin{aligned}
\left|\int \varphi(z,\lambda)f_1(z)\overline{g_1(z)}\, d\sigma_1\right| &= \left|\int\left(\int \varphi(z,w)e_\lambda(w)\overline{e_\lambda(w)}\, d|w|\right) f_1(z)\overline{g_1(z)}\, d\sigma_1\right| \\
&= \left|\iint \varphi(z,w)f_1(z)e_\lambda(w)\overline{g_1(z)e_\lambda(w)}\, d\sigma\right| \\
&= |\langle S_\varphi f_1(z)e_\lambda(w), g_1(z)e_\lambda(w)\rangle| \\
&\le \|S_\varphi\|\|f_1\|\|g_1\|,
\end{aligned}$$

where $e_\lambda(w) = (1-|\lambda|^2)^{1/2}\left(1-\overline{\lambda}w\right)$. Hence

$$\|S_\varphi\| \ge \|S_{\varphi(\cdot,\lambda)}\| = \|\varphi(z,\lambda) + q_1(z)H^\infty(z)\|, \tag{4.1}$$

for any λ in $\mathbb{D}$ by Proposition 2.1 in [8].

Let λ be in $\mathfrak{Z}(q_1)$. Then $e_\lambda(z)$ is in $H^2(z) \ominus q_1(z)H^2(z)$. For any f_2 and g_2 in $H^2(w)$, by the same calculation as the above,

$$\left|\int \varphi(\lambda,w)f_2(w)\overline{g_2(w)}\, d|w|\right| \le \|S_\varphi\|\|f_2\|\|g_2\|.$$

Hence

$$\|S_\varphi\| \geq \|T_{\varphi(\lambda,\cdot)}\| = \|\varphi(\lambda, w)\|. \tag{4.2}$$

(4.1) and (4.2) conclude (i).

By calculations similar to those in the proof of (i), we have inequalities in (ii).

5 $(q_1H^2 + q_2H^2) \cap H^\infty = q_1H^\infty + q_2H^\infty$?

In this section, we study the following problem (cf. [5]):

$$(q_1(z)H^2 + q_2(w)H^2) \cap H^\infty = q_1(z)H^\infty + q_2(w)H^\infty \ ? \tag{5.1}$$

Theorem 5.1 *If $q_1(z)$ is a finite Blaschke product, then the equality* (5.1) *holds.*

Proof Let $\deg(q_1(z)) = k < +\infty$. Since

$$q_1(z)H^2 + q_2(w)H^2 = q_1(z)H^2 \oplus (H^2(z) \ominus q_1(z)H^2(z)) \otimes q_2(w)H^2(w),$$

we have $f = q_1(z)g \oplus q_2(w) \sum_{i=1}^k \oplus e_i(z)h_i(w)$ for any f in $(q_1(z)H^2 + q_2(w)H^2) \cap H^\infty$, where $\{e_i\}_{i=1}^k$ is an orthonormal basis of $H^2(z) \ominus q_1(z)H^2(z)$, g is in H^2 and every $h_i(w)$ is in $H^2(w)$. We note that every $e_i(z)$ is a bounded rational function. Since

$$\begin{aligned}
\int_{\mathbb{T}} f\overline{e_i(z)}\, d\sigma_1 &= \int \left(q_1(z)g \oplus q_2(w) \sum_{j=1}^k e_j(z)h_j(w) \right) \overline{e_i(z)}\, d\sigma_1 \\
&= \int q_1(z)g\overline{e_i(z)}\, d\sigma_1 + \sum_{j=1}^k q_2(w)h_j(w)\langle e_j(z), e_i(z)\rangle_{H^2(z)} \\
&= q_2(w)h_i(w),
\end{aligned}$$

we have

$$|h_i(w)| = |q_2(w)h_i(w)| = \left| \int f\overline{e_i(z)}\, d\sigma_1 \right| \leq \|f\|_\infty \|e_i(z)\|_\infty.$$

Therefore every $h_i(w)$ and g are bounded. Hence $(q_1(z)H^2+q_2(w)H^2)\cap H^\infty \subseteq q_1(z)H^\infty + q_2(w)H^\infty$. Trivially, $(q_1(z)H^2 + q_2(w)H^2) \cap H^\infty \supseteq q_1(z)H^\infty + q_2(w)H^\infty$. Hence (5.1) holds.

Next we give another approach to this problem. Consider two bounded linear maps δ_1 and δ_2 defined as follows:

$$\begin{aligned}
\delta_1 &: H^2 \to H^2 \oplus H^2,\ \delta_1(f) = (-q_2(w)f, q_1(z)f) \\
\delta_2 &: H^2 \oplus H^2 \to q_1(z)H^2 + q_2(w)H^2,\ \delta_2(f, g) = q_1(z)f + q_2(w)g.
\end{aligned}$$

Then the following sequence is exact:

$$0 \longrightarrow H^2 \xrightarrow{\delta_1} H^2 \oplus H^2 \xrightarrow{\delta_2} q_1(z)H^2 + q_2(w)H^2 \longrightarrow 0.$$

Indeed, it is easy to check that Im $\delta_1 \subseteq \ker \delta_2$. We show that $\ker \delta_2 \subseteq$ Im δ_1. Let (f, g) be in $\ker \delta_2$. Then

$$\langle (q_1(z)f + q_2(w)g)\, F, G\rangle = \langle 0, G\rangle = 0$$

for any F and G in H^2. Set $F = 1$ and let G be an element in $H^2 \ominus q_1(z)H^2$. Then

$$\langle (q_1(z)f + q_2(w)g)1, G\rangle = \langle q_2(w)g, G\rangle = 0.$$

Hence $q_2(w)g$ is in $q_1(z)H^2$, that is, $q_2(w)g = q_1(z)g'$ for some function g' in H^2. Furthermore, $g = T^*_{q_2(w)}q_2(w)g = T^*_{q_2(w)}q_1(z)g' = q_1(z)T^*_{q_2(w)}g'$. Setting $h = T^*_{q_2(w)}g'$, we have $0 = q_1(z)f + q_2(w)g = q_1(z)f + q_2(w)\,(q_1(z)h)$ and $f = -q_2(w)h$. Thus $(f, g) = (-q_2(w)h, q_1(z)h) = \delta_1(h)$.

Thus, $\varphi = q_1(z)f_1 + q_2(w)g_1 = q_1(z)f_2 + q_2(w)g_2$ if and only if there exists a function h in H^2 such that $f_1 - f_2 = -q_2(w)h$ and $g_1 - g_2 = q_1(z)h$. By this observation we have the following:

Theorem 5.2 *The equality* (5.1) *holds if and only if, for any function φ in H^∞ having a form $\varphi = q_1(z)f + q_2(w)g$ for some functions f and g in H^2, there exists a function h in H^2 such that both $f - q_2(w)h$ and $g + q_1(z)h$ are bounded.*

6 A Carathéodory type interpolation problem in two variables

In this section, we consider the following interpolation problem:

- Given $n+1$ analytic functions $u_0(z), \dots, u_n(z)$ in $H^\infty(z)$, can one find a function φ in H^∞ whose power series begins with $u_0(z) + \cdots + u_j(z)w^j + \cdots + u_n(z)w^n$ and $\|\varphi\|_\infty \le 1$?

We set $\mathcal{N} = H^2/w^{n+1}H^2 = H^2(z) \otimes \mathcal{N}_w$. Then $\mathcal{N}$ can be identified with $H^2(z) \oplus \cdots \oplus H^2(z)$ (the direct sum of $n+1$ copies of $H^2(z)$). Assume that an operator T on $\mathcal{N}$ commutes with both T_z and S_w. Then

$$T \begin{pmatrix} T_z & & 0 \\ & \ddots & \\ 0 & & T_z \end{pmatrix} = \begin{pmatrix} T_z & & 0 \\ & \ddots & \\ 0 & & T_z \end{pmatrix} T \quad \text{on} \quad H^2(z) \oplus \cdots \oplus H^2(z).$$

Hence $T_{ij}T_z = T_zT_{ij}$ for $T = (T_{ij})$. Then, for every T_{ij}, there exists a function $\varphi_{ij}(z)$ in $H^\infty(z)$ such that $T_{\varphi_{ij}(z)} = T_{ij}$. Since T commutes with S_w, that is, T commutes with the truncated shift operator on $\mathcal{N}_w$ with respect to the orthonormal basis $\{1, \dots, w^n\}$, T has the following form on $H^2(z) \oplus \cdots \oplus H^2(z)$:

$$T = C(T_{\varphi_0(z)}, \dots, T_{\varphi_n(z)}) := \begin{pmatrix} T_{\varphi_0(z)} & 0 & 0 & \cdots & 0 \\ T_{\varphi_1(z)} & T_{\varphi_0(z)} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ T_{\varphi_n(z)} & T_{\varphi_{n-1}(z)} & T_{\varphi_{n-2}(z)} & \cdots & T_{\varphi_0(z)} \end{pmatrix},$$

where $\varphi_j(z)$ denotes $\varphi_{j0}(z)$, for short.

Theorem 6.1 *Let $u_0(z), \dots, u_n(z)$ be given $n+1$ analytic functions in $H^\infty(z)$. Then $\|C(T_{u_0(z)}, \dots, T_{u_n(z)})\| \le 1$ if and only if there exists a function ψ in $\mathbb{H}^\infty_w$ whose power series begins with $u_0(z) + \cdots + u_j(z)w^j + \cdots + u_n(z)w^n$ and $\|\psi\|_\infty \le 1$.*

Proof Suppose that $u_0(z), \dots, u_n(z)$ are $n+1$ analytic functions in $H^\infty(z)$. We set $\mathcal{N} = H^2/(w^{n+1}H^2) = H^2(z) \otimes (H^2(w) \ominus w^{n+1}H^2(w))$. Then $T = C(T_{u_0(z)}, \dots, T_{u_n(z)})$ is a bounded linear operator on $H^2(z) \oplus \cdots \oplus H^2(z)$ (the direct sum of n copies of $H^2(z)$). It is easy to check T commutes with T_z and S_w Setting $\varphi = T \cdot 1$, we have $\varphi = u_0(z) + \cdots + u_n(z)w^n$ and we showed $T = S_\varphi$ in the proof of Lemma 3.1.

If $\|T\| \le 1$ then $\|S_\varphi\| \le 1$. Since $\|S_\varphi\| = \|\varphi + w^{n+1}\mathbb{H}^\infty_w\|$ by (i) in Theorem 4.1, there exists a function in ψ in $\mathbb{H}^\infty_w$ such that $\|\psi\|_\infty \le 1$. Then the power series of ψ begins with $u_0(z) + \cdots + u_n(z)w^n$.

Conversely, if there exists a function ψ in $\mathbb{H}^\infty_w$ such that whose power series begins with $u_0(z) + \cdots + u_n(z)w^n$ and $\|\psi\|_\infty \le 1$. Then $\psi - \varphi$ is in $w^{n+1}\mathbb{H}^\infty_w$ where we set $\varphi = u_0(z) + \cdots + u_n(z)w^n$, $\|S_\varphi\| = \|\varphi + w^{n+1}\mathbb{H}^\infty_w\|_\infty \le \|\psi\|_\infty \le 1$ by (i) in Theorem 4.1 and S_φ can be identified with $C(T_{u_0(z)}, \dots, T_{u_n(z)})$ on $H^2(z) \oplus \cdots \oplus H^2(z)$. This completes the proof.

7 A Nevanlinna-Pick type interpolation problem in two variables

In this section we study the following interpolation problem:

- Let $\lambda_1, \dots, \lambda_n$ be n distinct points in $\mathbb{D}$ and $u_1(w), \dots, u_n(w)$ be n analytic functions in $H^\infty(w)$. Then, can one find a function φ in H^∞ such that $\|\varphi\|_\infty \le 1$ and $\varphi(\lambda_j, w) = u_j(w)$ for $j = 1, \dots, n$?

To begin with, we prove the following lemma:

Lemma 7.1 *Let $\lambda_1, \dots, \lambda_n$ be n points in $\mathbb{D}$ and let φ be a function in $\mathbb{H}^\infty_z$. If $\varphi(\lambda_j, w) = 0$ w-a.e. for every λ_j then φ is a function in $q(z)\mathbb{H}^\infty_z$, where $q(z) = \prod_{j=1}^n b_{\lambda_j}(z)$.*

Proof For any integer l,

$$\varphi_l(z) := \int_{\mathbb{T}} \varphi(z, w)\overline{w}^l \, d\sigma_2$$

can be extended analytically to $\mathbb{D}$ with respect to the variable z. Since $\varphi(\lambda_j, w) = 0$ w-a.e.,

$$\varphi_l(\lambda_j) = \int_{\mathbb{T}} \varphi_l(z)\overline{k_{\lambda_j}(z)} \, d\sigma_1 = \int_{\mathbb{T}}\int_{\mathbb{T}} \varphi(z, w)\overline{k_{\lambda_j}(z)w^l} \, d\sigma = \int_{\mathbb{T}} \varphi(\lambda_j, w)\overline{w}^l \, d\sigma_2 = 0.$$

Hence there exists a function $\psi(z)$ in $H^2(z)$ such that $\varphi_l(z) = q(z)\psi_l(z)$. Therefore

$$\varphi = \sum_{l\in\mathbb{Z}} \varphi_l(z)w^l = \sum_{l\in\mathbb{Z}} q(z)\psi_l(z)w^l = q(z)\sum_{l\in\mathbb{Z}} \psi_l(z)w^l.$$

Therefore φ is in $q(z)\mathbb{H}^2_z$. Since $\varphi/q(z)$ is a bounded function on $\mathbb{T}^2$, we have the conclusion.

Theorem 7.1 *Let $\lambda_1, \ldots, \lambda_n$ be n distinct points in $\mathbb{D}$ and $u_1(w), \ldots, u_n(w)$ be n functions in $H^\infty(w)$ such that $\|u_j(w)\|_\infty \leq 1$ for $j = 1, \ldots, n$. Then the quadratic form*

$$\sum_{1\leq i,j\leq n} \frac{\left\langle \left(1 - T_{u_j(w)}T^*_{u_i(w)}\right) f_i(w), f_j(w)\right\rangle}{1 - \overline{\lambda_i}\lambda_j}$$

is non-negative for any $f_1(w), \ldots, f_n(w)$ in $H^2(w)$ if and only if there exists a function φ in $\mathbb{H}^\infty_z$ satisfying $\|\varphi\|_\infty \leq 1$ and $\varphi(\lambda_j, w) = u_j(w)$ for $j = 1, \ldots, n$.

Proof We define an operator as follows:

$$A^* k_{\lambda_i}(z)f(w) = k_{\lambda_i}(z)T^*_{u_i(w)}f(w),$$

for every $0 \leq i \leq n$ and $f(w)$ in $H^2(w)$. Then A is a bounded linear operator on $(H^2(z) \ominus q(z)H^2(z)) \otimes H^2(w)$, where $q(z) = \prod_{i=1}^n b_{\lambda_i}(z)$. Since every $T^*_{u_i(w)}$ commutes with T^*_w and

$$\begin{aligned} S^*_z A^* k_{\lambda_i}(z)f(w) &= S^*_z k_{\lambda_i}(z)T^*_{u_i(w)}f(w) \\ &= \overline{\lambda_i} k_{\lambda_i}(z)T^*_{u_i(w)}f(w) \\ &= A^*\overline{\lambda_i} k_{\lambda_i}(z)f(w) \\ &= A^* S^*_z k_{\lambda_i}(z)f(w) \end{aligned}$$

Hence A commutes with S_z and T_w. By Theorem 3.3, there exists ψ in H^∞ such that $A = S_\psi$, and we have $\psi(\lambda_i, w) = u_i(w)$. Indeed,

$$\begin{aligned} \left\langle S_\psi k_{\lambda_i}(z)f(w), k_{\lambda_j}(z)g(w)\right\rangle &= \left\langle \psi(z,w)k_{\lambda_i}(z)f(w), k_{\lambda_j}(z)g(w)\right\rangle \\ &= \langle k_{\lambda_i}(\lambda_j)\psi(\lambda_j, w)f(w), g(w)\rangle \end{aligned}$$

and

$$\begin{aligned} \left\langle S_\psi k_{\lambda_i}(z)f(w), k_{\lambda_j}(z)g(w)\right\rangle &= \left\langle Ak_{\lambda_i}(z)f(w), k_{\lambda_j}(z)g(w)\right\rangle \\ &= \left\langle k_{\lambda_i}(z)f(w), A^* k_{\lambda_j}(z)g(w)\right\rangle \\ &= \left\langle k_{\lambda_i}(z)f(w), k_{\lambda_j}(z)T^*_{u_j(w)}g(w)\right\rangle \\ &= \left\langle k_{\lambda_i}(z)T_{u_j(w)}f(w), k_{\lambda_j}(z)g(w)\right\rangle \\ &= \langle k_{\lambda_i}(\lambda_j)u_j(w)f(w), g(w)\rangle, \end{aligned}$$

for any $f(w)$ and $g(w)$ in $H^2(w)$. Hence $\psi(\lambda_i, w) = u_i(w)$. Moreover

$$\begin{aligned}
&\left\langle (I - AA^*) \sum_i k_{\lambda_i}(z) f_i(w), \sum_j k_{\lambda_j}(z) f_j(w) \right\rangle \\
= &\sum_{i,j} \frac{\langle f_i(w), f_j(w) \rangle}{1 - \overline{\lambda_i}\lambda_j} - \sum_{i,j} \frac{\left\langle T^*_{u_i(w)} f_i(w), T^*_{u_j(w)} f_j(w) \right\rangle}{1 - \overline{\lambda_i}\lambda_j} \\
= &\sum_{i,j} \frac{\left\langle \left(1 - T_{u_j(w)} T^*_{u_i(w)}\right) f_i(w), f_j(w) \right\rangle}{1 - \overline{\lambda_i}\lambda_j}
\end{aligned}$$

Hence we have $\|A\| \leq 1$ if and only if the quadratic form

$$\sum_{1 \leq i,j \leq n} \frac{\left\langle \left(1 - T_{u_j(w)} T^*_{u_i(w)}\right) f_i(w), f_j(w) \right\rangle}{1 - \overline{\lambda_i}\lambda_j}$$

is non-negative for any $f_1(w), \ldots, f_n(w)$ in $H^2(w)$.

We suppose that $\|A\| \leq 1$. Since $\|A\| = \|S_\psi\| = \|\psi + q(z)\mathbb{H}^\infty_z\|_\infty$ by (i) in Theorem 4.1, there exists a function φ in $\mathbb{H}^\infty_z$ such that $\|\varphi\|_\infty \leq 1$ and φ is in $\psi + q(z)\mathbb{H}^\infty_z$. Then $\varphi(\lambda_i, w) = \psi(\lambda_i, w) = u_j(w)$ for $1 \leq i \leq n$.

Conversely, we suppose that there exists a function φ in $\mathbb{H}^\infty_z$ satisfying $\|\varphi\|_\infty \leq 1$ and $\varphi(\lambda_j, w) = u_j(w)$ for $1 \leq j \leq n$. Setting

$$\psi(z, w) = \sum_{l=1}^{n} u_l(w) \prod_{j \neq l} \frac{(z - \lambda_j)}{\lambda_l - \lambda_j} \text{ and } q(z) = \prod_{i=1}^{n} b_{\lambda_i}(z),$$

we have $\varphi(z, w) - \psi(z, w)$ is an element of $q(z)\mathbb{H}^\infty_z$ by Lemma 7.1 and ψ is in H^∞. Let A be the operator S_ψ acting on $(H^2(z) \ominus q(z)H^2(z)) \otimes H^2(w)$. Then, by (i) in Theorem 4.1, we have

$$\|A\| = \|S_\psi\| = \|\psi + q(z)\mathbb{H}^\infty_z\|_\infty \leq \|\varphi\|_\infty \leq 1.$$

Further, it is easy to check that

$$A^* k_{\lambda_i}(z) f_(w) = k_{\lambda_i}(z) T^*_{u_i(w)} f(w).$$

Hence the quadratic form

$$\sum_{1 \leq i,j \leq n} \frac{\left\langle \left(1 - T_{u_j(w)} T^*_{u_i(w)}\right) f_i(w), f_j(w) \right\rangle}{1 - \overline{\lambda_i}\lambda_j}$$

is non-negative for any $f_1(w), \ldots, f_n(w)$ in $H^2(w)$. This completes the proof.

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Takahiko Nakazi
Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060-0810, Japan
e-mail: nakazi@math.sci.hokudai.ac.jp

Michio Seto
Department of Mathematics
Kanagawa University
Yokohama 221-8686, Japan
e-mail: seto@kanagawa-u.ac.jp