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Integral Operators on a Subspace of Holomorphic Functions on the Disc

by

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Abstract. Let $H(D)$ be an algebra of all holomorphic functions on the open unit disc D and X a subspace of $H(D)$. When g is a function in $H(D)$, put

$$J_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta \text{ and } I_g(f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (z \in D)$$

for f in X . In this paper, we study $J[X] = \{g \in H(D) ; J_g(f) \in X \text{ for all } f \text{ in } X\}$ and $I[X] = \{g \in H(D) ; I_g(f) \in X \text{ for all } f \text{ in } X\}$. We apply the results to concrete spaces. For example, we study $J[X]$ and $I[X]$ when X is a weighted Bloch space, a Hardy space or a Privalov space.

§1. Introduction

Let D denote the open unit disc in the complex plane $\mathcal{C}$ and $H = H(D)$ the set of all holomorphic functions on D . For a given g in H , define three operators :

$$\begin{aligned} (M_g f)(z) &= g(z)f(z) & (f \in H, z \in D) \\ (J_g f)(z) &= \int_0^z f(\zeta)g'(\zeta)d\zeta & (f \in H, z \in D) \end{aligned}$$

and

$$(I_g f)(z) = \int_0^z f'(\zeta)g(\zeta)d\zeta \quad (f \in H, z \in D).$$

Then $(J_g f)(z) + (I_g f)(z) = (M_g f)(z) - g(0)f(0)$. If $g(z) = z$ then J_g is the Volterra integral operator and if $g(z) = \log 1/(1 - z)$ then J_g is the Cesàro operator.

In this paper we assume that X is a subspace of H which contains constants. X_1 denotes the set $\{f \in H ; f' \in X\}$. For each subspace X put

$$\begin{aligned} M[X] &= \{g \in H ; M_g(X) \subseteq X\}, \\ J[X] &= \{g \in H ; J_g(X) \subseteq X\} \end{aligned}$$

and

$$I[X] = \{g \in H ; I_g(X) \subseteq X\}.$$

We define that $J^{n+1}[X] = J[J^n[X]]$ and $I^{n+1}[X] = I[I^n[X]]$ for $n \geq 1$ where $J^1[X] = J[X]$ and $I^1[X] = I[X]$. For X and Y which are subspaces of H , XY denotes a subspace of H which is generated by a product of a function in X and one in Y . Let Y^n be a subspace of H which is generated by finite n products of functions in a subspace Y of H . For a subspace X of H , $B(X)$ denotes the set of all bounded linear operators on X .

Now we give a lot of examples of X . For $0 < p \leq \infty$, H^p is the usual Hardy space on D , N is the Nevalinna class and N_+ is the Smirnov class on D . These are F-spaces, and N and N_+ are algebras. It is known that $J[H^p] = \text{BMOA}$ (see [2], [1]), $z \notin J[N]$ [5] and $z \notin J[N_+]$ [7]. The Bloch space $\mathcal{B}$ is defined to be a Banach space in H with the norm

$$\|f\| = \sup_{z \in D} (1 - |z|^2)|f'(z)| + |f(0)|.$$

Then $\mathcal{B}$ contains H^∞ properly. Recently, R.Yoneda [8] described $J[\mathcal{B}]$ and he [9] also proved that $I[\mathcal{B}] = H^\infty$. It is well known that $M[H^p] = H^\infty$.

In Section 2, we assume only that X is a subspace of H . Theorem 1 implies that $J[X]^n \subset X$ for any $n \geq 1$. In Section 3, we study $J[X]$ and $I[X]$ when X is an invariant subspace of H or a subalgebra of H . Theorem 2 implies that if $H^\infty X \subset X$ and $J[X]$ contains z then $J[X] \supseteq H_1^\infty$. In Section 4, assuming that X is a F-space we show that $J[X]$ is contained in some weighted Bloch space and $I[X] \subset H^\infty$. In Section 5, we define a weighted Bloch space $\mathcal{B}_\omega$ and we describe $J[\mathcal{B}_\omega]$. In Section 6, we study $J\left[\bigcap_{t < p} H^t\right]$ and $I\left[\bigcap_{t < p} H^t\right]$. In Section 7, we show that $J[N^p]$ is a subalgebra of N^p which contains N_1^p , where N^p is a Privalov space.

§2. Subspace

In this section, we study $M[X]$, $J[X]$ and $I[X]$ assuming only that X is a subspace of H .

Lemma 1. *Let X be a subspace of H and f, g in H .*

(1) $I_g I_f = I_{gf} = I_f I_g$ on X

(2) $I_g J_f = J_f M_g$ on X

Proof. (1) For $k \in X$,

$$((I_g I_f)k)(z) = \int_0^z (I_f k)'(\zeta) g(\zeta) d\zeta = \int_0^z k'(\zeta) f(\zeta) g(\zeta) d\zeta = (I_{fg} k)(z)$$

(2) For $k \in X$,

$$\begin{aligned} ((I_g J_f)k)(z) &= \int_0^z (J_f k)'(\zeta) g(\zeta) d\zeta = \int_0^z k(\zeta) f'(\zeta) g(\zeta) d\zeta \\ &= (J_f(gk))(z) = (J_f M_g k)(z). \end{aligned}$$

Theorem 1. *Let X be a subspace of H with constants. Then $J[X]$ is a subspace of X with constants and $J[X]^n \subset X$.*

Proof. If $g \in J[X]$ then $J_g(1) = g - g(0) \in X$ and so $g \in X$ because $1 \in X$. Hence $J[X]$ is a subspace of X with constants.

Assuming $J[X]^n \subset X$, we will show that $J[X]^{n+1} \subset X$. Suppose that $g \in J[X]$ and $\{g_j\}_{j=1}^n \subset J[X]$. In order to prove that $g \prod_{j=1}^n g_j$ belongs to X , we will use the following

equalities.

$$\begin{aligned} & \int_0^z g(\zeta) \left(\prod_{j=1}^n g_j \right)'(\zeta) d\zeta \\ &= g(z) \left(\prod_{j=1}^n g_j \right)(z) - g(0) \left(\prod_{j=1}^n g_j \right)(0) - \int_0^z g'(\zeta) \left(\prod_{j=1}^n g_j \right)(\zeta) d\zeta \end{aligned}$$

and

$$\int_0^z g(\zeta) \left(\prod_{j=1}^n g_j \right)'(\zeta) d\zeta = \sum_{\ell=1}^n \int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta) d\zeta.$$

By hypothesis on induction, $\prod_{j=1}^n g_j \in X$ and so $\int_0^z g'(\zeta) \left(\prod_{j=1}^n g_j \right)(\zeta) d\zeta \in X$ because $g \in J[X]$. By hypothesis on induction, for $\ell = 1, \dots, n$ $g \prod_{j \neq \ell} g_j \in X$ and so $\int_0^z (g(\zeta) \prod_{j \neq \ell} g_j(\zeta)) g'_\ell(\zeta) d\zeta \in X$ because $g_\ell \in J[X]$. By the above two equalities, $g \prod_{j=1}^n g_j$ belongs to X . This implies that $J[X]^{n+1} \subset X$.

Proposition 1. *Let X be a subspace of H with constants. Then $I[X]$ is a subalgebra of H .*

Proof. If $k \in I[X]$ and $g \in I[X]$ then it is easy to see that $I_k I_g = I_{kg}$ (see Proposition 3). Hence $I_k I_g(X) = I_k(I_g(X)) \subseteq I_k(X) \subseteq X$ and so kg belongs to $I[X]$. It is clear that $I[X]$ is a subspace of H .

Proposition 2. *Suppose X is a subspace of H with constants.*

- (1) $M[X]$ is an algebra in X .
- (2) $J[X] \cap M[X] = I[X] \cap M[X]$.
- (3) $J[X] \cap I[X] \subseteq M[X]$.
- (4) $J[X] \subset M[X]$ if and only if $J[X] \subset I[X]$. Similarly $I[X] \subset M[X]$ if and only if $I[X] \subset J[X]$.

Proof. (1) is clear. (2) and (3) follow from the equality : $J_g f + I_g f = M_g f - g(0)f(0)$. (4) If $J[X] \subset M[X]$ then by (2) $J[X] \subset I[X]$. Conversely if $J[X] \subset I[X]$ then by (3) $J[X] \subset M[X]$.

§3. Invariant subspace and subalgebra

In this section, we study $J[X]$ and $I[X]$ when X is an invariant subspace or a subalgebra of H .

Theorem 2. Suppose that X is a subspace of H with constants and $kX \subset X$ for any k in H^∞ .

(1) If g_0 is an arbitrary function in $J[X]$, then $J[X]$ contains $\{g \in H ; |g'(z)| \leq |g'_0(z)|(z \in D)\}$.

(2) If $J[X]$ contains z then it contains H_1^∞ .

(3) Suppose $J[X]$ contains z . If $\{g_n\}$ is in $J[X]$ and $g'_n \rightarrow g'$ uniformly on D then g belongs to $J[X]$.

(4) $zJ[X] \subset J[X]$ if and only if $J_z(J[X]X) \subset X$.

(5) $J[X] \cap H^\infty \subset I[X]$ and hence $I[X]$ contains H_1^∞ if $z \in J[X]$.

Proof. (1) If $g \in H$ and $|g'(\zeta)| \leq |g'_0(\zeta)|(\zeta \in D)$, then $g'(g'_0)^{-1} \in H^\infty$ and so $fg'(g'_0)^{-1} \in X$ for any $f \in X$. Hence for any $f \in X$

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)g'(\zeta)g'_0(\zeta)^{-1}g'_0(\zeta)d\zeta$$

belongs to X because $fg'(g'_0)^{-1} \in X$ and $g_0 \in J[X]$. This implies that g belongs to $J[X]$.

(2) Since $z \in J[X]$, by (1) and the definition of H_1^∞ , H_1^∞ is contained in $J[X]$.

(3) If $g'_n \rightarrow g'$ uniformly on D , then $(g - g_n)' \in H^\infty$. Hence $f(g - g_n)' \in X$ for any $f \in X$. Therefore g belongs to $J[X]$ because $z \in J[X]$ and

$$\int_0^z f(\zeta)g'(\zeta)d\zeta = \int_0^z f(\zeta)(g(\zeta) - g_n(\zeta))'d\zeta + \int_0^z f(\zeta)g'_n(\zeta)d\zeta.$$

(4) follows trivially from the following equality :

$$\int_0^z f(\zeta)(\zeta g(\zeta))'d\zeta = \int_0^z f(\zeta)g(\zeta)d\zeta + \int_0^z f(\zeta)\zeta g'(\zeta)d\zeta$$

for $f \in X$ and $g \in J[X]$.

(5) By the equality : $I_g(f) = fg - (fg)(0) - J_g(f)$, if $g \in J[X] \cap H^\infty$ and $f \in X$ then $I_g(f)$ belongs to X because $gX \subset X$.

Proposition 3. If X is a subalgebra of H which contains constants then $M[X] = X$, $J[X]$ is also a subalgebra of X and $J[X] = I[X] \cap X$.

Proof. $M[X] = X$ is clear. If both g and h are in $J[X]$, then by Theorem 1 both fh and fg belongs to X for any $f \in X$ because X is an algebra. Hence gh belongs to $J[X]$ by the following equality : $J_{gh}(f) = J_g(fh) + J_h(fg)$ for any $f \in X$. This implies that $J[X]$ is a subalgebra of X by Theorem 1. From (2) of Proposition 2 $J[X] = I[X] \cap X$ follows.

§4. F-space

Let X be an F-space in H with an invariant metric d . For each a in D , put for f in X

$$\mathcal{E}_a f = f(a) \quad \text{and} \quad \mathcal{D}_a f = f'(a).$$

In this section we assume that both $\mathcal{E}_a$ and $\mathcal{D}_a$ are bounded on X . Put

$$S(a) = \sup\{|\mathcal{E}_a(f)| ; f \in X, d(f, 0) \leq 1\}$$

and

$$s(a) = \sup\{|\mathcal{D}_a(f)| ; f \in X, d(f, 0) \leq 1\},$$

then $S(a) < \infty$ and $s(a) < \infty$ if $a \in D$. Suppose v is a nonnegative function on D . For a function f in H put

$$\|f\|_\omega = \sup_{z \in D} \omega(z)|f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H ; \|f\|_\omega < \infty\}.$$

If ω is bounded, $\mathcal{B}_\omega$ contains all holomorphic functions on the closed unit disc $\bar{D}$.

Proposition 4. *If X is an F -space such that $S(a) < \infty$ and $s(a) < \infty$ for each $a \in D$, then $M[X]$, $J[X]$ and $I[X]$ belongs to $B[X]$.*

Proof. We will prove only that $J[X] \subset B[X]$ because the other statements are similar. By the closed graph theorem, it is enough to prove that for $\phi \in J[X]$ if $f_n \rightarrow f$ in X and $J_\phi(f_n) \rightarrow F$ then $J_\phi(f) = F$. Since $S(a) < \infty$, $f_n(a) \rightarrow f(a)$ ($a \in D$). Since $s(a) < \infty$, $f_n(a)\phi'(a) \rightarrow F'(a)$ ($a \in D$). Thus $f(a)\phi'(a) = F'(a)$ and so $J_\phi(f) = F$ because $F(0) = 0$.

Theorem 3. *Let X be an F -space in H with an invariant metric d . Suppose that $\sup_{|a| \leq 1-\varepsilon} S(a) < \infty$ for any $\varepsilon > 0$. Then $J[X] \subset \mathcal{B}_{\omega_0} \cap X$ and $I[X] \subset H^\infty$, where $\omega_0 = 1/sS$.*

Proof. If $g \in J[X]$ then by Proposition 4, for any $f \in X$ $d(J_g f, 0) \leq \|J_g\|d(f, 0)$. Since $J_g f \in X$, by definition of $\mathcal{D}_z$ $|\mathcal{D}_z(J_g f)| \leq s(z)d(J_g f, 0)$ ($z \in D$). Hence

$$s(z)^{-1}|f(z)||g'(z)| \leq \|J_g\|d(f, 0) \quad (z \in D)$$

and so

$$s^{-1}(z)S^{-1}(z)|g'(z)| \leq \|J_g\| \quad (z \in D).$$

By Theorem 1 g belongs to $\mathcal{B}_{\omega_0} \cap X$ where $\omega_0 = 1/sS$. If $g \in I[X]$ then by Proposition 4, for any $f \in X$ $d(I_g f, 0) \leq \|I_g\|d(f, 0)$. Since $I_g f \in X$, by definition of $\mathcal{D}_z$ $|\mathcal{D}_z(I_g f)| \leq s(z)d(I_g f, 0)$ ($z \in D$). Hence

$$s(z)^{-1}|f'(z)||g(z)| \leq \|I_g\|d(f, 0) \quad (z \in D)$$

and so

$$|g(z)| \leq \|I_g\| \quad (z \in D).$$

Proposition 5. *Let X be a subspace of H with constants which is of finite dimension. Then $J[X] = I[X] = M[X] = \mathcal{C}$.*

Proof. Suppose $\{f_j\}_{j=1}^n$ is a basis in X with $f_1 \equiv 1$. We will show that $J[X] = \mathcal{C}$. If $g \in J[X]$ then by Theorem 1 $g^\ell \in X$ for any $\ell \geq 0$ and so there exist $\{\alpha_j^\ell\}_1^n \subset \mathcal{C}$ such that $g^\ell = \sum_{j=1}^n \alpha_j^\ell f_j$. Hence there exist $\{b_\ell\}_{\ell=0}^n \subset \mathcal{C}$ such that $\sum_{\ell=0}^n b_\ell g^\ell = 0$. This implies that g is just constant because g is analytic. Therefore $J[X] = \mathcal{C}$. We will show that $I[X] = \mathcal{C}$. Put $X_1 = \{f' ; f \in X\}$. If $g \in I[X]$ then by Proposition 1 $g^\ell X_1 \subset X_1$ for any $\ell \geq 1$ and so there exist $\{\alpha_j^\ell\}_1^n \subset \mathcal{C}$ such that $g^\ell f'_2 = \sum_{j=2}^n \alpha_j^\ell f'_j$. By the same argument above gf'_2 is constant. Similary it follows that $\{gf'_j\}_2^n$ are constants and so g is constant because $\{f'_j\}_\ell^n$ is a basis in X_1 . Therefore $I[X] = \mathcal{C}$.

§5. Weighted Bloch space

Let ω be a positive bounded function on D . For a function f in H put

$$\|f\|_\omega = \sup_{z \in D} \omega(z) |f'(z)| + |f(0)|$$

and

$$\mathcal{B}_\omega = \{f \in H ; \|f\|_\omega < \infty\}.$$

Since ω is bounded, $\mathcal{B}_\omega$ contains all holomorphic functions on the closed unit disc $\bar{D}$. $\mathcal{B}_\omega$ is called a weighted Bloch space. A weight ω is called measurable when $\omega(at)$ is measurable on $[0,1]$ for each a in D . Put $\varepsilon(r) = \inf\{\omega(z) ; |z| \leq r\}$ and $r < 1$.

Lemma 2. *If $\varepsilon(r) > 0$ for $0 \leq r < 1$ then $\mathcal{B}_\omega$ is a Banach space with norm $\|\cdot\|_\omega$.*

Proof. Suppose that $\{f_n\}$ is a Cauchy sequence in $\mathcal{B}_\omega$. For any $\varepsilon > 0$, there exist a positive integer n_0 such that $\|f_n - f_m\|_\omega < \varepsilon$ if $n, m \geq n_0$. Hence if $r < 1$ and $z \in D_r = \{z ; |z| < r\}$ then

$$|f'_n(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)}.$$

By the normal family argument, there exists a function $f' \in H(D_r)$ such that $f'_n \rightarrow f'$ uniformly on D_r . Hence as $n \rightarrow \infty$,

$$|f'(z) - f'_m(z)| \leq \frac{\varepsilon}{\omega(z)} \leq \frac{\varepsilon}{\varepsilon(r)} \quad (z \in D_r).$$

Since r is arbitrary, f' belongs to $H(D)$ and

$$\omega(z) |f'(z) - f'_m(z)| \leq \varepsilon \quad (z \in D)$$

if $m \geq n_0$. Since $f_m(0) \rightarrow f(0)$, $\|f - f_m\|_\omega \rightarrow 0$.

Theorem 4. Let ω be a measurable, $\varepsilon(r) > 0$ for $0 \leq r < 1$ and $X = \mathcal{B}_\omega$. Then

$$\mathcal{B}_{\omega S} = J[\mathcal{B}_\omega] \text{ and } I[\mathcal{B}_\omega] \subset H^\infty$$

where $S(z) = \sup\{|f(z)|; f \in \mathcal{B}_\omega, \|f\|_\omega \leq 1\}$. Moreover $\|J_g\| = \|g\|_{\omega S}$ for each g in $J[\mathcal{B}_\omega]$ with $g(0) = 0$.

Proof. By Theorem 1, $J[\mathcal{B}_\omega] \subseteq \mathcal{B}_\omega$. If $g \in J[\mathcal{B}_\omega]$ then $\|J_g f\|_\omega \leq \|J_g\| \|f\|_\omega$ ($f \in \mathcal{B}_\omega$) and so $\omega(z) |f(z)| \cdot |g'(z)| \leq \|J_g\| \cdot \|f\|_\omega$. Hence

$$\omega(z) S(z) |g'(z)| \cdot \frac{|f(z)|}{S(z)} \leq \|J_g\| \cdot \|f\|_\omega$$

and so

$$\omega(z) S(z) |g'(z)| \leq \|J_g\|.$$

Therefore g belongs to $\mathcal{B}_{\omega S}$ and $\|g\|_{\omega S} \leq \|J_g\| + |g(0)|$. Thus $J[\mathcal{B}_\omega] \subseteq \mathcal{B}_{\omega S}$. Note that $\mathcal{B}_{\omega S} \subseteq \mathcal{B}_\omega$ because $S(z) \geq 1$ ($z \in D$). Conversely if $g \in \mathcal{B}_{\omega S}$ then

$$\omega(z) |J_g(f)'(z)| = \omega(z) S(z) |g'(z)| \cdot \frac{|f(z)|}{S(z)} \leq \|g\|_{\omega S} \|f\|_\omega$$

and so g belongs to $J[\mathcal{B}_\omega]$. Thus

$$\|g\|_{\omega S} \leq \|J_g\| + |g(0)| \leq \|g\|_{\omega S} + |g(0)|.$$

In Theorem 4, if ω is an absolute value of some analytic function and a radial function, R.Yoneda ([8],[9]) showed those under some special technical conditions on ω .

§6. Hardy space

For $0 < p \leq \infty$, H^{p-} denotes $\bigcap_{t < p} H^t$ and $H^{\infty-}$ is written as H^ω . For $0 < p < \infty$, when $W = |h|^p$ for an outer function h in H^p , $H^p(W)$ denotes a weighted Hardy space that is, the closure of H^∞ in $L^p(W d\theta/2\pi)$.

Lemma 3 is well known (cf. [3, Theorem 5.12]). In Proposition 6 it is known ([1],[2]) that $J[H^p] = \text{BMOA}$. Hence our result is weaker than that. However if $J[H^p] = \text{BMOA}$ then our result shows that $I[H^p] = H^\infty$.

Lemma 3. (1) For $0 < p < 1$, if f is a function in H^p then $\int_0^z f(\zeta) d\zeta$ belongs to $H^{p/1-p}$. (2) If f is a function in H^1 then $\int_0^z f(\zeta) d\zeta$ belongs to H^∞ .

Proposition 6. For $0 < p < \infty$, $H_1^\infty \subset J[H^p] \subset H^\omega$ and $zJ[H^p] \subset J[H^p]$. Moreover $M[H^p] = H^\infty$ and $I[H^p] = J[H^p] \cap H^\infty$.

Proof. By Lemma 3, $z \in J[H^p]$ and so by (2) of Theorem 2 $H_1^\infty \subset J[H^p]$. Theorem 1 implies that $J[H^p] \subset H^\omega$. By (5) of Theorem 2, $J[H^p] \cap H^\infty \subset I[H^p]$. Theorem 3 implies that $I[H^p] \subset H^\infty$. Hence $I[H^p] \cap H^p \subset H^p$ and so (4) of Proposition 2 $I[H^p] \subset J[H^p]$. It is well known that $M[H^p] = H^\infty$. By (2) of Proposition 2 $I[H^p] = J[H^p] \cap H^\infty$. By (4) of Theorem 2, to prove that $zJ[H^p] \subset J[H^p]$ it is sufficient to show that $J_z(J[H^p]H^p) \subset H^p$. Since $J[H^p]H^p \subset H^{p-}$, by Lemma 2, $J_z(J[H^p]H^p) \subset H^p$.

Theorem 5. For $0 < p < \infty$, $\bigcap_{t < 1} H^t \subset J[H^{p-}] \subset H^\omega$ and so $\log(1-z)^{-1}$ belongs to $J[H^{p-}]$. Moreover $zJ[H^{p-}] \subset J[H^{p-}]$, $M[H^{p-}] = H^\infty$ and so $J[H^{p-}] \cap H^\infty = I[H^{p-}] \cap H^\infty$. When $p = \infty$, $J[H^\omega] = I[H^\omega] \cap H^\omega$ and $J[H^\omega]$ is a subalgebra of H^ω which contains H_1^∞ .

Proof. By Theorem 1, $J[H^{p-}] \subset H^\omega$. We will show that $\bigcap_{t < 1} H_1^t \subset J[H^{p-}]$. If $g \in \bigcap_{t < 1} H_1^t$ then g' belongs to H^{1-} . If $f \in H^{p-}$ then f belongs to H^t for any $0 < t < p$. If $0 < s < t/(t+1)$ then $t/s > 1$ and $1/(t/s) + 1/(t/t-s) = 1$. By the Hölder inequality,

$$\begin{aligned} & \int_0^{2\pi} |f(e^{i\theta})g'(e^{i\theta})|^s d\theta/2\pi \\ & \leq \left(\int_0^{2\pi} |f(e^{i\theta})|^t d\theta/2\pi \right)^{\frac{s}{t}} \left(\int_0^{2\pi} |g'(e^{i\theta})|^{\frac{st}{t-s}} d\theta/2\pi \right)^{\frac{s-t}{t}} \end{aligned}$$

and so fg' belongs to $\bigcap_{s < t/(t+1)} H^s$. By Lemma 3, $\int_0^z f(\zeta)g'(\zeta)d\zeta$ belongs to $H^{\frac{s}{1-s}}$. As $s \rightarrow t/(t+1)$, $s/(1-s) \rightarrow t$ and so $\int_0^z f(\zeta)g'(\zeta)d\zeta$ belongs to H^{t-} . As $t \rightarrow p$, $\int_0^z f(\zeta)g'(\zeta)d\zeta$ belongs to H^{p-} . Thus $J_g[H^{p-}] \subseteq H^{p-}$ and so $\bigcap_{t < 1} H_1^t \subset J[H^{p-}]$. By (4) of Theorem 2, if we show that $J_z(J[H^{p-}]H^{p-}) \subset H^{p-}$ then it follows that $zJ[H^{p-}] \subset J[H^{p-}]$. Since $J[H^{p-}]H^{p-} \subset H^{p-}$, by Lemma 4 $J_z(J[H^{p-}]H^{p-}) \subset H^{p-}$. It is known that $M[H^{p-}] = H^\infty$. The last statement is a result of (2) of Proposition 2.

When $p = \infty$, by Proposition 3 $J[H^\omega] = I[H^\omega] \cap H^\omega$ and $J[H^\omega]$ is a subalgebra of H^ω . Theorem 2 implies $J[H^\omega] \supset H_1^\infty$.

Theorem 6. Let $1 \leq p < \infty$ and $W = |h|^p$ for some outer function h in H^p . Then $\{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta \text{ and } k \in H^{\frac{p}{p-1}}\} \subset J[H^p(W)] \subset H^\omega(W)$. $M[H^p(W)] = H^\infty$ and $J[H^p(W)] \cap H^\infty = I[H^p(W)]$. There exists a weight W such that z does not belong to $J[H^p(W)]$.

Proof. If $g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta$ and $k \in H^{\frac{p}{p-1}}$, then

$$h(z)\{J_g(h^{-1}f)\}(z) = h(z) \int_0^z f(\zeta)k(\zeta)d\zeta$$

and so $hJ_g h^{-1}f$ belongs to H^p for all $f \in H^p$ by Lemma 3 because $fk \in H^1$. Therefore $\{g \in H; g(z) = \int_0^z h(\zeta)k(\zeta)d\zeta \text{ and } k \in H^{\frac{p}{p-1}}\} \subset J[H^p(W)]$. By Theorem 1, $J[H^p(W)] \subset$

$\bigcap_{p<\infty} H^p(W)$. In fact, since $g^n h \in H^p$ for any $n \geq 1$, $gh^{1/n} \in H^{np}$ and so g belongs to $H^{np}(W)$. If $\phi \in M(H^p(W))$ then $\phi(h^{-1}H^p) \subset h^{-1}H^p$ and so $\phi H^p \subset H^p$. Hence $\phi \in M(H^p) = H^\infty$. Therefore $M(H^p(W)) = H^\infty$ and so by (2) of Proposition 2 $J[H^p(W)] \cap H^\infty = I[H^p(W)] \cap H^\infty$. For $a \in D$ it is easy to see that

$$\sup\{|f(a)| ; f \in H^p(W) \text{ and } \|f\|_{W,p} \leq 1\} = (1 - |a|^2)^{-1/p} |h(a)|^{-p} < \infty$$

and so by Theorem 3 $I[H^p(W)] \subset H^\infty$. Thus $J[H^p(W)] \cap H^\infty = I[H^p(W)]$. If $J_z(H^p(W)) \subseteq H^p(W)$ for any W with $\log W \in L^1(d\theta/2\pi)$ then $J_z(N_+) \subseteq N_+$. For by a theorem of H.Helson [6] N_+ is the union of all $H^p(W)$ as W ranges over the set of weights with sumable $\log W$. Hence there exists a weight W such that $z \notin J[H^p(W)]$. Because it is known that $J_z(N_+) \not\subset N_+$ [7].

§7. Privalov space

We denote by N^p , for $1 \leq p < \infty$, the set of all functions f in H which satisfy

$$\sup_{0 < r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta < \infty.$$

When $p = 1$, N^p is just N . Then

$$\bigcup_{p>0} H^p \subset \bigcap_{p>1} N^p \text{ and } \bigcup_{p>1} N^p \subset N_+ \subset N^1 = N.$$

Proposition 7. *Let $X = N_+$ or N . Then $J[X]$ is a subalgebra of X and $J[X] = I[X] \cap X$. If $(f')^{-1}$ is in H^∞ then f does not belong to $J[X]$.*

Proof. It is known that N_+ and N are subalgebras of H . Hence the first part of this proposition is a result of Theorem 1 and Proposition 3. By [5] and [7], $z \notin J[X]$ and so the second part follows from (1) of Theorem 2.

In Proposition 7, it is known ([5],[7]) that $z \notin J[X]$. Hence $I[X] \not\ni z$. We don't know whether $J[X] = \mathcal{C}$ and $I[X] = \mathcal{C}$.

Theorem 7. *If $1 < p < \infty$ then $J[N^p]$ is a subalgebra of N^p which contains N_1^p , and $J[N^p] = I[N^p] \cap N^p$.*

Proof. Suppose $1 < p < \infty$ and $g \in N_1^p$. If $f \in N^p$ then

$$\begin{aligned} & \left\{ \int_0^{2\pi} (\log^+ |(J_g f)(re^{i\theta})|)^p d\theta / 2\pi \right\}^{1/p} \\ &= \left\{ \int_0^{2\pi} \left(\log^+ \left| \int_0^r f(te^{i\theta}) g'(te^{i\theta}) dt \right| \right)^p d\theta / 2\pi \right\}^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \int_0^{2\pi} \left(\log^+ \int_0^{1-} |f(te^{i\theta})g'(te^{i\theta})| dt \right)^p d\theta/2\pi \right\}^{1/p} \\
&\leq \left\{ \int_0^{2\pi} \left(\log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| + \log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p d\theta/2\pi \right\}^{1/p} \\
&\leq \left\{ \int_0^{2\pi} \left(\log^+ \sup_{0 \leq t < 1} |f(te^{i\theta})| \right)^p d\theta/2\pi \right\}^{1/p} + \left\{ \int_0^{2\pi} \left(\log^+ \sup_{0 \leq t < 1} |g'(te^{i\theta})| \right)^p d\theta/2\pi \right\}^{1/p}.
\end{aligned}$$

Put $u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-\theta) \log^+ |f(e^{it})| dt$, then $u(r, \theta) \geq \log^+ |f(re^{i\theta})|$. Since $\log^+ |f(e^{it})| \in L^p$, by a theorem of Hardy and Littlewood (cf. [3, Proposition 1.8]), $\sup_{0 \leq r < 1} u(r, \theta)$ belongs to L^p and so $\log^+ \sup_{0 \leq r < 1} |f(re^{i\theta})|$ belongs to L^p . Similarly we can prove that $\log^+ \sup_{0 \leq r < 1} |g'(re^{i\theta})|$ belongs to L^p . Thus $J_g f$ belongs to N^p . Hence $N_1^p \subset J[N^p]$. It is known that N^p is a subalgebra of H . Hence, by Proposition 3 $J[N^p]$ is a subalgebra of N^p and $J[N^p] = I[N^p] \cap N^p$.

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