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Finite Dimensional Semisimple Q -Algebras

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Abstract A Q -algebra can be represented as an operator algebra on an infinite dimensional Hilbert space. However we don't know whether a finite n -dimensional Q -algebra can be represented on a Hilbert space of dimension n except $n = 1, 2$. It is known that a two dimensional Q -algebra is just a two dimensional commutative operator algebra on a two dimensional Hilbert space. In this paper we study a finite n -dimensional semisimple Q -algebra on a finite n -dimensional Hilbert space. In particular we describe a three dimensional Q -algebra of the disc algebra on a three dimensional Hilbert space. Our studies are related to the Pick interpolation problem for a uniform algebra.

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1. Introduction

Let A be a uniform algebra on a compact Hausdorff space X . If I is a closed ideal of A , then the quotient algebra A/I is a commutative Banach algebra with unit. In this paper, if a Banach algebra $\mathcal{B}$ is isometrically isomorphic to A/I , then $\mathcal{B}$ is called a Q -algebra. (F. Bonsall and J. Duncan called $\mathcal{B}$ an IQ -algebra. (cf. [1], p.270)) B. Cole (cf. [1], p.272) showed that any Q -algebra is an operator algebra on a Hilbert space H , that is, there exists an isometric isomorphism to an operator algebra on H . Let μ be a probability measure on X and $H^2(\mu)$ the closure of A in $L^2(\mu)$. $H^2(\mu) \cap I^\perp$ denotes the annihilator of I in $H^2(\mu)$. Let P be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^\perp$ and for any $f \in A$ put

$$S_f^\mu \phi = P(f\phi), \quad (\phi \in H^2(\mu) \cap I^\perp).$$

Then $S_{f+k}^\mu = S_f^\mu$ for $k \in I$ and $\|S_f^\mu\| \leq \|f+I\|$. S^μ is the map of A/I on operators on $H^2(\mu) \cap I^\perp$ which sends $f+I \rightarrow S_f^\mu$ for each f in A . Hence S^μ is a contractive homomorphism from A into $B(H^2(\mu) \cap I^\perp)$ where $B(H^2(\mu) \cap I^\perp)$ is the set of all bounded linear operators on $H^2(\mu) \cap I^\perp$. The kernel of S^μ contains I . Then we say that S^μ gives a contractive representation of A/I into $B(H^2(\mu) \cap I^\perp)$. If $\|S_f^\mu\| = \|f+I\|$, ($f \in A$) then $\ker S^\mu = I$ and we say that S^μ gives an isometric representation of A/I on $H^2(\mu) \cap I^\perp$.

Problem 1. Prove that any finite n -dimensional Q -algebra can be represented on a Hilbert space of finite dimension n .

If S^μ is isometric then we solve Problem 1. In fact, T. Nakazi and K. Takahashi (cf. [9]) solved Problem 1 for $n = 2$ in this way. It seems to be unknown for $n \geq 3$.

Problem 2. Describe a finite n -dimensional Q -algebra in finite n -dimensional commutative operator algebras with unit on a Hilbert space of finite dimension n .

Problem 2 is clear for $n = 1$ and it was proved by S.W. Drury (cf. [4]) and T. Nakazi (cf. [8]) that a 2-dimensional commutative operator algebra with unit on a Hilbert space is just a Q -algebra. J. Holbrook (cf. [6]) proved that von Neumann's inequality

$$\|p(T)\| \leq \|p\|_\infty$$

can fail for some polynomials p in 3 variables, where $T = (T_1, T_2, T_3)$ is a triple of commuting contractions on $\mathbf{C}^4$, and T_1, T_2, T_3 are simultaneously diagonalizable. Then we can construct a 4-dimensional commutative matrix algebra with unit on $\mathbf{C}^4$, which is not a Q -algebra. If $n \geq 4$, then this implies that the set of all n -dimensional Q -algebra A/I is smaller than the set of all set of all n -dimensional commutative operator algebras with unit on an n -dimensional Hilbert space. If $n = 3$, then Problem 2 has not been solved yet. In this paper, we concentrate on a semisimple commutative Banach algebra and we study Problem 2. In Section 2, we will prove several general results of finite dimensional semisimple Q -algebras that will be used in the latter sections. In Section 3, we will study arbitrary n -dimensional semisimple Q -algebras for $n = 2, 3$. In Section 4, we will study the isometric representation of A/I . In Section 5, we will describe completely 3-dimensional semisimple Q -algebras of the disc algebra in 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

2. Semisimple and commutative matrix algebra

In this section, we study 3-dimensional semisimple commutative operator algebras on a 3-dimensional Hilbert space. In particular, we study when two such operator algebras are isometric or unitary equivalent. S. McCullough and V. Paulsen (cf. [7], Proposition 2.2) proved the similar result of Proposition 2.3. We use Lemma 2.1 to prove Proposition 2.2 and Proposition 2.3.

Lemma 2.1. *Let $n \geq 2$ and let H be an n -dimensional Hilbert space which is spanned by $k_1, k_2, \dots, k_n$. Let*

$$\psi_1 = \frac{k_1}{\|k_1\|}, \quad \psi_j = \frac{k_j - \sum_{i=1}^{j-1} \langle k_j, \psi_i \rangle \psi_i}{\|k_j - \sum_{i=1}^{j-1} \langle k_j, \psi_i \rangle \psi_i\|} \quad (2 \leq j \leq n).$$

Then $\{\psi_1, \dots, \psi_n\}$ is an orthonormal basis for H . Let $P_1, \dots, P_n$ be the idempotent operators on H such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For $1 \leq m \leq n$, let $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$, $(1 \leq i, j \leq n)$. Then $P_m = (a_{ij}^{(m)})_{1 \leq i, j \leq n}$ is an $n \times n$ matrix such that

$$P_1 = \begin{pmatrix} B_1 \\ O \end{pmatrix}, \dots, P_m = \begin{pmatrix} O & B_m \\ O & O \end{pmatrix}, \dots, P_n = \begin{pmatrix} O & B_n \end{pmatrix},$$

where B_m is an $m \times (n - m + 1)$ matrix such that

$$B_1 = \begin{pmatrix} 1 & \dots & a_{1n}^{(1)} \end{pmatrix}, \dots, B_m = \begin{pmatrix} a_{1m}^{(m)} & \dots & a_{1n}^{(m)} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ 1 & \dots & a_{mn}^{(m)} \end{pmatrix}, \dots, B_n = \begin{pmatrix} a_{1n}^{(n)} \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

Then $a_{mm}^{(m)} = 1$, and for $m \geq 2$,

$$a_{im}^{(m)} = \frac{\langle k_m, \psi_i \rangle}{\|k_m - \sum_{h=1}^{m-1} \langle k_m, \psi_h \rangle \psi_h\|},$$

and for $m + 1 \leq j \leq n$,

$$a_{ij}^{(m)} = \frac{-\sum_{h=m}^{j-1} \langle k_j, \psi_h \rangle a_{ih}^{(m)}}{\|k_j - \sum_{h=1}^{j-1} \langle k_j, \psi_h \rangle \psi_h\|}.$$

Since this lemma is proved by elementary calculations, the proof is omitted. It is well known that any n -dimensional semisimple commutative Banach algebra with unit I is spanned by commuting idempotents $P_1, \dots, P_n$ satisfying $P_1 + \dots + P_n = I$.

Proposition 2.2. In Lemma 2.1, for $1 \leq m \leq n$, $\text{rank} P_m = 1$, and $\mathcal{B} = \text{span}\{P_1, \dots, P_n\}$ is an n -dimensional semisimple commutative operator algebra with unit on H . Then $n \times n$ matrix $(a_{ij}^{(m)})$ for P_m with respect to $\{\psi_1, \dots, \psi_n\}$ is $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$, and

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & a_{12}^{(1)} & \cdot & \cdot & \cdot & \cdot & a_{1n}^{(1)} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \quad P_2 = (a_{ij}^{(2)}) = \begin{pmatrix} 0 & a_{12}^{(2)} & a_{13}^{(2)} & \cdot & \cdot & \cdot & a_{1n}^{(2)} \\ 0 & 1 & a_{23}^{(2)} & \cdot & \cdot & \cdot & a_{2n}^{(2)} \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

$$\dots, \quad P_n = (a_{ij}^{(n)}) = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{1n}^{(n)} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{2n}^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & a_{n-1n}^{(n)} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix}.$$

In Lemma 2.1, $a_{ij}^{(m)}$ is written using $k_1, \dots, k_n$ and $\psi_1, \dots, \psi_n$.

Proof. By the assumption of Lemma 2.1, $P_i k_i = k_i$ and $P_i k_j = 0$ if $i \neq j$. Hence $\text{rank} P_m = 1$. If $i \neq j$, then $P_i P_j k_m = \delta_{jm} P_i k_j = 0$, ($1 \leq m \leq n$). Since $H = \text{span}\{k_1, \dots, k_n\}$, this implies that $P_i P_j = 0$ if $i \neq j$. Hence $\mathcal{B}$ is commutative. Since $P_i^2 k_m = \delta_{im} P_i k_m = P_i k_m$, ($1 \leq m \leq n$), it follows that $P_i^2 = P_i$. Hence $\mathcal{B}$ is semisimple and n -dimensional. Since $(P_1 + \dots + P_n) k_m = P_m k_m = k_m$, ($1 \leq m \leq n$), it follows that $P_1 + \dots + P_n = I$. Hence $\mathcal{B}$ has a unit I . This completes the proof.

Proposition 2.3. Let H be a 3-dimensional Hilbert space which is spanned by k_1, k_2, k_3 . Let $\langle \cdot, \cdot \rangle$ denote the inner product, and let $\| \cdot \|$ denote the norm of H .

$$\psi_1 = \frac{k_1}{\|k_1\|}, \quad \psi_2 = \frac{k_2 - \langle k_2, \psi_1 \rangle \psi_1}{\|k_2 - \langle k_2, \psi_1 \rangle \psi_1\|}, \quad \psi_3 = \frac{k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

Then ψ_1, ψ_2, ψ_3 is an orthonormal basis in H . Let P_i be the idempotent operator on H such that $P_i k_i = k_i$, $P_i k_j = 0$ if $i \neq j$. For $m = 1, 2, 3$, the 3×3 matrix $(a_{ij}^{(m)})$ for P_m with respect to $\{\psi_1, \psi_2, \psi_3\}$ is $a_{ij}^{(m)} = \langle P_m \psi_j, \psi_i \rangle$. Then

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = (a_{ij}^{(2)}) = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = (a_{ij}^{(3)}) = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}}, \quad y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|},$$

$$z = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

Proof. By Proposition 2.2, there exist x, y such that

$$P_1 = (a_{ij}^{(1)}) = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By Lemma 2.1,

$$x = a_{12}^{(1)} = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}},$$

and

$$y = a_{13}^{(1)} = \frac{-\sum_{h=1}^2 \langle k_3, \psi_h \rangle a_{1h}^{(1)}}{\|k_3 - \sum_{h=1}^2 \langle k_3, \psi_h \rangle \psi_h\|} = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

By Proposition 2.2, there exist z, w such that

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & w \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & -w - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

because $P_1 + P_2 + P_3 = I$. By Lemma 2.1,

$$z = a_{23}^{(2)} = \frac{-\sum_{h=2}^2 \langle k_3, \psi_h \rangle a_{2h}^{(2)}}{\|k_3 - \sum_{h=1}^2 \langle k_3, \psi_h \rangle \psi_h\|} = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|}.$$

By Lemma 2.1,

$$a_{12}^{(1)} = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}} = -a_{12}^{(2)}.$$

Hence

$$w = a_{13}^{(2)} = \frac{-\sum_{h=2}^2 \langle k_3, \psi_h \rangle a_{1h}^{(2)}}{\|k_3 - \sum_{h=1}^2 \langle k_3, \psi_h \rangle \psi_h\|} = \frac{-\langle k_3, \psi_2 \rangle a_{12}^{(2)}}{\|k_3 - \sum_{h=1}^2 \langle k_3, \psi_h \rangle \psi_h\|} = z a_{12}^{(2)} = -z a_{12}^{(1)} = -xz.$$

This completes the proof.

Theorem 2.4. *Let P_1, P_2, P_3 be idempotent operators defined in Proposition 2.3. Let H' be a 3-dimensional Hilbert space. Let $\mathcal{B}'$ be a 3-dimensional semisimple commutative operator algebra on H' . Then, there are idempotent operators Q_1, Q_2, Q_3 on H' , an orthonormal basis $\psi'_1, \psi'_2, \psi'_3$ in H' and complex numbers x_0, y_0, z_0 such that $\mathcal{B}' = \text{span}\{Q_1, Q_2, Q_3\}$ and, as matrices relative to $\psi'_1, \psi'_2, \psi'_3$,*

$$Q_1 = \begin{pmatrix} 1 & x_0 & y_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -x_0 & -x_0 z_0 \\ 0 & 1 & z_0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & x_0 z_0 - y_0 \\ 0 & 0 & -z_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let τ be the map of $\mathcal{B}$ on $\mathcal{B}'$ such that

$$\tau(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3) = \lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3, \quad (\lambda_1, \lambda_2, \lambda_3 \in \mathbf{C}).$$

(1) τ is isometric if and only if

$$\begin{aligned} |x|^2 + |y|^2 &= |x_0|^2 + |y_0|^2, \\ (1 + |x|^2)(1 + |z|^2) &= (1 + |x_0|^2)(1 + |z_0|^2), \\ |x|^2 + xz\bar{y} &= |x_0|^2 + x_0z_0\bar{y}_0. \end{aligned}$$

(2) τ is induced by a unitary map from H to H' if and only if there are complex numbers u_1, u_2, u_3 such that

$$|u_1| = |u_2| = |u_3| = 1, \quad u_1x = u_2x_0, \quad u_1y = u_3y_0, \quad u_2z = u_3z_0.$$

Then $|x| = |x_0|$, $|y| = |y_0|$, $|z| = |z_0|$, $xz\bar{y} = x_0z_0\bar{y}_0$.

Proof. (1) By the theorem of B. Cole and J. Wermer (cf. [3]), τ is isometric if and only if, writing tr for trace,

$$\text{tr}(P_i^*P_j) = \text{tr}(Q_i^*Q_j), \quad (1 \leq i, j \leq 3).$$

If τ is isometric, then

$$\begin{aligned} 1 + |x|^2 + |y|^2 &= \text{tr}(P_1^*P_1) = \text{tr}(Q_1^*Q_1) = 1 + |x_0|^2 + |y_0|^2, \\ (1 + |x|^2)(1 + |z|^2) &= \text{tr}(P_2^*P_2) = \text{tr}(Q_2^*Q_2) = (1 + |x_0|^2)(1 + |z_0|^2), \\ |x|^2 + xz\bar{y} &= \text{tr}(P_1^*P_2) = \text{tr}(Q_1^*Q_2) = |x_0|^2 + x_0z_0\bar{y}_0. \end{aligned}$$

Conversely, if three equalities in (1) hold, then

$$\begin{aligned} \text{tr}(P_1^*P_1) &= 1 + |x|^2 + |y|^2 = 1 + |x_0|^2 + |y_0|^2 = \text{tr}(Q_1^*Q_1), \\ \text{tr}(P_2^*P_2) &= (1 + |x|^2)(1 + |z|^2) = (1 + |x_0|^2)(1 + |z_0|^2) = \text{tr}(Q_2^*Q_2), \\ \text{tr}(P_1^*P_2) &= |x|^2 + xz\bar{y} = |x_0|^2 + x_0z_0\bar{y}_0 = \text{tr}(Q_1^*Q_2), \\ \text{tr}(P_2^*P_3) &= \overline{xz}(y - xz) - |z|^2 = \overline{x_0z_0}(y_0 - x_0z_0) - |z_0|^2 = \text{tr}(Q_2^*Q_3), \\ \text{tr}(P_3^*P_1) &= y(\overline{xz - y}) = y_0(\overline{x_0z_0 - y_0}) = \text{tr}(Q_3^*Q_1), \\ \text{tr}(P_3^*P_3) &= 1 + |z|^2 + |xz - y|^2 = 1 + |z_0|^2 + |x_0z_0 - y_0|^2 = \text{tr}(Q_3^*Q_3). \end{aligned}$$

(2) Suppose τ is induced by a unitary map $U = (u_{ij})$, $(1 \leq i, j \leq 3)$ from H to H' . Since $UP_1 = Q_1U$, it follows that $u_{21} = u_{31} = 0$. Since $UP_2 = Q_2U$, it follows that $u_{32} = 0$. Hence U is an upper triangular matrix. Since the columns of U are pairwise orthogonal, U is a diagonal matrix. Hence there are complex numbers u_1, u_2, u_3 such that u_1, u_2, u_3 are diagonal element of U , and $|u_1| = |u_2| = |u_3| = 1$. Since $UP_1 = Q_1U$, it follows that $u_1x = u_2x_0$, $u_1y = u_3y_0$. Since $UP_2 = Q_2U$, it follows that $u_2z = u_3z_0$. The converse is also true. This completes the proof.

Example 2.5. Let $\mathcal{B}_0 = \text{span}\{P_1, P_2, P_3\}$, where

$$P_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

and let $\mathcal{B}_1 = \text{span}\{P_1^*, P_2^*, P_3^*\}$. This is an example which is established by W. Wogen (cf. [3]). He proved that $\mathcal{B}_0$ and $\mathcal{B}_1$ are isometrically isomorphic, and not unitarily equivalent. There is another example as the following. Let $\mathcal{B}_2 = \text{span}\{Q_1, Q_2, Q_3\}$, where

$$Q_1 = \begin{pmatrix} 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -\sqrt{2} & -\sqrt{2/3} \\ 0 & 1 & 1/\sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 0 & \sqrt{2/3} \\ 0 & 0 & -1/\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathcal{B}_0$ and $\mathcal{B}_2$ are 3-dimensional commutative operator algebras with unit. By the calculation,

$$\begin{aligned} |x|^2 + |y|^2 &= |x_0|^2 + |y_0|^2 = 2, \\ (1 + |x|^2)(1 + |z|^2) &= (1 + |x_0|^2)(1 + |z_0|^2) = 4, \\ |x|^2 + xz\bar{y} &= |x_0|^2 + x_0 z_0 \bar{y}_0 = 2. \end{aligned}$$

By (1) of Theorem 2.4, this implies that $\mathcal{B}_0$ and $\mathcal{B}_2$ are isometrically isomorphic. By (2) of Theorem 2.4, $\mathcal{B}_0$ and $\mathcal{B}_2$ are not unitarily equivalent.

3. One to one representation

In this section, we assume that A/I is n -dimensional and semisimple. Hence there exist $\tau_1, \dots, \tau_n$ in the maximal ideal space $M(A)$ of A such that $\tau_i \neq \tau_j$ ($i \neq j$) and $I = \bigcap_{j=1}^n \ker \tau_j$. S^μ gives a contractive representation of A/I into $B(H^2(\mu) \cap I^\perp)$ and $\dim H^2(\mu) \cap I^\perp \leq \dim A/I = n$. We study when S^μ is one to one from A/I to $B(H^2(\mu) \cap I^\perp)$. It is clear that S^μ is one to one if and only if $\dim H^2(\mu) \cap I^\perp = \dim A/I$. For $1 \leq j \leq n$, there exist $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \text{span}\{f_1 + I, \dots, f_n + I\}$. The following two quantities are important to study S^μ . For $1 \leq j \leq n$,

$$\rho_j = \sup\{|\tau_j(f)| ; f \in \bigcap_{l \neq j} \ker \tau_l, \|f\| \leq 1\}$$

and

$$\rho_j(\mu) = \sup\{|\tau_j(f)| ; f \in \bigcap_{l \neq j} \ker \tau_l, \|f\|_\mu \leq 1\},$$

where $\|f\|$ denotes the supnorm of f in A and $\|f\|_\mu = \langle f, f \rangle_\mu^{1/2} = (\int |f|^2 d\mu)^{1/2}$. Then it is easy to see that

$$\|f_j + I\| = \frac{1}{\rho_j}, \quad \|f_j + I\|_\mu = \frac{1}{\rho_j(\mu)},$$

and

$$\|f_j + I\| \geq \|S_{f_j}^\mu\| \geq \|f_j + I\|_\mu.$$

If S^μ is one to one then τ_j has a bounded extension to $H^2(\mu)$. In fact, if S^μ is one to one then $\dim H^2(\mu) \cap I^\perp = n$ and so $\dim H^2(\mu) \cap (\ker \tau_j)^\perp = 1$ for $1 \leq j \leq n$. Then for $1 \leq j \leq n$, there exists $k_j \in H^2(\mu)$ such that

$$\tau_j(f) = \langle f, k_j \rangle_\mu = \int_X f \bar{k}_j d\mu, \quad (f \in A).$$

Proposition 3.1. *There exists a one to one contractive representation S^μ of A/I .*

Proof. Since $\tau_j \in M(A)$, there exists a positive representing measure m_j of τ_j on X . Let $\mu = \sum_{j=1}^n m_j/n$. Then

$$|\tau_j(f)| = \left| \int_X f dm_j \right| \leq n \left(\int_X |f|^2 d\mu \right)^{1/2} = n \|f\|_\mu, \quad (f \in A).$$

Hence τ_j has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$ and $\tilde{\tau}_j \neq \tilde{\tau}_i$ ($j \neq i$). Since $H^2(\mu) \cap I^\perp = \bigcap_{j=1}^n \ker \tilde{\tau}_j$, $\dim H^2(\mu) \cap I^\perp = n$. Hence $H^2(\mu) \cap I^\perp = \text{span}\{k_1, \dots, k_n\}$. Suppose $S_f^\mu = 0$, then $\tau_j(f)k_j = (S_f^\mu)^*k_j = 0$. Hence $\tau_j(f) = 0$, ($1 \leq j \leq n$). Hence $f \in \bigcap_j \ker \tau_j = I$. This implies that S^μ is one to one from A/I to $B(H^2(\mu) \cap I^\perp)$. This completes the proof.

Theorem 3.2. *Suppose that S^μ is a one to one contractive representation of A/I . Let k_j be a function in $H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_\mu$ ($f \in A$), for $1 \leq j \leq n$. Then*

- (1) $H^2(\mu) \cap I^\perp = \text{span}\{k_1, \dots, k_n\}$ and $H^2(\mu) \cap I_j^\perp = \text{span}\{k_j\}$ where $I_j = \ker \tau_j$.
- (2) If $m_j = \|k_j\|_\mu^{-2} |k_j|^2 d\mu$ and $m = \sum_{j=1}^n m_j/n$, then m_j is a representing measure for τ_j for each $1 \leq j \leq n$, and we may assume that μ is absolutely continuous with respect to m .
- (3) $\|S_{f_j}^\mu\| = \|k_j\|_\mu \|f_j + I\|_\mu$ for $1 \leq j \leq n$.

Proof. (1) Since S^μ is one to one, τ_j has a unique bounded extension $\tilde{\tau}_j$ to $H^2(\mu)$. In fact, if S^μ is one to one then $\dim H^2(\mu) \cap I^\perp = n$ and so $\dim H^2(\mu) \cap (\ker \tau_j)^\perp = 1$ for $1 \leq j \leq n$. Then there exists $k_j \in H^2(\mu)$ such that $\tau_j(f) = \langle f, k_j \rangle_\mu$ ($f \in A$). If $g \in I$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I^\perp$ for each j . Since $k_1, \dots, k_n$ are linearly independent, $\{k_1, \dots, k_n\}$ is a basis of $H^2(\mu) \cap I^\perp$. If $g \in I_j$, then $0 = \tau_j(g) = \langle g, k_j \rangle$ and so $k_j \perp g$. Thus $k_j \in H^2(\mu) \cap I_j^\perp$ for each j . Hence k_j is a basis of $H^2(\mu) \cap I_j^\perp$.

(2) For $1 \leq j \leq n$,

$$\int_X f dm_j = \int_X f \frac{|k_j|^2}{\|k_j\|_\mu^2} d\mu = \frac{\langle f k_j, k_j \rangle_\mu}{\|k_j\|_\mu^2} = \frac{\tilde{\tau}_j(f k_j)}{\|k_j\|_\mu^2} = \frac{\tau_j(f) \tilde{\tau}_j(k_j)}{\|k_j\|_\mu^2} = \tau_j(f), \quad (f \in A).$$

Hence m_j is a representing measure for τ_j . Let $\mu = \mu^a + \mu^s$ be a Lebesgue decomposition by m . Then $H^2(\mu^a) \cap I^\perp = H^2(\mu) \cap I^\perp$ and so $H^2(\mu^s) \cap I^\perp = \{0\}$ where μ^a and μ^s are divided by their total masses. Hence $S_f^\mu = S_f^{\mu^a} \oplus S_f^{\mu^s} = S_f^{\mu^a} \oplus 0$ and so $\|S_f^\mu\| = \|S_f^{\mu^a}\|$ for $f \in A$.

(3) Since $\text{rank}(S_{f_j}^\mu)^* = 1$, there exists $x_j \in H^2(\mu) \cap I^\perp$ such that $(S_{f_j}^\mu)^* \phi = \langle \phi, x_j \rangle k_j = (k_j \otimes x_j) \phi$, ($\phi \in H^2(\mu) \cap I^\perp$). Then $\|S_{f_j}^\mu\| = \|(S_{f_j}^\mu)^*\| = \|k_j \otimes x_j\| = \|k_j\|_\mu \|x_j\|_\mu$. Let P be the orthogonal projection from $H^2(\mu)$ onto $H^2(\mu) \cap I^\perp$. Then

$$\langle P f_j, \phi \rangle = \langle S_{f_j}^\mu 1, \phi \rangle = \langle 1, (S_{f_j}^\mu)^* \phi \rangle = \langle x_j, \phi \rangle \langle 1, k_j \rangle = \langle x_j, \phi \rangle, \quad (\phi \in H^2(\mu) \cap I^\perp),$$

because $\langle 1, k_j \rangle = 1$. Hence $P f_j = x_j$. Hence

$$\|f_j + I\|_\mu = \|P f_j\|_\mu = \|x_j\|_\mu.$$

Hence $\|S_{f_j}^\mu\| = \|k_j\|_\mu \|f_j + I\|_\mu$. This completes the proof.

Let $G(\tau)$ denote the Gleason part of τ . If $G(\tau_i) = G(\tau_j)$, then we write $\tau_i \sim \tau_j$.

Proposition 3.3. *Suppose that $\dim H^2(\mu) \cap I^\perp = n$, $I^1 = \bigcap_{j \in N^1} \ker \tau_j$, $I^2 = \bigcap_{j \in N^2} \ker \tau_j$, $N^1 \cap N^2 = \emptyset$ and $N^1 \cup N^2 = \{1, 2, \dots, n\}$. Let $\#N^j$ denote the number of elements in N^j . If $\tau_j \not\sim \tau_k$ whenever $j \in N^1$ and $k \in N^2$, then $H^2(\mu) = H^2(\mu^1) \oplus H^2(\mu^2)$, $H^2(\mu) \cap I^\perp = (H^2(\mu^1) \cap (I^1)^\perp) \oplus (H^2(\mu^2) \cap (I^2)^\perp)$, $S_\phi^\mu = S_\phi^{\mu^1} \oplus S_\phi^{\mu^2}$ and $\dim H^2(\mu^j) \cap (I^j)^\perp = \#N^j$, where $\mu = \frac{\mu^1 + \mu^2}{2}$, $\mu^1 \perp \mu^2$ and μ^j is a probability measure for $j = 1, 2$.*

Proof. By (1) of Theorem 3.2, $H^2(\mu) \cap I^\perp = \text{span}\{k_1, \dots, k_n\}$. We may assume that $N^1 = \{1, 2, \dots, l\}$ and $N^2 = \{l+1, \dots, n\}$. By (2) of Theorem 3.2, $m_j = \|k_j\|_\mu^{-2} |k_j|^2 d\mu$ is a representing measure for τ_j for each $1 \leq j \leq n$. Put $\lambda^1 = \frac{1}{l} \sum_{j=1}^l m_j$ and $\lambda^2 = \frac{1}{n-l} \sum_{j=l+1}^n m_j$ then $\lambda^1 \perp \lambda^2$ by definitions of N^1 and N^2 . Let $\mu = \mu_0^1 + \mu_0^2$ be a Lebesgue decomposition with respect to λ^1 such that $\mu_0^1 \ll \lambda^1$ and $\mu_0^2 \perp \lambda^1$. Put $\mu^1 = \mu_0^1 / \|\mu_0^1\|$ and $\mu^2 = \mu_0^2 / \|\mu_0^2\|$. This completes the proof.

4. Isometric representation

In this section, we assume that A/I is n -dimensional and semisimple. Hence there exist $\tau_1, \dots, \tau_n$ in the maximal ideal space $M(A)$ of A such that $\tau_i \neq \tau_j$ ($i \neq j$) and $I = \bigcap_{j=1}^n \ker \tau_j$. For $1 \leq j \leq n$, there exist $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \text{span}\{f_1 + I, \dots, f_n + I\}$. If S^μ is an isometric representation of A/I , then $\|S_{f_j}^\mu\| = \|f_j + I\|$ for $1 \leq j \leq n$. By (3) of Theorem 3.2, this implies that $\|f_j + I\| = \|k_j\|_\mu \|f_j + I\|_\mu$. Hence, if S^μ is an isometric representation of A/I , then $\|k_j\|_\mu = \|f_j + I\| / \|f_j + I\|_\mu$ for $1 \leq j \leq n$. Is the converse of this statement true? If $n = 2$, then the answer will be given in Proposition 4.4.

Theorem 4.1. *Suppose that $G(\tau_i) \cap G(\tau_j) \cap G(\tau_l) = \emptyset$ if i, j and l are different from each other. Then there exists an isometric representation S^μ of A/I .*

Proof. By Proposition 3.3, if $G(\tau_j) = \{\tau_j\}$, for all $1 \leq j \leq n$, then there exists an isometric representation S^{μ^j} of A/I_j where $I_j = \ker \tau_j$, and $\mu^i \perp \mu^j$. If $\mu = (\mu^1 + \dots + \mu^n)/n$, then $H^2(\mu) \cap I^\perp = (H^2(\mu^1) \cap I^\perp) \oplus \dots \oplus (H^2(\mu^n) \cap I^\perp)$ and $S_f^\mu = S_f^{\mu^1} \oplus \dots \oplus S_f^{\mu^n}$ ($f \in A$). Therefore, the theorem is proved in the case when $G(\tau_j) = \{\tau_j\}$, for all $1 \leq j \leq n$. It is sufficient to prove the theorem when $\tau_i \sim \tau_j$ for some i, j ($i \neq j$). Suppose $\tau_{2k-1} \sim \tau_{2k}$, ($1 \leq k \leq n_0$) and $G(\tau_l) = \{\tau_l\}$, ($2n_0 + 1 \leq l \leq n$) for some n_0 . Since $G(\tau_i) \cap G(\tau_j) \cap G(\tau_l) = \emptyset$, it follows that $\dim A/I_{ij} = 2$ where $I_{ij} = I_i \cap I_j = \ker \tau_i \cap \ker \tau_j$. By Corollary 1 in [9], there is a probability measure μ^{ij} such that $\|S_f^{\mu^{ij}}\| = \|f + I_{ij}\|$ for all $f \in A$. By Proposition 3.3, there are probability measures $\mu^{2k-1, 2k}$, ($1 \leq k \leq n_0$) and μ^l , ($2n_0 + 1 \leq l \leq n$) such that $\mu = (\mu^{12} + \mu^{34} + \dots + \mu^{2n_0-1, 2n_0} + \mu^{2n_0+1} + \dots + \mu^n)/(n - n_0)$, $H^2(\mu) \cap I^\perp = (H^2(\mu^{12}) \cap I_{12}^\perp) \oplus \dots \oplus (H^2(\mu^{2n_0-1, 2n_0}) \cap I_{2n_0-1, 2n_0}^\perp) \oplus (H^2(\mu^{2n_0+1}) \cap I_{2n_0+1}^\perp) \oplus \dots \oplus (H^2(\mu^n) \cap I_n^\perp)$, $S_f^\mu = S_f^{\mu^{12}} \oplus \dots \oplus S_f^{\mu^{2n_0-1, 2n_0}} \oplus S_f^{\mu^{2n_0+1}} \oplus \dots \oplus S_f^{\mu^n}$. Hence S^μ is an isometric representation of A/I .

where $I = (\cap_{k=1}^{n_0} I_{2k-1, 2k}) \cap (\cap_{l=2n_0+1}^n I_l)$. This completes the proof.

For example, we consider when $n = 3$ and $\tau_1 \sim \tau_2 \not\sim \tau_3$. Let $I_{12} = I_1 \cap I_2 = \ker \tau_1 \cap \ker \tau_2$. Then $\dim A/I_{12} = 2$. By Corollary 1 in [9], there is a probability measure μ^{12} such that $\|S_f^{\mu^{12}}\| = \|f + I_{12}\|$ for all $f \in A$. Let S^{μ^3} be the isometric representation of A/I_3 where $I_3 = \ker \tau_3$. Let $\mu = (\mu^{12} + \mu^3)/2$. Then $\mu^{12} \perp \mu^3$, $H^2(\mu) \cap I^\perp = (H^2(\mu^{12}) \cap I_{12}^\perp) \oplus (H^2(\mu^3) \cap I_3^\perp)$, $S_f^\mu = S_f^{\mu^{12}} \oplus S_f^{\mu^3}$, $(f \in A)$, $(S_f^{\mu^{12}})^* k_j = \overline{\tau_j(f)} k_j$, $(j = 1, 2)$, and $(S_f^{\mu^3})^* k_3 = \overline{\tau_3(f)} k_3$. Hence

$$\|S_f^\mu\| = \max(\|S_f^{\mu^{12}}\|, \|S_f^{\mu^3}\|) = \max(\|f + I_{12}\|, |\tau_3(f)|) = \sup_{\nu \in (A/I)^*, \|\nu\| \leq 1} \left| \int_X f d\nu \right| = \|f + I\|.$$

Hence S^μ is an isometric representation of A/I where $I = I_{12} \cap I_3$. By the theorem of T. Nakazi (cf. [8]), $\|f + I_{12}\|$ can be written using $\rho_1 = \sup\{|\tau_1(f)| ; f \in \ker \tau_2, \|f\| \leq 1\}$.

Corollary 4.2. *Let A be a uniform algebra and $I = \cap_{j=1}^n \ker \tau_j$ and $\tau_i \not\sim \tau_j (i \neq j)$. Then there exists an isometric representation S^μ of A/I , and $\|f + I\| = \max(|\tau_1(f)|, \dots, |\tau_n(f)|)$.*

Proof. Since $\tau_i \not\sim \tau_j (i \neq j)$, there exist probability measures $\mu^1, \dots, \mu^n$ such that $\mu = (\mu^1 + \dots + \mu^n)/n$, $\mu^i \perp \mu^j (i \neq j)$, $H^2(\mu) \cap I^\perp = (H^2(\mu^1) \cap I^\perp) \oplus \dots \oplus (H^2(\mu^n) \cap I^\perp)$, $S_f^\mu = S_f^{\mu^1} \oplus \dots \oplus S_f^{\mu^n}$. Since $(S_f^{\mu^j})^* k_j = \overline{\tau_j(f)} k_j$, and $(S_f^{\mu^j})^*$ is a rank 1 operator on $H^2(\mu) \cap (\ker \tau_j)^\perp = \text{span } \{k_j\}$, it follows that $\|S_f^{\mu^j}\| = \|(S_f^{\mu^j})^*\| = |\tau_j(f)|$. Then

$$\|S_f^\mu\| = \max(\|S_f^{\mu^1}\|, \dots, \|S_f^{\mu^n}\|) = \max(|\tau_1(f)|, \dots, |\tau_n(f)|) = \sup_{\nu \in (A/I)^*, \|\nu\| \leq 1} \left| \int_X f d\nu \right| = \|f + I\|.$$

This completes the proof.

Corollary 4.3. *Let A be a uniform algebra and $I = \cap_{j=1}^n \ker \tau_j$ and $\tau_i \not\sim \tau_j (i \neq j)$. Suppose that S^μ is an isometric representation of A/I . Then,*

- (1) $\mu = \sum_{j=1}^n \mu^j$, $\mu^i \perp \mu^j (i \neq j)$, $\mu^j \ll m^j$ where μ^j is a positive measure and m^j is some representing measure for τ_j .
- (2) $S_f^\mu = \sum_{j=1}^n S_f^{\mu^j} (f \in A)$ where μ^j is divided by its total variation and S^{μ^j} is an isometric representation of A/I_j , where $I_j = \ker \tau_j$.
- (3) S_f^μ is an isometric representation of a diagonal $n \times n$ matrix for any f in A .

Proof. By the proof of (2) of Theorem 3.2 and Theorem 4.1, (1), (2) and (3) holds.

If A/I is 2-dimensional and semisimple, then there exist τ_1, τ_2 in $M(A)$ such that $\tau_1 \neq \tau_2$ and $I = \ker \tau_1 \cap \ker \tau_2$. For $j = 1, 2$, there exists $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Then $f_j + I$ is idempotent in A/I and $A/I = \text{span}\{f_1 + I, f_2 + I\}$. If $n = 2$, then

$$\rho_1 = \sup\{|\tau_1(f)| ; f \in \ker \tau_2, \|f\| \leq 1\},$$

$$\rho_1(\mu) = \sup\{|\tau_1(f)| ; f \in \ker \tau_2, \|f\|_\mu \leq 1\}$$

where $\|f\|$ denotes the supnorm of f in A and $\|f\|_\mu = \langle f, f \rangle_\mu = (\int |f|^2 d\mu)^{1/2}$. Then ρ_1 is a Gleason distance between τ_1 and τ_2 , and $\|f_1 + I\| = 1/\rho_1$, $\|f_1 + I\|_\mu = 1/\rho_1(\mu)$. The following proposition is essentially known (cf. Lemma 3 of [9]).

Proposition 4.4. *If A/I is 2-dimensional and semisimple, then the following conditions are equivalent.*

- (1) S^μ is an isometric representation of A/I .
- (2) $\|k_1\|_\mu = \rho_1(\mu)/\rho_1$.
- (3) $\|k_1\|_\mu = \|f_1 + I\|/\|f_1 + I\|_\mu$.

Proof. By Theorem 3.2, (1) implies (3). By the above remark, (2) is equivalent to (3). It is sufficient to show that (3) implies (1). By Theorem 3.2, if (3) holds, then $\|S_{f_1}^\mu\| = \|f_1 + I\|$. By the above remark, this implies $\|S_{f_1}^\mu\| = 1/\rho_1$. By the theorem of T. Nakazi (cf. [8]), if $I = \{f \in A ; \tau_1(f) = \tau_2(f) = 0\}$, then

$$\begin{aligned} \|f + I\| &= \sqrt{\left| \frac{\tau_1(f) - \tau_2(f)}{2} \right|^2 \left(\frac{1}{\rho_1^2} - 1 \right) + \left(\frac{|\tau_1(f)| + |\tau_2(f)|}{2} \right)^2} \\ &+ \sqrt{\left| \frac{\tau_1(f) - \tau_2(f)}{2} \right|^2 \left(\frac{1}{\rho_1^2} - 1 \right) + \left(\frac{|\tau_1(f)| - |\tau_2(f)|}{2} \right)^2}. \end{aligned}$$

Since $\|S_{f_1}^\mu\| = 1/\rho_1$, it follows from the theorem of I. Feldman, N. Krupnik and A. Markus (cf. [5]) that

$$\|f + I\| = \|\tau_1(f)S_{f_1}^\mu + \tau_2(f)S_{f_2}^\mu\| = \|S_f^\mu\|.$$

This completes the proof.

T. Nakazi and K. Takahashi [9] proved that there exists an isometric representation of A/I in the case when $\dim A/I = 2$. The following theorem gives a concrete matrix representation of A/I .

Theorem 4.5. *Suppose A/I is 3-dimensional and semisimple. If $\tau_1 \sim \tau_2 \not\sim \tau_3$ and S^μ is an isometric representation of A/I , then A/I is isometric to $\{S_f^\mu ; f \in A\} = \text{span}\{S_{f_1}^\mu, S_{f_2}^\mu, S_{f_3}^\mu\}$, $S_f^\mu = \tau_1(f)S_{f_1}^\mu + \tau_2(f)S_{f_2}^\mu + \tau_3(f)S_{f_3}^\mu$, and*

$$(S_{f_1}^\mu)^* = \begin{pmatrix} 1 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^\mu)^* = \begin{pmatrix} 0 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^\mu)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$x = \frac{-\langle k_2, k_1 \rangle_\mu}{\sqrt{\|k_1\|_\mu^2 \|k_2\|_\mu^2 - |\langle k_1, k_2 \rangle_\mu|^2}}.$$

Proof. This follows from Lemma 2.1 and Theorem 4.1.

If $\mathcal{B} \subset B(H)$ and $\dim H = 3$, then

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

It follows from a 2-dimensional case that if $y = z = 0$, then $\mathcal{B}$ is a $\mathcal{Q}$ -algebra.

If the following condition (1) implies (2) for any distinct points $\tau_1, \dots, \tau_n \in M(A)$ and complex numbers $w_1, \dots, w_n$, then we say that A/I satisfies the Pick property.

- (1) $[(1 - w_i \overline{w_j})k_{ji}]_{i,j=1}^n \geq 0$, where $k_{ij} = \langle k_i, k_j \rangle_\mu$, and $\tau_j(f) = \langle f, k_j \rangle_\mu$, ($f \in A$).
- (2) There exists $f \in A$ such that $\tau_j(f) = w_j$, ($1 \leq j \leq n$) and $\|f + I\| \leq 1$.

The following proposition is essentially known.

Proposition 4.6. *Let A/I be an n -dimensional semisimple commutative Banach algebra. Then $S^\mu : A/I \rightarrow B(H^2(\mu) \cap I^\perp)$ is isometric if and only if A/I satisfies the Pick property.*

Proof. Suppose S^μ is isometric. For any $w_1, \dots, w_n \in \mathbf{C}$, there exists an $f \in A$ such that $\tau_j(f) = w_j$, ($1 \leq j \leq n$). Suppose $[(1 - w_i \overline{w_j})k_{ji}]_{i,j=1}^n \geq 0$. For any complex numbers $\alpha_1, \dots, \alpha_n$, let $k = \sum_{j=1}^n \alpha_j k_j$. Then $\|k\|_\mu^2 = \sum_{i,j=1}^n \overline{\alpha_i} \alpha_j k_{ji}$. Since $(S_f^\mu)^* k_j = \overline{\tau_j(f)} k_j$, $(S_f^\mu)^* k = \sum_{j=1}^n \alpha_j \overline{\tau_j(f)} k_j$. By (1),

$$\|k\|_\mu^2 - \|(S_f^\mu)^* k\|_\mu^2 = \sum_{i,j=1}^n \overline{\alpha_i} \alpha_j (1 - w_i \overline{w_j}) k_{ji} \geq 0.$$

Since $H^2(\mu) \cap I^\perp$ is spanned by $k_1, \dots, k_n$, this implies that $\|(S_f^\mu)^*\| \leq 1$. Since S^μ is isometric, $\|f + I\| = \|S_f^\mu\| \leq 1$. Therefore A/I satisfies the Pick property. Conversely, suppose A/I satisfies the Pick property and $\|S_f^\mu\| = 1$. Since $(S_f^\mu)^* k_j = \overline{\tau_j(f)} k_j$ and $\|(S_f^\mu)^*\| = 1$, it follows that

$$\sum_{i,j=1}^n \overline{\alpha_i} \alpha_j (1 - \tau_i(f) \overline{\tau_j(f)}) k_{ji} = \|k\|_\mu^2 - \|(S_f^\mu)^* k\|_\mu^2 \geq 0,$$

and hence $[(1 - \tau_i(f) \overline{\tau_j(f)})k_{ji}]_{i,j=1}^n \geq 0$. By the Pick property, there exists $g \in A$ such that $\|g + I\| \leq 1$ and $\tau_j(g) = \tau_j(f)$, ($1 \leq j \leq n$). Therefore $\|f + I\| = \|g + I\| \leq 1 = \|S_f^\mu\|$. Since the reverse inequality $\|S_f^\mu\| \leq \|f + I\|$ is always holds, $\|S_f^\mu\| = \|f + I\|$. This completes the proof.

5. Q -Algebras of a Disc Algebra

In this section, we assume that A is the disc algebra and $\dim A/I = 3$. For $f \in A$, let $\|f + I\| = \|f + I\|_{A/I}$. Since $M(A) = \bar{\mathbf{D}} = \{|z| \leq 1\}$, for each $1 \leq j \leq 3$, τ_j is just an evaluation functional at a point of $\bar{\mathbf{D}}$ and so we write that $\tau_1 = a$, $\tau_2 = b$ and $\tau_3 = c$, where a , b and c are in $\bar{\mathbf{D}}$. By Theorem 3.2, we may assume that a , b and c are in $\mathbf{D} = \{|z| < 1\}$. Theorem 5.2 shows that the set of all 3-dimensional semisimple Q -algebras of the disc algebra is a proper subset in the set of all 3-dimensional semisimple commutative operator algebras with unit on a Hilbert space of dimension 3. However Theorem 5.2 has not solved Problem 2 yet. We use Lemma 5.1 to prove Theorem 5.2. Let a, b, c be the distinct points in the open unit disc $\mathbf{D}$. Let $T(a, b, c)$ denote the subset of $\mathbf{C}^3$ which consists of all $(x, y, z) \in \mathbf{C}^3$ satisfying

$$1 + |x|^2 = \left| \frac{1 - \bar{b}a}{a - b} \right|^2, \quad 1 + |z|^2 = \left| \frac{1 - \bar{c}b}{b - c} \right|^2,$$

$$1 + |y|^2 \left| \frac{a - b}{1 - \bar{b}a} \right|^2 = \left| \frac{1 - \bar{a}c}{c - a} \right|^2.$$

This implies that $x \neq 0$, $y \neq 0$, and $z \neq 0$. $T(a, b, c)$ is characterized by saying that the absolute values of x, y, z are fixed and that their argument are arbitrary. In the following, we consider some inequalities of x, y , and z . For $j = 1, 2, 3$, there exists $f_j \in A$ such that $\tau_i(f_j) = \delta_{ij}$. Hence, $f_1(a) = f_2(b) = f_3(c) = 1$, and $f_1(b) = f_1(c) = f_2(a) = f_2(c) = f_3(a) = f_3(b) = 0$.

Lemma 5.1. *Let a, b, c be the distinct points in $\mathbf{D}$. Let $f \in A$. Let $I = \{g \in A ; g(a) = g(b) = g(c) = 0\}$. Let $d\mu = \frac{d\theta}{2\pi}$.*
(1) $S_f^\mu = f(a)S_{f_1}^\mu + f(b)S_{f_2}^\mu + f(c)S_{f_3}^\mu$, and

$$(S_{f_1}^\mu)^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^\mu)^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^\mu)^* = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x, y, z) \in T(a, b, c)$.

(2) $\|f + I\| = \|S_f^\mu\|$, ($f \in A$). That is, A/I is isometrically isomorphic to the 3-dimensional semisimple commutative operator algebra on $H^2(\mu) \cap I^\perp$ which is spanned by

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz - y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

for some $(x, y, z) \in T(a, b, c)$.

Proof. $H^2(\mu) \cap I^\perp$ is a 3-dimensional Hilbert space which is spanned by

$$k_1(z) = \frac{1}{1 - \bar{a}z}, \quad k_2(z) = \frac{1}{1 - \bar{b}z}, \quad k_3(z) = \frac{1}{1 - \bar{c}z}.$$

For orthonormal basis ψ_1, ψ_2, ψ_3 defined in Proposition 2.3,

$$\psi_1(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}, \quad \psi_2(z) = \gamma_2 \frac{z-a}{1-\bar{a}z} \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}, \quad \psi_3(z) = \gamma_3 \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \frac{\sqrt{1-|c|^2}}{1-\bar{c}z},$$

where

$$\gamma_2 = - \left(\frac{a-b}{1-\bar{a}b} \right)^{-1} \left| \frac{a-b}{1-\bar{a}b} \right|, \quad \gamma_3 = \left(\frac{a-c}{1-\bar{a}c} \right)^{-1} \left| \frac{a-c}{1-\bar{a}c} \right| \left(\frac{b-c}{1-\bar{b}c} \right)^{-1} \left| \frac{b-c}{1-\bar{b}c} \right|.$$

Since

$$k_2 - (k_2, \psi_1)\psi_1 = \frac{(\bar{b}-\bar{a})(z-a)}{(1-\bar{b}a)(1-\bar{a}z)(1-\bar{b}z)},$$

it follows that

$$\|k_2 - (k_2, \psi_1)\psi_1\| = \left| \frac{\bar{b}-\bar{a}}{1-\bar{b}a} \right| \frac{1}{\sqrt{1-|b|^2}}.$$

Hence

$$\psi_2 = \frac{k_2 - (k_2, \psi_1)\psi_1}{\|k_2 - (k_2, \psi_1)\psi_1\|} = \gamma_2 \frac{z-a}{1-\bar{a}z} \frac{\sqrt{1-|b|^2}}{1-\bar{b}z}.$$

Since

$$k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2 = \frac{(\bar{a}-\bar{c})(\bar{b}-\bar{c})(z-a)(z-b)}{(1-\bar{c}a)(1-\bar{c}b)(1-\bar{a}z)(1-\bar{b}z)(1-\bar{c}z)},$$

it follows that

$$\psi_3 = \frac{k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|} = \gamma_3 \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \frac{\sqrt{1-|c|^2}}{1-\bar{c}z}.$$

If we calculate x, y, z using the formulas in Proposition 2.3, then it follows that $(x, y, z) \in T(a, b, c)$. Then

$$x = \frac{-\langle k_2, k_1 \rangle}{\sqrt{\|k_1\|^2 \|k_2\|^2 - |\langle k_1, k_2 \rangle|^2}} = \frac{\frac{-1}{1-\bar{a}b}}{\sqrt{\frac{1}{(1-|a|^2)(1-|b|^2)} - \frac{1}{|1-\bar{a}b|^2}}} = \gamma_4 \frac{\sqrt{1-|a|^2} \sqrt{1-|b|^2}}{|a-b|},$$

where

$$\gamma_4 = - \frac{1-\bar{a}b}{|1-\bar{a}b|}.$$

Hence

$$1 + |x|^2 = \left| \frac{1-\bar{a}b}{a-b} \right|^2.$$

Since

$$-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x = \frac{\sqrt{1-|a|^2}}{1-\bar{c}a} \frac{1-\bar{a}b}{\bar{b}-\bar{a}} \frac{\bar{c}-\bar{b}}{1-\bar{b}c},$$

it follows that

$$y = \frac{-\langle k_3, \psi_1 \rangle - \langle k_3, \psi_2 \rangle x}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|} = \gamma_5 \frac{1-\bar{a}b}{\bar{a}-\bar{b}} \frac{\sqrt{1-|a|^2} \sqrt{1-|c|^2}}{|a-c|},$$

where

$$\gamma_5 = \left(\frac{a-b}{1-a\bar{b}} \right) \left| \frac{a-b}{1-a\bar{b}} \right|^{-1} \left(\frac{b-c}{1-b\bar{c}} \right)^{-1} \left| \frac{b-c}{1-b\bar{c}} \right| \frac{|1-\bar{c}a|}{1-\bar{c}a}.$$

Since

$$\langle k_3, \psi_2 \rangle = \frac{\bar{c} - \bar{a}}{\gamma_2} \frac{\sqrt{1-|b|^2}}{1-a\bar{c}},$$

it follows that

$$z = \frac{-\langle k_3, \psi_2 \rangle}{\|k_3 - \langle k_3, \psi_1 \rangle \psi_1 - \langle k_3, \psi_2 \rangle \psi_2\|} = \gamma_6 \frac{\sqrt{1-|b|^2} \sqrt{1-|c|^2}}{|b-c|},$$

where

$$\gamma_6 = \left(\frac{a-b}{1-a\bar{b}} \right) \left| \frac{a-b}{1-a\bar{b}} \right|^{-1} \left(\frac{c-a}{1-a\bar{c}} \right)^{-1} \left| \frac{c-a}{1-a\bar{c}} \right| \frac{|1-b\bar{c}|}{1-b\bar{c}}.$$

Since $|\gamma_2| = |\gamma_3| = |\gamma_4| = |\gamma_5| = |\gamma_6| = 1$, it follows that

$$1 + |y|^2 \left| \frac{a-b}{1-\bar{b}a} \right|^2 = \left| \frac{1-\bar{a}c}{c-a} \right|^2, \quad 1 + |z|^2 = \left| \frac{1-\bar{c}b}{b-c} \right|^2.$$

Hence, (1) follows. It is sufficient to prove (2). By the theorem of D. Sarason (cf. [2], p.125, [10], Vol.1, p.231, [11]), $\|f + I\| = \|S_f^\mu\|$. Then $(S_{f_1}^\mu)^* k_1 = k_1$, $(S_{f_1}^\mu)^* k_2 = (S_{f_1}^\mu)^* k_3 = 0$, $(S_{f_2}^\mu)^* k_2 = k_2$, $(S_{f_2}^\mu)^* k_3 = (S_{f_2}^\mu)^* k_1 = 0$, $(S_{f_3}^\mu)^* k_3 = k_3$, and $(S_{f_3}^\mu)^* k_1 = (S_{f_3}^\mu)^* k_2 = 0$. By Proposition 2.3,

$$(S_{f_1}^\mu)^* = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_2}^\mu)^* = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad (S_{f_3}^\mu)^* = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $f - f(a)f_1 - f(b)f_2 - f(c)f_3 \in I$ and $I(H^2(\mu) \cap I^\perp) \subset IH^2(\mu) \subset H^2(\mu) \cap I^\perp$, it follows that

$$(S_f^\mu - S_{f(a)f_1+f(b)f_2+f(c)f_3}^\mu)\psi = S_{f-f(a)f_1-f(b)f_2-f(c)f_3}^\mu\psi = 0, \quad (\psi \in I_\mu^\perp).$$

Hence

$$S_f^\mu = S_{f(a)f_1+f(b)f_2+f(c)f_3}^\mu = f(a)S_{f_1}^\mu + f(b)S_{f_2}^\mu + f(c)S_{f_3}^\mu.$$

This completes the proof.

For example, if $(a, b, c) = (0, \frac{1}{2}, \frac{1}{3})$ and $(x, y, z) = (-\sqrt{3}, 4\sqrt{2}, -2\sqrt{6})$, then the algebra $\text{span}\{P_1, P_2, P_3\}$ is isometrically isomorphic to A/I which is a Q -algebra of a disc algebra.

Theorem 5.2. *Let a, b, c be the distinct points in $\mathbf{D}$. Let $f \in A$. Let $d\mu = \frac{d\theta}{2\pi}$. Let $I = \{g \in A ; g(a) = g(b) = g(c) = 0\}$. If a 3-dimensional semisimple commutative operator algebra $\mathcal{B}$ on $H^2(\mu) \cap I^\perp$ is isometrically isomorphic to A/I , then $\mathcal{B}$ is unitarily equivalent to the 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space H spanned by P_1, P_2, P_3 such that*

$$P_1 = \begin{pmatrix} 1 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & -x & -xz \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & xz-y \\ 0 & 0 & -z \\ 0 & 0 & 1 \end{pmatrix},$$

where x, y, z satisfy (1) $\sim$ (3).

$$(1) \quad xyz \neq 0,$$

$$(2) \quad \frac{1}{\sqrt{1+|y|^2}} < \frac{1}{\sqrt{1+|x|^2}} + \frac{1}{\sqrt{1+|z|^2}},$$

$$(3) \quad |y| > \frac{|xz|}{\sqrt{1+|z|^2}+1},$$

Proof. By the theorem of B. Cole and J. Wermer (cf. [3]) and (2) of Theorem 2.4, we may assume that H is spanned by the orthonormal basis ψ_1, ψ_2, ψ_3 which are calculated in the proof of Lemma 5.1. By Lemma 5.1, there are complex numbers x, y, z satisfying $(x, y, z) \in T(a, b, c)$. Since

$$1+|x|^2 = \left| \frac{1-\bar{b}a}{a-b} \right|^2 > 1, \quad 1+|z|^2 = \left| \frac{1-\bar{c}b}{b-c} \right|^2 > 1,$$

$$1+|y|^2 \left| \frac{a-b}{1-\bar{b}a} \right|^2 = \left| \frac{1-\bar{a}c}{c-a} \right|^2 > 1,$$

(1) follows. Let

$$\rho(z, w) = \left| \frac{z-w}{1-\bar{w}z} \right|.$$

Then

$$\rho(a, b) = \frac{1}{\sqrt{1+|x|^2}}, \quad \rho(b, c) = \frac{1}{\sqrt{1+|z|^2}}, \quad \rho(c, a) = \sqrt{\frac{1+|x|^2}{1+|x|^2+|y|^2}} > \frac{1}{\sqrt{1+|y|^2}}.$$

Since $\rho(c, a) \leq \rho(a, b) + \rho(b, c)$, (2) follows. Let

$$d(z, w) = \frac{1}{2} \log \frac{1+\rho(z, w)}{1-\rho(z, w)}.$$

Since $d(c, a) \leq d(a, b) + d(b, c)$,

$$\frac{\sqrt{1+|x|^2+|y|^2} + \sqrt{1+|x|^2}}{\sqrt{1+|x|^2+|y|^2} - \sqrt{1+|x|^2}} \leq \frac{\sqrt{1+|z|^2}+1}{\sqrt{1+|z|^2}-1} \cdot \frac{\sqrt{1+|x|^2}+1}{\sqrt{1+|x|^2}-1}.$$

Hence

$$\frac{\sqrt{1+|x|^2}+1}{|y|} < \frac{\sqrt{1+|x|^2+|y|^2} + \sqrt{1+|x|^2}}{|y|} \leq \frac{\sqrt{1+|z|^2}+1}{|z|} \cdot \frac{\sqrt{1+|x|^2}+1}{|x|}.$$

this implie (3). This completes the proof.

Example 5.3. In Example 2.5, $\mathcal{B}_0$ is isometrically isomorphic to $\mathcal{B}_2$. Since $y_0 = 0$, it follows from Theorem 5.2 that $\mathcal{B}_2$ is not isometrically isomorphic to a 3-dimensional semisimple Q -algebra A/I where A is a disc algebra. Hence $\mathcal{B}_0$ is also not isometrically isomorphic to a Q -algebra A/I . Therefore $\mathcal{B}_0$ and $\mathcal{B}_2$ is the example to show that the set of all 3-dimensional semisimple Q -algebra A/I where A is a disc algebra is smaller than the set of all 3-dimensional commutative operator algebras with unit on a 3-dimensional Hilbert space.

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