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UNIQUENESS OF CONSTANT WEAKLY ANISOTROPIC MEAN CURVATURE IMMERSION OF SPHERE S^2 IN $\mathbb{R}^3$

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ABSTRACT. We prove that the constant anisotropic mean curvature immersion of sphere S^2 in $\mathbb{R}^3$ is unique, provided that anisotropy is weak in the sense that the energy density function is close to isotropic one.

1. Introduction

G. Wulff (1901)[W] formulated the following generalized isoperimetric problem "Find a set minimizing the surface energy with fixed volume" and conjectured that the answer is a dilation of the Wulff shape. A. Dinghas (1944)[D] gave a formal proof. J. Taylor (1978)[T] gave a rigorous proof for very general surface energies based on geometric measure theory. The minimizer of the generalized isoperimetric problem (Wulff's problem) is also unique up to translation (see [DP], [F], [FM], [KoP], [M] and [S] for detail). The anisotropic mean curvature of the boundary of the Wulff shape is constant([So], [G]). However, the converse problem seems to be open unless the energy is isotropic. The following problem (uniqueness conjecture of Wulff shape) is proposed by the first author [G] in 2002 and also by F. Morgan [M]: if an embedded compact hypersurface has a constant anisotropic mean curvature, is the hypersurface a boundary of the Wulff shape up to translation and dilation?

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It is well known that in the isotropic case, only round sphere is the constant mean curvature embedded compact surface ([A]) in any dimensional Euclidean space and the constant mean curvature immersion of sphere S^2 in $\mathbb{R}^3$ (proved by Hopf in 1951, see [Ho]). In the proof of Aleksandrov(1956), the isotropy is essential to use the moving plane method. On the other hand, in higher dimensions, the result for immersed case fails even for the isotropic case, as shown by the constant mean curvature immersions of Wente and Kapouleas([We], [K]). In this paper, we consider a similar problem where the surfaces have anisotropic structure depending on their normal direction. Particularly, we give a positive answer to the uniqueness conjecture of Wulff shape for the immersions of sphere if the surface energy density is close to isotropic one.

Related to our result, Morgan [M] proved that in $\mathbb{R}^2$, an immersed closed rectifiable curve in equilibrium for fixed area must be the Wulff shape, where smoothness assumptions on the anisotropic structure are not needed. A local version of constant anisotropic mean curvature (CAMC) curve with boundary in plane is considered by Mucha and Rybka in [MR]. They proved that the local CAMC curves can be glued up and a uniquely defined closed curve can be obtained and up to a translation it is the Wulff shape.

Note that the mean curvature is the change ratio of surface area per change of volume enclosed by the surface. If the surface has anisotropic structure, it is natural to consider surface energy instead of area. Let γ denote the anisotropic structure of a surface M . Suppose $\gamma : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ satisfies

$$(1.1) \quad \begin{aligned} & (1) \text{twice differentiable in } \mathbb{R}^3 \setminus \{0\}; \\ & (2) \text{positive homogeneity of degree 1 : } \forall r \geq 0, \quad \gamma(rp) = r\gamma(p); \\ & (3) \text{strict convexity : } \exists c_0^\gamma > 0, \text{ s.t. } \forall p \in S^2, \forall \eta \in \mathbb{R}^3 \setminus \{0\}, \text{ if } \eta \cdot p = 0, \text{ then} \\ & \quad \eta \cdot (\eta \nabla_p^2 \gamma(p)) \geq c_0^\gamma |\eta|^2. \end{aligned}$$

These conditions are equivalent to say that the Frank diagram

$$\text{Frank}\gamma = \{p \in \mathbb{R}^3 : \gamma(p) \leq 1\}$$

has C^2 boundary with positive principal curvatures(see [G]).

The restriction $\gamma|_{S^2}$ of the function γ on the image of the Gauss map n of the surface M , is called the surface energy density and the surface energy of M is

defined by

$$\int_M \gamma(n) d\sigma,$$

where $d\sigma$ denotes the surface element. The change ratio of the surface energy per change of volume enclosed by M is called the anisotropic mean curvature of M , which is a generalization of the mean curvature.

Suppose $G \subset \mathbb{R}^2$ is a domain,

$$x(u, v) : (u, v) \in G \mapsto x(u, v) \in \mathbb{R}^3$$

and $M := \{x(u, v) : \forall (u, v) \in G\}$ is a surface in $\mathbb{R}^3$. Moreover, suppose x is conformal, i.e.

$$(1.2) \quad x_u \cdot x_v = 0, \quad |x_u|^2 = |x_v|^2.$$

To obtain the constant anisotropic mean curvature (CAMC) equations, we consider

$$(1.3) \quad \begin{aligned} \mathcal{H}(\Phi_t(M)) &= \int_{\Phi_t(M)} \gamma(n_{\Phi_t(M)}(y)) d\mathcal{H}^2(y) \\ &= \int_G \gamma(n_{\Phi_t(M)}(y(u, v))) |y_u \times y_v| dudv \end{aligned}$$

where for any smooth vector field $X(x)$ of $\mathbb{R}^3$ satisfying $X(x) = 0$ on ∂G ,

$$\Phi_t(M) := \{x + tX(x) : \forall x \in M\},$$

and $n_{\Phi_t(M)}(y)$ is the unit normal vector of surface $\Phi_t(M)$ at $y \in \Phi_t(M)$. We use $n(x)$ denoting the unit normal vector of surface M at $x \in M$.

By direct calculation, we have

$$(1.4) \quad \begin{aligned} &\frac{d}{dt} \Big|_{t=0} \mathcal{H}(\Phi_t(M)) \\ &= \int_G \nabla_p \gamma(n(x(u, v))) \cdot [X_u(x(u, v)) \times x_v(u, v) + x_u(u, v) \times X_v(x(u, v))] dudv \\ &= (-1) \int_G (X(x(u, v)) \cdot n(x(u, v))) H(x(u, v)) |x_u(u, v) \times x_v(u, v)| dudv, \end{aligned}$$

where the anisotropic mean curvature of surface M at x is defined by

$$(1.5) \quad H(x(u, v)) = (-1) \frac{x_u(u, v) \times \partial_v \nabla_p \gamma(n(x(u, v))) - x_v(u, v) \times \partial_u \nabla_p \gamma(n(x(u, v)))}{|x_u(u, v) \times x_v(u, v)|}.$$

Notice that this quantity can be expressed in concise form as in [G, p.61]

$$H(x) = -\operatorname{div}_M(\nabla_p \gamma(n(x))),$$

where div_M is the surface divergence. Denote

$$(1.6) \quad \gamma(p) = |p| + \epsilon \gamma_1(p), \quad \forall p \in \mathbb{R}^3.$$

Then (1.5) can be written as

$$(1.7) \quad \frac{1}{2} \Delta x = H(x)(x_u \times x_v) + \epsilon(x_u \times \partial_v \nabla_p \gamma_1(n(x)) - x_v \times \partial_u \nabla_p \gamma_1(n(x))), \quad \forall (u, v) \in G.$$

Suppose $H(x) \equiv 1$ on M . We get the CAMC equations

$$(1.8) \quad (x_u \times \partial_v \nabla_p \gamma(n(x)) - x_v \times \partial_u \nabla_p \gamma(n(x))) + x_u \times x_v = 0, \quad \forall (u, v) \in G.$$

It can also be written as

$$\frac{1}{2} \Delta x = x_u \times x_v + \epsilon(x_u \times \partial_v \nabla_p \gamma_1(n(x)) - x_v \times \partial_u \nabla_p \gamma_1(n(x))), \quad \forall (u, v) \in G.$$

If $\epsilon = 0$, (1.8) is the CMC equations. It is well known that only round sphere is the embedding compact CMC surface or the constant mean curvature immersion of sphere S^2 in $\mathbb{R}^3$, and Wente torus are immersed CMC surfaces in $\mathbb{R}^3$. In this paper we shall use the global inverse function theorem to prove the similar results for small ϵ .

Main Theorem. *Suppose*

$$\inf_{|p|=1} \gamma(p) \geq c_1^\gamma (> 0), \quad \sup_{|p|=1} |\nabla_p \gamma_1(p)| \leq c_3^\gamma, \quad \sup_{|p|=1} |\nabla_p^2 \gamma_1(p)| \leq c_4^\gamma.$$

There is $r_0 = r_0(c_0^\gamma, c_3^\gamma, c_4^\gamma) (> 0)$ depending only on c_0^γ , c_3^γ and c_4^γ such that if

$$-r_0 < \epsilon < r_0$$

then the CAMC immersion of sphere S^2 in $\mathbb{R}^3$ is unique.

Remark. Koiso and Palmer [KoP] have showed that for a large class of rotationally symmetric energy functionals, the only stable equilibria supported on parallel

planes are either cylinders or a part of the Wulff shape. Particularly, in the case of $\gamma(p) = \gamma(p_3)$ for all $p \in S^2$, the only closed embedded CAMC surface is the Wulff shape up to translation and homothety. In [KoP1], they proved that except for CMC surfaces, there exist no closed surfaces with non zero volume whose mean curvature is a linear function of their height above a horizontal plane. In [Z], without the assumption of the weak anisotropy the uniqueness of closed embedded CAMC surface in R^3 is proved.

Let us explain the strategies of the proof. Define

$$(1.9) \quad P(x) = \left\{ \frac{1}{2} \Delta x - x_u \times x_v \right\} : \left\{ (x_u \times \partial_v \nabla_p \gamma_1(n(x)) - x_v \times \partial_u \nabla_p \gamma_1(n(x))) \right\}.$$

Here for any two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$,

$$a : b = (a_1/b_1, a_2/b_2, a_3/b_3).$$

Denote

$$(1.10) \quad \begin{aligned} P(x) &= a(x) : b(x), \\ a(x) &= \frac{1}{2} \Delta x - x_u \times x_v \\ b(x) &= (x_u \times \partial_v \nabla_p \gamma_1(n(x)) - x_v \times \partial_u \nabla_p \gamma_1(n(x))). \end{aligned}$$

Note that from (1.2) and (1.1)(2),

$$a(x) // b(x) // (x_u \times x_v).$$

If $P(x)$ is restricted on conformal map x , then $P(x)$ can be regarded as a nonlinear operator from $C^{2+\alpha}(G, \mathbb{R}^3)$ to $C^\alpha(G, \mathbb{R})$. The equation (1.8) can be written as

$$(1.11) \quad P(x) = \epsilon.$$

We calculate the differential operator $DP(\bar{x})y$ of P at $\bar{x}$. Let $x = \bar{x} + ty$. Because

$$(1.12) \quad P(x)b(x) = a(x),$$

we have

$$(1.13) \quad \begin{aligned} \frac{d}{dt} \Big|_{t=0} \{P(\bar{x} + ty)b(\bar{x} + ty)\} &= \frac{d}{dt} \Big|_{t=0} \{a(\bar{x} + ty)\}, \\ (DP(\bar{x})y)b(\bar{x}) + P(\bar{x})Db(\bar{x})y &= Da(\bar{x})y. \end{aligned}$$

Suppose $\bar{x}$ is a solution of the CMC equations (i.e. (1.8) with $\epsilon = 0$). The differential operator of P at $\bar{x}$ is

$$\begin{aligned}
 (DP(\bar{x})y) & \{ \bar{x}_u \times \partial_v \nabla_p \gamma_1(n(\bar{x})) + \bar{x}_v \times \partial_u \nabla_p \gamma_1(n(\bar{x})) \} \\
 (1.14) \quad & = \left\{ \frac{1}{2} \Delta y - y_u \times \bar{x}_v - \bar{x}_u \times y_v \right\}, \\
 & \quad \forall y \in C^{2+\alpha}(G).
 \end{aligned}$$

Notice that $DP(\bar{x})$ being invertible implies that there is a neighborhood V of 0 in $\mathbb{R}$, such that for any $\epsilon \in V$ there is a unique solution $x \in C^{2+\alpha}(G)$ of (1.8) in a neighborhood of $\bar{x}$.

Furthermore, we shall use the following global implicit function theorem to obtain the global uniqueness of CAMC for all ϵ satisfying the condition of the main theorem.

Global Implicit Function Theorem. *[CH] Suppose Y, Z are metric spaces, Y is arcwise connected and Z is simply connected. If $\Phi : Y \rightarrow Z$ is continuous and proper, and local invertible at the all points of Y , then Φ is a homeomorphism from Y to Z .*

In section 2, the space of CAMC immersions of sphere S^2 in $\mathbb{R}^3$ is defined, which satisfies the conditions of the global implicit function theorem. In section 3, we shall prove the local invertibility of the map P . In section 4, we introduce the definition of weak CAMC surfaces and give some regularity results for the weak CAMC surfaces which will be used in section 5 to prove that the map P is proper.

2. The metric space of CAMC immersions of sphere S^2 in $\mathbb{R}^3$

We consider the case $G = S^2$ where G is a parameter space of a surface. We shall use local conformal coordinates on $G = S^2$. In particular, we shall always identify S^2 with $\mathbb{C} \cup \{\infty\}$ via the stereographic projection to give a conformal structure on S^2 .

Let

$$(2.1) \quad Z = \{ \epsilon \in C^\alpha(S^2, \mathbb{R}) : d_Z(\epsilon, 0) < r \}$$

where $r > 0$,

$$(2.2) \quad d_Z(\epsilon, \epsilon') = \|\epsilon - \epsilon'\|_{C^\alpha(S^2)} + |\epsilon(p) - \epsilon'(p)|, \quad p = (1, 0, 0),$$

and the metric of S^2 used to define the distance of $C^\alpha(S^2, \mathbb{R})$ is induced from the Euclidean space $\mathbb{R}^3$.

The domain of the operator P

$$Dom(P) := \{x \in C^{2+\alpha}(S^2, \mathbb{R}^3) : x \text{ is an immersion of } S^2 \text{ in } \mathbb{R}^3\}$$

and the metric of S^2 used to define the distance of $C^{2+\alpha}(S^2, \mathbb{R}^3)$ is induced from the Euclidean space $\mathbb{R}^3$.

Definition 2.1. *For any $x, \bar{x} \in Dom(P)$, we call that x and $\bar{x}$ are equivalent, denoted by $x \sim \bar{x}$, if there is a vector $b \in \mathbb{R}^3$, such that*

$$(2.3) \quad M = M' + b, \quad M := Image(x), \quad M' := Image(\bar{x}).$$

Define $Y = \{x : [x] \in Dom(P)/\sim, x \text{ is conformal and } P(x) \in Z\}$.

Remark. (1) If $Image(x) = Image(\bar{x})$, we can think that x is equivalent to $\bar{x}$ up to a change of parameter space. Notice that for a fixed CAMC surface in a fixed coordinate space x , by selecting different parameter space, there are many different equivalent expression.

(2) If x is a solution of the CAMC equation with respect to $\gamma(\cdot)$, then for any constant vector $b \in \mathbb{R}^3$, $x + b$ is also a solution of the CAMC equation with respect to $\gamma(\cdot)$. On the other hand for all rotation R of $\mathbb{R}^3$, Rx is also a solution of the CAMC equation with respect to $\gamma(R^{-1}\cdot)$. So x is equivalent to Rx in the sense that up to a rotation R , $(x, \gamma(\cdot))$ is equivalent to $(Rx, \gamma(R^{-1}\cdot))$. Notice that the value of $P(x)$ with $\gamma(\cdot)$ is equal to the value of $P(Rx)$ when $\gamma(\cdot)$ is replaced by $\gamma(R^{-1}\cdot)$.

Let

$$(2.4) \quad \begin{aligned} d(x, \bar{x}) &= \sum_{0 \leq j \leq 2} \|x - \bar{x}\|_{C^{j+\alpha}(S^2)} + \|x - \bar{x}\|_{C(S^2)}, \\ d_Y(x, \bar{x}) &= \inf_{x \in [x], \bar{x} \in [\bar{x}]} \sum_{0 \leq j \leq 2} \|x - \bar{x}\|_{C^{j+\alpha}(S^2)} + \|x - \bar{x}\|_{C(S^2)}. \end{aligned}$$

It is easy to see that d and d_Y are metrics of $Dom(P)$ and Y respectively.

Proposition 2.2. *For each $y \in Dom(P)$, there is a conformal representation, that is, there is $x \in Dom(P)$ such that x is conformal and $x \sim y$. The sets Y and Z are metric space with the distance function d_Y and d_Z respectively. Moreover, Y is arcwise connected and Z is simply connected.*

Proof. Step 1. For each $y \in Y$, for example, from [J] any $C^{2+\alpha}$ immersed surface M of S^2 in $\mathbb{R}^3$ has a $C^{2+\alpha}$ conformal representation.

Step 2. Y is arcwise connected.

Since for any $x \in \text{Dom}(P)$, there is $\delta > 0$ and a neighborhood $O(x)$ of x in $C^{2+\alpha}(S^2, \mathbb{R}^3)$:

$$O(x) = \{y \in C^{2+\alpha}(S^2, \mathbb{R}^3) : d(x, y) < \delta\},$$

such that $O(x) \subset \text{Dom}(P)$. So $\text{Dom}(P)$ is locally arcwise connected. Then Y is locally arcwise connected with respect to d_Y .

On the other hand, $\text{Dom}(P)$ is connected, since there is no non-trivial subset which is open and closed in $\text{Dom}(P)$. Then $\text{Dom}(P)$ is arcwise connected. Thus Y is arcwise connected with respect to d_Y .

Step 3. Note that if $\epsilon \in Z$ and $\epsilon' \in Z$, then

$$t\epsilon + (1-t)\epsilon' \in Z, \quad \forall t \in [0, 1].$$

So Z is simply connected. $\square$

3. Local invertibility

It is clear that P is a continuous map from Y to Z .

Let

$$Z_1 = \{\epsilon \equiv \text{constant} \in (-r, r)\} \subset Z,$$

and

$$Y_1 = \{x \in Y : P(x) \in Z_1\}.$$

Remark that Z_1 is simply connected. Note that S^2 is the unique CAMC immersion of sphere S^2 in $\mathbb{R}^3$ w.r.t. $\epsilon = 0$, which is locally arcwise connected to each possible components in Y_1 . So Y_1 is arcwise connected.

In this section, we shall prove that for any $\bar{x} \in Y_1$, the differential operator of P at $\bar{x}$

$$(3.1) \quad DP(\bar{x}) : Y \rightarrow Z$$

is an invertible linear operator, where

$$(3.2) \quad \|x\|_Y := d_Y(x, 0), \quad \|\epsilon\|_Z := d_Z(\epsilon, 0).$$

Then by the local inverse function theorem (c.f. for example, [H]), P is invertible in a neighborhood of $\bar{x}$ in Y .

First we note that from (1.13),

$$(3.3) \quad DP(\bar{x})y = 0 \Rightarrow Da(\bar{x})y - P(\bar{x})Db(\bar{x})y = 0,$$

that is,

$$(3.4) \quad \begin{aligned} & \frac{1}{2}\Delta y - y_u \times \bar{x}_v - \bar{x}_u \times y_v \\ &= \epsilon \{ y_u \times \partial_v \nabla_p \gamma_1(n(\bar{x})) + \bar{x}_u \times \partial_v \left(\frac{y_u \times \bar{x}_v + \bar{x}_u \times y_v}{|\bar{x}_u \times \bar{x}_v|} \cdot \nabla_p^2 \gamma_1(n(\bar{x})) \right) \right. \\ & \quad \left. - y_v \times \partial_u \nabla_p \gamma_1(n(\bar{x})) - \bar{x}_v \times \partial_u \left(\frac{y_u \times \bar{x}_v + \bar{x}_u \times y_v}{|\bar{x}_u \times \bar{x}_v|} \cdot \nabla_p^2 \gamma_1(n(\bar{x})) \right) \right\}. \end{aligned}$$

Lemma 3.1. *Assume that $\epsilon \in Z_1$ and*

$$(3.5) \quad \sup_{|p|=1} |\nabla_p \gamma_1(p)| \leq c_3^\gamma, \quad \sup_{|p|=1} |\nabla_p^2 \gamma_1(p)| \leq c_4^\gamma.$$

There is $r_0 = r_0(c_3^\gamma, c_4^\gamma) > 0$ such that if

$$r < r_0,$$

then there is no non-constant solution of the equations (3.4) in Y .

Remark. As in the remark after the Definition 2.1, if $\bar{x}$ is a solution of the CAMC equation with respect to $\gamma(\cdot)$, then for any constant vector $b \in \mathbb{R}^3$, $\bar{x} + b$ is also a solution of the CAMC equation with respect to $\gamma(\cdot)$. Moreover $y + b$ is also a solution of (3.4). The operator $DP(\bar{x})$ is invertible in the sense that up to a translation, (3.4) only has the zero solution. On the other hand for all rotation R of $\mathbb{R}^3$, $R\bar{x}$ is also a solution of the CAMC equation with respect to $\gamma(R^{-1}\cdot)$. In this case Ry is also a solution of (3.4) where $\bar{x}$ and $\gamma_1(\cdot)$ are replaced by $R\bar{x}$ and $\gamma_1(R^{-1}\cdot)$. Notice that $P(\bar{x})$ with $\gamma(p)$ is equal to $P(R\bar{x})$ where $\gamma(p)$ is replaced by $\gamma(R^{-1}p)$. The operators $DP(\bar{x})$ and $DP(R\bar{x})$ are invertible for $\gamma(\cdot)$ and $\gamma(R^{-1}\cdot)$ respectively.

Proof. For $z_0 \in \mathbb{C}$ and $\sigma > 0$, let $\eta \in C_0^\infty(B_{2\sigma}(z_0))$ be a cut-off function with

$$(3.6) \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \quad \text{on} \quad B_\sigma(z_0), \quad |\nabla \eta| \leq \frac{C}{\sigma}.$$

Multiplying (3.4) by $\varphi = \eta^2 y$ and integrating by parts, and noting that

$$\begin{aligned}
& \int (\bar{x}_v \times y_u + y_v \times \bar{x}_u) \cdot (\eta^2 y) = \int (\bar{x}_v \times (\eta y)_u + (\eta y)_v \times \bar{x}_u) \cdot (\eta y) \\
& = -2 \int ((\eta y)_u \times (\eta y)_v) \cdot \bar{x}, \\
& (\eta y) \cdot \Delta(\eta y) = (\eta \Delta y + \nabla \eta \cdot \nabla y) \cdot (\eta y) + \operatorname{div}(\eta |y|^2 \nabla \eta) - (\nabla \eta \cdot \nabla(\eta y)) \cdot y \\
& = \eta^2 y \Delta y - |\nabla \eta|^2 |y|^2 + \operatorname{div}(\eta |y|^2 \nabla \eta),
\end{aligned}$$

we get

$$\begin{aligned}
(3.7) \quad & \left(\frac{1}{4} - |\epsilon|(c_3^\gamma + c_4^\gamma) - \max_{B_{2\sigma}(z_0)} |\bar{x}(z)| \right) \int |\nabla(\eta y)|^2 \\
& \leq \frac{C}{\sigma^2} (|\epsilon|(c_3^\gamma + c_4^\gamma)) \int_{B_{2\sigma}(z_0) \setminus B_\sigma(z_0)} |y|^2.
\end{aligned}$$

If

$$(3.8) \quad r(c_3^\gamma + c_4^\gamma) < \frac{1}{4},$$

and

$$(3.9) \quad \max_{B_{2\sigma}(z_0)} |\bar{x}(z)| < \frac{1}{4} - r(c_3^\gamma + c_4^\gamma).$$

The inequality (3.9) is called a Caccioppoli's type inequality (c.f. [Gi]).

From the Definition 2.1, there is a CAMC surface in $[\bar{x}]$ (still denoted by $\bar{x}$) such that $\bar{x}(z_0) = 0$. So if σ is small enough, then we have (3.9).

On the other hand, it is easy to see that for any vector $b \in \mathbb{R}^3$, if y is a solution of (3.4), then $y - b$ is also a solution of (3.4). So we have

$$(3.10) \quad \int |\nabla(\eta(y - b))|^2 \leq \frac{C|\epsilon|(c_3^\gamma + c_4^\gamma)}{\sigma^2} \int_{B_{2\sigma}(z_0) \setminus B_\sigma(z_0)} |y - b|^2.$$

Particularly, we may take $b = y(z_1)$ for a fixed point $z_1 \in B_{2\sigma}(z_0) \setminus B_\sigma(z_0)$. Then the Poincaré inequality can be applied to the right of (3.10). Note that $\eta = 1$ in $B_\sigma(z_0)$ we get

$$(3.11) \quad \int_{B_\sigma(z_0)} |\nabla y|^2 \leq C|\epsilon|(c_3^\gamma + c_4^\gamma) \int_{B_{2\sigma}(z_0) \setminus B_\sigma(z_0)} |\nabla y|^2,$$

where the constant C is independent of σ . If we take ϵ such that

$$C|\epsilon|(c_3^\gamma + c_4^\gamma) \leq \frac{1}{2^3},$$

then we get

$$(3.12) \quad \int_{B_\sigma(z_0)} |\nabla y|^2 \leq \frac{1}{2^3} \int_{B_{2\sigma}(z_0) \setminus B_\sigma(z_0)} |\nabla y|^2.$$

For $k = 1, 2, 3, \dots$, using (3.12) step by step, we get

$$(3.13) \quad |B_{\sigma/2^k}(z_0)|^{-1} \int_{B_{\sigma/2^k}(z_0)} |\nabla y|^2 \leq |B_{\sigma/2^k}(z_0)|^{-1} \left(\frac{1}{2^3}\right)^{k+1} \int_{B_{2\sigma}(z_0) \setminus B_\sigma(z_0)} |\nabla y|^2.$$

Taking $k \rightarrow \infty$ in (3.13), we get $|\nabla y(z_0)| = 0$. Thus y must be constant. $\square$

The Lemma 3.1 implies

Corollary 3.2. *P is locally invertible in Y at $\bar{x} \in Y_1 \subset Y$.*

4. Regularity of weak CAMC surfaces

In next section we shall see that the limit of CAMC surfaces is a CAMC surface in weak sense. So we need to discuss when a weak CAMC surface is a regular one.

Define

$$(4.1) \quad E(x) = \int \gamma(n(x)) |x_u \times x_v| + Q(x) \cdot (x_u \times x_v) dudv$$

for $x \in H^{1,2}(S^2, \mathbb{R}^3)$, where Q is to be determined later.

Definition 4.1. *A map $x \in H^{1,2}(S^2, \mathbb{R}^3)$ is called a weak CAMC surface if it is weakly conformal*

$$x_u \cdot x_v = 0, \quad |x_u|^2 = |x_v|^2, \quad \text{for a.e. } (u, v) \in S^2,$$

and the first variation of the functional E at x equals to zero

$$(4.2) \quad \int (x_v \times \nabla_p \gamma(n(x))) \cdot \varphi_u - (x_u \times \nabla_p \gamma(n(x))) \cdot \varphi_v + (\operatorname{div}_x Q(x)) (x_u \times x_v) \cdot \varphi dudv = 0,$$

for all $\varphi \in H^{1,2}(S^2, \mathbb{R}^3)$, with $Q(x) = \frac{1}{3}x$ and $\operatorname{div}_x Q(x) = 1$.

Theorem 4.2. *Suppose*

$$(4.3) \quad \inf_{|p|=1} \gamma(p) \geq c_1^\gamma (> 0), \quad \sup_{|p|=1} \frac{|\nabla_p \gamma(p)|}{\gamma(p)} \leq c_2^\gamma (< \infty).$$

Then the weak CAMC surfaces are continuous.

Proof. Step 1. Let $\rho > 0$, $\beta > 0$ and ψ be a cut-off function with

$$(4.4) \quad \psi(t) = \begin{cases} 1, & t \leq \frac{1}{2\rho^{\beta-1}} \\ 0, & t \geq \frac{1}{\rho^{\beta-1}} \end{cases}, \quad \psi'(t) \leq 0.$$

For any fixed $z_0 \in \mathbb{C} \cup \{\infty\}$ and $\sigma > 0$, fix z_1 : $|z_1 - z_0| < \sigma$, let

$$(4.5) \quad \rho' = \inf_{z=u+iv \in \partial B(z_0, \sigma)} |x(z) - x(z_1)|.$$

Define the density of x at z_1 by

$$(4.6) \quad I(\rho) := \int_{B(z_0, \sigma)} \gamma(n(x)) |x_u \times x_v| \psi\left(\frac{|x - x(z_1)|}{\rho^\beta}\right) dudv.$$

Note that

$$(4.7) \quad 2I(\rho) - \rho I'(\rho) = \int_{B(z_0, \sigma)} \gamma(n(x)) |x_u \times x_v| \left\{ 2\psi + \beta \frac{\psi' |x - x(z_1)|}{\rho^\beta} \right\} dudv.$$

Take

$$\varphi(z) = \psi\left(\frac{|x(z) - x(z_1)|}{\rho^\beta}\right) (x(z) - x(z_1))$$

in (4.2). We have

$$\begin{aligned} & - \int (x_u \times x_v) \cdot (x - x(z_1)) \psi dudv \\ &= \int (x_v \times \nabla_p \gamma(n(x))) \cdot \left\{ \psi x_u + \psi' \frac{(x - x(z_1)) \cdot x_u}{\rho^\beta |x - x(z_1)|} (x - x(z_1)) \right\} \\ & - (x_u \times \nabla_p \gamma(n(x))) \cdot \left\{ \psi x_u + \psi' \frac{(x - x(z_1)) \cdot x_v}{\rho^\beta |x - x(z_1)|} (x - x(z_1)) \right\} dudv \\ &= \int \gamma(n(x)) |x_u \times x_v| \left\{ 2\psi + \psi' \left\{ \frac{(x - x(z_1)) \cdot x_u}{\rho^\beta |x - x(z_1)|} (x - x(z_1)) \cdot (x_v \times \nabla_p \gamma(n(x))) \right. \right. \\ & \quad \left. \left. - \frac{(x - x(z_1)) \cdot x_v}{\rho^\beta |x - x(z_1)|} (x - x(z_1)) \cdot (x_u \times \nabla_p \gamma(n(x))) \right\} \right\} dudv, \end{aligned}$$

where $p \cdot \nabla_p \gamma(p) = \gamma(p)$ is invoked. Then

$$(4.8) \quad \begin{aligned} & - \int (x_u \times x_v) \cdot (x - x(z_1)) \psi dudv \\ & \geq \int \gamma(n(x)) |x_u \times x_v| \left\{ 2\psi + 2c_2^\gamma \psi' \frac{|x - x(z_1)|}{\rho^\beta} \right\} dudv. \end{aligned}$$

Take

$$(4.9) \quad \beta = 2c_2^\gamma.$$

From (4.7) and (4.8),

$$\begin{aligned}
 (4.10) \quad & 2I(\rho) - \rho I'(\rho) \\
 & \leq - \int (x_u \times x_v) \cdot (x - x(z_1)) \psi dudv \\
 & \leq \frac{\rho}{c_1^\gamma} I(\rho).
 \end{aligned}$$

So we have the monotonicity inequality

$$(4.11) \quad \frac{I(\rho')}{\rho'^2} \geq e^{\left(\frac{\rho^2 - \rho'^2}{2c_1^\gamma}\right)} \frac{I(\rho)}{\rho^2}, \quad \rho' \geq \forall \rho > 0.$$

Step 2. Note that from (4.3)-(4.4), (4.6), we have

$$(4.12) \quad \lim_{\rho \rightarrow 0} \frac{I(\rho)}{\rho^2} \geq \lim_{\rho \rightarrow 0} \frac{c_1^\gamma \rho^{-2}}{2} \int_{\{|x - x(z_1)| \leq \rho/2\}} |\nabla x|^2 dudv \geq \frac{c_1^\gamma \pi}{8},$$

where the estimate of the energy from below was proved by Grüter (see [Gr, Proposition 2.5]).

Then

$$\begin{aligned}
 (4.13) \quad & \inf_{z=u+iv \in \partial B(z_0, \sigma)} |x(z) - x(z_1)| = \rho' \\
 & \leq 4 \left(\frac{I(\rho')}{c_1^\gamma \pi} \right)^{1/2} e^{\left(\frac{\rho'^2}{4c_1^\gamma}\right)} \rightarrow 0, \quad \text{as } \sigma \rightarrow 0,
 \end{aligned}$$

and for any $z_1 \in B(z_0, \sigma)$, $z_2 \in B(z_0, \sigma)$, there are $z'_1 \in \partial B(z_0, \sigma)$ and $z'_2 \in \partial B(z_0, \sigma)$, such that

$$(4.14) \quad |x(z_1) - x(z_2)| \leq |x(z_1) - x(z'_1)| + |x(z_2) - x(z'_2)| + |x(z'_1) - x(z'_2)|.$$

As in [Gr] and [J, Chapter 2], for $x \in H^{1,2}(S^2)$ with the finite integration $\int_{S^2} |\nabla x|^2$, by Courant-Lebesgue lemma (see [J, p.2]), for any $\delta \in (0, 1)$ there is $\sigma \in (\delta, \sqrt{\delta})$ such that for all $z'_1 \in \partial B(z_0, \sigma)$ and $z'_2 \in \partial B(z_0, \sigma)$, we can estimate $|x(z'_1) - x(z'_2)|$ only by the integration $\int_{S^2} |\nabla x|^2$ and σ . Thus we have the continuity of x . $\square$

For $\epsilon \in (-r, r)$, let

$$(4.15) \quad \gamma^\epsilon(p) = |p| + \epsilon \gamma_1(p).$$

Lemma 4.3. *Assume that*

$$(4.16) \quad \sup_{|p|=1} |\nabla_p \gamma_1(p)| \leq c_3^\gamma, \quad \sup_{|p|=1} |\nabla_p^2 \gamma_1(p)| \leq c_4^\gamma.$$

Suppose x is a continuous weak CAMC surface with respect to γ^ϵ , and

$$(4.17) \quad |\epsilon| < \frac{c_0^\gamma}{200c_3^\gamma},$$

then x is the $C^{2+\alpha}$ regular CAMC surface.

Proof. Step 1. For any $\eta \in H^{1,2}(S^2, \mathbb{R}^3)$, for any open set $\Omega \subset \mathbb{C}$, from (1.8), we have

$$(4.18) \quad \int_{\Omega} (x_v \times \partial_u \nabla_p \gamma(n(x))) \cdot \eta - (x_u \times \partial_v \nabla_p \gamma(n(x))) \cdot \eta \, dudv = \int_{\Omega} (x_u \times x_v) \cdot \eta \, dudv.$$

Taking $\eta = (x_u \xi)_u$ and $(x_v \xi)_v$ in (4.18), we see that the sum of them is

$$(4.19) \quad \begin{aligned} & \int_{\Omega} \xi [\partial_u (x_u \times x_v) \cdot \partial_u \nabla_p \gamma + \partial_v (x_u \times x_v) \cdot \partial_v \nabla_p \gamma] \, dudv \\ &= \int_{\Omega} (x_u \times x_v) \cdot \Delta x \xi \, dudv \\ &+ 2 \int_{\Omega} (x_{uv} \times x_{uu} - x_{uv} \times x_{vv}) \cdot n(x) \xi \, dudv \\ &+ 2\epsilon \int_{\Omega} (x_{uv} \times x_{uu} - x_{uv} \times x_{vv}) \cdot \nabla_p \gamma_1 \xi \, dudv \\ &+ \int_{\Omega} [\xi_v (x_v \times x_{vu}) - \xi_u (x_v \times x_{vv}) + \xi_v (x_u \times x_{uu}) - \xi_u (x_u \times x_{uv})] \cdot n(x) \, dudv \\ &+ \epsilon \int_{\Omega} [\xi_v (x_v \times x_{vu}) - \xi_u (x_v \times x_{vv}) + \xi_v (x_u \times x_{uu}) - \xi_u (x_u \times x_{uv})] \cdot \nabla_p \gamma_1 \, dudv. \end{aligned}$$

Since x is conformal, from (1.2),

$$\begin{aligned} x_{uu} \cdot x_v &= -x_u \cdot x_{uv}, & x_{uv} \cdot x_v &= -x_u \cdot x_{vv}, \\ x_u \cdot x_{uu} &= x_v \cdot x_{uv}, & x_u \cdot x_{uv} &= x_v \cdot x_{vv}, \\ |x_u \times x_v| &= |x_u| |x_v| = \frac{1}{2} |\nabla x|^2, \end{aligned}$$

and from $\Delta x / (x_u \times x_v)$,

$$x_{uu} \cdot x_u = -x_{vv} \cdot x_u, \quad x_{uu} \cdot x_v = -x_{vv} \cdot x_v.$$

So we have

$$(4.20) \quad (x_{uv} \times x_{uu} - x_{uv} \times x_{vv}) \cdot n(x) = \frac{(-1) |\nabla |\nabla x|^2|^2}{8 |\nabla x|^2}.$$

Moreover, noting that $\Delta x/n(x)$, we have

$$\begin{aligned}
& [\xi_v(x_v \times x_{vu}) - \xi_u(x_v \times x_{vv}) + \xi_v(x_u \times x_{uu}) - \xi_u(x_u \times x_{uv})] \cdot n(x) \\
&= [-\xi_v \partial_v(x_u \times x_v) - \xi_u \partial_u(x_u \times x_v) + \xi_v x_u \times \Delta x + \xi_u \Delta x \times x_v] \cdot n(x) \\
&= [-\xi_v \partial_v(x_u \times x_v) - \xi_u \partial_u(x_u \times x_v)] \cdot n(x) \\
&= \left(-\frac{1}{2}\right) \frac{\xi_v \partial_v |x_u \times x_v|^2 + \xi_u \partial_u |x_u \times x_v|^2}{|x_u \times x_v|} \\
&= (-1) \{ \xi_v \partial_v |\nabla x|^2 + \xi_u \partial_u |\nabla x|^2 \}.
\end{aligned}$$

Using the strict convexity (1.1)(3), for any non-negative ξ ,

$$\begin{aligned}
(4.21) \quad & c_0^\gamma \int_{\Omega} |\nabla n(x)|^2 \xi |\nabla x|^2 dudv + \frac{1}{8} \int_{\Omega} \frac{|\nabla |\nabla x|^2|^2}{|\nabla x|^2} \xi dudv \\
& + \int_{\Omega} \xi_v \partial_v |\nabla x|^2 + \xi_u \partial_u |\nabla x|^2 dudv \\
& \leq \left| \int_{\Omega} (x_u \times x_v) \cdot \Delta x \xi dudv \right| \\
& + 2\epsilon \int_{\Omega} (x_{uv} \times x_{uu} - x_{uv} \times x_{vv}) \cdot \nabla_p \gamma_1 \xi dudv \\
& + \epsilon \int_{\Omega} [\xi_v(x_v \times x_{vu}) - \xi_u(x_v \times x_{vv}) + \xi_v(x_u \times x_{uu}) - \xi_u(x_u \times x_{uv})] \cdot \nabla_p \gamma_1 dudv|.
\end{aligned}$$

Note that $x_u \cdot n(x) = x_v \cdot n(x) = 0$,

$$\Delta x = (\Delta x \cdot n(x))n(x) = -(x_u \cdot \partial_u n(x) + x_v \cdot \partial_v n(x))n(x).$$

Then

$$(4.22) \quad |\Delta x|^2 \leq 2|x_u|^2 |\nabla n(x)|^2.$$

Note also that

$$\begin{aligned}
x_{uv} &= (x_{uv} \cdot x_u) \frac{x_u}{|x_u|^2} + (x_{uv} \cdot x_v) \frac{x_v}{|x_v|^2} + (x_{uv} \cdot (x_u \times x_v)) \frac{x_u \times x_v}{|x_u \times x_v|^2} \\
&= (x_v \cdot x_{vv}) \frac{x_u}{|x_u|^2} + (x_u \cdot x_{uu}) \frac{x_v}{|x_v|^2} - (x_u \cdot \partial_v n(x))n(x).
\end{aligned}$$

Then

$$(4.23) \quad |x_{uv}|^2 \leq 2|\Delta x|^2 + |x_u|^2 |\nabla n(x)|^2 \leq 5|x_u|^2 |\nabla n(x)|^2.$$

Thus by using (4.16)-(4.17) and (4.21)-(4.23), for any non-negative

$$\xi \in H^{1,2}(S^2, \mathbb{R}^+)$$

we have

$$\begin{aligned}
(4.24) \quad & c_0^\gamma \int_{\Omega} |\nabla n(x)|^2 \xi |\nabla x|^2 dudv + \frac{1}{8} \int_{\Omega} \frac{|\nabla |\nabla x|^2|^2}{|\nabla x|^2} \xi dudv \\
& + \int_{\Omega} \xi_v \partial_v |\nabla x|^2 + \xi_u \partial_u |\nabla x|^2 dudv \\
& \leq \int_{\Omega} \left\{ \frac{1}{2} |\nabla x|^3 |\nabla n(x)| + 10 |\epsilon| c_3^\gamma |\nabla x|^2 |\nabla n(x)|^2 \right\} \xi + |\epsilon| c_3^\gamma |\nabla \xi| |\nabla x| |\nabla^2 x| dudv.
\end{aligned}$$

Take $V = |\nabla x|^2$, and $\xi = V^s \zeta^2$. Then there is $\delta > 0$ such that for any ϵ satisfying (4.17), for all $s \in \mathbb{N}$ and for all $\zeta \in H^{1,2}(S^2, [-1, 1])$

$$\begin{aligned}
(4.25) \quad & \delta \int_{\Omega} |\nabla^2 x|^2 V^s \zeta^2 dudv + s \int_{\Omega} |\nabla V|^2 V^{s-1} \zeta^2 dudv \\
& \leq C \int_{\Omega} V^{s+1} |\nabla \zeta|^2 dudv + C_\delta \int_{\Omega} V^{s+2} \zeta^2 dudv,
\end{aligned}$$

where the constant C_δ only depends on δ .

Step 2. Take $\Omega = B(z_0, \sigma)$. Since

$$\begin{aligned}
& \int_{B(z_0, \sigma)} V^{s+2} \zeta^2 dudv = \int_{B(z_0, \sigma)} V^{s+1} \zeta^2 |\nabla x|^2 dudv \\
& = (-1) \int_{B(z_0, \sigma)} (x - x(z_0)) \{ V^{s+1} \zeta^2 \Delta x + 2(s+1) V^s \zeta^2 \nabla x \cdot (\nabla x \cdot \nabla^2 x) \\
& \quad + 2 V^{s+1} \zeta \nabla \zeta \cdot \nabla x \} dudv \\
& \leq C_1 \max_{B(z_0, \sigma)} |x - x(z_0)| \int_{B(z_0, \sigma)} \{ V^{s+2} \zeta^2 + V^s |\nabla^2 x|^2 \zeta^2 + V^{s+1} |\nabla \zeta|^2 \} dudv.
\end{aligned}$$

From the proof of the Theorem 4.2, for any $\mu > 0$ there is a $\sigma_0 > 0$ which depends only on μ and $\int |\nabla x|^2 dudv$, such that for $\sigma \leq \sigma_0$, we have

$$C_\delta C_1 \max_{B(z_0, \sigma)} |x - x(z_0)| < \mu,$$

and

$$\begin{aligned}
& \int_{B(z_0, \sigma)} V^{s+2} \zeta^2 dudv \leq C \mu \int_{B(z_0, \sigma)} V^s |\nabla^2 x|^2 \zeta^2 + V^{s+1} |\nabla \zeta|^2 dudv \\
& \leq C \mu \int_{B(z_0, \sigma)} V^{s+1} |\nabla \zeta|^2 + V^{s+2} \zeta^2 dudv. \quad (\text{by (4.25)})
\end{aligned}$$

If we take μ small enough then

$$(4.26) \quad \int_{B(z_0, \sigma)} V^{s+2} \zeta^2 dudv \leq C \int_{B(z_0, \sigma)} V^{s+1} |\nabla \zeta|^2 dudv.$$

Step 3. Denote $w = V\zeta^2$. Then by (4.25) we have for all $q > 1$

$$\begin{aligned}
 (4.27) \quad & \int_{B(z_0, \sigma)} |\nabla w|^2 dudv = \int_{B(z_0, \sigma)} |(\nabla V)\zeta^2 + 2V\zeta\nabla\zeta|^2 dudv \\
 & \leq \int_{B(z_0, \sigma)} |\nabla V|^2 \zeta^4 + 4V^2 \zeta^2 |\nabla\zeta|^2 dudv \\
 & \leq C \int_{B(z_0, \sigma)} V^2 \zeta^2 |\nabla\zeta|^2 + V^2 |\nabla\zeta|^2 + V^3 \zeta^2 dudv \\
 & \leq C \left(\int_{B(z_0, \sigma)} V^{3q'} dudv \right)^{1/q'} |\Omega|^{1/q}.
 \end{aligned}$$

From the a priori estimates (4.26)-(4.27), by using a calculus inequality given in the Lemma 5.2 of Chapter 2 of [LU], we can estimate the C_{loc}^1 norm of x . Moreover we can prove that the continuous weak CAMC surfaces are $C^{2+\alpha}$ regular. $\square$

5. Properness

We prove that the map P is proper, i.e., for any compact subset of Z_1 , its inverse image is compact in Y . For $\epsilon \in (-r, r)$, let

$$(5.1) \quad \gamma^\epsilon(p) := |p| + \epsilon\gamma_1(p), \quad \forall p \in \mathbb{R}^3.$$

Lemma 5.1. *For any $K > 0$, let Y_K denote the subset of Y satisfying*

$$(5.2) \quad \int_{S^2} |\nabla x^\epsilon|^2 \leq K.$$

Assume that

$$(5.3) \quad \sup_{|p|=1} |\nabla_p \gamma_1(p)| \leq c_3^\gamma.$$

Suppose $\epsilon_j \rightarrow \epsilon$ in Z_1 , $\{x^{\epsilon_j}\}_j \subset Y_K$ are CAMC surfaces with respect to γ^{ϵ_j} . Then there is a subsequence, denoted also by $\{x^{\epsilon_j}\}_j$, and $x \in H^{1,2}$ such that

$$(5.4) \quad \nabla x^{\epsilon_j} \rightarrow \nabla x, \quad \text{strongly in } L^2.$$

Moreover, x is a weak CAMC surface with respect to γ^ϵ .

Proof. By the assumption (5.2)

$$\int_{S^2} |\nabla x^{\epsilon_j}|^2 \leq K, \quad \text{for all } j.$$

From (4.26) and the Step 2 of the proof of Lemma 4.3, there is a constant $C(K) < \infty$ which depends only on K such that

$$(5.5) \quad \int_{S^2} |\nabla^2 x^{\epsilon_j}|^2 \leq C(K).$$

Then there is a subsequence still denoted by $\{\epsilon_j\}$ and $x \in L^2$ such that

$$(5.6) \quad \nabla x^{\epsilon_j} \rightarrow \nabla x, \quad \text{strongly in } L^2.$$

Because for any $\varphi \in C_0^\infty$,

$$\int (x_u^{\epsilon_j} \times x_v^{\epsilon_j} - x_u \times x_v) \cdot \varphi \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

then

$$(5.7) \quad x_u^{\epsilon_j} \times x_v^{\epsilon_j} \rightarrow x_u \times x_v \quad \text{in the sense of distribution.}$$

On the other hand, for any $\varphi \in H^{1,2}$, by the weak conformality and $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we have

$$(5.8) \quad \begin{aligned} & \int \{x_u^{\epsilon_j} \times n(x^{\epsilon_j}) - x_u \times n(x)\} \cdot \varphi_v dudv \\ &= \int \{-x_v^{\epsilon_j} + x_v\} \cdot \varphi_v dudv \\ &\rightarrow 0, \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Note that there is a subsequence $\{x^{\epsilon_{j'}}\}_{j'}$, such that

$$(5.9) \quad \nabla x^{\epsilon_{j'}} \rightarrow \nabla x, \quad \text{for a.e. } (u, v) \in S^2.$$

Then x is weakly conformal.

Because $\gamma_1 \in C^2(\mathbb{R}^3 \setminus \{0\})$,

$$(5.10) \quad \nabla_p \gamma_1(n(x^{\epsilon_j})) \rightarrow \nabla_p \gamma_1(n(x)), \quad \text{for a.e. } (u, v) \in S^2,$$

by (5.3) and Lebesgue convergence theorem, for any $\varphi \in H^{1,2}$

$$(5.11) \quad \begin{aligned} & \lim_{j'} \int \{x_u^{\epsilon_{j'}} \times \nabla_p \gamma^{\epsilon_{j'}}(n(x^{\epsilon_{j'}})) - x_u \times \nabla_p \gamma^\epsilon(n(x))\} \cdot \varphi_v dudv = 0, \\ & \lim_{j'} \int \{x_v^{\epsilon_{j'}} \times \nabla_p \gamma^{\epsilon_{j'}}(n(x^{\epsilon_{j'}})) - x_v \times \nabla_p \gamma^\epsilon(n(x))\} \cdot \varphi_u dudv = 0. \end{aligned}$$

The convergence of (5.7) and (5.11) imply that x satisfies (4.2) for all $\varphi \in C_0^\infty$. For any $\varphi \in H^{1,2}$, there is $\varphi_k \in C_0^\infty$ such that

$$(5.12) \quad \varphi_k \rightarrow \varphi \quad \text{strongly in } H^{1,2}.$$

By (4.2),

$$(5.13) \quad \begin{aligned} & \left| \int (x_u \times x_v) \cdot (\varphi_k - \varphi_j) dudv \right| \\ & \leq 2c_3^\gamma \int |\nabla x| |\nabla(\varphi_k - \varphi_j)| dudv \\ & \leq 2c_3^\gamma \|\nabla x\|_{L^2} \|\nabla(\varphi_k - \varphi_j)\|_{L^2} \end{aligned}$$

By the Lebesgue convergence theorem,

$$\left| \int (x_u \times x_v) \cdot (\varphi - \varphi_j) dudv \right| \leq 2c_3^\gamma \|\nabla x\|_{L^2} \|\nabla(\varphi - \varphi_j)\|_{L^2},$$

so x satisfies (4.2) for all $\varphi \in H^{1,2}$, and x is a weak CAMC surface in the sense of Definition 4.1. $\square$

Proposition 5.2. *Suppose (4.3), (4.16)-(4.17). Then the weak CAMC surface x obtained in Lemma 5.1 is $C^{2+\alpha}$ regular. Moreover, for any $K > 0$, $P : Y_1 \cap Y_K \rightarrow Z_1$ is proper.*

Proof. From Lemma 5.1, if $\epsilon_j \rightarrow \epsilon$ in Z_1 and $\{x^{\epsilon_j}\}_j \subset Y_K$ are CAMC surfaces with respect to γ^{ϵ_j} , then there is a subsequence, denoted also by $\{x^{\epsilon_j}\}_j$, and $x \in H^{1,2}$ such that

$$\nabla x^{\epsilon_j} \rightarrow \nabla x, \quad \text{strongly in } L^2,$$

and x is a weak CAMC surface with respect to γ^ϵ . Moreover from Theorem 4.2 and Lemma 4.3, $x \in C^{2+\alpha}$. Note that Corollary 3.2 implies that the $C^{2+\alpha}$ CAMC surface w.r.t. γ^ϵ is locally unique. So P is proper. $\square$

6. Proof of Main Theorem

Since S^2 is compact and for any CAMC immersion M of sphere S^2 there is a homeomorphism of S^2 onto M , we see that M is compact. For any $p \in S^2$, there is an open neighborhood $O(p)$ of p in S^2 such that

$$\int_{O(p)} |\nabla x|^2 < \infty,$$

where x is the conformal map from S^2 to M . Then from the compactness of S^2 ,

$$\int_{S^2} |\nabla x|^2 < \infty.$$

So there is $K > 0$ such that

$$\int_{S^2} |\nabla x|^2 < K$$

and $x \in Y_K$ (c.f. Lemma 5.1, the definition of Y_K). Thus from Proposition 2.2, Corollary 3.2 and Proposition 5.2, M is unique.

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