Invariant Subspaces Of Toeplitz Operators And Uniform Algebras

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Abstract Let T_ϕ be a Toeplitz operator on the one variable Hardy space H^2. We show that if T_ϕ has a nontrivial invariant subspace in the set of invariant subspaces of T_z then ϕ belongs to H^∞. In fact, we also study such a problem for the several variables Hardy space H^2.

§1. Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. A probability measure m (on X) denotes a representing measure for some nonzero complex homomorphism. The abstract Hardy space $H^p = H^p(m)$, $1 \le p \le \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak* closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$.

Let P be the orthogonal projection from L^2 onto H^2. For ϕ in L^∞, put

$$T_\phi f = P(\phi f) \quad (f \in H^2)$$

and then T_ϕ is called a Toeplitz operator. In this paper, we are interested in invariant subspaces of Toeplitz operators. Put $\mathcal{A} = \{T_\phi \ ; \ \phi \in H^\infty\}$ and $\mathcal{A}^* = \{T_\phi^* \ ; \ \phi \in H^\infty\}$. Lat T_ϕ denotes the set of all invariant subspaces of T_ϕ, Lat $\mathcal{A} = \cap \ \{\text{Lat } T_\phi \ ; \ \phi \in H^\infty\}$ and Lat $\mathcal{A}^* = \cap\{\text{Lat } T_\phi^* ; \phi \in H^\infty\}$. We don't know whether arbitrary T_ϕ has a nontrivial invariant subspace. When ϕ is in H^∞ and H^∞ has a nonconstant unimodular function q, T_ϕ has a nontrivial invariant subspace $M = qH^2$. Hence Lat $T_\phi \neq \{\langle 0 \rangle, \ H^2\}$.

Let K be the orthogonal complement of $\bar{H}^2$ in L^2. Then $L^2 = H^2 \oplus \bar{K}$. $I(H^\infty)$ denotes the set of all unimodular functions in H^∞. A function in $I(H^\infty)$ is called an inner function. For a subset Y in $L^\infty, Y^\perp$ denotes $\{g \in L^1 \ ; \ \int g\bar{f} dm = 0 \quad (f \in Y)\}$.

In this paper we study the following four natural questions :

Question 1. *If* Lat $T_\phi \supseteq$ Lat $\mathcal{A}$ *then does* T_ϕ *belong to* $\mathcal{A}$ *?*

Question 2. *Suppose that* H^∞ *is a weak* closed maximal algebra in* L^∞. *If* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$ *then is* Lat $T_\phi = \{\langle 0 \rangle, \ H^2\}$ *?*

Question 3. *Is* Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A} = \{\langle 0 \rangle, H^2\}$ *?*

Question 4. *Can we describe* Lat $T_\phi \cap$ Lat $\mathcal{A}$ *or equivalently* Lat $T_\phi \cap$ Lat $\mathcal{A}^*$?

In this paper, we will answer these four questions positively when A is the disc algebra. In fact, for Question 1 we can do it for more general uniform algebras. However for Question 2 we could not answer even for simple uniform algebras. Question 3 can be answered for almost all uniform algebras.

In this paper $H^p(D^n)$ denotes the Hardy space on the polydisc D^n and $H^p(\Omega)$ denotes the Hardy space on a finitely connected domain Ω. $L^p_a(D)$ denotes the Bergman space on D and put $N^2 = L^2(D) \ominus \{L^2_a(D) \oplus \bar{z}\bar{L}^2_a(D)\}$. H^p_0 denotes the set of $\{f \in H^p \ ; \ \int f dm = 0\}$. $H^p(\Gamma)$ denotes the usual Hardy space on the dual group $\hat{\Gamma}$ where Γ is an ordered subgroup of the reals.

§2. Lat $\mathcal{A} \subseteq$ Lat T_ϕ

In this section we study Question 1. Theorem 1 shows that Question 1 can be answered positively for very general uniform algebras.

Lemma 1. *Let M be a closed subspace of H^2. $M \in$ Lat T_ϕ if and only if $\phi M \subset M \oplus \bar{K}$.*

Proof. By definition of a Toeplitz operator, this is clear.

Lemma 2. *If ϕ is a function in L^∞ and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then $\phi = \phi_0 + \bar{k}_0$ where $\phi_0 \in H^2$ and $\bar{k}_0 \in \cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\}$.*

Proof. Since $L^2 = H^2 \oplus \bar{K}$, there exist $h \in H^2$ and $k \in K$ such that $\phi = h + \bar{k}$. If $q \in I(H^\infty)$ then $qH^2 \in$ Lat $\mathcal{A}$ and so by Lemma 1 $\phi q = qh + q\bar{k} \in qH^2 + \bar{K}$. Since $T_\phi q \in qH^2$ and $qh \in qH^2$, $P(q\bar{k}) \in qH^2$. Hence $q\bar{k} = q\ell + \bar{t}$ where $\ell \in H^2$ and $\bar{t} \in \bar{K}$. Therefore $\bar{k} = \ell + \overline{qt}$ and $\ell = \bar{k} - \overline{qt} \in H^2 \cap \bar{K} = \langle 0 \rangle$. Hence $\ell = 0$ and $\bar{k} = \overline{qt}$. This implies that k belongs to $\bar{q}\bar{K}$ for any $q \in I(H^\infty)$.

Theorem 1. *Suppose that $\cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\} = \langle 0 \rangle$. If ϕ is a function in L^∞ and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then ϕ belongs to H^∞.*

Proof. Lemmas 1 and 2 imply the theorem trivially.

Corollary 1. *Suppose that $H^2 = H^2(\mathbf{T}^N)$. If ϕ is a function in L^∞ and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then ϕ belongs to H^∞.*

Proof. K is an invariant subspace under multiplications by the coordinates functions $z_1, \cdots, z_n$. $\cap \{z_1^{\ell_1} \cdots z_n^{\ell_n} K\ ;\ (\ell_1, \cdots, \ell_n) \geq (0, \cdots, 0)\}$ is a reducing subspace and so $\cap\ z_n^{\ell_1} \cdots z_n^{\ell_n} K = \chi_E L^2$ for some characteristic function χ_E. Since $\chi_E L^2$ is orthogonal to $\bar{H}^2$, $\chi_E = 0$ and so $\langle 0 \rangle = \cap \bar{z}_1^{\ell_1} \cdots \bar{z}_n^{\ell_n} \bar{K} = \cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\}$.

Corollary 2. *Suppose that $H^2 = H^2(\Omega)$. If ϕ is a function in L^∞ and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then ϕ belongs to H^∞.*

Proof. Let Z be the Ahlfors function for Ω then $\mid Z \mid = 1$ on $\partial\Omega = X$ (see [3]). $\bigcap_{n=0}^{\infty} \bar{Z}^n \bar{K}$ is invariant under the multiplications by Z and $\bar{H}^\infty$. Since H^∞ is a weak$*$ maximal subalgebra of L^∞, $\bigcap_{n=0}^{\infty} \bar{Z}^n \bar{K} = \chi_E L^2$. Since $\chi_E L^2$ is orthogonal to H^2, $\chi_E = 0$ and so $\cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\} = \{0\}$.

Corollary 3. *Let A be a Dirichlet algebra (see [4]). If ϕ is a function in L^∞ and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then ϕ belongs to H^∞.*

Proof. Since H^∞ is a uniform algebra which has the annulus property ([2],[6]) on a totally disconnected space, by [2, Theorem 1] the set of quotients of inner functions is norm dense in the set of unimodular functions in L^∞. In this situation, $\bar{K} = \bar{H}_0^2$ and $Y = \cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\} \subset \bar{H}^2$. $\bar{q}Y = Y$ for any q in $I(H^\infty)$ and so $\bar{q}_1 q_2 Y \subseteq Y$ for any q_1, q_2 in $I(H^\infty)$. Hence $\phi Y \subseteq Y$ for any unimodular function ϕ in L^∞. Hence $Y = \chi_E L^2$ for the characteristic function χ_E for some set E. Since $Y \subset \bar{H}^2$, Y must be $\{0\}$.

Proposition 1. *Suppose that $H^2 = L_a^2(D)$, ϕ is a function in L^∞ and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ. *Then the following are valid.*

(1) ϕ belongs to $L_a^2(D) + N^2$.

(2) If $\phi = f + \ell$ where $f \in H^\infty$ and $\ell \in N^2$ then Lat $T_\ell \supseteq$ Lat $\mathcal{A}$.

Proof. (1) Since $zL_a^2 \in$ Lat T_ϕ by hypothesis, $\mathbb{C} \in$ Lat $T_\phi^* =$ Lat $T_{\bar{\phi}}$ and so $\bar{\phi} = \bar{c} + \bar{k}$ where $c \in \mathbb{C}$ and $k \in zL_a^2(D) + N^2$. Hence $\phi \in L_a^2(D) + N^2$. (2) If $\phi = f + \ell$ and $M \in$ Lat $\mathcal{A}$ then $\phi M \subset M + \bar{K}$. Hence $(f+\ell)g = fg + \ell g \in M + \bar{K}$ for any $g \in M$. Since $fg \in M$, $\ell g \in M + \bar{K}$ for any $g \in M$ and so $\ell M \subset M + \bar{K}$. Thus $M \in$ Lat T_ℓ.

A bounded operator B is called reflexive if whenever C is a bounded operator and Lat $B \subseteq$ Lat C then C belongs to the closed algebra (in weak operator topology) generated by B. When B is subnormal, it is known that B is reflexive [7]. Hence if f is a nonzero function in H^∞ and Lat $T_\phi \supseteq$ Lat T_f then T_ϕ belongs to the closed algebra generated by T_f. Hence T_ϕ belongs to $\mathcal{A}$. Usually Lat $\mathcal{A} \underset{\neq}{\subset}$ Lat T_f and so this does not answer Question 1. However if there exists a function f in H^∞ such that Lat $T_f =$ Lat $\mathcal{A}$ then the above result about subnormal operators answers Question 1. Hence when $H^2 = H^2(T)$, if Lat $T_\phi \supseteq$ Lat $\mathcal{A}$ then T_ϕ belongs to $\mathcal{A}$ because Lat $T_z =$ Lat $\mathcal{A}$. Therefore Corollary 1 is not new for $N = 1$. Similarly Question 1 can be answered for $H^2 = L_a^2(D)$. Hence Proposition 1 is a very weak result.

§3. Lat $T_\phi \underset{\neq}{\subset}$ Lat $\mathcal{A}$

In this section we study Question 2. Theorem 2 shows that Question 2 can be answered positively for the disc algebra. In fact, it gives a few results for more general uniform algebras about Question 2.

Lemma 3. *Let Q be a function in $I(H^\infty)$. Then* $\bar{K} = \sum_{n=0}^{\infty} \oplus (\bar{K} \ominus \bar{Q}\bar{K})\bar{Q}^n \oplus \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K}$.

Proof. Since $|Q| = 1$ *a.e.* and $\bar{Q}\bar{K} \subset \bar{K}, \bar{Q}$ is an isometry on $\bar{K}$. Hence this is well known and called a Wold decomposition.

Theorem 2. *Suppose that* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$*. If* $M \in$ Lat T_ϕ *and* $\cap \{\bar{Q}^n \bar{K} \ ; \ Q \in \mathcal{I}\} = \{0\}$ *for some subset* $\mathcal{I}$ *in* $I(H^\infty)$ *then there exists a nonconstant* Q *in* $\mathcal{I}$ *such that* $M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle$ *or* $\phi M \subseteq M$*.*

Proof. If $M \in$ Lat T_ϕ then by Lemma 1 there exist $f \in M$, $g \in M$ and $k \in K$ such that $\phi f = g + \bar{k}$. If $\phi M \not\subseteq M$ then we may assume that $k \neq 0$. For any fixed $Q \in \mathcal{I}$, by Lemma 3 $\bar{K} = \left\{ \sum_{n=0}^{\infty} \oplus (\bar{K} \ominus \bar{Q}\bar{K})\bar{Q}^n \right\} \oplus \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K}$ and so

$$\bar{k} = \sum_{n=0}^{\infty} k_n \bar{Q}^n + k_\infty$$

where $k_n \in \bar{K} \ominus \bar{Q}\bar{K}$ $(n = 0, 1, 2, \cdots)$ and $k_\infty \in \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K}$. Then $Q\bar{k} = Qk_0 + \sum_{n=1}^{\infty} k_n \bar{Q}^{n-1} + Qk_\infty$ and by Lemma 1 $Q\bar{k}$ belongs to $M + \bar{K}$ because $\phi f = g + \bar{k}$ and $QM \subset M$.

Suppose that there does not exist a nonconstant function Q in $\mathcal{I}$ such that $M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle$. Then we will get a contradiction. By what was proved above, Qk_0 belongs to $M \cap (H^2 \ominus QH^2) = \{0\}$. Hence $k_0 \equiv 0$. Next we consider $Q^2 \bar{k}$ and then $k_1 \equiv 0$ follows. Proceeding similarly we can show that $\bar{k} = k_\infty$. By hypothesis, this implies that $\bar{k} \equiv 0$ because Q is arbitrary in $\mathcal{I}$. This contradiction implies that there exists $Q \in I$ such that $M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle$.

Corollary 4. *Suppose that* $H^2 = H^2(\mathbf{T}^N)$, ϕ *is a function in* L^∞ *and* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$*. If* $M \in$ Lat T_ϕ *and* $M \neq \langle 0 \rangle$ *then* M *contains a nonzero function which is* $(N-1)$*-variable. Hence if* $N = 1$ *then* $M = H^2$*.*

Proof. It is known that if $\phi M \subseteq M$ then $\phi \in H^\infty$. Hence we may assume that $\phi M \not\subseteq M$. Put $\mathcal{I} = \{z_1, \cdots, z_N\}$ then $\mathcal{I}$ satisfies the condition of Theorem 2. By Theorem 2, there exists z_j such that $1 \leq j \leq N$ and $(H^2 \ominus z_j H^2) \cap M \neq \{0\}$. Since $H^2 \ominus z_j H^2 = H^2(z'_j, T^{N-1})$ where $z = (z_j, z'_j)$, M contains a nonzero $(N-1)$-variable function.

Corollary 5. *Suppose that* $H^2 = H^2(\Omega)$*,* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$ *and* Z *is the Alfors function for* Ω *(see [3]). If* $M \in$ Lat T_ϕ *and* $M \neq \langle 0 \rangle$ *then* $M \cap (H^2 \ominus ZH^2) \neq \langle 0 \rangle$*.*

Proof. Put $\mathcal{I} = \{Z\}$ then $\mathcal{I}$ satisfies the condition of Theorem 2. It is known that if $\phi M \subseteq M$ then $\phi \in H^\infty$. Hence we may assume that $\phi M \not\subseteq M$.

Proposition 2. *If* T_ϕ *is subnormal and* Lat $T_\phi \subseteq$ Lat $\mathcal{A}$ *then* T_ϕ *commutes with* $\mathcal{A}$ *and so* $T_\phi f = P(\phi_0 f)$ $(f \in H^\infty)$ *for some* ϕ_0 *in* H^2*. If* A *is a uniform algebra which approximates in modulus on* X *then* ϕ *belongs to* $H^2 \cap L^\infty$*.*

Proof. If T_ϕ is subnormal and Lat $T_\phi \subseteq$ Lat $\mathcal{A}$ then it is known [7] that $\mathcal{A}$ is contained in the closed algebra generated by T_ϕ. Hence T_ϕ commutes with $\mathcal{A}$. Let $\phi_0 =$

$T_\phi 1$ then $T_\phi f = T_\phi T_f 1 = T_f T_\phi 1 = P(\phi_0 f)$ for $f \in H^\infty$. Since $\|\phi_0 f\|_2 \leq \|T_\phi\| \|f\|_2$ ($f \in H^\infty$),

$$\left| \int_X \phi_0 f \bar{g} dm \right| \leq \|\phi\|_\infty \|f\|_2 \|g\|_2 \quad (f, g \in H^\infty).$$

Hence

$$\left| \int_X \phi_0 \mid f \mid^2 dm \right| \leq \|\phi\|_\infty \|f^2\|_1.$$

Since A approximates in modulus on X, ϕ_0 belongs to $H^2 \cap L^\infty$. It is easy to see that $\phi = \phi_0$.

Corollary 6. *Suppose that* $H^2 = H^2(\mathbf{T}^N)$ *or* $H^2 = H^2(\Omega)$. *If* T_ϕ *is subnormal then* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$ *or* ϕ *belongs to* H^∞.

Proof. A uniform algebra A approximates in modulus on X, that is , for every positive continuous function g on X and $\varepsilon > 0$, there is an f in A with $|g - |f|| < \varepsilon$ if the set of unimodular elements of A separates points of X (see [6, Lemma 4.12]). Since the coordinate functions $z_1, \cdots, z_n$ separate T^N, the polydisc algebra approximates in modulus on T^N. If T_ϕ is subnormal on $H^2(T^N)$ and Lat $T_\phi \subseteq$ Lat $\mathcal{A}$ then by Proposition 2 ϕ belongs to $H^2(T^N) \cap L^\infty = H^\infty(T^N)$. If $A = H^\infty(\Omega)$ then by [3, Lemma 4.8] $I(H^\infty(\Omega))$ separates $X =$ the maximal ideal space of $L^\infty(\partial D)$. Hence Corollary 6 for $H^2 = H^2(\Omega)$ follows from Proposition 2.

Proposition 3. *If* Lat $T_\phi \subseteq$ Lat $\mathcal{A}$, *then* Lat $T_\phi^* \cap$ Lat $T_\phi \subset$ Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A}$.

Proof. If $M \in$ Lat T_ϕ^* then $M^\perp \in$ Lat T_ϕ and so $M^\perp \in$ Lat $\mathcal{A}$ because Lat $T_\phi \subseteq$ Lat $\mathcal{A}$. Hence $M \in$ Lat $\mathcal{A}^*$ and so Lat $T_\phi^* \subseteq$ Lat $\mathcal{A}^*$.

By Proposition 3, when Let $\mathcal{A}^* \cap$ Lat $\mathcal{A} = \{\langle 0 \rangle, H^2\}$, if Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$ then T_ϕ does not have a nontrivial reducing subspace. Hence if T_ϕ is normal then Lat $T_\phi \not\subset$ Lat $\mathcal{A}$. Therefore it is important to know that Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A} = \{\langle 0 \rangle, H^2\}$, that is, $\mathcal{A}$ is irreducible.

§4. Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A}$

In this section we study Question 3. Theorem 3 shows that Question 3 can be answered positively for usual uniform algebras. Recall $\mathcal{A}^* = \{T_\phi^* \ ; \ \phi \in H^\infty\}$.

Theorem 3. *If* $M \in$ Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A}$ *then* $M \subset \chi_E L^2 \subset M + \bar{K}$ *where* $E = \cup\{\text{supp } f \ ; \ f \in M\}$. *Hence if* $E = X$ *then* $M = H^2$.

Proof. If $\phi \in L^\infty$ then by the Stone-Weierstrass theorem for any $\varepsilon > 0$ there exist $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$ in H^∞ such that $\| \ \phi - \sum_{j=1}^n f_j \bar{g}_j \ \|_\infty < \varepsilon$. Since $T_{f_j \bar{g}_j} M \subset M$ for

$j=1,\cdots,n,\ T_\phi M\subset M$. By Lemma 1 $\phi M\subset M\oplus\bar{K}$. Thus $\chi_E L^2\subset M\oplus\bar{K}$. If $E=X$ then $L^2=M\oplus\bar{K}$ and so $M=H^2$.

Corollary 7. *Suppose that there does not exist a nonzero function in H^2 such that $m(\{x\in X\ ;\ f(x)=0\})\neq 0$. If $M\in$ Lat $\mathcal{A}^*\cap$ Lat $\mathcal{A}$ then $M=\langle 0\rangle$ or H^2.*

§5. Lat $T_\phi\cap$ Lat $\mathcal{A}$

In this section we study Question 4. We don't know whether Lat $T_\phi\neq\{\langle 0\rangle,H^2\}$. However we show that Lat $T_\phi\cap$ Lat $\mathcal{A}=\{\langle 0\rangle,H^2\}$ if $\phi\notin H^\infty$ and $H^2=H^2(\mathbf{T})$. For any M in Lat T_ϕ, put

$$K_M=\{k\in K\ ;\ \bar{k}=\phi f-g \text{ for some } f \text{and } g\in M\}$$

, then $K_M\subseteq K$ and $\phi M\subset M+\bar{K}_M$ (see Lemma 1).

Theorem 4. *If $M\in$ Lat $T_\phi\cap$ Lat $\mathcal{A}$ then $K_M\times(H^2\ominus M)\subseteq(H^\infty)^\perp$ and $T_k^*(H^\infty)\subseteq M$ for any k in K_M.*

Proof. By the remark above, if $M\in$ Lat $T_\phi\cap$ Lat $\mathcal{A}$ then $\phi M\subset M+\bar{K}_M$. If $k\in K_M$ then by its definition there exist f and g such that $\phi f=g+\bar{k}$. For any $\ell\in H^\infty$, $\phi f\ell=g\ell+\bar{k}\ell\in M+\bar{K}_M$ and so $P(\bar{k}\ell)\in M$. Since

$$\bar{k}\ell=P(\bar{k}\ell)+(I-P)(\bar{k}\ell)\in M+\bar{K}_M,$$

if $s\in H^2\ominus M$ then $\langle\bar{k}\ell,s\rangle=\langle P(\bar{k}\ell),s\rangle=0$. Hence ks belongs to $(H^\infty)^\perp$ and so $K_M\times(H^2\ominus M)\subseteq(H^\infty)^\perp$. The above proof implies that $T_k^*(H^\infty)\subseteq M$.

Corollary 8. *Suppose that $H^2=H^2(\Omega)$, $\not{C}\backslash\Omega$ has n components and $\phi\notin H^\infty$. If $M\in$ Lat $T_\phi\cap$ Lat $\mathcal{A}$ then $\dim(H^2\ominus M)\le n$.*

Proof. By Theorem 4

$$K_M\times(H^2\ominus M)\subseteq(H^\infty)^\perp\cap(\bar{H}^\infty)^\perp=(H^\infty+\bar{H}^\infty)^\perp\cap L^1$$

and $\dim(H^\infty+\bar{H}^\infty)^\perp\cap L^1=n$ because $\not{C}\backslash\Omega$ has n components. If $K_M=\langle 0\rangle$ then $\phi M\subset M$. It is known [4] that L^∞ is generated by ϕ and H^∞ in the weak* topology. Hence $M\in$ Lat $\mathcal{A}\cap$ Lat $\mathcal{A}^*=\{\langle 0\rangle,H^2\}$ by Corollary 7 and so $M=H^2$. It is clear that if $K_M\neq\langle 0\rangle$ then $\dim(H^2\ominus M)\le n$.

Corollary 9. *If $H^2=H^2(\mathbf{T})$ and $\phi\notin H^\infty$ then* Lat $T_\phi\cap$ Lat $\mathcal{A}=\{\langle 0\rangle,H^2\}$.

Proof. When Ω is the open unit disc, $H^2(\Omega)=H^2(\mathbf{T})$ and so by Corollary 8 Lat $T_\phi\cap$ Lat $\mathcal{A}=\{\langle 0\rangle,H^2\}$.

Corollary 10. *Let A be a Dirichlet algebra. If $\phi \notin H^\infty$ then* Lat $T_\phi \cap$ Lat $\mathcal{A} = \{\langle 0\rangle, H^2\}$.

Proof. It is known that $(\bar{H}^\infty)^\perp \cap (H^\infty)^\perp = \langle 0\rangle$. The corollary is a result of Theorem 4.

In general, it seems to be difficult to describe Lat $T_\phi \cap$ Lat $\mathcal{A}$. When $H^2 = H^2(\Omega)$ and $\bar{\phi} \in H^\infty$, Lat $T_\phi \cap$ Lat $\mathcal{A} = \{\langle 0\rangle, H^2\}$ by Corollary 8. In fact, if $M \in$ Lat $T_\phi \cap$ Lat $\mathcal{A}$ then $\bar{\phi}(H^2 \ominus M) \subseteq H^2 \ominus M$. Since $\dim(H^2 \ominus M) < \infty$ by Corollary 8, M must be equal to H^2. When $H^2 = H^2(T^2)$ and $\phi = \bar{z}$, Lat $T_\phi \cap$ Lat $\mathcal{A} = \{\langle 0\rangle, qH^2(w,T); q = q(w)$ is a one variable inner function$\}$ where z and w are the indepedent variables on T^2. In fact, if $M \in$ Lat $T_\phi \cap$ Lat $\mathcal{A}$ then $T_z^* M_1$ is orthogonal to M where $M_1 = M \ominus zM$. Since $T_z^* M_1 \subset M$, $T_z^* M_1 = \langle 0\rangle$ and so $M_1 \subset H^2(w,T)$. Corollary 10 shows that Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A} = \{\langle 0\rangle, H^2\}$ if A is a Dirichlet algebra.

References

1. R. G. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, 1972

2. R. G. Douglas and W. Rudin, Approximation by inner functions, Pacific J.M. 31(1969), 313-320

3. S. Fisher, Function Theory On Planar Domains, John Wiley and Son, New York, 1983

4. T. Gamelin, Uniform Algebras, Prentice-Hall, Englewood Cliffs, N.J.. 1969

5. T. W. Gamelin, J.B.Garnett, L.A.Rubel, and A.L.Shields, On badly approximable functions, J.Approx.Theory 17(1976), 280-296

6. I. Glicksberg, Measures orthogonal to algebras and sets of antisymmetry, Trans. Amer. Math.Soc. 105(1962), 415-435

7. R. F. Olin and J.E.Thomson, Algebras of subnormal operators, J.Funct.Anal. 37(1980), 271-301

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