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Global Existence of Small Classical Solutions to Nonlinear Schrödinger Equations

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Abstract

We study the global Cauchy problem for nonlinear Schrödinger equations with cubic interactions of derivative type in space dimension $n \geq 3$. The global existence of small classical solutions is proved in the case where every real part of the first derivatives of the interaction with respect to first derivatives of wavefunction is derived by a potential function of quadratic interaction. The proof depends on the energy estimate involving the quadratic potential and on the endpoint Strichartz estimates.

1 Introduction

In this paper we consider the Cauchy problem for nonlinear Schrödinger equations of the form

$$i\partial_t u + \frac{1}{2}\Delta u = F(u, \nabla u), \tag{1.1}$$

where u is a complex-valued function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, Δ is the Laplacian in $\mathbb{R}^n$, $\partial_t = \partial/\partial t$, $\nabla = (\partial_1, \cdots, \partial_n)$, $\partial_j = \partial/\partial x_j$, and F is a smooth function on $\mathbb{C}\times\mathbb{C}^n$ vanishing of third order at the origin. Here we do not assume analyticity of F and we consider the derivatives in the real sense. For instance, for $(z,p) \in \mathbb{C}\times\mathbb{C}^n$, F' is defined as a linear operator on $\mathbb{C}\times\mathbb{C}^n$:

$$F'(z,p)(\xi,q) = \frac{\partial F}{\partial z}\xi + \frac{\partial F}{\partial p}q + \frac{\partial F}{\partial \bar z}\bar\xi + \frac{\partial F}{\partial \bar p}\bar q$$

for $(\xi,q) \in \mathbb{C}\times\mathbb{C}^n$, where we have used the standard notation such as

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

for $z = x + iy$. Accordingly, it is sometimes convenient to regard F as a function of $(z,p,\bar z,\bar p) \in \mathbb{C}\times\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}^n$. Moreover, we use the notation $\partial F/\partial u$, $\partial F/\partial(\nabla u)$, $\partial F/\partial \bar u$, $\partial F/\partial(\nabla \bar u)$ for the derivatives at $(u, \nabla u, \bar u, \nabla \bar u)$ and the associated complex-valued functions on $\mathbb{R}\times\mathbb{R}^n$ as well.

There is a large literature on the Cauchy problem for (1.1). See for instance [1-12,16-18,20,24-32] and references therein. The classical energy method naturally requires that the real part of every component of $\partial F/\partial(\nabla u)$ vanishes. We write the condition as

$$\mathrm{Re}\ \frac{\partial F}{\partial(\nabla u)} = 0. \tag{1.2}$$

With the condition (1.2), Klainerman [18], Klainerman and Ponce [20], and Shatah [28], proved the global existence of classical solutions for small Cauchy data with sufficient regularity and decay at infinity. Here the decay at infinity is imposed on the Cauchy data ϕ in such a way that $\phi \in H^m_{p'}(\mathbb{R}^n)$ with an integer $m > n/2$, $p > 2$ for instance, which provides explicit time-decay of solutions in $L^p(\mathbb{R}^n)$. Here $H^s_q = H^s_q(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^q(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^q$ is the Sobolev space in terms of Bessel potential and p' is the exponent dual to p defined by $1/p + 1/p' = 1$.

Chihara [3, 4] and Hayashi, Miao, and Naumkin [8] removed the condition (1.2) by using smoothing operators and first order partial differential operators which have special commutation relations with $i\partial_t + (1/2)\Delta$. In [3, 4, 8, 19], decay at infinity is imposed on the Cauchy data ϕ in such a way that $x^\alpha\phi \in H^m$ with an integer $m > n/2$, $|\alpha| \le 2$ for instance, which also provides explicit time-decay of solutions in L^p for some $p > 2$

through the first order partial differential operators above. Here $H^s = H^s_2$ is the standard Sobolev space.

The purpose in this paper is to remove those assumptions related to decay at infinity of the Cauchy data ϕ and reduce the required regularity down to $n/2+2$ (limit excluded) such as $\phi \in H^s$ with $s > n/2+2$. The condition $s > n/2+2$ is most natural in the framework of classical solutions. Instead, we need an assumption on the structure of nonlinearity which is weaker than (1.2).

To state the main result precisely, we introduce some notation. Throughout this paper we denote by σ any real number larger than $n/2$ and by 2^* the Sobolev exponent $2n/(n-2)$. It is well known that $H^\sigma \hookrightarrow L^\infty$ and $H^\sigma_{2^*} \hookrightarrow W^1_\infty$, where $W^m_p = \{f \in L^p : \partial^\alpha f \in L^p \text{ for all } \alpha \text{ with } |\alpha| \le m\}$. The main assumption on F is the following:

(H) There exists a function $\theta \in C^2((0,\infty);\mathbb{R})$ with $\theta(0) = 0$ such that

$$\text{Re } \frac{\partial F}{\partial(\nabla u)} = \nabla(\theta(|u|^2)). \tag{1.3}$$

Theorem. *Let $n \ge 3$ and let $\sigma > n/2$. Let $s = \sigma + 2$. Let F be smooth function vanishing of third order at the origin and satisfying (H). Then there exists $\delta > 0$ such that for any $\phi \in H^s$ with $\|\phi; H^s\| \le \delta$ the equation (1.1) has a unique global solution $u \in (C_w \cap L^\infty)(\mathbb{R}; H^s) \cap C(\mathbb{R}; H^{s-1}) \cap L^2(\mathbb{R}; H^{s-1}_{2^*})$ with $u(0) = \phi$. Moreover, there exist $\phi_\pm \in H^s$ such that*

$$\|u(t) - U(t)\phi_\pm; H^{s-1}\| \to 0$$

as $t \to \pm\infty$, where $U(t) = \exp(i\frac{t}{2}\Delta)$ is the free propagator.

Remark 1. *When $\theta = 0$, the assumption (H) reduces to (1.2). An example of F satisfying (1.2) is given by*

$$F(u, \nabla u) = i\sum_{j=1}^{n}(a_j|u|^2 + b_j|\partial_j u|^2)\partial_j u + F_0(u, \nabla u),$$

where $a_j, b_j \in \mathbb{R}$ and F_0 satisfies

$$\frac{\partial F_0}{\partial(\nabla u)} = 0.$$

Remark 2. *If F has the form*

$$F(u, \nabla u) = \lambda(\nabla u)^2\bar{u} + \mu|\nabla u|^2 u + F_1(u, \nabla u)$$

with $\lambda, \mu \in \mathbb{R}$ and F_1 satisfying (1.2), then θ defined by $\theta(\rho) = \left(\lambda + \frac{\mu}{2}\right)\rho$ satisfies (H).

Remark 3. *If F has the form*

$$F(u,\nabla u)=\frac{\lambda(\nabla u)^2\bar{u}+\mu|\nabla u|^2u}{1+|u|^2}+F_1(u,\nabla u)$$

with $\lambda,\mu\in\mathbb{R}$ and F_1 satisfying (1.2), then θ defined by $\theta(\rho)=\left(\lambda+\dfrac{\mu}{2}\right)\log(1+\rho)$ satisfies (H). The case where $\mu=0$ and $F_1=0$ appears as a model of Schrödinger map [2, 24]. See also [11, 15, 29, 30].

In the theorem above, assumptions on the Cauchy data are given exclusively on the basis of the usual Sobolev spaces. There is no assumption of additional decay at infinity previously imposed in terms of L^q spaces with $1\le q\le 2$ [18, 20, 28] and weighted Sobolev spaces [3, 4, 8, 19]. Moreover, the required regularity is minimal as far as the classical solutions are concerned. Those two ingredients are new even when (H) is replaced by more restrictive assumption (1.2).

We prove the theorem in the next section. The proof depends on the a priori estimates of two kinds. At the level of H^{s-1}, we use the endpoint Strichartz estimates [14], which lose first derivatives at L^2 level but still ensure square integrability in time of solutions with values in $H^{s-1}_{2^*}$. At the level of H^s, we estimate loss of derivatives by means of "gauge transformation" given by the multiplication by $\exp(\pm\theta(|u|^2))$, which enables us to provide a priori estimates of H^s norm of gauge transformed solutions. There appear coefficients bounded by a constant multiple of the $H^{s-1}_{2^*}$ norm squared, time integrability of which has been ensured by the argument at the level of H^{s-1}.

The importance of the endpoint Strichartz estimates in cubic nonlinearities has been noticed in [21, 22, 23], though this paper seems to be the first application of the endpoint Strichartz estimates to nonlinear Schrödinger equations of the form (1.1). "Gauge transformation" technique has been exploited in [9, 10, 26, 27, 32], though this paper seems to be the first that shows how the endpoint Strichartz estimates come into play in the a priori energy estimates with transformed derivatives.

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2 Proof of the theorem.

For simplicity, we treat the case where F is a cubic polynomial. We restrict our attention to the case $t > 0$ since the case $t < 0$ is treated analogously. Let $\phi \in H^s$. For $\varepsilon > 0$ we consider the regularized equation

$$i\partial_t u_\varepsilon + \frac{1}{2}(1 - i\varepsilon)\Delta u_\varepsilon = F(u_\varepsilon, \nabla u_\varepsilon) \tag{2.1}$$

in $(0, \infty) \times \mathbb{R}^n$ with $u_\varepsilon(0, x) = \phi(x),\ x \in \mathbb{R}^n$. By the standard method we see that there exists a unique local solution $u_\varepsilon \in C([0, T_\varepsilon); H^s) \cap C^1((0, T_\varepsilon) : H^{s-2})$. Here $T_\varepsilon > 0$ may be taken $T_\varepsilon = \infty$ if we can show an a priori estimate in H^s for local solutions.

From now on we abbreviate the subscript ε to write $u = u_\varepsilon$ for simplicity. We write the equation (2.1) in the integral form:

$$u(t) = U_\varepsilon(t)\phi - i\int_0^t U_\varepsilon(t - t')F(u(t'), \nabla u(t'))dt', \tag{2.2}$$

where $U_\varepsilon(t) = \exp(i\frac{t}{2}(1 - i\varepsilon)\Delta)$. We note here that the regularizing factor $\exp(\frac{t\varepsilon}{2}\Delta)$ is a contraction semigroup in L^p for any p with $1 \le p < \infty$ and therefore the propagator U_ε has the same Strichartz estimates as those of the usual Schrödinger group on positive time intervals of the form $[0, T]$ with $T > 0$. We now apply the endpoint Strichartz estimates to (2.2) on the interval $[0, T]$ to obtain

$$\begin{aligned}
&\|u; L^\infty(L^2) \cap L^2(L^{2^*})\| \equiv \max(\|u; L^\infty(L^2)\|,\ \|u; L^2(L^{2^*})\|) \\
&\le C\|\phi; L^2\| + C\|F(u, \nabla u); L^1(L^2)\| \\
&\le C\|\phi; L^2\| + C\|u; L^2(W^1_\infty)\|^2\|u; L^\infty(H^1)\| \\
&\le C\|\phi; L^2\| + C\|u; L^2(H^\sigma_{2^*})\|^2\|u; L^\infty(H^1)\|,
\end{aligned} \tag{2.3}$$

where we have used Hölder's inequalities in space and time and the Sobolev embedding $H^\sigma_{2^*} \hookrightarrow W^1_\infty$.

We differentiate (2.2) to have

$$\begin{aligned}
&\partial_j u(t) = U_\varepsilon(t)\partial_j\phi \\
&-i\int_0^t U_\varepsilon(t - t')\left(\frac{\partial F}{\partial u}\partial_j u + \frac{\partial F}{\partial \bar{u}}\partial_j \bar{u} + \frac{\partial F}{\partial(\nabla u)}\partial_j \nabla u + \frac{\partial F}{\partial(\nabla \bar{u})}\partial_j \nabla \bar{u}\right)(t')dt'.
\end{aligned} \tag{2.4}$$

By the endpoint Strichartz estimates, fractional Leibniz rule, and estimates on composite

functions in the case where F is not a polynomial, we obtain

$$
\begin{aligned}
&\|\partial_j u; L^\infty(H^\sigma) \cap\ L^2\ (H^\sigma_{2*})\| \\
&\le C\|\partial_j \phi; H^\sigma\| +\ C\ \|\frac{\partial F}{\partial u}\partial_j u; L^1(H^\sigma)\| +\ C\ \|\frac{\partial F}{\partial \bar{u}}\partial_j u; L^1(H^\sigma)\| \\
&\qquad + C\ \|\frac{\partial F}{\partial(\nabla u)}\partial_j \nabla u; L^1(H^\sigma)\| + C\|\frac{\partial F}{\partial(\nabla \bar{u})}\partial_j \nabla \bar{u}; L^1(H^\sigma)\| \\
&\le C\ \|\phi; H^{\sigma+1}\| +\ C\ \|u; L^2(W^1_\infty)\|^2 \|u; L^\infty(H^{\sigma+2}_2)\| \\
&\qquad + C\ \|u; L^2(W^1_\infty)\| \|u; L^2(H^{\sigma+1}_{2^*})\| \|u; L^\infty(H^2_n)\| \\
&\le C\ \|\phi; H^{s-1}\| +\ C\ \|u; L^2(H^{\sigma+1}_{2^*})\|^2 \|u; L^\infty(H^s)\|,
\end{aligned}
\tag{2.5}
$$

where we have used Hölder's inequality in space and time and embeddings $H^\sigma_{2*} \hookrightarrow W^1_\infty$ and $H^{\sigma+2} \hookrightarrow H^2_n$.

By (2.3) and (2.5), we have

$$
\begin{aligned}
&\|u; L^\infty(H^{s-1}) \cap L^2(H^{s-1}_{2^*})\| \\
&\le C\|\phi; H^{s-1}\| + C\|u; L^2(H^{s-1}_{2^*})\|^2 \|u; L^\infty(H^s)\|.
\end{aligned}
\tag{2.6}
$$

By (2.1) or (2.4), we have

$$
\left(i\partial_t + \frac{1}{2}(1 - i\varepsilon)\Delta\right)\partial_j u = \frac{\partial F}{\partial u}\partial_j u + \frac{\partial F}{\partial \bar{u}}\partial_j \bar{u} + \frac{\partial F}{\partial(\nabla u)}\partial_j \nabla u + \frac{\partial F}{\partial(\nabla \bar{u})}\partial_j \nabla \bar{u}. \tag{2.7}
$$

With $\Gamma = (1-\Delta)^{(s-1)/2}$, we have by (H) and (2.7)

(2.8)

$$
\begin{aligned}
&\frac{d}{dt}\|e^{-\theta(|u|^2)}\partial_j\Gamma u; L^2\|^2 \\
&= \ 2\mathrm{Im}\ (\left(i\partial_t + \frac{1}{2}\Delta\right)(e^{-\theta(|u|^2)}\partial_j\Gamma u),\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&= \ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}(-\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\theta(|u|^2)\right]\partial_j\Gamma u + \left(i\partial_t + \frac{1}{2}\Delta\right)\partial_j\Gamma u \\
&\qquad\qquad -\nabla(\theta(|u|^2))\cdot\nabla\partial_j\Gamma u), e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&= \ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}(-\left[\left(i\partial_t + \frac{1}{2}(1-i\varepsilon)\Delta\right)\theta(|u|^2)\right]\partial_j\Gamma u + \left(i\partial_t + \frac{1}{2}(1-i\varepsilon)\Delta\right)\partial_j\Gamma u \\
&\qquad\qquad -\left(\mathrm{Re}\ \frac{\partial F}{\partial(\nabla u)}\right)\cdot\nabla\partial_j\Gamma u), e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&- \varepsilon\mathrm{Re}\ (e^{-\theta(|u|^2)}[\Delta\theta(|u|^2)]\partial_j\Gamma u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ \varepsilon\mathrm{Re}\ (e^{-\theta(|u|^2)}\Delta\partial_j\Gamma u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&= \ -\ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}[\left(i\partial_t + \frac{1}{2}(1-i\varepsilon)\Delta\right)\theta(|u|^2)]\partial_j\Gamma u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}\Gamma\left(\frac{\partial F}{\partial u}\partial_j u + \frac{\partial F}{\partial \bar{u}}\partial_j\bar{u}\right),\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ 2\mathrm{Re}\ (e^{-\theta(|u|^2)}\left(\mathrm{Im}\frac{\partial F}{\partial(\nabla u)}\right)\cdot\nabla\partial_j\Gamma u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}\left[\Gamma, \frac{\partial F}{\partial(\nabla u)}\right]\cdot\nabla\partial_j u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}\left(\frac{\partial F}{\partial(\nabla\bar{u})}\right)\cdot\nabla\partial_j\Gamma\bar{u},\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ 2\mathrm{Im}\ (e^{-\theta(|u|^2)}\left[\Gamma, \frac{\partial F}{\partial(\nabla\bar{u})}\right]\cdot\nabla\partial_j\bar{u},\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&- \varepsilon\mathrm{Re}\ (e^{-\theta(|u|^2)}[\Delta\theta(|u|^2)]\partial_j\Gamma u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u) \\
&+ \varepsilon\mathrm{Re}\ (e^{-\theta(|u|^2)}\Delta\partial_j\Gamma u,\ e^{-\theta(|u|^2)}\partial_j\Gamma u),
\end{aligned}
$$

where $(\cdot,\cdot)$ and $[\cdot,\cdot]$ denote the scalar product in L^2 and the commutator of operators, respectively. We denote by I,$\cdots$ VIII the first,$\cdots$, eighth terms on the RHS of the last equality in (2.8) and consider those contributions separately. For I, we compute

$$\begin{aligned}
&\left(i\partial_t + \frac{1}{2}(1-i\varepsilon)\Delta\right)\theta(|u|^2)\\
&= \ 2(1-i\varepsilon)\theta''(|u|^2)(\mathrm{Re}(\bar{u}\nabla u))^2\\
&+ 2i\theta'(|u|^2)\ \mathrm{Im}\ (\bar{u}F(u,\nabla u))\\
&+ (1-i\varepsilon)\theta'(|u|^2)|\nabla u|^2\\
&+ \theta'(|u|^2)u\Delta\bar{u}
\end{aligned}$$

to obtain

$$\begin{aligned}
|\mathrm{I}| \ &\le CM(\|u;W^1_\infty\|^4 + \|u;L^\infty\|\|u;W^2_\infty\| + \|u;W^1_\infty\|^2)\|e^{-\theta(|u|^2)}\partial_j\Gamma u;L^2\|^2\\
&\le CM(1+\|u;H^{\sigma+1}\|^2)\|u;H^{\sigma+1}_{2^*}\|^2\|e^{-\theta(|u|^2)}\nabla\Gamma u;L^2\|^2 \qquad (2.9)\\
&= CM(1+\|u;H^{s-1}\|^2)\|u;H^{s-1}_{2^*}\|^2\ \|e^{-\theta(|u|^2)}\nabla\Gamma u;L^2\|^2,
\end{aligned}$$

where

$$M = \sum_{j=0}^{2}\sup\{|\theta^{(j)}(\rho)|;\rho \le C\|u;L^\infty(H^\sigma)\|^2\}$$

and we have used the embedding $H^{\sigma+1}\hookrightarrow W^1_\infty$ and $H^{\sigma+1}_{2^*}\hookrightarrow W^2_\infty$.

As in (2.5), we estimate II as

$$\begin{aligned}
|\mathrm{II}| \ &\le 2e^M\left(\|\frac{\partial F}{\partial u}\partial_j u;H^{s-1}\| + \|\frac{\partial F}{\partial\bar{u}}\partial_j\bar{u};H^{s-1}\|\right)\|e^{-\theta(|u|^2)}\partial_j\Gamma u;L^2\|\\
&\le Ce^M\|u;H^{s-1}_{2^*}\|^2\|\partial_j u;H^{s-1}\|\|e^{-\theta(|u|^2)}\nabla\Gamma u;L^2\| \qquad (2.10)\\
&\le Ce^{2M}\|u;H^{s-1}_{2^*}\|^2\|e^{-\theta(|u|^2)}\nabla\Gamma u;L^2\|^2.
\end{aligned}$$

For III, we integrate by parts to obtain

$$\begin{aligned}
|\mathrm{III}| \ &= |(\nabla\cdot\left(e^{-2\theta(|u|^2)}\mathrm{Im}\frac{\partial F}{\partial(\nabla u)}\right),\ |\partial_j\Gamma u|^2)|\\
&\le CM(\|u;W^1_\infty\|^5 + \|u;W^1_\infty\|^2\|u;W^2_\infty\|)\|e^{-\theta(|u|^2)}\partial_j\Gamma u;L^2\|^2\\
&\le CM(\|u;H^{\sigma+1}\|^3\|u;H^\sigma_{2^*}\|^2 + \|u;H^{\sigma+1}\|\|u;H^\sigma_{2^*}\|\|u;H^{\sigma+1}_{2^*}\|) \qquad (2.11)\\
&\qquad\cdot\|e^{-\theta(|u|^2)}\nabla\Gamma u;L^2\|^2\\
&\le CM(\|u;H^{s-1}\|^3 + \|u;H^{s-1}\|)\|u;H^{s-1}_{2^*}\|^2\|e^{-\theta(|u|^2)}\nabla\Gamma u;L^2\|^2.
\end{aligned}$$

We apply Kato-Ponce's commutator estimate [13] to IV to obtain

$$
\begin{aligned}
|\mathrm{IV}| &\le 2e^M \| \left[\Gamma, \frac{\partial F}{\partial(\nabla u)}\right] \partial_j \nabla u; L^2 \| \| e^{-\theta(|u|^2)} \partial_j \Gamma u; L^2 \| \\
&\le Ce^M \left(\| \nabla \cdot \frac{\partial F}{\partial(\nabla u)}; L^\infty \| \| \partial_j \nabla u; H^{s-2} \| + \| \frac{\partial F}{\partial(\nabla u)}; H^{s-1} \| \| \partial_j \nabla u; L^\infty \| \right) \\
&\qquad \cdot \| e^{-\theta(|u|^2)} \nabla \Gamma u; L^2 \| \\
&\le Ce^M (\| u; W^1_\infty \| \| u; W^2_\infty \| \| \nabla u; H^{s-1} \| \\
&\qquad + \| u; H^\sigma_{2^*} \| \| \nabla u; H^{s-1} \| \| u; H^{s-1}_{2^*} \|) \| e^{-\theta(|u|^2)} \nabla \Gamma u; L^2 \| \\
&\le Ce^{2M} \| u; H^{s-1}_{2^*} \|^2 \| e^{-\theta(|u|^2)} \nabla \Gamma u; L^2 \|^2,
\end{aligned}
\tag{2.12}
$$

where we have used the usual Sobolev inequality for $\dot{H}^1 \hookrightarrow L^{2^*}$ when $\partial F/\partial(\nabla u)$ involves terms like u^2. We estimate V and VI in the same way as in (2.11), (2.12), and (2.9), respectively. Moreover, VII is estimated in the same way as in (2.9) for any ε with $0 < \varepsilon \le 1$. To estimate VIII, we write

$$
\mathrm{VIII} = \frac{\varepsilon}{2} (\Delta(e^{-2\theta(|u|^2)}) \cdot \partial_j \Gamma u, \partial_j \Gamma u) - \varepsilon \| e^{-\theta(|u|^2)} \nabla \partial_j \Gamma u; L^2 \|^2,
$$

where the first term on the RHS is estimated in the same way as in (2.9).

Combining those estimates above, we obtain

$$
\begin{aligned}
&\frac{d}{dt} \| e^{-\theta(|u|^2)} \partial_j \Gamma u; L^2 \|^2 + \varepsilon \| e^{-\theta(|u|^2)} \nabla \partial_j \Gamma u; L^2 \|^2 \\
&\le Ce^{2M} (1 + \| u; H^{s-1} \|^3) \| u; H^{s-1}_{2^*} \|^2 \| e^{-\theta(|u|^2)} \nabla \Gamma u; L^2 \|^2,
\end{aligned}
$$

where C is independent of $\varepsilon \in (0, 1]$.

Taking summation with respect to j and integrating, we have

$$
\begin{aligned}
&\| e^{-\theta(|u|^2)} \nabla \Gamma u; L^\infty(L^2) \| \\
&\le \| e^{-\theta(|u|^2)} \nabla \Gamma \phi; L^2 \| \exp(Ce^{2M} (1 + \| u; L^\infty(H^{s-1}) \|^3) \| u; L^2(H^{s-1}_{2^*}) \|^2).
\end{aligned}
\tag{2.14}
$$

By (2.3) and (2.14)

$$
\begin{aligned}
&\| u; L^\infty(H^s) \| \\
&\le Ce^{2M} \| \phi; H^s \| \exp(Ce^{2M} (1 + \| u; L^\infty(H^{s-1}) \|^3) \| u; L^2(H^{s-1}_{2^*}) \|^2),
\end{aligned}
\tag{2.15}
$$

where C is independent of $\varepsilon \in (0, 1]$. We now choose $\delta, \eta > 0$ sufficiently small to ensure that

$$
C\delta + C\eta^3 \le \eta \tag{2.16}
$$

$$
Ce^{2M} \delta \exp(Ce^{2M} (1 + \eta^3) \eta^2) \le \eta \tag{2.17}
$$

Then for any $\phi \in H^s$ with $\|\phi; H^s\| \le \delta$ the corresponding solution u_ε of the regularized equation (2.1) satisfies

$$\max(\|u_\varepsilon; L^\infty(H^s)\|,\ \|u_\varepsilon; L^2(H^{s-1}_{2^*})\|) \le \eta. \tag{2.18}$$

This implies that u_ε extends to a global solution belonging to $L^\infty(0,\infty;H^s)\cap L^2(0,\infty;H^{s-1}_{2^*})$. Moreover, by a compactness argument it follows that (1.1) has a global solution $u \in (C_w \cap L^\infty)(0,\infty;H^s)\cap L^2(0,\infty;H^{s-1}_{2^*})$ with $u(0)=\phi$ satisfying

$$\max(\|u; L^\infty(H^s)\|,\ \|u; L^2(H^{s-1}_{2^*})\|) \le \eta. \tag{2.19}$$

We now consider the uniqueness of solutions u of (1.1) satisfying $u(0)=\phi \in H^s$ with $\|\phi; H^s\| \le \delta$ and (2.19). Let u and v be those two solutions. We consider the difference $u-v$ in H^2. For that purpose we estimate

$$\begin{aligned}
&\frac{d}{dt}\|u-v; L^2\|^2 \\
&= 2\mathrm{Im}\ ((i\partial_t + \frac{1}{2}\Delta)(u-v), u-v) \\
&= 2\mathrm{Im}\ (F(u,\nabla u) - F(v,\nabla v), u-v) \\
&\le C(\|u; W^1_\infty\|^2 + \|v; W^1_\infty\|^2)\|u-v; H^1\|\|u-v; L^2\| \\
&\le C(\|u; H^{s-1}_{2^*}\|^2 + \|v; H^{s-1}_{2^*}\|^2)\|u-v; H^1\|^2.
\end{aligned} \tag{2.20}$$

In the same way as in (2.8), we obtain

(2.21)

$$
\begin{aligned}
&\frac{d}{dt}\|e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v; L^2\|^2 \\
&= 2\ \mathrm{Im}\ (\left(i\partial_t + \frac{1}{2}\Delta\right)(e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v), e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&- 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\theta(|u|^2)\right]\partial_j\partial_k u - e^{-\theta(|v|^2)}\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\theta(|v|^2)\right]\partial_j\partial_k v, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\partial_j\partial_k v - \nabla\theta(|u|^2)\cdot\nabla\partial_j\partial_k u\right] \\
&\qquad -e^{-\theta(|v|^2)}\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\partial_j\partial_k v - \nabla\theta(|v|^2)\cdot\nabla\partial_j\partial_k v\right], e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ \mathrm{Im}\ (e^{-\theta(|u|^2)}(\nabla\theta(|u|^2))^2\partial_j\partial_k u - e^{-\theta(|v|^2)}(\nabla\theta(|v|^2))^2\partial_j\partial_k v,\ e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&- 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\theta(|u|^2)\right]\partial_j\partial_k u - e^{-\theta(|v|^2)}\left[\left(i\partial_t + \frac{1}{2}\Delta\right)\theta(|v|^2)\right]\partial_j\partial_k v, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}\partial_k\left(\frac{\partial F}{\partial u}\partial_j u + \frac{\partial F}{\partial \bar{u}}\partial_j\bar{u}\right) - e^{-\theta(|v|^2)}\partial_k\left(\frac{\partial F}{\partial v}\partial_j v + \frac{\partial F}{\partial \bar{u}}\partial_j\bar{v}\right), \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Re}\ (e^{-\theta(|u|^2)}\partial_k\left(\mathrm{Im}\ \frac{\partial F}{\partial(\nabla u)}\right)\cdot\nabla\partial_j u - e^{-\theta(|v|^2)}\partial_k\left(\mathrm{Im}\ \frac{\partial F}{\partial(\nabla v)}\right)\cdot\nabla\partial_j v, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Re}\ (e^{-\theta(|u|^2)}\left(\mathrm{Im}\frac{\partial F}{\partial(\nabla u)}\right)\cdot\nabla\partial_j\partial_k u - e^{-\theta(|v|^2)}\left(\mathrm{Im}\frac{\partial F}{\partial(\nabla u)}\right)\cdot\nabla\partial_j\partial_k v, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}\partial_k\left(\frac{\partial F}{\partial(\nabla\bar{u})}\right)\cdot\nabla\partial_j\bar{u} - e^{-\theta(|v|^2)}\partial_k\left(\frac{\partial F}{\partial(\nabla\bar{v})}\right)\cdot\nabla\partial_j\bar{v}, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}\left(\frac{\partial F}{\partial(\nabla\bar{u})}\right)\cdot\nabla\partial_j\partial_k\bar{u} - e^{-\theta(|v|^2)}\left(\frac{\partial F}{\partial(\nabla\bar{v})}\right)\cdot\nabla\partial_j\partial_k\bar{v}, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v) \\
&+ 2\ \mathrm{Im}\ (e^{-\theta(|u|^2)}(\nabla\theta(|u|^2))^2\partial_j\partial_k u - e^{-\theta(|v|^2)}(\nabla\theta(|v|^2))^2\partial_j\partial_k v, \\
&\qquad e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v).
\end{aligned}
$$

As before, we denote by I,$\cdots$,VII the first,$\cdots$, seventh terms on the RHS of the last equality of (2.21), respectively, and consider those contributions separately.

In the same way as in the derivation of (2.13), we obtain

$$\begin{aligned}(2.22)\qquad &\frac{d}{dt}\|e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v; L^2\|^2\\ \le\ & Ce^{CM}(1+\|u;H^{s-1}\|^3+\|v:H^{s-1}\|^3)(\|u;H^{s-1}_{2^*}\|^2+\|v;H^{s-1}_{2^*}\|^2)\\ &\cdot\sum_{l,m=1}^{n}(\|e^{-\theta(|u|^2)}\partial_l\partial_m u - e^{-\theta(|v|^2)}\partial_l\partial_m v;L^2\|^2+\|\partial_l\partial_m u-\partial_l\partial_m v;L^2\|^2),\end{aligned}$$

where we have used the inequality

$$\begin{aligned}&|e^{-\theta(|u|^2)}-e^{-\theta(|v|^2)}|\\ &=|\int_0^1 e^{-\lambda\theta(|u|^2)-(1-\lambda)\theta(|v|^2)}d\lambda(\theta(|u|^2)-\theta(|v|^2))|\\ &\le CMe^{CM}(|u|+|v|)|u-v|.\end{aligned}$$

Moreover, from the identity

$$\begin{aligned}\partial_j\partial_k u-\partial_j\partial_k v\ &= e^{\theta(|u|^2)}(e^{-\theta(|u|^2)}\partial_j\partial_k u - e^{-\theta(|v|^2)}\partial_j\partial_k v)\\ &\quad -\int_0^1 e^{\lambda(\theta(|u|^2)-\theta(|v|^2))}d\lambda(\theta(|u|^2)-\theta(|v|^2))\partial_j\partial_k v\end{aligned}$$

we have

$$\begin{aligned}(2.23)\qquad \|\partial_j\partial_k u-\partial_j\partial_k v;L^2\|\ &\le e^M\|e^{-\theta(|u|^2)}\partial_j\partial_k u-e^{-\theta(|v|^2)}\partial_j\partial_k v:L^2\|\\ &\quad +Ce^{CM}(\|u;L^\infty(H^s)\|^2+\|v;L^\infty(H^s)\|^2)\|u-v;L^2\|.\end{aligned}$$

Therefore, (2.22) and (2.23) imply, for any $t>0$

$$\begin{aligned}(2.24)\qquad &\sum_{j,k=1}^{n}\|e^{-\theta(|u|^2)}\partial_j\partial_k u-e^{\theta(|v|^2)}\partial_j\partial_k v;L^2\|^2\\ &\le Ce^{CM}(1+\|u;L^\infty(H^{s-1})\|^3+\|v;L^\infty(H^{s-1})\|^3)\\ &\quad\cdot\int_0^t(\|u;H^{s-1}_{2^*}\|^2+\|v;H^{s-1}_{2^*}\|^2)\sum_{j,k=1}^{n}\|e^{-\theta(|u|^2)}\partial_j\partial_k u-e^{\theta(|v|^2)}\partial_j\partial_k v;L^2\|^2dt'\\ &+Ce^{CM}(1+\|u;L^\infty(H^s)\|^7+\|v;L^\infty(H^s)\|^7)\\ &\quad\cdot\int_0^t(\|u;H^{s-1}_{2^*}\|^2+\|v;H^{s-1}_{2^*}\|^2)\|u-v;L^2\|^2dt'.\end{aligned}$$

We define

$$N(t)=\sum_{j,k=1}^{n}\|e^{-\theta(|u|^2)}\partial_j\partial_k u-e^{\theta(|v|^2)}\partial_j\partial_k v;L^2\|^2+\|u-v;L^2\|^2.$$

Then, by (2.20) and (2.24),

$$N(t) \le C(\eta)e^{CM}\int_0^t (\|u; H^{s-1}_{2^*}\|^2 + \|v; H^{s-1}_{2^*}\|^2)N(t')dt', \tag{2.25}$$

where $C(\eta)$ is a constant depending on η and we have used the inequality

$$\begin{aligned}\|u-v; H^1\|^2 &\le C\|u-v; L^2\|^2 + C\sum_{j,k=1}^n \|\partial_j\partial_k u - \partial_j\partial_k v; L^2\|^2 \\ &\le C(\eta)e^{CM}N(t),\end{aligned}$$

which follows from (2.23).

By Gronwall's lemma, $N(t) = 0$ for any $t > 0$. This proves the uniqueness.

The existence of asymptotic states $\phi_\pm \in H^{s-1}$ follows from the standard argument based on the Strichartz estimates (see for instance [1, 12, 25]). By the unitarity of the free propagator $U(t)$ in H^s and the fact that $u \in L^\infty(H^s)$, we see that $\phi_\pm \in H^s$.

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