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Author(s)	Sasaki, Hironobu
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REMARK ON THE SCATTERING PROBLEM FOR THE KLEIN-GORDON EQUATION WITH POWER NONLINEARITY

HIRONOBU SASAKI*
DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY,
060-0810, JAPAN.

ABSTRACT. We consider the scattering problem for the nonlinear Klein-Gordon equation. The nonlinear term of the equation behaves like a power term. We show that we can define the scattering operator on a suitable 0-neighborhood of the weighted Sobolev space, which improves the known results in some sense. Our proof is based on the Strichartz type estimates.

1. INTRODUCTION

This paper is concerned with the scattering problem for the nonlinear Klein-Gordon equation of the form

$$\partial_t^2 u - \Delta u + u = F(u) \quad (1.1)$$

in $\mathbb{R} \times \mathbb{R}^n$. Here, u is a real or complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\partial_t = \partial/\partial t$ and Δ is the Laplacian in $\mathbb{R}^n$. For some $p > 1$, the term $F : \mathbb{C} \rightarrow \mathbb{R}$ satisfies

$$F(0) = 0, \quad (1.2)$$

$$|F(u_1) - F(u_2)| \lesssim |u_1 - u_2|(|u_1| + |u_2|)^{p-1}, \quad (1.3)$$

where $a \lesssim b$ denotes $a \leq Cb$. If $F \equiv 0$, then we call (1.1) and its solution *the free Klein-Gordon equation* and *a free solution*, respectively. An example of the nonlinearity is $F = \pm|u|^{p-1}u$.

Before we mention the scattering problem, we review the Cauchy problem for the nonlinear Klein-Gordon equation

$$\begin{cases} \partial_t^2 u - \Delta u + u = F(u), & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = \varepsilon \phi_0^1(x), & x \in \mathbb{R}^n, \\ \partial_t u(0, x) = \varepsilon \phi_0^2(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.4)$$

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The local theory (see, e.g., Reed [17] and Sogge [20]) gives that (1.4) is locally well-posed in $H^1(\mathbb{R}^n)$ if we have

$$1 < p < \begin{cases} \infty & \text{if } n = 1, 2, \\ 1 + 4/(n-2) & \text{if } n \geq 3. \end{cases}$$

If there exists some scalar potential $\mathcal{V}(u)$ such that $\nabla \mathcal{V}(u) = -F(u)$ and $\mathcal{V}(0) = 0$, then we have the energy conservation law

$$E(0) = E(t) = \int_{\mathbb{R}^n} \left(\frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 + \mathcal{V}(u) \right) dx. \quad (1.5)$$

given by the same arguments as in Ginibre–Velo [6] (see also Ozawa [15]). Using (1.5) and the same argument as in Cazenave [4], we see that the time-local solution to (1.4) becomes a time-global solution if $V \geq 0$, or if $\varepsilon > 0$ is sufficiently small. For more general F , Lindblad–Sogge [7] proved that (1.4) has a unique, time-global solution if $1 \leq n \leq 3$, $1 + 2/n < p$ and $\varepsilon > 0$ is sufficiently small.

In order to treat the scattering problem, we mention some notation. Put $\omega = \sqrt{1 - \Delta}$. For $s, w \in \mathbb{R}$ and $1 \leq r \leq \infty$, we define the weighted Sobolev space $H_r^{s,w}$ by $\omega^{-s}(1 + |x|^2)^{-w/2} L^r(\mathbb{R}^n)$, and we set $H_p^s = H_p^{s,0}$ and $H^s = H_2^s$. We denote $(H^{s,w} \cap H^1) \oplus (H^{s-1,w} \cap L^2)$ by $X(s, w)$. A norm of a Banach space A is expressed by $\|\cdot\|_A$. For a positive number δ and a Banach space A , we denote the ball $\{a \in A; \|a\|_A \leq \delta\}$ by $B(\delta; A)$.

An operator S is defined as the mapping

$$S : B(\delta; \Sigma) \ni \phi_- \mapsto \phi_+ \in X(1, 0)$$

if the following condition holds for some $\delta > 0$ and for some Banach space $\Sigma \subset X(1, 0)$.

Let Z be some suitable subspace of $C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$, and let $u_{\pm}(t)$ be free solutions whose initial data are $\phi_{\pm}$, respectively. For any $\phi_- \in B(\delta; \Sigma)$, there uniquely exist a function $u \in Z$ and a data $\phi \in X(1, 0)$ such that

$$u(t) = u_-(t) + \int_{-\infty}^t \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau)) d\tau \quad (1.6)$$

and

$$\|u(t) - u_{\pm}(t)\|_{H^1} + \|\partial_t u(t) - \partial_t u_{\pm}(t)\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (1.7)$$

We call that “ (S, Σ) is well-defined” if we can define the operator $S : B(\delta; \Sigma) \rightarrow X(1, 0)$ for some δ . Let us call the space Z “the solution space” for (1.1).

There is a large literature on the scattering problem for (1.1). In particular, it was shown by Strauss [21] and Pecher [16] that $(S, X(1, 0))$ is well-defined if

$$p_1 \equiv 1 + \frac{4}{n} \leq p \begin{cases} < \infty & \text{if } n = 1, 2, \\ \leq 1 + \frac{4}{n-2} & \text{if } n \geq 3. \end{cases}$$

Furthermore, Mochizuki and Motai [10] proved that there exists some Σ such that (S, Σ) is well-defined if

$$\gamma(n) \equiv \frac{n + 2 + \sqrt{n^2 + 12n + 4}}{2n} < p < p_1.$$

Now, we shall restrict Σ to the Hilbert space $X(s, w)$. By using the method of [10] and using the L^p - L^q estimate in Brenner [2], we see that $(S, X(s, w))$ is well-defined if $\gamma(n) < p < p_1$,

$$s > \frac{n}{2} + \frac{2}{np} - \frac{n}{2p}, \quad w > \frac{1}{p} \quad (1.8)$$

(for the proof, see Appendix A). For other results of the scattering problem for (1.1), see [3, 9, 11, 13, 14, 19, 22]. In view of the conditions of s and w , there is a gap between the two cases $p \geq p_1$ and $p < p_1$. In this paper, we shall fill the gap. Our result is the following:

Theorem 1.1. *Assume that $n \geq 1$ and $\gamma(n) < p < p_1$. Suppose that F satisfies (1.2) and (1.3). Then there exist some space $Z \subset \mathcal{H}$ and some $s(p) < \frac{n}{2} + \frac{2}{np} - \frac{n}{2p}$ such that if $s > s(p)$ and $w > \frac{2}{p-1} - \frac{n}{2}$, then there exist some positive numbers η_0 and η_- satisfying the following properties:*

- (i) *For $\phi_0 \in B(\eta_0; X(s, w))$, there uniquely exist $u \in Z$ and $\phi_{\pm} \in X(1, 0)$ such that u is a time-global solution to*

$$u(t) = u_0(t) + \int_0^t \frac{\sin(t - \tau)\omega}{\omega} F(u(\tau)) d\tau$$

and satisfies (1.7), where u_0 is a free solution whose initial data is ϕ_0 .

Moreover, the operators $B(\eta_0; X(s, w)) \ni \phi_0 \mapsto \phi_{\pm} \in X(1, 0)$ are well-defined, injective and continuous.

- (ii) *For $\phi_- \in B(\eta_-; X(s, w))$, there uniquely exist $u \in Z$ and $\phi_+ \in X(1, 0)$ such that u is a time-global solution to (1.6), and we have (1.7).*

Moreover, the scattering operator $S : B(\eta_-; X(s, w)) \ni \phi_- \mapsto \phi_+ \in X(1, 0)$ is well-defined, injective and continuous.

Remark 1. The above (i) implies that we can fill the gap of the conditions of s . If we put $h(p) = 2/(p - 1) - n/2$, then we have $h(p) < 1/p$ for $\gamma(n) < p < p_1$. Thus, the conditions $s > s(p)$ and $w > h(p)$ are weaker than (1.8). From $h(p_1) = 0$, we can fill the gap of the conditions of w in some sense.

Remark 2. From the inequality (3.5) and Remark 5 below, we see that the time-global solutions u given by the above theorem becomes time-global solutions to (1.1) in the H^{-1} -sense.

Remark 3. The similar result for the Klein-Gordon equation with a cubic convolution

$$\partial_t^2 u - \Delta u + u = \pm \int_{\mathbb{R}^n} |y|^{-\sigma} |u(t, x - y)|^2 u(t, x) dy \quad (1.9)$$

has been given by Sasaki [18]. In view of the selection of the solution space, the case (1.1) is more difficult than the case (1.9).

The contents of this paper are as follows: In Section 2, we give the key lemma to select the solution space. For this purpose, we use the concept of the complex interpolation method. In Section 3, applying the key lemma, we show Theorem 1.1 by using an estimate in the Besov space and by applying the contraction mapping principle. We next remark that the densely scattering operator for (1.1) can be defined on some suitable domain.

2. PRELIMINARIES

In this section, we prepare some lemmas which will be applied to prove Theorem 1.1. We remark that if $n \geq 1$ and $\gamma(n) < p < p_1$, then we have

$$-2 < np + \frac{4p}{1-p} < 0. \quad (2.1)$$

For $1 \leq q, r \leq \infty$, we set

$$\frac{1}{\dot{r}} = \frac{1}{2} + \frac{2}{n} \left(1 - \frac{p}{q}\right), \quad \dot{\rho} = \frac{n+2}{n} \left(1 - \frac{p}{q}\right).$$

The following lemma is essential to use the Strichartz type estimates:

Lemma 2.1. *Let $n \geq 1$ and $\gamma(n) < p < p_1$. Then there exist some $1 \leq q, r \leq \infty$ satisfying the following properties.*

(1) *We have*

$$\begin{aligned} \max\left(0, \frac{1}{2} - \frac{1}{n}\right) &< \frac{1}{r} < \frac{1}{2}, \quad \frac{1}{2} < \frac{p}{r} < \min\left(1, \frac{1}{2} + \frac{1}{n}\right), \\ 0 &< \frac{n}{2} \left(\frac{1}{2} - \frac{1}{r}\right) < \frac{1}{q} < n \left(\frac{1}{2} - \frac{1}{r}\right), \quad 0 < 1 - \frac{p}{q} < n \left(\frac{p}{r} - \frac{1}{2}\right), \\ \rho = \rho(p) &\equiv \frac{n+2}{2} \left(\frac{p}{r} - \frac{1}{r}\right) < 1, \end{aligned} \quad (2.2)$$

$$\frac{2}{q} + \frac{n}{r} + 2 = \frac{2p}{q} + \frac{np}{r}. \quad (2.3)$$

(2) *Let*

$$s_1 = \max\left\{\frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2}\right) + \frac{\dot{\rho}}{p}, \dot{\rho}\right\}.$$

Then there exists some $\beta \in (0, 1]$ such that

$$\frac{1}{\dot{r}} = \left(\frac{1}{r} - \beta \frac{s_1 - \dot{\rho}}{n}\right) + (p-1) \left(\frac{1}{r} - \beta \frac{s_1}{n}\right), \quad (2.4)$$

$$\frac{1}{2} < \frac{1}{\dot{r}} < \min\left(1, \frac{1}{2} + \frac{1}{n}\right), \quad \frac{1}{r} - \beta \frac{s_1}{n} > 0, \quad \frac{1}{\dot{r}} + \frac{s_1 - \dot{\rho}}{n} < 1. \quad (2.5)$$

Furthermore, if $1 \leq n \leq 3$, then we have

$$\frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right) + \frac{\dot{\rho}}{p} \geq \dot{\rho}. \quad (2.6)$$

Using the Strichartz type estimates in Nakamura–Ozawa [12] and [18], we see from Lemma 2.1 that the following estimates hold:

Corollary 2.2. *Let $n \geq 1$ and $\gamma(n) < p < p_1$. Suppose that q and r satisfy (1) and (2) in Lemma 2.1. Put*

$$W = L^q(\mathbb{R}; B_r^{s_1}), \quad \widetilde{W} = L^{q/p}(\mathbb{R}; B_{r/p}^{s_1-1+p}), \quad \dot{W} = L^{q/p}(\mathbb{R}; B_{\dot{r}}^{\dot{\rho}}), \quad (2.7)$$

where B_α^s means the Besov space $B_{\alpha,2}^s(\mathbb{R}^n)$ (for the definition, see, e.g., Bergh–Löfström [1]). Then we have

$$\left\| \int_{-\infty}^t \omega^{-1} e^{\pm i(t-\tau)\omega} f d\tau \right\|_W \lesssim \|f\|_{\widetilde{W}},$$

$$\left\| \int_{-\infty}^t \omega^{-1} e^{\pm i(t-\tau)\omega} f d\tau \right\|_{L^\infty(\mathbb{R}; H^1)} \lesssim \|f\|_{\dot{W}}.$$

Furthermore, we have

$$\|e^{\pm it\omega} \varphi\|_{W \cap L^\infty(\mathbb{R}; H^1)} \lesssim \|\varphi\|_{H(s_0, w)}$$

if

$$s_0 > \frac{n+2}{nq}, \quad w > \frac{2}{p-1} - \frac{n}{2}.$$

Proof of Lemma 2.1. We show the lemma dividing into two cases.
(Case I; $1 \leq n \leq 3$.) Having in mind that

$$\frac{1}{np} \left(\frac{2}{p-1} - \frac{n}{2} \right) < \frac{1}{2},$$

we see from $1 + 2/n < \gamma(n)$ and (2.1) that there exist some $\varepsilon, \theta \in (0, 1)$ such that

$$np + \frac{4p}{1-p} = 4p\varepsilon + 2(\theta - 1), \quad (2.8)$$

$$\frac{4p}{p-1} - 6 + 4p\varepsilon < 0 \quad \text{if } n = 1, \quad (2.9)$$

$$\frac{n+2}{n} \left(\frac{1}{p} + (p-1)\varepsilon \right) < 1, \quad (2.10)$$

$$\varepsilon < \min \left\{ \frac{n}{(n+2)p(p-1)} \left(\frac{2}{p-1} - \frac{n}{2} \right), \frac{1}{2p} \right\}, \quad (2.11)$$

$$\frac{2p\varepsilon}{n} + \frac{n+2}{n^2}\varepsilon + \frac{1}{np} \left(\frac{2}{p-1} - \frac{n}{2} \right) < \frac{1}{2}. \quad (2.12)$$

We put

$$\frac{1}{q} = \frac{1}{p} - \varepsilon, \quad \frac{1}{r} = \frac{1}{2} - \frac{1+\theta}{pn}.$$

We shall prove only

$$0 < \frac{1}{r} \quad \text{and} \quad \frac{p}{r} < 1 \quad \text{for } n = 1, \quad (2.13)$$

$$\frac{1}{2} < \frac{p}{r}, \quad (2.14)$$

$$\frac{n}{2} \left(\frac{1}{2} - \frac{1}{r} \right) < \frac{1}{q}, \quad (2.15)$$

$$1 - \frac{p}{q} < n \left(\frac{p}{r} - \frac{1}{2} \right), \quad (2.16)$$

(2.2) and (2.4), because the others are easily given by (2.11), (2.12) and the assumptions of p and n . If $n = 1$, it follows from (2.9) that

$$\frac{4p}{p-1} - 6 + 4p\varepsilon < 0 < \frac{4p}{p-1} - 4 + 4p\varepsilon.$$

By (2.8) with $n = 1$, we see that $2(\theta + 1) < p < 2(\theta + 2)$, which is equivalent to (2.13).

From (2.8) and $p < p_1$, we have

$$np = (n + 2\theta + 2) + \left(\frac{4p}{p-1} - (n + 4) + 4p\varepsilon \right) > (n + 2\theta + 2).$$

Therefore, we obtain (2.14). By (2.1) and (2.8), we have $4p\varepsilon + 2(\theta - 1) < 0$. Thus, (2.15) holds. The property (2.2) is given by (2.3) and (2.10).

We see that (2.16) is equivalent to $n(p - 1) - 2(1 + \theta) - 2p\varepsilon > 0$. On the other hand, it follows from (2.8) and $p < p_1$ that

$$n(p - 1) - 2(1 + \theta) - 2p\varepsilon = \frac{4p}{p-1} - (n + 4) + 2p\varepsilon > 0.$$

Thus, (2.16) holds.

Since (2.4) is equivalent to

$$\frac{1}{2} + \frac{2}{n} \left(1 - \frac{p}{q} \right) = \frac{p}{r} - \frac{\beta(ps_1 - \dot{\rho})}{n},$$

it follows from (2.3) that

$$\frac{1}{2} + \frac{1}{n} = \frac{2p}{n(p-1)} - \frac{\beta(ps_1 - \dot{\rho})}{n}.$$

Thus, we have

$$\beta \left(s_1 - \frac{\dot{\rho}}{p} \right) = \frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right).$$

Therefore, there exists $\beta \in (0, 1]$ satisfying (2.4).

(Case II; $n \geq 4$.) From (2.1), there exists some $\theta \in (0, 1)$ such that

$$np + \frac{4p}{1-p} = 2(\theta - 1). \quad (2.17)$$

We put

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{2}, \quad \frac{1}{r} = \frac{1}{2} - \frac{1-\theta}{pn} - \frac{\theta}{n}.$$

We shall show only (2.14), (2.16) and (2.2), because the others are immediately given by the assumptions of n and p , and by following the proof for Case I.

Since $np > n + 2\theta + 2$ holds even if $n \geq 4$, it follows from $p < 2$ that

$$np > n + 2(p-1)\theta + 2,$$

which is equivalent to (2.14).

We can immediately see that (2.16) holds if we show that

$$p\theta < np - 2 - n. \quad (2.18)$$

It follows from (2.17) that (2.18) is equivalent to

$$(np^2 - (n+2)p - 2)(p-2) + 2(np - (n+4)) < 0$$

which is true if $\gamma(n) < p < p_1$ and $n \geq 4$.

Applying (2.17) to $\rho(p)$, we have

$$\rho(p) = \frac{n+2}{4n} \left\{ -n \left(p - \left(\frac{3}{2} + \frac{1}{n} \right) \right)^2 + \frac{(n+2)^2}{4n} \right\}.$$

Since $3/2 + 1/n \geq 1 + 4/n$ for $n \geq 6$, we obtain

$$\sup_{p \in (\gamma(n), p_1)} \rho(p) \leq \begin{cases} (n+2)^3/16n^2 & \text{if } n = 4, 5, \\ (n^2-4)/n^2 & \text{if } n \geq 6. \end{cases} < 1.$$

□

We next introduce some properties of s_1 defined in Lemma 2.1.

Proposition 2.3. *Let $n \geq 1$ and $\gamma(n) < p < p_1$. Suppose that q and r satisfy (1) and (2) in Lemma 2.1. Let s_1 be a number defined in Lemma 2.1. Then we have $s_1 < 1$,*

$$\frac{n+2}{nq} + s_1 < \frac{n}{2} + \frac{2}{np} - \frac{n}{2p} \quad (2.19)$$

and

$$\lim_{p \uparrow p_1} \left(\frac{n+2}{nq} + s_1 \right) < 1. \quad (2.20)$$

Proof. The property (2.20) can be easily shown. We prove the remaining properties dividing into two cases.

(Case I; $1 \leq n \leq 3$.) By (2.11), $p > 2$ and $2/(p-1) - n/2 < 1/p$, we obtain

$$\begin{aligned} \frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right) + \frac{\dot{p}}{p} &= \frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right) + \frac{n+2}{n} \varepsilon \\ &< \frac{1}{p-1} \left(\frac{2}{p-1} - \frac{n}{2} \right) < 1. \end{aligned}$$

From (2.6), we have $s_1 < 1$.

Moreover, we have

$$\frac{n+2}{nq} + \frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right) + \frac{\dot{p}}{p} < \frac{n}{2} + \frac{2}{np} - \frac{n}{2p} \quad (2.21)$$

for $p > \gamma(n)$. From (2.6), we see that the left hand side of (2.21) is equal to $(n+2)/nq + s_1$. Hence, we obtain (2.19).

(Case II; $n \geq 4$.) From $p > 1$, we obtain $\dot{p} < 1$. By (2.17), we have $(1-\theta)/p = 2/(p-1) - n/2$. Thus, we see that

$$\begin{aligned} \frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right) + \frac{\dot{p}}{p} &= \frac{1}{p} \cdot \frac{1-\theta}{p} + \frac{1}{p} \cdot \frac{n+2}{n} \theta \left(1 - \frac{p}{2} \right) \\ &< \frac{n+2}{n} \cdot \frac{1}{p^2}, \end{aligned}$$

where we have used $1/p > 1 - p/2$ in the last inequality. Since $(n+2)/np^2 < 1$ for $p > \gamma(n)$, we see that $s_1 < 1$.

We have (2.21) even if $n \geq 4$. Since $-(p-1)(p-2) \leq 1/4$ for all $p \in \mathbb{R}$, we obtain

$$-\frac{n+2}{n}(p-1)(p-2) < p$$

for $p > 1$. By (2.18), we have for $\gamma(n) < p < p_1$,

$$-\frac{n+2}{n}\theta(p-1)(p-2) < np - n - 2,$$

which is equivalent to

$$\frac{n+2}{nq} + \frac{n+2}{n} \left(1 - \frac{p}{q} \right) < \frac{n}{2} + \frac{2}{np} - \frac{n}{2p}.$$

Hence, we have (2.19). □

3. PROOF OF THEOREM 1.1

In this section, we shall show Theorem 1.1. We prove only (ii) because (i) can be shown similarly. Suppose that $n \geq 1$, $\gamma(n) < p < p_1$, and that F satisfies (1.2)–(1.3). Let q and r be positive numbers satisfying Lemma 2.1,(1) and (2). We again define ρ and s_1 by

$$\rho = \frac{n+2}{n} \left(\frac{p}{r} - \frac{1}{r} \right), \quad s_1 = \max \left\{ \frac{1}{p} \left(\frac{2}{p-1} - \frac{n}{2} \right) + \frac{\dot{\rho}}{p}, \dot{\rho} \right\}.$$

We assume that $\beta \in (0, 1]$ satisfies (2.4). The spaces W , $\widetilde{W}$ and $\dot{W}$ are defined as (2.7). We denote $W \cap C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$ by Z . Put

$$s(p) = \frac{n+2}{nq} + s_1.$$

From Proposition 2.3, we see that

$$s(p) < n/2 + 2/np - n/2p$$

and

$$\lim_{p \uparrow p_1} s(p) < 1.$$

Set

$$\Phi(u)(t) = u_-(t) + \int_{-\infty}^t \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau)) d\tau.$$

Let us show that Φ is a contraction on some suitable complete metric space. For $s > s(p)$ and $w > 2/(p-1) - n/2$, it follows from Corollary 2.2 and Appendix B that

$$\|\Phi(u)|Z\| \lesssim \|\phi|X(s, w)\| + \|F(u)|\widetilde{W}\| + \|F(u)|\dot{W}\|. \quad (3.1)$$

To estimate $F(u)$, we remark useful results (for the proof, see, e.g., Cazenave [5] and [1]).

Proposition 3.1. *Let $0 < \sigma < 1$ and $1 \leq \alpha, \alpha_1, \alpha_2 \leq \infty$. If $1/\alpha = 1/\alpha_1 + (p-1)/\alpha_2$, then we have*

$$\|F(u)|B_\alpha^\sigma\| \lesssim \|u|B_{\alpha_1}^\sigma\| \|u|L^{\alpha_2}\|. \quad (3.2)$$

Proposition 3.2. *Let $s \in \mathbb{R}$ and $1 \leq \alpha, \eta \leq \infty$.*

- (1) *If $1 < \alpha < 2$, then we have $B_{\alpha, \alpha}^s \hookrightarrow H_\alpha^s \hookrightarrow B_{\alpha, 2}^s$.*
- (2) *If $2 < \alpha < \infty$, then we have $B_{\alpha, 2}^s \hookrightarrow H_\alpha^s \hookrightarrow B_{\alpha, \alpha}^s$.*
- (3) *If $s_1 \in \mathbb{R}$, $1 \leq \alpha_1 \leq \infty$ and*

$$\frac{1}{\alpha_1} \geq \frac{1}{\alpha} \geq \frac{1}{\alpha_1} - \frac{s_1 - s}{n},$$

then we have $B_{\alpha_1, \eta}^{s_1} \hookrightarrow B_{\alpha, \eta}^s$.

Let us go back to the proof of Theorem 1.1. By Lemma 2.1 and Proposition 3.2, we have

$$B_r^{s_1} \hookrightarrow B_r^{s_1-1+\rho}, L^r, B_{r_1}^{\dot{\rho}}, L^{r_2},$$

where $1/r_1 = 1/r - \beta(s_1 - \dot{\rho})/n$ and $1/r_2 = 1/r - \beta s_1/n$. It follows from (3.2) and the above embeddings that

$$\begin{aligned} \|F(u)|\widetilde{W}\| &\lesssim \|F(u)|L^{q/p}B_{r/p}^{s_1}\| \\ &\lesssim \|u|W\| \|u|L^q L^r\|^{p-1} \\ &\lesssim \|u|W\|^p. \end{aligned}$$

Similarly, we see from (2.4) that

$$\begin{aligned} \|F(u)|\dot{W}\| &\lesssim \|u|L^q B_{r_1}^{\dot{\rho}}\| \|u|L^q L^{r_2}\|^{p-1} \\ &\lesssim \|u|W\|^p. \end{aligned}$$

Therefore, by (3.1), we have

$$\|\Phi(u)|Z\| \leq C_0 (\|\phi|X(s, w)\| + \|u|W\|^p), \quad (3.3)$$

where C_0 is dependent only on n and p .

We define $\ddot{r}$ by

$$\frac{1}{\ddot{r}} = \frac{1}{\dot{r}} + \beta \frac{s_1 - \dot{\rho}}{n}.$$

From (2.5), $s_1 \geq \dot{\rho}$ and Proposition 3.2, we see that

$$L^{\ddot{r}} \hookrightarrow B_{\dot{r}}^{-1+\dot{\rho}}.$$

By the above embedding, it follows from Corollary 2.2 that

$$\begin{aligned} \|\Phi(u_1) - \Phi(u_2)|L^q L^r \cap L^\infty L^2\| &\lesssim \|F(u_1) - F(u_2)|L^{q/p}(B_{r/p}^{-1+\rho} \cap B_{\dot{r}}^{-1+\dot{\rho}})\| \\ &\lesssim \|F(u_1) - F(u_2)|L^{q/p} L^{r/p}\| + \|F(u_1) - F(u_2)|L^{q/p} L^{\ddot{r}}\|. \end{aligned}$$

By (2.4) and the definition of $\ddot{r}$, we see from (1.3) that

$$\begin{aligned} \|\Phi(u_1) - \Phi(u_2)|L^q L^r \cap L^\infty L^2\| &\leq C_0 \|u_1 - u_2|L^q L^r\| \left(\max_{j=1,2} \|u_j|W\| \right)^{p-1}. \end{aligned} \quad (3.4)$$

Let $\|\phi_-|X(s, w)\| \leq \delta$. From

$$\|F(u)(t_2) - F(u)(t_2)|H^{-1}\| \lesssim \|u(t_2) - u(t_2)|H^1\| \left(\sup_{t \in \mathbb{R}} \|u(t)|H^1\|^2 \right) \quad (3.5)$$

and Appendix B, we find that $\Phi u \in C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$. By (3.3) and (3.4), we see that if $\delta > 0$ is sufficiently small, then Φ is a contraction on a complete metric space $B(\delta; Z)$ with a metric $d(u_1, u_2) = \|u_1 - u_2|L^q L^r \cap L^\infty L^2\|$. Thus, there exists unique solution to

(1.6) on $B(2C_0\|\phi_-\|X(s, w); Z)$. We can prove that the solution is also unique on Z (see Nakamura [11]).

We take

$$\phi_+ = \phi_- + \int_{-\infty}^{+\infty} \begin{pmatrix} (-\omega^{-1} \sin t\omega)F(u(t)) \\ (\cos t\omega)F(u(t)) \end{pmatrix} dt,$$

where u is the unique solution to (1.6). Then we see that $(S, X(s, w))$ is well-defined (for the proof, see, e.g., [18]). This completes the proof.

Remark 4. Using the method to prove Theorem 1.1, we can define the densely defined scattering operator on some 0-neighborhood of $\underline{Z}$ (see also [18]). Here,

$\underline{Z} = \{v \in Z; \text{there exist } f, g \in \mathcal{S}'(\mathbb{R}^n) \text{ such that}$

$$v(t) = (\cos t\omega)f + (\omega^{-1} \sin t\omega)g, \omega^{-1}\partial_t u(t) \in C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)\}.$$

The detail is the following:

Proposition 3.3. *Assume that $n \geq 1$ and $\gamma(n) < p < p_1$. Then there exist some positive numbers δ_0, δ_+ and δ_- satisfying the following properties:*

- (1) *If $u_0 \in B(\delta_0; \underline{Z})$, then there uniquely exist $u \in Z$ and $u_+, u_- \in \underline{Z}$ such that u is a time-global solution to*

$$u(t) = u_0(t) + \int_0^t \frac{\sin(t-\tau)\omega}{\omega} f(u(\tau)) d\tau \quad (3.6)$$

and we have (1.7). Furthermore, the operators $V_{\pm} : B(\delta_0; \underline{Z}) \ni u_0 \mapsto u_{\pm} \in \underline{Z}$ are well defined, injective and continuous.

- (2) *If $u_{\pm} \in B(\delta_{\pm}; \underline{Z})$, then there uniquely exist $u \in Z$ and $u_0 \in \underline{Z}$ such that u is a time-global solution to (1.4) and we have (1.7).*

Furthermore, the wave operators $W_{\pm} : B(\delta_{\pm}; \underline{Z}) \ni u_{\pm} \mapsto u_0 \in \underline{Z}$ are well defined, injective and continuous.

- (3) *The numbers $\delta_{\pm}$ satisfy $B(\delta_-; \underline{Z}) \subset B(\delta_0; \underline{Z})$, $W_-(B(\delta_-; \underline{Z})) \subset B(\delta_0; \underline{Z})$ and $B(\delta_+; \underline{Z}) \subset V_+ \circ W_-(B(\delta_0; \underline{Z}))$. In particular, the densely defined scattering operator $V_+ \circ W_- : B(\delta_-; \underline{Z}) \rightarrow \underline{Z}$ is well defined, injective and continuous.*

APPENDIX A. MOCHIZUKI-MOTAI'S METHOD

In this Appendix, we show the following property introduced in Section 1:

Proposition A.1. *Assume that F satisfies Condition (PW). Let $n \geq 1$ and $\gamma(n) < p < p_1$. If $s > \frac{n}{2} + \frac{2}{np} - \frac{n}{2p}$ and $w > \frac{1}{p}$, then $(S, X(s, w))$ is well-defined.*

Proof of Proposition A.1. We follow the method of Mochizuki–Motai [10], and we use the L^p - L^q estimate in Brenner [2] instead of the estimate in Marshall–Strauss–Wainger [8]. Put

$$\Phi = u_- + \int_{-\infty}^t \frac{\sin(t-\tau)\omega}{\omega} F(u(\tau)) d\tau.$$

Assume that

$$s > \frac{n}{2} + \frac{2}{np} - \frac{n}{2p}, \quad w > \frac{1}{p}.$$

Then there exist some $0 < s_1 < 1$ and $2 < r < \infty$ such that

$$\begin{aligned} s_1 &\geq n\left(\frac{1}{r} - \frac{1}{2p}\right), \quad \frac{1}{2} - \frac{1}{n+2} < \frac{1}{r} < \frac{1}{2} - \frac{1}{np}, \\ 0 < \tilde{r} < r < pr' < 2p, \quad s &= s_0 + s_1, \quad w > \frac{n}{2} - \frac{n}{r}, \end{aligned}$$

where $s_0 = (n+2)(1/2 - 1/r)$ and $1/\tilde{r} = 1/r' - (p-1)/2p$. Therefore, we see from Proposition 3.2 that

$$B_r^{s_1} \hookrightarrow L^\beta(\mathbb{R}^n) \quad \text{for any } \beta \in [r, 2p], \quad (\text{A.1})$$

$$X(s_1 + s_0, w) \hookrightarrow B_{r'}^{s_1+s_0} \oplus B_{r'}^{s_1+s_0-1}, \quad (\text{A.2})$$

$$\frac{np}{2} - \frac{np}{r} > 1. \quad (\text{A.3})$$

Put $\langle t \rangle = (1 + |t|^2)^{1/2}$. By the L^p - L^q estimate in [8] and (A.2), it follows from $s_0 < 1$ that

$$\begin{aligned} \|\Phi u|B_r^{s_1}\| &\lesssim \|u_-|B_r^{s_1}\| + \int_{-\infty}^t \langle t - \tau \rangle^{-n(1/2-1/r)} \|F(u(\tau))|B_{r'}^{s_1+s_0-1}\| d\tau \\ &\lesssim \langle t \rangle^{-n(1/2-1/r)} \|\phi_-|X(s_1 + s_0, w)\| \\ &\quad + \int_{-\infty}^t \langle t - \tau \rangle^{-n(1/2-1/r)} \|F(u(\tau))|B_{pr'}^{s_1}\| d\tau. \end{aligned}$$

Since $0 < \tilde{r} < r < pr'$ and

$$\frac{1}{r'} = \frac{p-1}{2p} + \frac{1}{\tilde{r}} = \frac{p-1}{pr'} + \frac{1}{r'},$$

there exists $\alpha \in (pr', 2p)$ such that

$$\frac{1}{r'} = \frac{p-1}{\alpha} + \frac{1}{r}.$$

Hence, we see from (3.2) and (A.1) that

$$\|F(u(\tau))|B_{r'}^{s_1}\| \lesssim \|u(\tau)|B_r^{s_1}\| \|u(\tau)\|_\alpha^{p-1} \lesssim \|u(\tau)|B_r^{s_1}\|^p.$$

If we define a norm $[\cdot]$ by

$$[u] = \sup_{t \in \mathbb{R}} \left(\langle t \rangle^{n(1/2-1/r)} \|u|B_r^{s_1}\| \right),$$

then it follows from (A.3) that

$$\begin{aligned} [\Phi u] &\lesssim \|\phi_-|X(s_1 + s_0, w)\| + [u]^p \int_{-\infty}^t \langle t - \tau \rangle^{-n(1/2-1/r)} \langle \tau \rangle^{-np(1/2-1/r)} d\tau. \\ &\lesssim \|\phi_-|X(s_1 + s_0, w)\| + [u]^p. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \|\Phi u|H^1\| &\lesssim \|\phi_-|X(1, 0)\| + \int_{-\infty}^t \|F(u(\tau))\|_2 d\tau \\ &\lesssim \|\phi_-|X(1, 0)\| + \int_{-\infty}^t \|u(\tau)\|_{2p}^p d\tau \\ &\lesssim \|\phi_-|X(1, 0)\| + [u]^p. \end{aligned}$$

Using the same argument as the proof of Theorem 1.1, we see that the solution space is

$$\{v \in \mathcal{H}; \|v|_{\mathcal{H}}\| + [v] < \infty\},$$

and that $(S, X(s_0 + s_1, w))$ is well-defined. $\square$

APPENDIX B. INTEGRAL EQUATIONS

In this section, we shall list some properties for the integral type of (1.1).

We first prepare some notation which are used in this section. We put $\omega = \sqrt{1 - \Delta}$ and $U(t) = e^{\pm it\omega}$. For any $t \in \mathbb{R}$, we set $J(t) = (0, t)$ or $(-\infty, t)$ or (t, ∞) . For an open interval I , and for $g \in L^1(I; H^{-1})$ and $t \in \mathbb{R}$, we set

$$\Psi_{\pm}(t) = \Psi_{\pm}[I; g](t) = \int_I \omega^{-1} U(t - \tau) g(\tau) d\tau.$$

Throughout this section, we assume that $g \in L^1(I; H^{-1})$ satisfies

$$g \in C(I; H^{-1}) \cap L(\rho, (1/q', 1/r')). \quad (B.1)$$

Here, (q, r) satisfies that $(q, r) \neq (2, \infty)$ and

$$\max(0, \frac{1}{2} - \frac{1}{n}) < \frac{1}{r} < \frac{1}{2}, \quad \frac{2}{q} = \frac{n}{2} - \frac{n}{r}, \quad 2\rho = \frac{n+2}{2} - \frac{n+2}{r}.$$

We are ready to state the first property.

Proposition B.1. *Suppose that I is a fixed open interval and that g satisfies (B.1). For any $t \in \mathbb{R}$, we have*

$$\begin{aligned} U(t)\Psi_{\pm}[I; g](0) &= \Psi_{\pm}[I; g](t) \quad \text{in } H^1, \\ \omega U(t)\Psi_{\pm}[I; g](0) &= \Psi_{\pm}[I; \omega g](t) \quad \text{in } L^2(\mathbb{R}^n). \end{aligned}$$

Proof. We shall prove only the first equality because the second one is similarly shown. From (B.1), we see from the Strichartz estimate in [12] that

$$\sup_t \left\| \int_I \omega^{-1} \mathcal{U}(t - \tau) g(\tau) d\tau \right\|_{H^1} \leq C \|g\|_{L(\rho, (1/q', 1/r'))} < \infty, \quad (\text{B.2})$$

and that

$$\begin{aligned} \left\| \Psi[I; g](t) - \int_{(-R, R) \cap I} \omega^{-1} \mathcal{U}(t - \tau) g(\tau) d\tau \right\|_{H^1} \\ = \left\| \int_{\{|\tau| > R\} \cap I} \omega^{-1} \mathcal{U}(t - \tau) g(\tau) d\tau \right\|_{H^1} \\ = \|g\|_{L^{q'}(\{|\tau| > R\} \cap I; B_{r'}^{\rho})} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \end{aligned}$$

Thus, we have for any $t \in \mathbb{R}$,

$$\Psi_{\pm}^R \equiv \Psi_{\pm}[(-R, R) \cap I; g](t) \rightarrow \Psi[I; g]_{\pm}(t) \quad \text{in } H^1 \text{ as } R \rightarrow \infty. \quad (\text{B.3})$$

On the other hand, $\mathcal{U}(t)$ is a bounded mapping from H^{-1} into itself. Hence, we see from (B.1) that

$$\mathcal{U}(t) \Psi_{\pm}^R(0) = \Psi_{\pm}^R(t).$$

Therefore, it follows from (B.3) that

$$\mathcal{U}(t) \Psi_{\pm}(0) = \mathcal{U}(t) \lim_{R \rightarrow \infty} \Psi_{\pm}^R(0) = \lim_{R \rightarrow \infty} \mathcal{U}(t) \Psi_{\pm}^R(0) = \lim_{R \rightarrow \infty} \Psi_{\pm}^R(t) = \Psi_{\pm}(t).$$

□

Let us consider the integral equation

$$u(t) = v(t) + \int_{J(t)} \frac{\sin(t - \tau)\omega}{\omega} g(\tau) d\tau \quad (\text{B.4})$$

for $t \in \mathbb{R}$, where $u(t)$ is a real or complex-valued unknown function, $v(t) \in \mathcal{H}$, f is a given nonlinear term. By the above proposition, we can rewrite (B.4) to

$$\begin{aligned} u(t) = v(t) + \frac{1}{2i} \left\{ \mathcal{U}(t) \Psi_+[J(t_0); g](0) - \mathcal{U}(-t) \Psi_-[J(t_0); g](0) \right\} \\ + \frac{1}{2i} \left\{ \Psi_+[(t_0, t); g](0) - \Psi_-[(t_0, t); g](0) \right\}, \end{aligned} \quad (\text{B.5})$$

where $t_0 \in \mathbb{R} \setminus \{t\}$.

The following result is essential to prove Theorems 1.1:

Proposition B.2. *Assume that u is a time-global solution to (B.4). Suppose that g satisfies (B.1). Then we have $u \in \mathcal{H}$ and*

$$\partial_t u(t) = \partial_t v(t) + \int_{J(t)} (\cos(t - \tau)\omega) g(\tau) d\tau \quad \text{in } L^2(\mathbb{R}^n). \quad (\text{B.6})$$

Proof. Since $\Psi_{\pm}(0) \in H^1$, we see that

$$u^I(t) \equiv v(t) + \frac{1}{2i} \left\{ U(t) \Psi_+[J(t_0); g](0) - U(-t) \Psi_-[J(t_0); g](0) \right\} \in \mathcal{H}$$

and

$$\partial_t u^I(t) = \partial_t v(t) + \frac{\omega}{2} \left\{ U(t) \Psi_+[J(t_0); g](0) - U(-t) \Psi_-[J(t_0); g](0) \right\}.$$

Put $u^{II} = u - u^I$. Then

$$\|u^{II}(t) - u^{II}(t_0)\|_{H^1} = \|u^{II}(t)\| \leq C \|g\|_{L^{q'}((t_0, t); B_r^p)} \rightarrow 0 \quad \text{as } t_0 \rightarrow t.$$

Hence, $u^{II} \in C(\mathbb{R}; H^1)$.

It follows from (B.1) that

$$\begin{aligned} & \left\| (t - t_0)^{-1} \left\{ \Psi_{\pm}[(t_0, t); g](t) - \Psi_{\pm}[(t_0, t); g](t_0) \right\} - \omega^{-1} g(t_0) \right\|_2 \\ &= \left\| (t - t_0)^{-1} \int_{t_0}^t \left\{ U(t - \tau) \omega^{-1} g(\tau) - \omega^{-1} g(t_0) \right\} d\tau \right\|_2 \\ &\leq \sup_{\tau \in (t_0, t)} \|U(t - \tau) g(\tau) - g(t_0)\|_{H^{-1}} \\ &\leq \sup_{\tau \in (t_0, t)} \|U(t - \tau) g(\tau) - U(t - \tau) g(t_0)\|_{H^{-1}} \\ &\quad + \sup_{\tau \in (t_0, t)} \|U(t - \tau) g(t_0) - g(t_0)\|_{H^{-1}} \\ &\leq \sup_{\tau \in (t_0, t)} \left\{ \|g(\tau) - g(t_0)\|_{H^{-1}} + \|U(t - \tau) g(t_0) - g(t_0)\|_{H^{-1}} \right\} \\ &\rightarrow 0 \quad \text{as } |t - t_0| \rightarrow 0. \end{aligned}$$

Thus, we have $\Psi_{\pm}[(t_0, t); g] \in C^1(\mathbb{R}; L^2(\mathbb{R}^n))$ and

$$\partial_t \Psi_{\pm}[(t_0, t); g](t_0) = \omega^{-1} g(t_0) \quad \text{in } L^2(\mathbb{R}^n).$$

Therefore, we obtain $u = u^I + u^{II} \in C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^n))$ and (B.6) holds. Using (B.4) and (B.6), we see from the same argument as in (B.2) that $u \in \mathcal{H}$. $\square$

Remark 5. Assume that u is a time-global solution to (B.4). Suppose that g satisfies (B.1). Repeating the same argument as the proof of the above proposition, we see that $u \in \mathcal{H} \cap \mathcal{B}^2(\mathbb{R}; H^{-1})$ and

$$\partial_t u - \Delta u + u = g(t) \quad \text{in } H^{-1}.$$

REFERENCES

- [1] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, Berlin, 1976.
- [2] P. Brenner, *On $L_p - L_{\dot{p}}$ estimates for the wave equation*, Math. Z. **145** (1975), 251–254.
- [3] P. Brenner, *On scattering and everywhere defined scattering operators for nonlinear Klein-Gordon equations*, J. Differential Equations **56** (1985), 310–344.
- [4] T. Cazenave, *Uniform estimates for solutions of nonlinear Klein-Gordon equations*, J. Funct. Anal. **60** (1985), 36–55.
- [5] T. Cazenave, *Semilinear Schrödinger Equations*, American Mathematical Society, 2003.
- [6] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equation I*, J. Funct. Anal. **32** (1979), 1–32.
- [7] H. Lindblad and C. Sogge, *Restriction theorems and semilinear Klein-Gordon equations in $(1 + 3)$ -dimensions*, Duke Math. J., **85** (1996), 227–252.
- [8] B. Marshall, W. Strauss and S. Wainger, *$L^p - L^q$ estimates for the Klein-Gordon equation*, J. Math. pure et appl., **59** (1980), 417–440.
- [9] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. RIMS, Kyoto Univ. **12** (1976), 169–189.
- [10] K. Mochizuki and T. Motai, *On small data scattering for some nonlinear wave equations*, in “Patterns and Waves -qualitative analysis of nonlinear differential equations -”, pp.543–560, Stud. Math. Appl., **18**, North-Holland, Amsterdam, 1986.
- [11] M. Nakamura, *A miniature scattering theory for nonlinear Klein-Gordon Equations*, Kyushu J. Math. **57** (2003), 255–263.
- [12] M. Nakamura and T. Ozawa, *The Cauchy problem for nonlinear Klein-Gordon equations in the Sobolev spaces*, Publ. Res. Inst. Math. Sci. **37** (2001), 255–293.
- [13] K. Nakanishi, *Energy scattering for non-linear Klein-Gordon and Schrodinger equations in spatial dimensions 1 and 2*, J. Funct. Anal. **169** (1999), 201–225.
- [14] K. Nakanishi, *Scattering theory for nonlinear Klein-Gordon equation with Sobolev critical power*, Internat. Math. Res. Notices **1999** (1999), 31–60.
- [15] T. Ozawa, *Remark on proofs of conservation laws for nonlinear Schrödinger equations*, Cal. Var. PDE. **25** (2006), 403–408.
- [16] H. Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Mathematische Zeitschrift **185** (1984), 261–270.
- [17] M. Reed, *Abstract Nonlinear Wave Equations*, Springer-Verlag, New York, 1976.
- [18] H. Sasaki, *Small data scattering for the Klein-Gordon equation with cubic convolution nonlinearity*, Discrete Contin. Dynam. Systems **15** (2006), 973–981 .
- [19] H. Sasaki, *Convergence of scattering operators for the Klein-Gordon equation with a nonlocal nonlinearity*, Differential Integral Equations **19** (2006), 877–889.
- [20] C. Sogge, *Lectures on Nonlinear Wave Equations*, International Press, Cambridge, MA, 1995.

- [21] W. Strauss, *Nonlinear scattering theory at low energy: sequel*, J. Funct. Anal. **43** (1981), 281–293.
- [22] W. Baoxiang, *On existence and scattering for critical and subcritical nonlinear Klein-Gordon equations in H^s* , Nonlinear Anal. Theory, Methods & Applications, **31** (1998), 573–587.

E-mail address: hisasaki@math.sci.hokudai.ac.jp