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GLOBAL EXISTENCE OF SOLUTIONS FOR A REACTION-DIFFUSION SYSTEM

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ABSTRACT. We show the global existence of solutions of a reaction-diffusion system with the nonlinear terms $|x|^{\sigma_j} u^{p_j} v^{q_j}$. The proof is based on the existence of super-solutions and the comparison principle. We also prove that uniqueness of the global solutions holds in the superlinear case by contraction argument. Our conditions for the global existence are optimal in view of the nonexistence results proved by Yamauchi [12].

1. INTRODUCTION

We consider the Cauchy problem for the reaction-diffusion system:

$$\begin{cases} u_t - \Delta u = |x|^{\sigma_1} u^{p_1} v^{q_1}, & x \in \mathbf{R}^N, \ t > 0, \\ v_t - \Delta v = |x|^{\sigma_2} u^{p_2} v^{q_2}, & x \in \mathbf{R}^N, \ t > 0, \\ u(x, 0) = u_0(x) \geq 0, \neq 0, & x \in \mathbf{R}^N, \\ v(x, 0) = v_0(x) \geq 0, \neq 0, & x \in \mathbf{R}^N, \end{cases} \quad (1.1)$$

where $p_j, q_j \geq 0, \sigma_j \geq 0$ ($j = 1, 2$), and $p_1, q_2 \neq 1$.

Combustion process of single solid chemical material is expressed as a reaction-diffusion system (see, e.g., [2]). Two unknown functions in this original system represent the temperature and mass of the material. The nonlinearity in the original system consists of powers and exponential forms of the unknown functions. Our problem (1.1) describes a simplified model to investigate the mechanism of this type of nonlinearity.

Before stating our main results, we first recall known results for the single equation : $u_t - \Delta u = u^p$. Let N be the space dimension. Fujita [5] proved the existence of global solutions to the equation if $p > 1 + 2/N$ for exponential decaying small initial data. He also proved the nonexistence of global solutions if $p < 1 + 2/N$. In the critical case, $p = 1 + 2/N$, the nonexistence

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is proved by Hayakawa [6], Kobayashi, Sirao and Tanaka [7] and Weissler [11]. On the other hand, in the sublinear case, i.e. $0 < p < 1$, it is shown by Aguirre and Escobedo [1] that every solution to the equation exists globally in time.

For nonlinear terms with variable coefficients, Pinsky [10] showed the existence and nonexistence of global solutions to the equation: $u_t - \Delta u = a(x)u^p$, where $p > 1$ and $a(x)$ behaves like $|x|^m$ with $m > -2$ for large $|x|$.

We next consider a system:

$$\begin{cases} u_t - \Delta u = F_1(x, u, v), \\ v_t - \Delta v = F_2(x, u, v). \end{cases}$$

Escobedo and Herrero [3] studied the system with the nonlinear terms $F_1 = v^p$ and $F_2 = u^q$ with $p, q > 0$ for nonnegative, continuous and bounded initial data. Their results are divided into three cases: (i) $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) < N/2$, (ii) $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) \geq N/2$, (iii) $pq < 1$. When $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) < N/2$, there exist global solutions for small initial data. For large initial data, some solutions blow up in finite time. When $pq > 1$ and $(\max(p, q) + 1)/(pq - 1) \geq N/2$, there exist no global solutions. When $pq < 1$, every solution exists globally in time.

In case $F_1 = |x|^{\sigma_1}v^p$ and $F_2 = |x|^{\sigma_2}u^q$ with $p, q > 1$, $0 \leq \sigma_j < N(p_j + q_j - 1)$, $j = 1, 2$, Mochizuki and Huang [9] showed the existence and nonexistence of global solutions and the asymptotic behavior of the global solution. Let $\alpha = \{(p(\sigma_2 + 2) + (\sigma_1 + 2))/2\}/\{(pq - 1)\}$ and $\beta = \{(q(\sigma_1 + 2) + (\sigma_2 + 2))/2\}/\{2(pq - 1)\}$. They proved that if $0 < \max(\alpha, \beta) < N/2$, then global solution exists for small initial data and does not exist for large initial data. On the other hand, if $\max(\alpha, \beta) \geq N/2$, then there exist no global solutions.

For $F_j = u^{p_j}v^{q_j}$ with $p_j, q_j \geq 0$, $j = 1, 2$, $0 < p_1 + q_1 \leq p_2 + q_2$, Escobedo and Levine [4] showed the following results. Let $\alpha = (q_1 - q_2 + 1)/\{p_2q_1 - (1 - p_1)(1 - q_2)\}$ and $\beta = (p_1 - p_2 + 1)/\{p_2q_1 - (1 - p_1)(1 - q_2)\}$.

(i) Let $p_1 \leq 1$. If $0 \leq \max(\alpha, \beta) < N/2$, then global solution exists for small initial data and does not exist for large initial data. If $\max(\alpha, \beta) < 0$, then every solution exists globally in time. If $\max(\alpha, \beta) \geq N/2$, then there exist no global solutions.

(ii) Let $p_1 > 1$. If $p_1 + q_1 > 1 + 2/N$, then global solution exists for small initial data and does not exist for large initial data. If $p_1 + q_1 \leq 1 + 2/N$, then there exist no global solutions.

In this paper, we consider (1.1) as a generalization of these nonlinear terms. We prove the existence of global solutions to (1.1) under some conditions on $N, \sigma_j, p_j, q_j, j = 1, 2$, which are optimal in view of the nonexistence results proved by Yamauchi [12]. We emphasize that if $\sigma_j = 0, j = 1, 2$ or $p_1 = q_2 = 0$, our conditions for the global existence coincide with those of [4] or [9], respectively.

Since our problem includes the sublinear case, p_j or $q_j < 1$, the contraction argument does not always work. Thus, we prove the global existence for (1.1) by finding super-solutions and the comparison principle following [4]. However, the method of [4] does not seem directly applicable to our problem because of the presence of the variable coefficients $|x|^{\sigma_j}$. In order to find super-solutions for the strongly coupled nonlinear terms, we need to introduce exponential functions in t . Moreover, if the nonlinear terms are strongly coupled, the solutions of the problem do not always decay at infinity in space. Thus, we improve the estimates used in [9] to derive new ones for those solutions. We also show that for the superlinear case $p_j, q_j > 1$, $j = 1, 2$, uniqueness of the global solutions holds. The crucial part to show the result for the superlinear case is how to choose solution spaces and weighted norms. In view of that, our problem is more complicated than that of [9] since we treat the strongly coupled cases.

Our plan of this paper is as follows. In Section 2, we state several notation and main results. In Section 3, we show the local existence of classical solutions of the system (1.1). In Sections 4 and 5, we show the existence of global solutions by comparison principle and contraction argument, respectively.

2. MAIN RESULTS

We begin with stating some notation. Put

$$\begin{cases} \alpha = \frac{q_1(\sigma_2 + 2) + (1 - q_2)(\sigma_1 + 2)}{2\{p_2q_1 - (1 - p_1)(1 - q_2)\}}, \\ \beta = \frac{p_2(\sigma_1 + 2) + (1 - p_1)(\sigma_2 + 2)}{2\{p_2q_1 - (1 - p_1)(1 - q_2)\}}, \end{cases} \quad (2.1)$$

$$\begin{cases} \delta_1 = \frac{q_1\sigma_2 + (1 - q_2)\sigma_1}{p_2q_1 - (1 - p_1)(1 - q_2)}, \\ \delta_2 = \frac{p_2\sigma_1 + (1 - p_1)\sigma_2}{p_2q_1 - (1 - p_1)(1 - q_2)}. \end{cases} \quad (2.2)$$

For $a \in \mathbf{R}$, we define the function spaces:

$$I^a = \{w \in C(\mathbf{R}^N); w(x) \geq 0, \limsup_{|x| \rightarrow \infty} |x|^a w(x) < \infty\},$$

$$L_a^\infty = \{w \text{ is measurable function on } \mathbf{R}^N; \\ w(x) \geq 0, \|w\|_{\infty, a} \equiv \|\langle x \rangle^a w(x)\|_\infty < \infty\},$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$. We also define

$$E_T = \{(u, v); [0, T] \rightarrow L_{\delta_1}^\infty \times L_{\delta_2}^\infty, u, v \geq 0, \|(u, v)\|_{E_T} < \infty\},$$

where

$$\|(u, v)\|_{E_T} = \sup_{t \in [0, T]} (\|u(t)\|_{\infty, \delta_1} + \|v(t)\|_{\infty, \delta_2}).$$

We state our main results. Throughout this paper, we assume that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$.

Theorem 2.1. *Let $p_1 < 1$, $q_2 < 1$.*

- (i) *If $0 < \max(\alpha, \beta) < N/2$, then there exist global classical solutions of (1.1) for small initial data.*
- (ii) *If $\max(\alpha, \beta) < 0$, then every classical solution of (1.1) exists globally in time.*

Theorem 2.2. *Let $p_1 > 1$, $q_2 < 1$. If $\alpha < N/2$ and $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then there exist global classical solutions of (1.1) for small initial data.*

For $p_1 < 1$, we can rewrite Theorems 2.1 and 2.2 in the manner of [4] as follows.

Corollary 2.3. *Assume that*

$$\frac{p_1 + q_1 - 1}{\sigma_1 + 2} \leq \frac{p_2 + q_2 - 1}{\sigma_2 + 2}, \quad (2.3)$$

and let $p_1 < 1$, $q_2 \neq 1$.

- (i) *If $0 < \max(\alpha, \beta) < N/2$, then there exist global classical solutions of (1.1) for small initial data.*
- (ii) *If $\max(\alpha, \beta) < 0$, then every classical solution exists globally.*

Theorem 2.4. *Let $p_1 > 1$, $q_2 > 1$. If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$ and $p_2 + q_2 > 1 + (2 + \sigma_2)/N$, then there exist global classical solutions of (1.1) for small initial data.*

Also for $p_1 > 1$, we can rewrite Theorems 2.2 and 2.4 in the way of [4].

Corollary 2.5. *Assume (2.3), and let $p_1 > 1$, $q_2 \neq 1$. If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then there exist global classical solutions of (1.1) for small initial data.*

Remark 2.6. We remark that the condition $p_1 + q_1 > 1 + (2 + \sigma_1)/N$ in Corollary 2.5 consists of only the exponents in one equation. This condition is the same as that for the global existence for the single equation $u_t - \Delta u = |x|^{\sigma_1} u^{p_1 + q_1}$. See [10] for details.

In the superlinear case, we have a uniqueness result.

Theorem 2.7. *Assume (2.3) and let $p_j > 1$, $q_j > 1$ ($j = 1, 2$). Assume that $(u_0, v_0) \in I^a \times I^b$ ($(\sigma_1 + 2)/(p_1 + q_1 - 1) < a, b < N$). If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then there exist unique global solutions of (1.1) for small initial data.*

3. LOCAL EXISTENCE THEOREM

In this section, we show the local existence of classical solutions of (1.1).

Theorem 3.1. *Let δ_1 and δ_2 be defined in (2.2). Assume that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. Then there exist classical solutions $(u(t), v(t)) \in E_T$ for the system (1.1) for some $T > 0$.*

Proof. We first construct local solutions to the system of integral equations associated to (1.1):

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds, \quad (3.1)$$

$$v(t) = e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_2} u(s)^{p_2} v(s)^{q_2} ds, \quad (3.2)$$

where

$$e^{t\Delta}f(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy.$$

It is sufficient to show Propositions 3.2 and 3.4 below to prove Theorem 3.1. The local existence of solutions of (3.1) and (3.2) is given by the following proposition.

Proposition 3.2. *Let δ_1 and δ_2 be defined in (2.2). Assume that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. Then there exist $(u(t), v(t)) \in E_T$ satisfying the integral equations (3.1) and (3.2) for some $T > 0$.*

To prove the proposition, we define $\{u_n(x, t)\}$ and $\{v_n(x, t)\}$ ($n = 1, 2, \dots$) inductively by:

$$\begin{aligned} u_{n+1}(t) &= e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_1} u_n(s)^{p_1} v_n(s)^{q_1} ds, \\ v_{n+1}(t) &= e^{t\Delta}v_0 + \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_2} u_n(s)^{p_2} v_n(s)^{q_2} ds, \\ u_1 &= e^{t\Delta}u_0, \\ v_1 &= e^{t\Delta}v_0. \end{aligned}$$

We use the following estimates by weighted norms and uniform estimates for the solutions.

Lemma 3.3. (1) *Let $\delta, a \in \mathbf{R}, \sigma \geq 0$ and $\delta + a + \sigma = 0$. Then we have*

$$\|e^{t\Delta}|x|^\sigma \langle x \rangle^a\|_{\infty, \delta} \leq \begin{cases} C(1+t)^{\max(-\delta, 0, \delta-N)/2}, & (\delta \neq N), \\ C \log(2+t), & (\delta = N). \end{cases} \quad (3.3)$$

(2) *Suppose that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$. Then there exist $K > 0$ and $T > 0$ such that*

$$\begin{aligned} \sup_{t \in [0, T]} \|u_n(t)\|_{\infty, \delta_1} &< K, \\ \sup_{t \in [0, T]} \|v_n(t)\|_{\infty, \delta_2} &< K \end{aligned}$$

for all n .

Proof. (1) We show the estimates dividing into three cases: (i) $0 \leq \delta \leq N$, (ii) $\delta < 0$, and (iii) $\delta > N$.

For the case $0 \leq \delta \leq N$, using Lemma 2.1 in [9], we can see that

$$\begin{aligned} \|e^{t\Delta}|x|^\sigma \langle x \rangle^a\|_{\infty, \delta} &\leq \|e^{t\Delta} \langle x \rangle^{\sigma+a}\|_{\infty, \delta} \\ &\leq \begin{cases} C, & (0 \leq \delta < N), \\ C \log(2+t), & (\delta = N). \end{cases} \end{aligned}$$

In the case $\delta < 0$, we have

$$\begin{aligned} &\int_{\mathbf{R}^N} \exp\left(-\frac{|y|^2}{4t}\right) |x-y|^\sigma \langle x-y \rangle^a dy \\ &\leq \int_{\mathbf{R}^N} \exp\left(-\frac{|y|^2}{4t}\right) \langle x-y \rangle^{\sigma+a} dy \\ &\leq C \int_{|y| \leq |x|/2} \exp\left(-\frac{|y|^2}{4t}\right) \langle x \rangle^{\sigma+a} dy + C \int_{|y| > |x|/2} \exp\left(-\frac{|y|^2}{4t}\right) \langle y \rangle^{\sigma+a} dy \\ &\leq Ct^{N/2} \langle x \rangle^{\sigma+a} + Ct^{N/2} (1+t)^{(\sigma+a)/2}. \end{aligned}$$

Finally, for $\delta > N$, we have

$$\begin{aligned} &\int_{\mathbf{R}^N} \exp\left(-\frac{|x-y|^2}{4t}\right) |y|^\sigma \langle y \rangle^a dy \\ &\leq \int_{|y| \leq |x|/2} \exp\left(-\frac{|x|^2}{16t}\right) \langle y \rangle^{\sigma+a} dy + C \int_{|y| > |x|/2} \exp\left(-\frac{|x-y|^2}{4t}\right) \langle x \rangle^{\sigma+a} dy \\ &\leq C|x|^{N/2} \langle x \rangle^{\sigma+a} \langle x \rangle^{-(N+\sigma+a)} \exp\left(-\frac{|x|^2}{16t}\right) + Ct^{N/2} \langle x \rangle^{\sigma+a} \\ &\leq Ct^{N/2} \langle x \rangle^{\sigma+a} (1+t)^{-(N+\sigma+a)/2} + Ct^{N/2} \langle x \rangle^{\sigma+a}. \end{aligned}$$

(2) We first estimate $\|e^{t\Delta} u_0\|_{\infty, \delta_1}$. By (3.3), we have

$$\begin{aligned} \|e^{t\Delta} u_0\|_{\infty, \delta_1} &\leq \|u_0\|_{\infty, \delta_1} \|e^{t\Delta} \langle \cdot \rangle^{-\delta_1}\|_{\infty, \delta_1} \\ &\leq \begin{cases} C\|u_0\|_{\infty, \delta_1}, & (0 \leq \delta_1 < N), \\ C\|u_0\|_{\infty, \delta_1} (1+T)^{\kappa_1}, & (\text{otherwise}), \end{cases} \end{aligned}$$

where $\kappa_1 = \kappa_1(\delta_1, N) > 0$.

From above, there exists a constant $\tilde{C} = \tilde{C}(T) > 0$ satisfying

$$\begin{aligned} \sup_{t \in [0, T]} \|e^{t\Delta} u_0\|_{\infty, \delta_1} &\leq \tilde{C} \|u_0\|_{\infty, \delta_1}, \\ \sup_{t \in [0, T]} \|e^{t\Delta} v_0\|_{\infty, \delta_2} &\leq \tilde{C} \|v_0\|_{\infty, \delta_2} \end{aligned}$$

for fixed $T > 0$.

We next estimate the nonlinear terms. Define $\Phi_1(u, v)$ and $\Phi_2(u, v)$ by

$$\begin{aligned}\Phi_1(u, v)(t) &= \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds, \\ \Phi_2(u, v)(t) &= \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_2} u(s)^{p_2} v(s)^{q_2} ds.\end{aligned}$$

Applying (3.3) again, we obtain

$$\begin{aligned}\sup_{t \in [0, T]} \|\Phi_1(u_n, v_n)(t)\|_{\infty, \delta_1} &\leq C(T) \sup_{t \in [0, T]} \|u_n(t)\|_{\infty, \delta_1}^{p_1} \sup_{t \in [0, T]} \|v_n(t)\|_{\infty, \delta_2}^{q_1}, \\ \sup_{t \in [0, T]} \|\Phi_2(u_n, v_n)(t)\|_{\infty, \delta_2} &\leq C(T) \sup_{t \in [0, T]} \|u_n(t)\|_{\infty, \delta_1}^{p_2} \sup_{t \in [0, T]} \|v_n(t)\|_{\infty, \delta_2}^{q_2},\end{aligned}$$

where $C(T) \downarrow 0$ ($T \downarrow 0$). Indeed, we can see

$$\begin{aligned}\Phi_1(u, v)(t) &\leq \sup_{t \in [0, T]} \|u(t)\|_{\infty, \delta_1}^{p_1} \sup_{t \in [0, T]} \|v(t)\|_{\infty, \delta_2}^{q_1} \int_0^t \|e^{(t-s)\Delta} |\cdot|^{\sigma_1} \langle \cdot \rangle^{-p_1 \delta_1 - q_1 \delta_2}\|_{\infty, \delta_1} ds \\ &\leq \begin{cases} C \sup_{t \in [0, T]} \|u(t)\|_{\infty, \delta_1}^{p_1} \sup_{t \in [0, T]} \|v(t)\|_{\infty, \delta_2}^{q_1} T, & (0 \leq \delta_1 < N), \\ C \sup_{t \in [0, T]} \|u(t)\|_{\infty, \delta_1}^{p_1} \sup_{t \in [0, T]} \|v(t)\|_{\infty, \delta_2}^{q_1} T(1+T)^{\kappa_2}, & (\text{otherwise}), \end{cases}\end{aligned}$$

where $\kappa_2 = \kappa_2(\sigma_1, p_1, q_1, \delta_1, \delta_2, N) > 0$.

Here, we put $R = \max(\|u_0\|_{\infty, \delta_1}, \|v_0\|_{\infty, \delta_2})$. Taking large $K > 0$ and small $T > 0$ such that

$$K > 2\tilde{C}R, \quad C(T) < \frac{K - \tilde{C}R}{K^{p_1+q_1} + K^{p_2+q_2}},$$

we obtain the desired estimates. This completes the proof. $\square$

Returning to the proof of Proposition 3.2, one can see from Lemma 3.3(2) that

$$\begin{aligned}\sup_{t \in [0, T]} \|\langle \cdot \rangle^{\delta_1} u_n(t)\|_{\infty} &< K, \\ \sup_{t \in [0, T]} \|\langle \cdot \rangle^{\delta_2} v_n(t)\|_{\infty} &< K\end{aligned}$$

for all n . The monotonicity of the heat kernel gives

$$u_n \leq u_{n+1}, \quad v_n \leq v_{n+1}$$

for all n . Therefore, there exist $\tilde{u}(x, t) = \lim_{n \rightarrow \infty} u_n(x, t)$ and $\tilde{v}(x, t) = \lim_{n \rightarrow \infty} v_n(x, t)$ on $\mathbf{R}^N \times [0, T]$, and we have

$$\begin{aligned}\sup_{t \in [0, T]} \|\tilde{u}(t)\|_{\infty, \delta_1} &\leq K, \\ \sup_{t \in [0, T]} \|\tilde{v}(t)\|_{\infty, \delta_2} &\leq K.\end{aligned}$$

Moreover, from Lebesgue's monotone convergence theorem, we can easily see that $(\tilde{u}, \tilde{v})$ are local solutions of (3.1) and (3.2). This completes the proof of Proposition 3.2. $\square$

Next, we improve the regularity of the local solutions given in Proposition 3.2.

Proposition 3.4. *Let $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$, and let $(u, v) \in E_T$ be solutions of (3.1) and (3.2) in $\mathbf{R}^N \times (0, T]$. Assume that there exists a constant $C > 0$ such that*

$$\|u(t)\|_{\infty, \delta_1} < C, \quad \|v(t)\|_{\infty, \delta_2} < C \quad (0 \leq t \leq T). \quad (3.4)$$

Then (u, v) are classical solutions of (1.1) in $\mathbf{R}^N \times (0, T)$.

Proof. From the assumptions, we can easily see that $|x|^{\sigma_j} u^{p_j} v^{q_j}$ ($j = 1, 2$) are locally θ -Hölder continuous in x ($0 < \theta \leq 1$); that is, for any $\varepsilon > 0$ and for any compact set $K \subset \mathbf{R}^N$, there exists a constant $C > 0$ such that

$$|x_1|^{\sigma_j} u^{p_j} v^{q_j}(x_1, t) - |x_2|^{\sigma_j} u^{p_j} v^{q_j}(x_2, t)| < C|x_1 - x_2|^\theta,$$

for any $t \in (\varepsilon, T)$, and $x_1, x_2 \in K$. It follows from the Hölder continuity and the standard regularity argument in [8] that (u, v) are classical solutions. $\square$

This completes the proof of Theorem 3.1. $\square$

4. PROOF OF THEOREMS 2.1, 2.2, AND 2.4

In proving Theorems 2.1, 2.2, and 2.4, we use a comparison theorem and the existence of super-solutions. First, we show the following comparison theorem.

Proposition 4.1. *Let $f(u, v)$ and $g(u, v)$ be strictly monotone increasing in u and v for $u, v \geq 0$. Assume that $\bar{u}, \bar{v}, \underline{u}, \underline{v}$ are non-negative and satisfy*

$$\begin{cases} \bar{u}_t - \Delta \bar{u} \geq |x|^{\sigma_1} f(\bar{u}, \bar{v}), \\ \bar{v}_t - \Delta \bar{v} \geq |x|^{\sigma_2} g(\bar{u}, \bar{v}), \\ \underline{u}_t - \Delta \underline{u} \leq |x|^{\sigma_1} f(\underline{u}, \underline{v}), \\ \underline{v}_t - \Delta \underline{v} \leq |x|^{\sigma_2} g(\underline{u}, \underline{v}), \end{cases} \quad \text{in } \mathbf{R}^N \times (0, T),$$

$$\begin{cases} \bar{u}(x, 0) - \underline{u}(x, 0) \geq 0, \neq 0, \\ \bar{v}(x, 0) - \underline{v}(x, 0) \geq 0, \neq 0. \end{cases} \quad x \in \mathbf{R}^N.$$

Then we have $\bar{u}(x, t) \geq \underline{u}(x, t)$ and $\bar{v}(x, t) \geq \underline{v}(x, t)$ on $\mathbf{R}^N \times (0, T)$.

Proof. Put

$$t_1 = \inf\{\tau \in (0, T); \exists x' \in \mathbf{R}^N \text{ s.t. } \bar{u}(\tau, x') < \underline{u}(\tau, x')\},$$

$$t_2 = \inf\{\tau \in (0, T); \exists x' \in \mathbf{R}^N \text{ s.t. } \bar{v}(\tau, x') < \underline{v}(\tau, x')\}.$$

If $\bar{u}(t, x) \geq \underline{u}(t, x)$ for any $(t, x) \in (0, T) \times \mathbf{R}^N$, then let $t_1 = \infty$. And if $\bar{v}(t, x) \geq \underline{v}(t, x)$ for any $(t, x) \in (0, T) \times \mathbf{R}^N$, then let $t_2 = \infty$.

We assume that $t_1 \leq t_2$ and $t_1 < T$. By the definition of t_1 and the continuity argument, we have

$$\bar{u}(t_1, x_0) = \underline{u}(t_1, x_0) \quad \text{for some } x_0 \in \mathbf{R}^N, \quad (4.1)$$

$$\bar{u}(t, x) \geq \underline{u}(t, x) \quad \text{for any } (t, x) \in (0, t_1) \times \mathbf{R}^N, \quad (4.2)$$

$$\bar{v}(t, x) \geq \underline{v}(t, x) \quad \text{for any } (t, x) \in (0, t_1) \times \mathbf{R}^N. \quad (4.3)$$

From the associated integral inequalities of $\bar{u}$ and $\underline{u}$, we have

$$\begin{aligned} & \bar{u}(t_1, x_0) - \underline{u}(t_1, x_0) \\ & \geq e^{t_1 \Delta} (\bar{u}_0(x_0) - \underline{u}_0(x_0)) + \int_0^{t_1} e^{(t_1-s)\Delta} |x_0|^{\sigma_1} (f(\bar{u}, \bar{v}) - f(\underline{u}, \underline{v})) ds \\ & > \int_0^{t_1} e^{(t_1-s)\Delta} |x_0|^{\sigma_1} (f(\bar{u}, \bar{v}) - f(\underline{u}, \underline{v})) ds. \end{aligned}$$

Positivity of the heat kernel and (4.1) imply that there exist $(s, y) \in (0, t_1) \times \mathbf{R}^N$ such that

$$(f(\bar{u}, \bar{v}) - f(\underline{u}, \underline{v}))(s, y) < 0.$$

This contradicts (4.2), (4.3) and the monotone increasing property of the function f .

In the case $t_1 \geq t_2$ and $t_2 < T$, we can derive a contradiction in the same way as above by replacing u with v and f with g . This completes the proof. $\square$

We next show the existence of super-solutions of (1.1) in several cases.

Proposition 4.2. (i) *Let $p_1 > 1$, $q_2 > 1$ or $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$, and let $p_1 + q_1 > 1$, $p_2 + q_2 > 1$. Assume that one of the following conditions is satisfied:*

- (A) $p_1, q_2 > 1$, $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, $p_2 + q_2 > 1 + (2 + \sigma_2)/N$.
- (B) $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, $\alpha < N/2$.
- (C) $p_1, q_2 < 1$, $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$, $\alpha, \beta < N/2$.

Then there exist $C_1, C_2, \alpha_1, \beta_1 > 0, t_0 > 1$ such that

$$\bar{u}(x, t) = C_1(t + t_0)^{\alpha_1 - \frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t + t_0)}\right), \quad (4.4)$$

$$\bar{v}(x, t) = C_2(t + t_0)^{\beta_1 - \frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t + t_0)}\right) \quad (4.5)$$

are super-solutions of (1.1).

(ii) *Let $p_1 > 1$, $q_2 > 1$ or $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$. And let $p_1 + q_1 > 1$, $p_2 + q_2 \leq 1$. Assume that one of the following conditions is satisfied:*

- (D) $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, $\alpha < N/2$,
- (E) $p_1, q_2 \leq 1$, $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$, $\alpha, \beta < N/2$.

Then there exist $C_1, C_2, \alpha_1, \beta_1 > 0, t_0 > 1, a > 0$ such that

$$\bar{u}(x, t) = C_1(t + t_0)^{\alpha_1 - \frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t + t_0)}\right), \quad (4.6)$$

$$\bar{v}(x, t) = C_2(t + t_0)^{\beta_1 - \frac{Na}{2}} \exp\left(-\frac{a|x|^2}{4(t + t_0)}\right) \quad (4.7)$$

are super-solutions of (1.1).

(iii) Let $p_1 < 1, q_2 < 1$ and $p_2q_1 - (1 - p_1)(1 - q_2) < 0$. Then there exist $C_1, C_2, k, a > 0$ such that

$$\bar{u}(x, t) = C_1 \langle x \rangle^{-2\delta_1} \exp(kt), \quad (4.8)$$

$$\bar{v}(x, t) = C_2 \langle x \rangle^{-2\delta_2} \exp(akt) \quad (4.9)$$

are super-solutions of (1.1).

Proof. (i) Put

$$\bar{u}(x, t) = C_1(t + t_0)^{\alpha_1 - \frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t + t_0)}\right), \quad (4.10)$$

$$\bar{v}(x, t) = C_2(t + t_0)^{\beta_1 - \frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t + t_0)}\right), \quad (4.11)$$

where $C_1, C_2, \alpha_1, \beta_1 > 0, t_0 > 1$. We can see that $(\bar{u}, \bar{v})$ are global super-solutions for small $C_1, C_2 > 0$ and large $t_0 > 1$, provided that

$$\begin{cases} \alpha_1 - N/2 - 1 > p_1(\alpha_1 - N/2) + q_1(\beta_1 - N/2) - \sigma_1/2, & \text{and} \\ \beta_1 - N/2 - 1 > p_2(\alpha_1 - N/2) + q_2(\beta_1 - N/2) - \sigma_2/2, \end{cases} \quad (4.12)$$

which (4.12) is equivalent to

$$(p_1 - 1)\alpha_1 + q_1\beta_1 < (p_1 + q_1 - 1)N/2 - (\sigma_1 + 2)/2, \quad \text{and} \quad (4.13)$$

$$p_2\alpha_1 + (q_2 - 1)\beta_1 < (p_2 + q_2 - 1)N/2 - (\sigma_2 + 2)/2. \quad (4.14)$$

Now, we shall show the existence of $\alpha_1, \beta_1 > 0$ on the (α_1, β_1) -plane in each case of Proposition 4.2.

Case (A): $p_1, q_2 > 1, p_1 + q_1 > 1 + (2 + \sigma_1)/N, p_2 + q_2 > 1 + (2 + \sigma_2)/N$.

Since the right hand sides of (4.13) and (4.14) are positive, we can take small $\alpha_1, \beta_1 > 0$ satisfying (4.13) and (4.14).

Case (B): $p_1 > 1 > q_2, p_1 + q_1 > 1 + (2 + \sigma_1)/N, \alpha < N/2$.

We remark that the intersection of

$$(p_1 - 1)\alpha_1 + q_1\beta_1 = (p_1 + q_1 - 1)N/2 - (\sigma_1 + 2)/2, \quad \text{and}$$

$$p_2\alpha_1 + (q_2 - 1)\beta_1 = (p_2 + q_2 - 1)N/2 - (\sigma_2 + 2)/2$$

is $(\alpha_1, \beta_1) = (N/2 - \alpha, N/2 - \beta)$. From the assumption, we can see that the intersection lies above the α_1 -axis and that the boundary of (4.13) lies above the origin. For $\varepsilon_1, \varepsilon_2 > 0$, put $(\alpha_1, \beta_1) = (\varepsilon_1, \{(p_1 + q_1 - 1)N/2 - (\sigma_1 + 2)/2\}/q_1 + \varepsilon_2)$. Then there exist small constants $\varepsilon_1, \varepsilon_2 > 0$ such that

(α_1, β_1) satisfy (4.13) and (4.14).

Case (C): $p_1, q_2 < 1$, $p_2q_1 - (1 - p_1)(1 - q_2) > 0$, $\alpha, \beta < N/2$.

From the assumption, we can see that the intersection lies in the first quadrant. Since $p_1, q_2 < 1$ and $p_2q_1 - (1 - p_1)(1 - q_1) > 0$, we have $(1 - p_1)/q_1 < p_2/(1 - q_2)$, that is, the angular coefficient of (4.14) is larger than that of (4.13). Hence, there exist small constants $\varepsilon_1, \varepsilon_2 > 0$ such that $(\alpha_1, \beta_1) = (N/2 - \alpha - \varepsilon_1, N/2 - \beta - \varepsilon_2)$ satisfy (4.13) and (4.14). $\square$

(ii) Case (D): $p_1 > 1 > q_2$, $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, $\alpha < N/2$.

Put $a > 0$ such that

$$\max \left\{ 0, \frac{(1 - p_1)N + (\sigma_1 + 2)}{q_1N} \right\} < a < \frac{p_2}{1 - q_2}. \quad (4.15)$$

In fact, since $q_2 < 1$, $p_2q_1 - (1 - p_1)(1 - q_2) > 0$ and $\alpha < N/2$, we have

$$\begin{aligned} & \frac{p_2}{1 - q_2} - \frac{(1 - p_1)N + (\sigma_1 + 2)}{q_1N} \\ &= \frac{1}{Nq_1(1 - q_2)} \{ Nq_1p_2 - N(1 - q_2)(1 - p_1) - (1 - q_2)(\sigma_1 + 2) \} \\ &= \frac{2\{p_2q_1 - (1 - p_1)(1 - q_2)\}}{Nq_1(1 - q_2)} \left\{ \frac{N}{2} - \frac{(1 - q_2)(\sigma_1 + 2)}{2(p_2q_1 - (1 - p_1)(1 - q_2))} \right\} \\ &\geq \frac{2\{p_2q_1 - (1 - p_1)(1 - q_2)\}}{Nq_1(1 - q_2)} \left(\frac{N}{2} - \alpha \right) \\ &> 0. \end{aligned}$$

Therefore, we can take $a > 0$ satisfying (4.15). Let

$$\bar{u}(x, t) = C_1(t + t_0)^{\alpha_1 - \frac{N}{2}} \exp \left(-\frac{|x|^2}{4(t + t_0)} \right), \quad (4.16)$$

$$\bar{v}(x, t) = C_2(t + t_0)^{\beta_1 - \frac{Na}{2}} \exp \left(-\frac{a|x|^2}{4(t + t_0)} \right), \quad (4.17)$$

where $C_1, C_2, \alpha_1, \beta_1 > 0$, $t_0 > 1$. We can see that $(\bar{u}, \bar{v})$ are global supersolutions provided that

$$\begin{cases} \alpha_1 - N/2 - 1 > p_1(\alpha_1 - N/2) + q_1(\beta_1 - Na/2) - \sigma_1/2, \text{ and} \\ \beta_1 - Na/2 - 1 > p_2(\alpha_1 - N/2) + q_2(\beta_1 - Na/2) - \sigma_2/2 \end{cases} \quad (4.18)$$

for small $C_1, C_2 > 0$ and large $t_0 > 1$. The condition (4.18) is equivalent to

$$(p_1 - 1)\alpha_1 + q_1\beta_1 < (p_1 + aq_1 - 1)N/2 - (\sigma_1 + 2)/2, \text{ and} \quad (4.19)$$

$$p_2\alpha_1 + (q_2 - 1)\beta_1 < (p_2 + aq_2 - a)N/2 - (\sigma_2 + 2)/2. \quad (4.20)$$

We remark that the intersection of

$$\begin{aligned} (p_1 - 1)\alpha_1 + q_1\beta_1 &= (p_1 + aq_1 - 1)N/2 - (\sigma_1 + 2)/2, \text{ and} \\ p_2\alpha_1 + (q_2 - 1)\beta_1 &= (p_2 + aq_2 - a)N/2 - (\sigma_2 + 2)/2 \end{aligned}$$

is $(\alpha_1, \beta_1) = (N/2 - \alpha, Na/2 - \beta)$. From the assumption $\alpha < N/2$, we see that the intersection lies above the α_1 -axis. From $a > \{(1-p_1)N + (\sigma_1 + 2)\}/q_1N$, we can easily see that the boundary of (4.19) lies above the origin. Hence, we can prove the existence of (α_1, β_1) satisfying (4.19) and (4.20) in the same way as in Case (B).

Case (E): $p_1, q_2 \leq 1$, $p_2q_1 - (1-p_1)(1-q_2) > 0$, $\alpha, \beta < N/2$.
Putting $a > 0$ satisfying

$$\max \left\{ \frac{1-p_1}{q_1}, \frac{2\beta}{N} \right\} < a < \frac{p_2}{1-q_2}, \quad (4.21)$$

we can prove in the same way as in Case (C). In fact, since $q_2 < 1$, $p_2q_1 - (1-p_1)(1-q_2) > 0$ and $\alpha < N/2$, we have

$$\begin{aligned} & \frac{p_2}{1-q_2} - \frac{2\beta}{N} \\ &= \frac{p_2N\{p_2q_1 - (1-p_1)(1-q_2)\} - (1-p_1)p_2(\sigma_1 + 2) - p_2q_1(\sigma_2 + 2)}{(1-q_2)\{p_2q_1 - (1-p_1)(1-q_2)\}} \\ &+ \frac{p_2q_1(\sigma_2 + 2) - (1-p_1)(1-q_2)(\sigma_2 + 2)}{(1-q_2)\{p_2q_1 - (1-p_1)(1-q_2)\}} \\ &= \frac{2p_2N}{1-q_2} \left(\frac{N}{2} - \alpha \right) + \frac{\sigma_2 + 2}{1-q_2} \\ &> 0, \end{aligned}$$

and since $p_1, q_2 \leq 1$, $p_2q_1 - (1-p_1)(1-q_2) > 0$, we have $(1-p_1)/q_1 < p_2/(1-q_2)$. Therefore, we can take $a > 0$ satisfying (4.21). $\square$

(iii) Let $a = \frac{p_2}{1-q_2}$. Put

$$\bar{u}(x, t) = C_1 \langle x \rangle^{-2\delta_1} \exp(kt), \quad (4.22)$$

$$\bar{v}(x, t) = C_2 \langle x \rangle^{-2\delta_2} \exp(akt), \quad (4.23)$$

where $C_1, C_2, k > 0$. We can see that $(\bar{u}, \bar{v})$ are global super-solutions for large $k > 0$. $\square$

We are now in a position to prove Theorems 2.1-2.3.

Proof of Theorems 2.1(i), 2.2 and 2.4. Let T^* be the maximal existence time of the classical solutions for (1.1). From the local existence theorem in Section 3, it is clear that $T^* \neq 0$. Assume $T^* < \infty$. If the initial data (u_0, v_0) are sufficiently small, then the solutions (u, v) are estimated above by the super-solutions in Proposition 4.2. Using Proposition 3.2 and Proposition 3.4, we can extend the solutions (u, v) with new initial data $(u(T^*), v(T^*))$ to time $T^{**} > T^*$. This contradicts the maximality of T^* . Hence $T^* = \infty$. $\square$

Proof of Theorem 2.1 (ii). The constants C_1 and $C_2 > 0$ in Proposition 4.2 (iii) have no restriction. Hence, the argument as above works for arbitrary initial data in $I^{\delta_1} \times I^{\delta_2}$. $\square$

5. PROOF OF THEOREM 2.7

First, we prepare some notation. For $\gamma > 0$, we put

$$\eta_\gamma(t, x) = e^{t\Delta} \langle x \rangle^{-\gamma}.$$

We define the Banach space E_η by

$$E_\eta = \{(u(t), v(t)); u, v \geq 0, \|(u, v)\|_{E_\eta} < \infty\}$$

with the norm

$$\|(u, v)\|_{E_\eta} = \left\| \frac{u}{\eta_a} \right\|_\infty + \left\| \frac{v}{\eta_b} \right\|_\infty,$$

where

$$\|w\|_\infty = \sup_{t \in (0, \infty)} \|w(t)\|_\infty.$$

Let $m = \|u_0\|_{\infty, a} + \|v_0\|_{\infty, b}$. We define

$$\Psi(u, v) = (\Psi_1(u, v), \Psi_2(u, v)),$$

where

$$\begin{aligned} \Psi_1(u, v) &= e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds, \\ \Psi_2(u, v) &= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_2} u(s)^{p_2} v(s)^{q_2} ds. \end{aligned}$$

In this section, we use the following lemma to show that $\Psi(u, v)$ is a contraction mapping of $B(E_\eta, 2m) = \{(u, v) \in E_\eta; \|(u, v)\|_{E_\eta} \leq 2m\}$ into itself.

Lemma 5.1. *Let $\gamma > 0$ and $0 \leq \delta \leq \gamma < N$. Then we have*

$$\|\eta_\gamma(t)\|_{\infty, \delta} \leq C(1+t)^{(\delta-\gamma)/2}$$

for $t > 0$.

Proof. See Lemma 2.1 in [9]. $\square$

Proof of Theorem 2.7. It is sufficient to show that Ψ is a mapping of $B(E_\eta, 2m)$ into itself. Assume that $(u_0, v_0) \in I^a \times I^b$. Then we can easily see that

$$\begin{aligned} |e^{t\Delta} u_0(x)| &\leq \|u_0\|_{\infty, a} |e^{t\Delta} \langle x \rangle^{-a}| \\ &= \|u_0\|_{\infty, a} \eta_a. \end{aligned}$$

Hence, we obtain

$$\|(e^{t\Delta} u_0, e^{t\Delta} v_0)\|_{E_\eta} \leq m.$$

We next estimate the nonlinear parts. Since $\sigma_1/(p_1 + q_1 - 1) < a, b < N$ and $\sigma_1 - a(p_1 - 1) - bq_1 < -2$, we have from Lemma 5.1

$$\begin{aligned}
& \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_1} u(s)^{p_1} v(s)^{q_1} ds \\
&= \int_0^t e^{(t-s)\Delta} |\cdot|^{\sigma_1} \eta_a(s)^{p_1-1} \eta_b(s)^{q_1} \eta_a(s) \left| \frac{u(s)}{\eta_a(s)} \right|^{p_1} \left| \frac{v(s)}{\eta_b(s)} \right|^{q_1} ds \\
&\leq \int_0^t e^{(t-s)\Delta} \|\eta_a(s)\|_{\infty, \frac{\sigma_1}{p_1+q_1-1}}^{p_1-1} \|\eta_b(s)\|_{\infty, \frac{\sigma_1}{p_1+q_1-1}}^{q_1} \eta_a(s) \left| \frac{u(s)}{\eta_a(s)} \right|^{p_1} \left| \frac{v(s)}{\eta_b(s)} \right|^{q_1} ds \\
&\leq C \int_0^t e^{(t-s)\Delta} (1+s)^{\{\sigma_1-a(p_1-1)-bq_1\}/2} \eta_a(s) ds \left\| \frac{u}{\eta_a} \right\|_{\infty}^{p_1} \left\| \frac{v}{\eta_b} \right\|_{\infty}^{q_1} \\
&= C \eta_a(t) \int_0^t (1+s)^{\{\sigma_1-a(p_1-1)-bq_1\}/2} ds \left\| \frac{u}{\eta_a} \right\|_{\infty}^{p_1} \left\| \frac{v}{\eta_b} \right\|_{\infty}^{q_1} \\
&\leq C \eta_a(t) (2m)^{p_1+q_1}.
\end{aligned}$$

Hence, we obtain

$$\|(\Psi_1(u, v), \Psi_2(u, v))\|_{E_\gamma} \leq 2m$$

for sufficiently small $m > 0$. $\square$

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