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HYPERSURFACES IN STATISTICAL MANIFOLDS

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Abstract. The condition for the curvature of a statistical manifold to admit a kind of standard hypersurface is given as a first step of the statistical submanifold theory. A complex version of the notion of statistical structures is also introduced.

1. Introduction

Geometry of statistical manifolds lies at the confluence of some research areas such as information geometry, affine differential geometry, and Hessian geometry. In this paper, we study hypersurfaces in statistical manifolds apart from each peculiar background. After giving an abstract definition of statistical structures, we will go around such areas briefly.

Throughout this paper, let M be an n-dimensional manifold, ∇ an affine connection, and g a Riemannian metric on M. We denote by $\Gamma(E)$ the set of sections of a vector bundle $E \to M$. For example, $\Gamma(TM^{(p,q)})$ means the set of tensor fields of type (p,q) on M. The torsion tensor field of ∇ is denoted by $T^\nabla \in \Gamma(TM^{(1,2)})$. All the objects are assumed to be smooth.

Definition 1.1. A pair (∇, g) is called *statistical structure* on M if (1) $(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = -g(T^\nabla(X,Y),Z)$ holds for $X,Y,Z \in \Gamma(TM)$, and (2) $T^\nabla = 0$.

Let ∇^g be the Levi-Civita connection of g. By definition, a pair (∇^g, g) is a statistical structure, which is called a *trivial* statistical structure.

The name of this structure comes from information geometry (See [2]). Let $p(\cdot,\theta) : (X, dx) \to (0,\infty)$ be a probability density parametrized

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by $\theta = (\theta^1, \dots, \theta^n) \in \Theta \subset \mathbb{R}^n$. For any constant $\alpha \in \mathbb{R}$, we set

$$g_\theta := \sum \{ \int_X \frac{\partial \log p}{\partial \theta^i}(x,\theta) \frac{\partial \log p}{\partial \theta^j}(x,\theta) p(x,\theta) dx \} d\theta^i d\theta^j,$$

and

$$\Gamma^{(\alpha)}_{ijk}(\theta) := \int_X \{ \frac{\partial^2 \log p}{\partial \theta^i \partial \theta^j}(x,\theta) + \frac{1-\alpha}{2} \frac{\partial \log p}{\partial \theta^i}(x,\theta) \frac{\partial \log p}{\partial \theta^j}(x,\theta) \} \frac{\partial \log p}{\partial \theta^k}(x,\theta) p(x,\theta) dx.$$

It is easy to see that g_θ is a positive semi-definite quadratic form on $T_\theta\Theta$. If g is a Riemannian metric on Θ, then $(\Theta, \nabla^{(\alpha)}, g)$ is a statistical manifold, where $\nabla^{(\alpha)}$ is an affine connection defined by $\Gamma^{(\alpha)}_{ijk} = g(\nabla^{(\alpha)}_{\frac{\partial}{\partial \theta^i}} \frac{\partial}{\partial \theta^j}, \frac{\partial}{\partial \theta^k})$. In fact, g is known as the *Fisher metric* and $\nabla^{(\alpha)}$ the Amari's α-connection with respect to $\{p(\cdot, \theta) \mid \theta \in \Theta\}$. These objects are useful for a geometric understanding of statistical inference.

On the other hand, geometry of affine hypersurfaces has been classically studied. Blaschke finished the first monograph on this subject in 1923. By the Codazzi equation, a pair of an induced connection and an affine fundamental form, i.e. a second fundamental form in this setting, gives rise to a statistical structure on a nondegenerate hypersurface. A statistical structure is called a *Codazzi structure* as well (See [8], [5]).

We may say that Hessian geometry is developed by Koszul, Shima and many mathematicians (See [9]). A flat affine manifold with a Riemannian metric locally expressed as the Hessian matrix of a function with respect to an affine coordinate system is called a *Hessian manifold*. We can consider Hessian manifolds as an important class of statistical manifolds. The study on regular convex cones is the origin of Hessian geometry. It is remarked that the tangent bundle of a Hessian manifold admits a Kähler metric in a natural way, and that Hessian geometry is closely related to Kähler geometry.

One of interesting examples of Hessian manifolds is a *special Kähler manifold*, which was introduced by the physicists de Wit and Van Proeyen in 1984. A number of mathematical researches of special Kähler manifolds followed (See [1], [3]). We consider special Kähler manifolds as a class of statistical manifolds and define a larger one, which will be called *holomorphic statistical manifolds* (Kurose [6], [7], Definition 2.4).

An isometric immersion preserving attached connections is called a statistical immersion (Definition 3.1). In this paper, we study such immersions, in particular, elementary properties of hypersurfaces in statistical manifolds of constant curvature as a first step. Fundamental equations for statistical submanifolds were given by Vos [11] in 1989.

As an application, he gave an interpretation of Bartlett's correction in terms of curvatures and other invariant quantities.

Isometric immersions of Kähler manifolds into real space forms have been studied as the Riemannian submanifold theory (See [10], [4]). We consider an analogue in the *statistical submanifold theoretical* setting. In Section 4, we prove that if a hypersurface in a statistical manifold of constant curvature carries a holomorphic statistical structure of constant holomorphic curvature, then the hypersurface is a special Kähler manifold and the ambient space is a Hessian manifold (Theorem 4.1).

In Section 5, when a hypersurface in a Hessian manifold of positive constant Hessian curvature carries a trivial Hessian structure, we can determine its shape operator (Theorem 5.1). In particular, a horosphere $f_0 : \mathbb{R}^n \ni {}^t(y^1, \dots, y^n) \mapsto {}^t(y^1, \dots, y^n, y_0) \in H$ can be characterized as such a hypersurface, where y_0 is a positive constant, and H is the $(n+1)$-dimensional upper half Hessian space of constant Hessian curvature 4 (Corollary 5.6).

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2. Preliminaries on Statistical Manifolds

For an affine connection ∇ and a Riemannian metric g on M, let ∇^* be the connection defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

for any $X, Y, Z \in \Gamma(TM)$, which is called the *dual connection* of ∇ with respect to g. If (∇, g) is a statistical structure on M, so is (∇^*, g).

Definition 2.1. A statistical structure (∇, g) is said to be *of constant curvature* $k \in \mathbb{R}$ if

$$R^\nabla(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}$$

holds, where $R^\nabla \in \Gamma(TM^{(1,3)})$ is the curvature tensor field of ∇. A statistical structure (∇, g) of constant curvature 0 is called a *Hessian structure*.

We can calculate that the curvature tensor filed R^{∇^*} of the dual connection satisfies

$$g(R^{\nabla^*}(X,Y)Z, W) = -g(Z, R^\nabla(X,Y)W).$$

Accordingly, if (∇, g) is a statistical structure of constant curvature k, then so is (∇^*, g). In particular, if (∇, g) is Hessian, so is (∇^*, g).

For a statistical structure (∇, g) we define the *difference tensor field* $K := K^{(\nabla,g)} \in \Gamma(TM^{(1,2)})$ as

$$K(X,Y) := \nabla_X Y - \nabla^g_X Y,$$

and the *first Koszul form* $\alpha := \alpha^{(\nabla,g)} \in \Gamma(TM^{(0,1)})$ as

$$\alpha(X) := -\operatorname{tr}\{Y \mapsto K(X,Y)\}.$$

The following formulas are obtained by direct calculation:

$$K(X,Y) = K(Y,X), \quad g(K(X,Y),Z) = g(Y,K(X,Z)), \tag{2.1}$$

$$\begin{aligned} &R^{\nabla}(X,Y)Z \\ &= R^{\nabla^g}(X,Y)Z + (\nabla^g K)(Y,Z;X) - (\nabla^g K)(Z,X;Y) \\ &\qquad +K(X,K(Y,Z)) - K(Y,K(Z,X)), \\ &R^{\nabla^*}(X,Y)Z \\ &= R^{\nabla^g}(X,Y)Z - (\nabla^g K)(Y,Z;X) + (\nabla^g K)(Z,X;Y) \\ &\qquad +K(X,K(Y,Z)) - K(Y,K(Z,X)), \\ &R^{\nabla^g}(X,Y)Z \\ &= R^{\nabla}(X,Y)Z - (\nabla K)(Y,Z;X) + (\nabla K)(Z,X;Y) \\ &\qquad +K(X,K(Y,Z)) - K(Y,K(Z,X)), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &(\nabla K)(Y,Z;X) - (\nabla K)(Z,X;Y) \\ &\quad = 2\{K(X,K(Y,Z)) - K(Y,K(Z,X))\} \\ &\qquad +\frac{1}{2}\{R^{\nabla}(X,Y)Z - R^{\nabla^*}(X,Y)Z\}. \end{aligned} \tag{2.3}$$

Combining $(2.2)_3$ and (2.3), we have

$$\begin{aligned} R^{\nabla^g}(X,Y)Z &= -\{K(X,K(Y,Z)) - K(Y,K(Z,X))\} \\ &= -\frac{1}{2}\{(\nabla K)(Y,Z;X) - (\nabla K)(Z,X;Y)\}, \end{aligned} \tag{2.4}$$

if (∇, g) is a Hessian structure.

Definition 2.2. A Hessian structure (∇, g) is said to be *of constant Hessian curvature* $c \in \mathbb{R}$ if

$$\left(\nabla_X K^{(\nabla,g)}\right)(Y,Z) = -\frac{c}{2}\{g(X,Y)Z + g(X,Z)Y\}$$

for any $X, Y, Z \in \Gamma(TM)$.

We can construct a Kähler metric g^T on the tangent bundle of a Hessian manifold M by the diagonal lifting (or the Sasaki lifting) and remark that M is of constant Hessian curvature c if and only if (TM, g^T) is of constant holomorphic sectional curvature $-c$ (See [9]). It is also remarked that the tangent bundle of a statistical manifold admits an almost Kähler metric in the same manner.

By (2.4), if a Hessian structure (∇, g) is of constant Hessian curvature c, then the Riemannian metric g is of constant curvature $-c/4$, that is,

$$R^{\nabla^g}(X,Y)Z = -\frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\}. \tag{2.5}$$

Example 2.3. Let $(H, \widetilde{g})$ be the upper half space of constant curvature -1:

$$H := \{y = {}^t(y^1, \dots, y^{n+1}) \in \mathbb{R}^{n+1} \mid y^{n+1} > 0\},$$
$$\widetilde{g} := (y^{n+1})^{-2} \sum_{A=1}^{n+1} dy^A dy^A.$$

We define an affine connection $\widetilde{\nabla}$ on H by the following relations:

$$\widetilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^{n+1}} = (y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}}, \quad \widetilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij}(y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}},$$
$$\widetilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^{n+1}} = \widetilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^j} = 0,$$

where $i, j = 1, \dots, n$. Then $(H, \widetilde{\nabla}, \widetilde{g})$ is a Hessian manifold of constant Hessian curvature 4.

We conjecture that $\widetilde{\nabla}$ is the only connection of constant Hessian curvature 4 globally defined on $(H, \widetilde{g})$. Moreover, we remark that $(H, \widetilde{\nabla}, \widetilde{g})$ expresses the statistical model of normal distributions when $\dim H = 2$. In fact, the normal distribution with mean μ and variance σ^2 is written as

$$N(x, \mu, \sigma^2) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}, \quad x \in \mathbb{R},\ \mu \in \mathbb{R},\ \sigma > 0.$$

Set $\Theta := \{(\theta^1, \theta^2) \in \mathbb{R}^2 \mid \theta_2 > 0\}$, $X := \mathbb{R}$ and $p(x, \theta) := N(x,\ \sqrt{2}\theta^1,\ (\theta^2)^2)$. Then the statistical manifold with respect to $\{p(\cdot, \theta) \mid \theta \in \Theta\}$ has a Riemannian metric $g = 2(\theta^2)^{-2} \sum d\theta^i d\theta^i$ of constant curvature $-1/2$ and a flat connection $\nabla^{(-1)}$ with

$$\nabla^{(-1)}_{\frac{\partial}{\partial \theta^1}} \frac{\partial}{\partial \theta^1} = 2(\theta^2)^{-1} \frac{\partial}{\partial \theta^2}, \quad \nabla^{(-1)}_{\frac{\partial}{\partial \theta^2}} \frac{\partial}{\partial \theta^2} = (\theta^2)^{-1} \frac{\partial}{\partial \theta^2},$$
$$\nabla^{(-1)}_{\frac{\partial}{\partial \theta^1}} \frac{\partial}{\partial \theta^2} = \nabla^{(-1)}_{\frac{\partial}{\partial \theta^2}} \frac{\partial}{\partial \theta^1} = 0.$$

Kurose [6] gave the following definition as a complex version of the notion of statistical structures (cf. [7]).

Definition 2.4. Let M be an almost complex manifold with almost complex structure $J \in \Gamma(TM^{(1,1)})$, and (∇, g) a statistical structure on M. We denote by ω the fundamental form with respect to J and g, that is, $\omega(X, Y) = g(X, JY)$. The triplet (∇, g, J) is called a *holomorphic statistical structure* on M if ω is a ∇-parallel 2-form.

Let (M, J) be an almost complex manifold, g a Hermitian metric, ω the fundamental 2-form, and ∇ an affine connection of torsion free. For $\theta \in \mathbb{R}$, we define $e^{\theta J} \in \Gamma(TM^{(1,1)})$, an affine connection ∇^θ and

$d^\nabla J \in \Gamma(\bigwedge^2 T^*M \otimes TM)$ by

$$\begin{aligned} e^{\theta J}X &:= \cos\theta X + \sin\theta JX, \\ \nabla^\theta_X Y &:= e^{\theta J}\nabla_X \left(e^{-\theta J}Y\right), \\ d^\nabla J(X,Y) &:= (\nabla_X J)Y - (\nabla_Y J)X. \end{aligned}$$

By direct calculation, we have the following formulas:

$$\begin{aligned} &(\nabla^\theta_X g)(Y,Z) - (\nabla^\theta_Y g)(X,Z) \\ &\qquad = g(T^{(\nabla^\theta)^*}(X,Y) - T^{(\nabla^\theta)}(X,Y), Z), \\ &(\nabla^\theta_X \omega)(Y, J^{-1}Z) = g(Y, (\nabla^\theta)^*_X Z - (\nabla^\theta)^J_X Z) \\ &\qquad = g(Y, e^{\theta J}(\nabla^*_X - \nabla^J_X)e^{-\theta J}Z), \\ &T^{\nabla^\theta}(X,Y) - T^\nabla(X,Y) = -\sin\theta e^{\theta J} d^\nabla J(X,Y), \end{aligned}$$

where $\nabla^J := \nabla^{\pi/2}$, that is, $\nabla^J_X Y := J\nabla_X(J^{-1}Y)$. Combining the above relations, we have

Proposition 2.5. *Let (M, ∇, g) be a statistical manifold, and J an almost complex structure compatible with g. Then the following four conditions are equivalent:*

(1) (∇, g, J) is a holomorphic statistical structure.

(2) $\nabla^* = \nabla^J$.

(3) $d^\nabla J = 0$.

(4) (∇^θ, g, J) is a holomorphic statistical structure for each θ.

Moreover, these conditions imply that

(5) (g, J) is Kählerian.

In fact, the last assertion is obtained as follows. The fundamental 2-form ω is closed, because it is ∇-parallel and ∇ is of torsion free. Since $2\nabla^g = \nabla + \nabla^* = \nabla + \nabla^J$, we have that J is ∇^g-parallel, and then J is integrable.

We remark that the statistical structure is trivial if $\nabla = \nabla^\theta$. It is also easy to show that the first Koszul form α of a holomorphic statistical manifold (M, ∇, g, J) vanishes. It follows from

$$\begin{aligned} K^{(\nabla^*, g)}(X,Y) &= -K^{(\nabla, g)}(X,Y), \\ K^{(\nabla, g)}(X, JY) &= -JK^{(\nabla, g)}(X,Y). \end{aligned} \tag{2.6}$$

In the same idea to the real case, we put the following

Definition 2.6. Let (M, ∇, g, J) be a holomorphic statistical manifold. It is said to be *of constant holomorphic curvature $k \in \mathbb{R}$* if the following holds:

$$\begin{aligned} R^\nabla(X,Y)Z = \frac{k}{4}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\} \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$.

A holomorphic statistical structure of holomorphic curvature 0 is nothing but a special Kähler manifold. There are interesting examples of such manifolds which are realized as improper affine hyperspheres constructed from holomorphic functions ([3]). By the following remark, you can also get holomorphic statistical structures, which are not special Kählerian.

Remark 2.7. Let (M, ∇, g, J) be a holomorphic statistical manifold with difference tensor field $K = K^{(\nabla, g)}$, and φ a function on M. Then $(\nabla^\varphi := \nabla + \varphi K, g, J)$ is also a holomorphic statistical structure on M. The relation of the curvature tensor fields are given by

$$\begin{aligned} &R^{\nabla^\varphi}(X,Y)Z \\ &= R^{\nabla}(X,Y)Z + \varphi^2\{K(X,K(Y,Z)) - K(Y,K(Z,X))\} \\ &\quad -\varphi\{(\nabla K)(Y,Z;X) - (\nabla K)(Z,X;Y)\} \\ &\qquad\qquad +d\varphi(Y)K(X,Z) - d\varphi(X)K(Y,Z). \end{aligned}$$

Remark 2.8. Let (M, g, J) be a Kähler manifold, and $K \in \Gamma(TM^{(1,2)})$ satisfying (2.1) and $(2.6)_2$. Then $(\nabla := \nabla^g + K, g, J)$ is a holomorphic statistical structure on M.

3. Statistical Hypersurfaces

Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical manifold, and $f : M \to \widetilde{M}$ an immersion. We define g and ∇ on M by

$$g = f^*\widetilde{g}, \quad g(\nabla_X Y, Z) = \widetilde{g}(\widetilde{\nabla}_X f_* Y, f_* Z) \tag{3.1}$$

for any $X, Y, Z \in \Gamma(TM)$, where the connection induced from $\widetilde{\nabla}$ by f on the induced bundle $f^*T\widetilde{M} \to M$ is denoted by the same symbol $\widetilde{\nabla}$. Then the pair (∇, g) is a statistical structure on M, which is called the one *induced by f from* $(\widetilde{\nabla}, \widetilde{g})$.

Definition 3.1. Let (M, ∇, g) and $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be two statistical manifolds. An immersion $f : M \to \widetilde{M}$ is called a *statistical* immersion if (∇, g) coincides with the induced statistical structure, namely, if (3.1) holds.

Concerning the theory on statistical submanifolds, we refer the reader to [11]. Fundamental equations for statistical immersions of general codimensions are obtained by Vos. This notion seems useful in statistical inference.

Suppose $f : (M, \nabla, g) \to (\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical immersion of codimension one, and $\xi \in \Gamma(f^*T\widetilde{M})$ a unit normal vector field of f. We define $h, h^* \in \Gamma(TM^{(0,2)})$, $A, A^* \in \Gamma(TM^{(1,1)})$ and $\tau, \tau^* \in \Gamma(TM^*)$ by the following Gauss and Weingarten formulas:

$$\begin{aligned}
&\widetilde{\nabla}_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi, \\
&\widetilde{\nabla}_X \xi = -f_* A^* X + \tau^*(X)\xi, \\
&\widetilde{\nabla}^*_X f_* Y = f_* \nabla^*_X Y + h^*(X,Y)\xi, \\
&\widetilde{\nabla}^*_X \xi = -f_* A X + \tau(X)\xi, \quad X, Y \in \Gamma(TM),
\end{aligned}$$

where $\widetilde{\nabla}^*$ is the dual connection of $\widetilde{\nabla}$ with respect to $\widetilde{g}$. It is easy to show that the connection induced from $\widetilde{\nabla}^*$ is the dual connection of ∇ with respect to g. Besides the following hold for any $X, Y \in \Gamma(TM)$:

$$\begin{aligned}
&h(X,Y) = g(AX,Y), \quad h^*(X,Y) = g(A^*X,Y), \\
&\tau(X) + \tau^*(X) = 0.
\end{aligned} \tag{3.2}$$

In addition, we define $II \in \Gamma(TM^{(0,2)})$ and $S \in \Gamma(TM^{(1,1)})$ by using the Riemannian Gauss and Weingarten formulas:

$$\begin{aligned}
&\nabla^{\widetilde{g}}_X f_* Y = f_* \nabla^g_X Y + II(X,Y)\xi, \\
&\nabla^{\widetilde{g}}_X \xi = -f_* S X.
\end{aligned}$$

In a standard way, we have the Gauss, Codazzi and Ricci equations for a statistical hypersurface in the case that the ambient space is of constant curvature $\widetilde{k}$:

$$\begin{aligned}
&R^{\nabla}(X,Y)Z = \widetilde{k}\{g(Y,Z)X - g(X,Z)Y\} \\
&\qquad\qquad + \{h(Y,Z)A^*X - h(X,Z)A^*Y\}, \\
&(\nabla_X h)(Y,Z) + \tau^*(X)h(Y,Z) \\
&\qquad\qquad = (\nabla_Y h)(X,Z) + \tau^*(Y)h(X,Z), \\
&(\nabla_X A^*)Y - \tau^*(X)A^*Y = (\nabla_Y A^*)X - \tau^*(Y)A^*X, \\
&h(X, A^*Y) - h(Y, A^*X) = d\tau^*(X,Y),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&R^{\nabla^*}(X,Y)Z = \widetilde{k}\{g(Y,Z)X - g(X,Z)Y\} \\
&\qquad\qquad + \{h^*(Y,Z)AX - h^*(X,Z)AY\}, \\
&(\nabla^*_X h^*)(Y,Z) + \tau(X)h^*(Y,Z) \\
&\qquad\qquad = (\nabla^*_Y h^*)(X,Z) + \tau(Y)h^*(X,Z), \\
&(\nabla^*_X A)Y - \tau(X)AY = (\nabla^*_Y A)X - \tau(Y)AX, \\
&h^*(X, AY) - h^*(Y, AX) = d\tau(X,Y),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&R^{\nabla^g}(X,Y)Z = R^{\nabla^{\widetilde{g}}}(X,Y)Z \\
&\qquad\qquad + II(Y,Z)SX - II(X,Z)SY, \\
&(\nabla^g_X II)(Y,Z) = (\nabla^g_Y II)(X,Z), \\
&(\nabla^g_X S)Y = (\nabla^g_Y S)X, \\
&II(X,SY) - II(Y,SX) = 0.
\end{aligned} \tag{3.5}$$

Furthermore, we will fix the notation here as follows:

$$\begin{aligned}
&K := K^{(\nabla,g)} \in \Gamma(TM^{(1,2)}),\ \widetilde{K} := K^{(\widetilde{\nabla},\widetilde{g})} \in \Gamma(T\widetilde{M}^{(1,2)}),\\
&b := h - II \in \Gamma(TM^{(0,2)}),\\
&B := A - S,\ B^* := A^* - S\ \in \Gamma(TM^{(1,1)}),\\
&\tau^\sharp \in \Gamma(TM)\ :\ g(\tau^\sharp, X) = \tau^*(X) \text{ for any } X \in \Gamma(TM),\\
&\nu := \widetilde{g}(\widetilde{K}(\xi,\xi),\xi) \in C^\infty(M).
\end{aligned}$$

By definition, we can get the following formulas: for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned}
&\widetilde{K}(f_*X, f_*Y) = f_*K(X,Y) + b(X,Y)\xi,\\
&\widetilde{K}(f_*X, \xi) = -f_*B^*X + \tau^*(X)\xi,\\
&\widetilde{K}(\xi,\xi) = f_*\tau^\sharp + \nu\xi,
\end{aligned}$$

and

$$\begin{aligned}
&(\widetilde{\nabla}_X\widetilde{K})(f_*Y, f_*Z)\\
&= f_*\{(\nabla_X K)(Y,Z) - b(Y,Z)A^*X\\
&\qquad +h(X,Y)B^*Z + h(X,Z)B^*Y\}\\
&\quad +\{h(X,K(Y,Z)) + (\nabla_X b)(Y,Z)\\
&\qquad +\tau^*(X)b(Y,Z) - \tau^*(Z)h(X,Y) - \tau^*(Y)h(X,Z)\}\xi,\\
&(\widetilde{\nabla}_X\widetilde{K})(f_*Y, \xi)\\
&= f_*\{K(Y,A^*X) - \tau^*(Y)A^*X - h(X,Y)\tau^\sharp\\
&\qquad -(\nabla_X B^*)Y + \tau^*(X)B^*Y\}\\
&\quad +\{-h(X,Y)\nu - h(X,B^*Y) + (\nabla_X\tau^*)(Y) + b(Y,A^*X)\}\xi,\\
&(\widetilde{\nabla}_X\widetilde{K})(\xi,\xi)\\
&= f_*\{\nabla_X\tau^\sharp - \nu A^*X - 2B^*A^*X - 2\tau^*(X)\tau^\sharp\}\\
&\quad +\{h(X,\tau^\sharp) - \nu\tau^*(X) + 2\tau^*(A^*X) + X\nu\}\xi.
\end{aligned} \tag{3.6}$$

4. Holomorphic Statistical Manifolds as Hypersurfaces

In this section, we prove the following

Theorem 4.1. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a $(2m+1)$-dimensional statistical manifold of constant curvature $\widetilde{k}$ with $m \geq 2$, and (M, ∇, g, J) a holomorphic statistical manifold of constant holomorphic curvature k. If there exists a statistical immersion $f : M \to \widetilde{M}$ of codimension 1, then the curvatures $\widetilde{k}$ and k vanish. Moreover, the shape operators satisfy the following relations for some function μ :*

$$\begin{aligned}
A^* &= \mu A,\\
\mu\{g(AY,Z)AX - g(AX,Z)AY\} &= 0,
\end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. In particular, if $\operatorname{rank} A = 2m$, *then $A^* = 0$.*

Proof. By (3.2) and the Gauss equation $(3.3)_1$, we have

$$\begin{aligned}&\frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &\qquad\qquad +2g(X,JY)JZ\} \\ &= \widetilde{k}\{g(Y,Z)X - g(X,Z)Y\} + g(AY,Z)A^*X - g(AX,Z)A^*Y.\end{aligned}$$

By taking the trace with respect to X, we get

$$AA^* - (\operatorname{tr} A^*)\, A = \{(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k\}I, \tag{4.1}$$

where I is the identity transformation. Hence,

$$AA^* = A^*A. \tag{4.2}$$

In fact, let $\{e_j\}$ be an orthogonal basis of (T_xM, g_x) such that $Ae_j = \lambda_j e_j$ for some $\lambda_j \in \mathbb{R}$. Setting $A^*e_j = \sum a_j^l e_l$, we have

$$\begin{aligned}0 &= AA^*e_j - (\operatorname{tr} A^*)\, Ae_j - \{(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k\}e_j \\ &= \left[\lambda_j a_j^j + (\operatorname{tr} A^*)\,\lambda_j - \{(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k\}\right] e_j + \sum_{l\neq j} \lambda_l a_j^l e_l,\end{aligned}$$

from which for $l \neq j$, we obtain $\lambda_l a_j^l = 0$. Accordingly, $(AA^* - A^*A)e_j = 0$ for all e_j.

¿From (4.2) we can choose an orthogonal basis $\{e_j\}$ such that

$$Ae_j = \lambda_j e_j, \quad A^*e_j = \lambda_j^* e_j,$$

for some $\lambda_j, \lambda_j^* \in \mathbb{R}$. Using this frame at (4.1), we get

$$0 = \lambda_j\lambda_j^* - \left(\sum \lambda_l^*\right)\lambda_j - \{(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k\},$$

and that

$$\begin{aligned}0 &= \sum \lambda_j\lambda_j^* - \sum \lambda_l^* \sum \lambda_j - 2m\{(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k\} \\ &\leq -2m\{(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k\}.\end{aligned}$$

We then conclude that

$$(2m-1)\widetilde{k} - \frac{1}{2}(m+1)k \leq 0, \tag{4.3}$$

and that the equality holds if and only if there exists a function μ such that $A^* = \mu A$.

In the following step, we shall prove that k vanishes when $m \geq 2$. Using the above frame, we write the Gauss equation as

$$\begin{aligned} 0 &= \frac{k}{4}\{g(e_2,Z)e_1 - g(e_1,Z)e_2 + g(Je_2,Z)Je_1 - g(Je_1,Z)Je_2 \\ &\qquad +2g(e_1,Je_2)JZ\} - \widetilde{k}\{g(e_2,Z)e_1 - g(e_1,Z)e_2\} \\ &\qquad -g(Ae_2,Z)A^*e_1 + g(Ae_1,Z)A^*e_2 \\ &= \{\frac{k}{4} - \widetilde{k} - \lambda_2\lambda_1^*\}g(e_2,Z)e_1 - \{\frac{k}{4} - \widetilde{k} - \lambda_1\lambda_2^*\}g(e_1,Z)e_2 \\ &\qquad +\frac{k}{4}\{g(Je_2,Z)Je_1 - g(Je_1,Z)Je_2 + 2g(e_1,Je_2)JZ\}. \end{aligned}$$

If Z is orthogonal to $\{e_1, e_2\}$, then it follows that

$$kg(Je_2,Z) = kg(Je_1,Z) = kg(e_1,Je_2) = 0.$$

In the case that $g(e_1,Je_2) \neq 0$, we get $k = 0$. In the other case, setting $Z = Je_1$, we arrive at the same result.

In the last step, we shall prove that $\widetilde{k}$ vanishes. Since $k = 0$, the Gauss equation is written as

$$0 = \widetilde{k}\{g(Y,Z)X - g(X,Z)Y\} + g(AY,Z)A^*X - g(AX,Z)A^*Y.$$

Setting $X = Z = e_i$ and $Y = e_j$ $(j \neq i)$, we have

$$\lambda_i\lambda_j^* = -\widetilde{k}, \quad j \neq i.$$

If $\widetilde{k} \neq 0$, then it implies that

$$\lambda_1^* = \cdots = \lambda_{2m}^* =: \lambda^* \neq 0, \quad \lambda_1 = \cdots = \lambda_{2m} =: \lambda \neq 0,$$

from which $A^* = \lambda^*\lambda^{-1}A$, that is, the equality of (4.3) holds. Therefore, it contradicts the assumption. □

5. Trivial Statistical Manifolds as Hypersurfaces

In this section, we prove the following

Theorem 5.1. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a Hessian manifold of constant Hessian curvature $\widetilde{c}$, and (M, ∇, g) a trivial Hessian manifold. If there is a statistical immersion $f : M \to \widetilde{M}$ of codimension one, then $\widetilde{c}$ is non-negative. Moreover, when $\widetilde{c}$ is positive, the Riemannian shape operator S of f is given by $S = \pm\frac{1}{2}\sqrt{\widetilde{c}}I$.*

Lemma 5.2. *Suppose $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a Hessian manifold of constant Hessian curvature $\widetilde{c} \neq 0$, (M, ∇, g) a trivial statistical manifold, and $f : M \to \widetilde{M}$ a statistical immersion of codimension one. Then τ^* vanishes.*

Proof. By (2.5) and the Gauss equation $(3.5)_1$, we have

$$0 = \frac{\widetilde{c}}{4}\{g(Y,Z)X - g(X,Z)Y\} + g(BY,Z)B^*X - g(BX,Z)B^*Y. \tag{5.1}$$

In the same way, a direct calculation shows

$$0 = \tau^*(X)b(Y,Z) - \tau^*(Y)b(X,Z), \tag{5.2}$$

$$0 = \tau^*(Y)B^*X - \tau^*(X)B^*Y, \tag{5.3}$$

$$0 = -g([B,B^*]X,Y). \tag{5.4}$$

We remark that B and B^* are simultaneously diagonalizable by (5.4).

In the case that B^* is of the form $\lambda^* I$, we have $0 = \lambda^*\{\tau^*(Y)X - \tau^*(X)Y\}$ by (5.3). Suppose $\lambda^* \neq 0$, τ^* vanishes. Otherwise, $\widetilde{c}$ vanishes from (5.1).

In the case that B^* is not of the form $\lambda^* I$, there are $\lambda_1^* \neq \lambda_2^*$ so that $B^*X_j = \lambda_j^* X_j$ where $g(X_i, X_j) = \delta_{ij}$, $i, j = 1, 2$. Besides there are λ_1, λ_2 so that $BX_j = \lambda_j X_j$. The equation (5.1) implies that

$$0 = (\frac{\widetilde{c}}{4} + \lambda_1^*\lambda_2)g(X_2,Z)X_1 - (\frac{\widetilde{c}}{4} + \lambda_2^*\lambda_1)g(X_1,Z)X_2,$$

and hence

$$\lambda_1^*\lambda_2 = \lambda_2^*\lambda_1 = -\frac{\widetilde{c}}{4} \neq 0.$$

By (5.3) we have $0 = \lambda_2^*\tau^*(X_1)X_2 - \lambda_1^*\tau^*(X_2)X_1$, which implies that τ^* vanishes. □

The equations (3.6) combined with Definition 2.2 yield that

$$\begin{aligned} -\frac{\widetilde{c}}{2}\{g(X,Y)Z + g(X,Z)Y\} &= (\nabla_X K)(Y,Z) - b(Y,Z)A^*X \\ &\quad + h(X,Y)B^*Z + h(X,Z)B^*Y, \end{aligned} \tag{5.5}$$

$$\begin{aligned} 0 = h(X,K(Y,Z)) + (\nabla_X b)(Y,Z) \\ + \tau^*(X)b(Y,Z) - \tau^*(Z)h(X,Y) - \tau^*(Y)h(X,Z), \end{aligned} \tag{5.6}$$

$$\begin{aligned} 0 = K(Y,A^*X) - \tau^*(Y)A^*X - h(X,Y)\tau^\sharp \\ + (\nabla_X B^*)Y + \tau^*(X)B^*Y, \end{aligned} \tag{5.7}$$

$$\begin{aligned} -\frac{\widetilde{c}}{2}g(X,Y) = -h(X,Y)\nu - h(X,B^*Y) \\ + (\nabla_X \tau^*)Y + b(Y,A^*X), \end{aligned} \tag{5.8}$$

$$0 = \nabla_X \tau^\sharp - \nu A^*X - 2B^*A^*X - 2\tau^*(X)\tau^\sharp, \tag{5.9}$$

$$0 = h(X, \tau^\sharp) - \nu\tau^*(X) + 2\tau^*(A^*X) + d\nu(X). \tag{5.10}$$

Taking the trace of (5.5) with respect to X, we have

$$-\widetilde{c}g(Y,Z) = -\operatorname{tr} A^* b(Y,Z) + h(B^*Z, Y) + h(B^*Y, Z). \tag{5.11}$$

On the other hand, taking the trace of (5.5) with respect to Y, we get

$$\begin{aligned} &-\frac{\widetilde{c}}{2}(n+1)g(X,Z) \\ &= -b(A^*X, Z) + h(X, B^*Z) + \operatorname{tr} B^* h(X,Z). \end{aligned} \tag{5.12}$$

This combined with (5.8) shows

$$(\nu - \operatorname{tr} B^*)h(X,Y) = \frac{\widetilde{c}}{2}(n+2)g(X,Y) + (\nabla_X \tau^*)(Y).$$

By Lemma 5.2, if $\widetilde{c} \neq 0$, we have

$$h = \frac{\widetilde{c}}{2}(n+2)(\nu - \operatorname{tr} B^*)^{-1} g, \tag{5.13}$$

and that h is nondegenerate.

Lemma 5.3. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a Hessian manifold of constant Hessian curvature $\widetilde{c} \neq 0$, (M, ∇, g) a trivial Hessian manifold, and $f : M \to \widetilde{M}$ a statistical immersion of codimension one. Then the following hold:*

$$\begin{aligned} &A^* = 0, \qquad B^* = -\frac{1}{2}\nu I, \qquad h = \widetilde{c}\nu^{-1} g, \\ &A = \widetilde{c}\nu^{-1} I, \quad B = \frac{1}{2}\nu^{-1}(2\widetilde{c} - \nu^2) I. \end{aligned}$$

Proof. Since ∇ and $\widetilde{\nabla}$ are flat, the equation $(3.3)_1$ implies that $0 = h(Y,Z)A^*X - h(X,Z)A^*Y$, and so $0 = h((\operatorname{tr} A^* I - A^*)Y, Z)$. Since h is nondegenerate, A^* vanishes.

¿From (5.11) and (5.13), we have

$$\begin{aligned} 0 &= \widetilde{c}g(Y,Z) + h(Y, B^*Z) + h(B^*Y, Z) \\ &= \widetilde{c}g(\{(n+2)(\nu - \operatorname{tr} B^*)^{-1} B^* + I\}Y, Z), \end{aligned}$$

and that

$$B^* = -(n+2)^{-1}(\nu - \operatorname{tr} B^*) I. \tag{5.14}$$

On the other hand, the equation (5.12) implies

$$\begin{aligned} 0 &= \frac{\widetilde{c}}{2}(n+1)g(X,Z) + h(X, B^*Z) + \operatorname{tr} B^* h(X,Z) \\ &= \frac{\widetilde{c}}{2} g\big(X, \big[(n+2)(\nu - \operatorname{tr} B^*)^{-1} B^* + \\ &\qquad \{(n+1) + (n+2)(\nu - \operatorname{tr} B^*)^{-1} \operatorname{tr} B^*\} I\big] Z\big), \end{aligned}$$

and hence

$$B^* = -(n+2)^{-1}\{(n+1)\nu + \operatorname{tr} B^*\} I. \tag{5.15}$$

Comparing (5.14) and (5.15), we have

$$\operatorname{tr} B^* = -\frac{1}{2}n\nu, \tag{5.16}$$

and conclude that $B^* = -\frac{1}{2}\nu I$.

Finally, from (5.13) and (5.16), we get $h = \widetilde{c}\nu^{-1}g$. We obtain $A = \widetilde{c}\nu^{-1}I$ from (3.2), and calculate that $B = B^* + (A - A^*) = \frac{1}{2}\nu^{-1}(2\widetilde{c} - \nu^2)I$. □

Proof of Theorem 5.1 . Assuming $\widetilde{c} \neq 0$, by (5.1) and Lemma 5.3, we have

$$\begin{aligned} 0 &= \frac{\widetilde{c}}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &\qquad +g(BY,Z)B^*X - g(BX,Z)B^*Y \\ &= \frac{1}{4}(\nu^2 - \widetilde{c})\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned}$$

and thus conclude that $\widetilde{c} = \nu^2$ is positive. As a result, we have that

$$S = A^* - B^* = \frac{1}{2}\nu I = \pm\frac{1}{2}\sqrt{\widetilde{c}}I.$$

□

Example 5.4. Let $(H, \widetilde{\nabla}, \widetilde{g})$ be the $(n+1)$-dimensional upper half Hessian space of constant Hessian curvature 4 as in Example 2.3. For a constant $y_0 > 0$, write the following immersion by f_0:

$$\mathbb{R}^n \ni {}^t(y^1, \dots, y^n) \mapsto {}^t(y^1, \dots, y^n, y_0) \in H.$$

Let (∇, g) be the statistical structure on $\mathbb{R}^n$ induced by f_0 from $(\widetilde{\nabla}, \widetilde{g})$. We then get that (∇, g) is a Hessian structure and $K^{(\nabla,g)} = 0$. In other words, f_0 is a statistical immersion of the trivial Hessian manifold $(\mathbb{R}^n, \nabla, g)$ into the upper half Hessian space $(H, \widetilde{\nabla}, \widetilde{g})$. It is easy to calculate that

$$\begin{aligned} &\xi = y_0\frac{\partial}{\partial y^{n+1}}, \\ &II = g, \quad S = I, \\ &h = 2g,\ h^* = 0,\ A^* = 0,\ A = 2I,\ \tau^* = \tau = 0. \end{aligned}$$

Remark 5.5. We denote by f_1 the other type expression of horospheres in the upper half space, that is,

$$f_1 : \mathbb{R}^n \ni z \mapsto \begin{bmatrix} 4r^2(|z-a|^2 + 4r^2)^{-1}(z-a) + a \\ 8r^3(|z-a|^2 + 4r^2)^{-1} \end{bmatrix} \in H,$$

where $r > 0$ and $a \in \mathbb{R}^n$. We remark that the image $f_1(\mathbb{R}^n)$ is the set $\{y \in \mathbb{R}^{n+1} \mid |y - \begin{bmatrix} a \\ r \end{bmatrix}| = r\} \setminus \{\begin{bmatrix} a \\ 0 \end{bmatrix}\}$, which is congruent to $f_0(\mathbb{R}^n)$ in the sense of Riemannian geometry. We can take $\xi = 8r^3(|z-a|^2 + 4r^2)^{-2}\left\{4r\sum(z^i - a^i)\dfrac{\partial}{\partial y^i} + (4r^2 - |z-a|^2)\dfrac{\partial}{\partial y^{n+1}}\right\}$ as a unit normal vector field, and calculate that $II = g$, $S = I$ and ∇^g is flat, of course, but that $\tau^* = -r^{-2}(|z-a|^2+4r^2)^{-3}\left(|z-a|^6 + 6|z-a|^4r^2 - 32r^6\right)\sum(z^i - a^i)dz^i \neq 0$.

Combining Example 5.4, Remark 5.5 and Theorem 5.1, we conclude the following

Corollary 5.6. *Let (M, ∇, g) be a connected trivial Hessian manifold of dimension n. If $f : (M, \nabla, g) \to (H, \widetilde{\nabla}, \widetilde{g})$ is a statistical immersion, then $f(M)$ is an open subset of $f_0(\mathbb{R}^n)$.*

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