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Author(s)	Ozawa, Tohru; Sasaki, Hironobu
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INEQUALITIES ASSOCIATED WITH DILATIONS

TOHRU OZAWA AND HIRONOBU SASAKI*

ABSTRACT. Some properties of distributions f satisfying $x \cdot \nabla f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, are studied. The operator $x \cdot \nabla$ is the generator of a semi-group of dilations. We first give Sobolev type inequalities with respect to the operator $x \cdot \nabla$. Using the inequalities, we also show that if $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $x \cdot \nabla f \in L^p(\mathbb{R}^n)$ and $|x|^{n/p}|f(x)|$ vanishes at infinity, then f belongs to $L^p(\mathbb{R}^n)$. One of the Sobolev type inequalities is shown to be equivalent to the Hardy inequality in $L^2(\mathbb{R}^n)$.

1. INTRODUCTION

In this paper, we study some properties of distributions $f \in \mathcal{D}'(\Omega)$ satisfying $x \cdot \nabla f \in L^p(\Omega)$. Here, $\Omega \subset \mathbb{R}^n$ is an open set, $\mathcal{D}'(\Omega)$ is the set of all distributions on Ω , $x \cdot \nabla = \sum_{j=1}^n x_j \partial_j$, $x = (x_1, \dots, x_n) \in \Omega$ and $\partial_j f$ is a weak derivative of f with respect to x_j . The operator $x \cdot \nabla$ is well-known as the generator of a semi-group of dilations $\{T(t)\}_{t \geq 0}$ defined by

$$(T(t)g)(x) = g(e^t x), \quad g : \mathbb{R}^n \rightarrow \mathbb{C}, \quad x \in \mathbb{R}^n.$$

Let us recall the Sobolev inequality. For a Banach space A , we denote the norm of A by $\|\cdot\|_A$. It is well-known that if $1 < p, p^* < \infty$ and

$$\frac{1}{p} = \frac{1}{p^*} - \frac{1}{n}, \tag{1.1}$$

then we have the Sobolev inequality:

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C(p) \|\nabla g\|_{L^{p^*}(\mathbb{R}^n)}. \tag{1.2}$$

Remark that the constant $C(p)$ in (1.2) is independent of g . For any $\lambda > 0$, we obtain

$$\lambda^{-n/p} \|h\|_{L^p(\mathbb{R}^n)} \leq \lambda^{-n/p^*+1} C(p) \|\nabla h\|_{L^{p^*}(\mathbb{R}^n)}$$

by substituting $g(x) = h(\lambda x)$ into (1.2). Therefore, we observe that (1.1) is a necessary condition.

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Throughout this paper, we consider the following Sobolev type inequality with respect to the operator $x \cdot \nabla$ instead of ∇ :

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C'(p) \|x \cdot \nabla g\|_{L^q(\mathbb{R}^n)}. \quad (1.3)$$

Substituting $g(x) = h(\lambda x)$ into (1.3), we observe that $p = q$ is a necessary condition to obtain (1.3). Later, we shall prove that (1.3) holds if $1 \leq p = q < \infty$, $f \in L^p$ and $x \cdot \nabla f \in L^p$.

To state our results, we list some notation which will be used later. For $1 \leq p \leq \infty$, we put $L^p = L^p(\mathbb{R}^n)$ and $\|\cdot\|_p = \|\cdot\|_{L^p}$. For $k = 0, 1, \dots$ and for an open set $\Omega \subset \mathbb{R}^n$, We denote by $C_c^\infty(\Omega)$ the set of all C^∞ -functions with compact support in Ω . For $1 \leq p \leq \infty$ and for an open set $\Omega \subset \mathbb{R}^n$, let $W_0^{1,p}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to

$$\|g\|_{W^{1,p}(\Omega)} = \|g\|_{L^p(\Omega)} + \|\nabla g\|_{L^p(\Omega)}.$$

Let $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ be an even, C^∞ -function satisfying

- $0 \leq \zeta \leq 1$,
- $\zeta(r) = 1$ if $|r| \leq 1$,
- $\zeta(r) = 0$ if $|r| \geq 2$.

We are ready to state our first result.

Theorem 1.1. *Let $n \geq 1$.*

(i) *Assume that $\Omega \subset \mathbb{R}^n$ is an open set. If $1 \leq p < \infty$, $f \in W_0^{1,p}(\Omega)$ and $x \cdot \nabla f \in L^p(\Omega)$, then we have*

$$\|f\|_{L^p(\Omega)} \leq \frac{p}{n} \|x \cdot \nabla f\|_{L^p(\Omega)}. \quad (1.4)$$

(ii) *If $1 \leq p < \infty$, $f \in L^p$ and $x \cdot \nabla f \in L^p$, then (1.4) holds. That is, we have*

$$\|f\|_p \leq \frac{p}{n} \|x \cdot \nabla f\|_p. \quad (1.5)$$

(iii) *If $f \in C^1(\mathbb{R}^n)$ and if there exist positive numbers ε and R such that*

$$\text{supp } f \subset \{x \in \mathbb{R}^n; \varepsilon \leq |x| \leq R\}, \quad (1.6)$$

then we have

$$\|f\|_\infty \leq \ln \left(\frac{R}{\varepsilon} \right) \|x \cdot \nabla f\|_\infty. \quad (1.7)$$

The proof of Theorem 1.1 will be given in Section 2. We now list three remarks on Theorem 1.1.

Remark 1. A constant function $f_C(x) := C \neq 0$ is a typical example of a function which does not satisfy (1.4). Remark that

$$\begin{aligned} f_C &\in W^{1,p}(\Omega) \setminus W_0^{1,p}(\Omega) \quad \text{if } 1 \leq p < \infty \text{ and } \Omega \text{ has a finite measure,} \\ f_C &\in L_{loc}^p \setminus L^p \quad \text{if } 1 \leq p < \infty. \end{aligned}$$

Remark 2. The condition (1.6) is a necessary condition in some sense. For $0 < \delta < 1/2$ and $0 < \varepsilon < R < \infty$, we define functions $f_{\delta,R}^1$, $f_{\delta,\varepsilon}^2$ and f_δ^3 by

$$\begin{aligned} f_{\delta,R}^1(x) &= \begin{cases} (\ln \delta)^{-1} \zeta \left(\frac{|x|}{R} \right) \ln \sqrt{\frac{|x|^2}{R^2} + \delta^2} & \text{if } 0 \leq |x| \leq R, \\ 0 & \text{otherwise,} \end{cases} \\ f_{\delta,\varepsilon}^2(x) &= \begin{cases} 0 & \text{if } 0 \leq |x| \leq \varepsilon, \\ (\ln \delta)^{-1} \zeta \left(\frac{\varepsilon}{|x|} \right) \left\langle \frac{x}{\varepsilon} \right\rangle^{-\delta} \ln \sqrt{\frac{\varepsilon^2}{|x|^2} + \delta^2} & \text{otherwise,} \end{cases} \\ f_\delta^3(x) &= \langle x \rangle^{-\delta}, \end{aligned}$$

respectively. Here, $\langle x \rangle = (1 + |x|^2)^{1/2}$. Then we see that $f_{\delta,R}^1$, $f_{\delta,\varepsilon}^2$ and f_δ^3 are C^∞ -functions vanishing at infinity and satisfy

$$\begin{aligned} \text{supp } f_{\delta,R}^1 &= \{x \in \mathbb{R}^n; 0 \leq |x| \leq R\}, \\ \text{supp } f_{\delta,\varepsilon}^2 &= \{x \in \mathbb{R}^n; |x| \geq \varepsilon\}, \\ \text{supp } f_\delta^3 &= \mathbb{R}^n. \end{aligned}$$

Furthermore, we observe that

$$\lim_{\delta \rightarrow 0+} \frac{\|g_\delta\|_\infty}{\|x \cdot \nabla g_\delta\|_\infty} = \infty, \quad g_\delta = f_{\delta,R}^1, f_{\delta,\varepsilon}^2, f_\delta^3.$$

Remark 3. The number p/n appearing in (1.5) is the best constant. In fact, if we take

$$f_\varepsilon(x) = \begin{cases} |x|^{-n/p+\varepsilon} & \text{for } |x| < 1, \\ |x|^{-n/p-\varepsilon} & \text{for } |x| \geq 1, \end{cases} \quad (1.8)$$

then we have

$$x \cdot \nabla f_\varepsilon(x) = \begin{cases} \left(-\frac{n}{p} + \varepsilon\right) |x|^{-n/p+\varepsilon} & \text{if } |x| < 1, \\ \left(-\frac{n}{p} - \varepsilon\right) |x|^{-n/p-\varepsilon} & \text{if } |x| > 1 \end{cases}$$

in the L^p -sense. Hence we see that

$$\frac{\|f_\varepsilon\|_p}{\|x \cdot \nabla f_\varepsilon\|_p} \geq 2^{\frac{1}{p}} \left\{ \left(\frac{n}{p} - \varepsilon\right)^p + \left(\frac{n}{p} + \varepsilon\right)^p \right\}^{-\frac{1}{p}} \rightarrow \frac{p}{n}$$

as $\varepsilon \rightarrow 0$. Moreover, if we take $g(x) = e^{-|x|^2}$, then we have

$$\frac{\|g\|_1}{\|x \cdot \nabla g\|_1} = \frac{1}{n}.$$

Remark 4. If $\Omega \subset \mathbb{R}^n$ is bounded, then (1.4) implies the usual Poincaré inequality. See [5, 8] for related results.

As the application of the inequality (1.5), we give the following three propositions: The first proposition is concerned with a function space $\mathfrak{L}^p := \{f \in L^p; x \cdot \nabla f \in L^p\}$ equipped with a semi-norm

$$\|f|_{\mathfrak{L}^p}\| := \|x \cdot \nabla f\|_p.$$

We shall prove that $\mathfrak{L}^p$ becomes a Banach space. Proposition 1.3, which is the second application, indicates that if a function $f \in L^p_{loc}$ with $x \cdot \nabla f \in L^p$ satisfies some decreasing condition, then f belongs to L^p . The third application Proposition 1.4 means that if $n \geq 3$ and $p = 2$, then (1.5) is equivalent to the Hardy inequality.

Proposition 1.2. *Let $n \geq 1$ and $1 \leq p < \infty$. Then the semi-normed space $(\mathfrak{L}^p, \|\cdot|_{\mathfrak{L}^p}\|)$ becomes a Banach space. Furthermore, the embedding operator $\iota : \mathfrak{L}^p \hookrightarrow L^p$ has the norm*

$$\|\iota|_{\mathfrak{L}^p \rightarrow L^p}\| = \frac{p}{n}. \quad (1.9)$$

Proof. By (1.5), we find that $f = 0$ if and only if $\|f|_{\mathfrak{L}^p}\| = 0$. Thus, $\|\cdot|_{\mathfrak{L}^p}\|$ is a norm of $\mathfrak{L}^p$. Let $\{f_m\}_{m=1}^\infty$ be a Cauchy sequence in $\mathfrak{L}^p$. From (1.5), there exist $f \in L^p$ and $g \in L^p$ such that

$$\lim_{m \rightarrow \infty} f_m = f, \quad \lim_{m \rightarrow \infty} x \cdot \nabla f_m = g$$

in L^p . For any $\varphi \in C_c^\infty(\mathbb{R}^n)$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^n} (x \cdot \nabla f) \varphi dx &= - \int_{\mathbb{R}^n} f \operatorname{div}(x \varphi) dx \\ &= - \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} f_m \operatorname{div}(x \varphi) dx \\ &= \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} (x \cdot \nabla f_m) \varphi dx \\ &= \int_{\mathbb{R}^n} g \varphi dx. \end{aligned}$$

Therefore, we have $x \cdot \nabla f = g \in L^p$ and $\mathfrak{L}^p$ is complete. By Remark 3, (1.9) holds. $\square$

In Theorem 1.1, one of the main assumptions is that $f \in L^p$. As we see in Remark 1 above, the last space is not generalized to L^p_{loc} . The following proposition shows a sufficient condition for the condition that $f \in L^p$.

Proposition 1.3. *Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $f \in \mathcal{D}'(\mathbb{R}^n)$ is measurable on $\mathbb{R}^n$ and that $x \cdot \nabla f \in L^p$. If there exists $\{\phi_l\}_{l=1}^\infty \subset C_c^\infty(\mathbb{R}^n)$ such that*

- (i) $\sup_{l \geq 1} \|\phi_l\|_\infty < \infty$,
- (ii) $\lim_{l \rightarrow \infty} \phi_l(x) = 1$ for a.e. x ,
- (iii) $\phi_l f \in L^p$,

$$(iv) \liminf_{l \rightarrow \infty} \left\| (x \cdot \nabla \phi_l) f \right\|_p = 0,$$

then $f \in L^p$.

Remark 5. In Section 3 below, we show some corollaries of the above proposition. In particular, we shall show that if $f \in L^p_{\text{loc}}$ satisfies $x \cdot \nabla f \in L^p$ and if $|x|^{n/p}|f(x)|$ vanishes at infinity, then f belongs to L^p .

Proof of Proposition 1.3. By (iii) and (1.5), we have

$$\|\phi_l f\|_p \leq \frac{p}{n} \left\{ \|(x \cdot \nabla \phi_l) f\|_p + \|\phi_l (x \cdot \nabla f)\|_p \right\}$$

for any $l \geq 1$. Since $x \cdot \nabla f \in L^p$, we see from (i), (ii) and the Lebesgue dominated theorem that

$$\lim_{l \rightarrow \infty} \|\phi_l (x \cdot \nabla f)\|_p = \|x \cdot \nabla f\|_p.$$

By (iv), we hence obtain

$$\liminf_{l \rightarrow \infty} \|\phi_l f\|_p \leq \frac{p}{n} \|x \cdot \nabla f\|_p.$$

It follows from (ii) and the Fatou lemma that

$$\begin{aligned} \|f\|_p^p &= \int_{\mathbb{R}^n} \liminf_{l \rightarrow \infty} |(\phi_l f)(x)|^p dx \\ &\leq \liminf_{l \rightarrow \infty} \int_{\mathbb{R}^n} |(\phi_l f)(x)|^p dx \\ &= \liminf_{l \rightarrow \infty} \|\phi_l f\|_p^p. \end{aligned}$$

Thus, we see that $f \in L^p$. □

The inequality (1.5) is equivalent to Hardy's inequality if $p = 2$. To be more specific, we have:

Proposition 1.4. *Let $n \geq 3$. Then the following two statements are equivalent:*

(a) *For any $f \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, we have*

$$\|f\|_2 \leq \frac{2}{n} \|x \cdot \nabla f\|_2. \quad (1.10)$$

(b) *For any $g \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, we have*

$$\left\| \frac{g}{|x|} \right\|_2 \leq \frac{2}{n-2} \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2. \quad (1.11)$$

Remark 6. The original Hardy inequality was given by [6]. Later, a lot of generalized Hardy inequalities are studied (see, e.g., [1, 2, 3, 4, 7, 9]). In particular, it was shown that

$$\left\| \frac{g}{|x|} \right\|_p \leq \frac{p}{n-p} \|\nabla g\|_p, \quad g \in W^{1,p}(\mathbb{R}^n), \quad p < n. \quad (1.12)$$

The constant $p/(n-p)$ is optimal. Since $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $W^{1,2}(\mathbb{R}^n)$ for $n \geq 3$, by using Theorem 1.1(ii), the above proposition and (1.12), we can see that (1.11) holds for any $W^{1,2}(\mathbb{R}^n)$. If we put $g_\varepsilon = |x|f_\varepsilon$, where f_ε have been given in Remark 3, then we have

$$\frac{\|g_\varepsilon/|x|\|_2}{\|(x/|x|) \cdot \nabla g_\varepsilon\|_2} \geq 2^{\frac{1}{2}} \left\{ \left(\frac{n}{2} - 1 - \varepsilon \right)^2 + \left(\frac{n}{2} - 1 + \varepsilon \right)^2 \right\}^{-\frac{1}{2}} \rightarrow \frac{2}{n-2}$$

as $\varepsilon \rightarrow 0$. Therefore, the constant $2/(n-2)$ is optimal.

Proof of Proposition 1.4. We first prove that the statement (a) implies (b). Put $g = |x|f$. Then we obtain

$$\|x \cdot \nabla f\|_2 = \left\| -\frac{g}{|x|} + \frac{x}{|x|} \cdot \nabla g \right\|_2^2 = \left\| \frac{g}{|x|} \right\|_2^2 - 2\Re \left\langle \frac{g}{|x|}, \frac{x}{|x|} \cdot \nabla g \right\rangle + \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2^2, \quad (1.13)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2 . Since

$$-2\Re \left\langle \frac{g}{|x|}, \frac{x}{|x|} \cdot \nabla g \right\rangle = - \left\langle \frac{x}{|x|^2}, \nabla |g|^2 \right\rangle = \left\langle \operatorname{div} \left(\frac{x}{|x|^2} \right), |g|^2 \right\rangle = \left\langle \frac{n-2}{|x|^2}, |g|^2 \right\rangle,$$

It follows from the statement (a) and (1.13) that

$$\left\| \frac{g}{|x|} \right\|_2^2 \leq \frac{4}{n^2} \left((n-1) \left\| \frac{g}{|x|} \right\|_2^2 + \left\| \frac{x}{|x|} \cdot \nabla g \right\|_2^2 \right),$$

which implies (1.11).

Conversely, we assume that (b) holds. We put $f = g/|x|$. Then we have

$$\left\| \frac{x}{|x|} \cdot \nabla (|x|f) \right\|_2^2 = \|f + x \cdot \nabla f\|_2^2 = \|f\|_2^2 + 2\Re \langle xf, \nabla f \rangle + \|x \cdot \nabla f\|_2^2$$

Since $2\Re \langle xf, \nabla f \rangle = -n\|f\|_2^2$, we see from the statement (b) that

$$\|f\|_2^2 \leq \frac{4}{(n-2)^2} (\|x \cdot \nabla f\|_2^2 - (n-1)\|f\|_2^2),$$

which implies (1.10). □

2. PROOF OF THEOREM 1.1

In this section, we give a proof of Theorem 1.1. For this purpose, we prepare some notation. For $\lambda > 0$ and for $g : \mathbb{R}^n \rightarrow \mathbb{C}$, $(\delta_\lambda g)(x) = g(\lambda x)$. For $\lambda > 0$ and for $\Omega \subset \mathbb{R}^n$, $\lambda\Omega = \{\lambda\omega; \omega \in \Omega\}$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a positive function such that $\int \varphi = 1$ and $\text{supp } \varphi \subset \{x \in \mathbb{R}^n; |x| \leq 1\}$. For $\lambda > 0$, we set $\zeta_\lambda(x) = \zeta(\lambda|x|)$ and $\varphi_\lambda = \lambda^{-n}\delta_{\lambda^{-1}}\varphi$.

Proposition 2.1. *Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is an open set. If $f \in C_c^\infty(\Omega)$, then we have (1.4).*

Proof. For $\phi : \Omega \rightarrow \mathbb{C}$, we put

$$\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \|\delta_\lambda \tilde{\phi}\|_{L^p(\Omega)}^p &= \int_\Omega |\tilde{\phi}(\lambda x)|^p dx \\ &= \lambda^{-n} \int_{\lambda\Omega} |\tilde{\phi}(x)|^p dx \\ &\leq \lambda^{-n} \int_\Omega |\tilde{\phi}(x)|^p dx \\ &= \lambda^{-n} \|\phi\|_{L^p(\Omega)}^p. \end{aligned} \tag{2.1}$$

For any $\lambda > 0$ and $x \in \Omega$,

$$\begin{aligned} (\delta_\lambda \tilde{f} - \tilde{f})(x) &= \tilde{f}(\lambda x) - \tilde{f}(x) \\ &= \int_1^\lambda \frac{\partial}{\partial \alpha} \tilde{f}(\alpha x) d\alpha \\ &= \int_1^\lambda \alpha^{-1} \left(\widetilde{\delta_\alpha x \cdot \nabla f} \right)(x) d\alpha. \end{aligned} \tag{2.2}$$

For any $\lambda > 1$, we see from (2.1) and (2.2) that

$$\begin{aligned} (1 - \lambda^{-n/p}) \|f\|_{L^p(\Omega)} &\leq \|\delta_\lambda \tilde{f} - \tilde{f}\|_{L^p(\Omega)} \\ &\leq \int_1^\lambda \alpha^{-1} \left\| \widetilde{\delta_\alpha x \cdot \nabla f} \right\|_{L^p(\Omega)} d\alpha \\ &\leq \int_1^\lambda \alpha^{-1-n/p} \|x \cdot \nabla f\|_{L^p(\Omega)} d\alpha \\ &= \frac{p}{n} (1 - \lambda^{-n/p}) \|x \cdot \nabla f\|_{L^p(\Omega)}, \end{aligned}$$

which implies (1.5) by letting $\lambda \rightarrow \infty$. □

Proof of Theorem 1.1. We first show (i). There exists some $\{f_l\}_{l=1}^\infty \subset C_c^\infty(\Omega)$ such that $f_l \rightarrow f$ in $W^{1,p}(\Omega)$ as $l \rightarrow \infty$. Let $\lambda > 0$. We observe from Proposition 2.1 that

$$\|\zeta_\lambda f_l\|_{L^p(\Omega)} \leq \frac{p}{n} \left\{ \|(x \cdot \nabla \zeta_\lambda) f_l\|_{L^p(\Omega)} + \|\zeta_\lambda (x \cdot \nabla f_l)\|_{L^p(\Omega)} \right\}$$

for all $l \geq 1$. We easily see that

$$\begin{aligned} \lim_{l \rightarrow \infty} \zeta_\lambda f_l &= \zeta_\lambda f, \\ \lim_{l \rightarrow \infty} (x \cdot \nabla \zeta_\lambda) f_l &= (x \cdot \nabla \zeta_\lambda) f, \\ \lim_{l \rightarrow \infty} \zeta_\lambda (x \cdot \nabla f_l) &= \lim_{l \rightarrow \infty} (\zeta_\lambda x) \cdot (\nabla f_l) = \zeta_\lambda (x \cdot \nabla f) \end{aligned}$$

in $L^p(\Omega)$. Hence we obtain

$$\|\zeta_\lambda f\|_{L^p(\Omega)} \leq \frac{p}{n} \left\{ \|(x \cdot \nabla \zeta_\lambda) f\|_{L^p(\Omega)} + \|\zeta_\lambda (x \cdot \nabla f)\|_{L^p(\Omega)} \right\}.$$

Since

$$|(x \cdot \nabla \zeta_\lambda)(x)| \leq \begin{cases} \|x \cdot \nabla \zeta\|_\infty & \text{if } \lambda^{-1} < |x| < 2\lambda^{-1}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

(1.5) holds as $\lambda \rightarrow 0$.

We next prove (ii). We have only to prove that

$$\lim_{\lambda \rightarrow 0} \zeta_\lambda (f * \varphi_\lambda) = f, \quad (2.4)$$

$$\lim_{\lambda \rightarrow 0} x \cdot \nabla (\zeta_\lambda (f * \varphi_\lambda)) = x \cdot \nabla f \quad (2.5)$$

in L^p because $f_\lambda \equiv \zeta_\lambda (f * \varphi_\lambda) \in C_c^\infty(\mathbb{R}^n)$. We obviously have (2.4).

It follows that

$$\begin{aligned} x \cdot \nabla (\zeta_\lambda (f * \varphi_\lambda)) &= (x \cdot \nabla \zeta_\lambda) (f * \varphi_\lambda) + \zeta_\lambda (x \cdot \nabla) (f * \varphi_\lambda) \\ &= (x \cdot \nabla \zeta_\lambda) (f * \varphi_\lambda) + \zeta_\lambda ((x \cdot \nabla f) * \varphi_\lambda) + \zeta_\lambda (f * \operatorname{div}(x \varphi_\lambda)) \\ &\equiv I_\lambda + II_\lambda + III_\lambda. \end{aligned}$$

By (2.3), we have $\lim_{\lambda \rightarrow 0} I_\lambda = 0$. Repeating the same argument of the proof of (2.4), we obtain $\lim_{\lambda \rightarrow 0} II_\lambda = 0$. Since $\int \operatorname{div}(x \varphi_\lambda) = 0$, we see that

$$(f * \operatorname{div}(x \varphi_\lambda))(x) = \int_{\mathbb{R}^n} (f(x - \lambda y) - f(x) \operatorname{div}(y \varphi(y))) dy.$$

Hence we obtain

$$\|f * \operatorname{div}(x \varphi_\lambda)\|_p \leq \sup_{|z| \leq \lambda} \|f(\cdot - z) - f\|_p \|\operatorname{div}(x \varphi)\|_1 \rightarrow 0$$

as $\lambda \rightarrow 0$. Thus, we have $\lim_{\lambda \rightarrow 0} III_\lambda = 0$.

Finally, we show (iii). Since $f \in C^1(\mathbb{R}^n)$, we have

$$f(e^\theta x) - f(x) = \int_0^\theta \frac{\partial}{\partial \alpha} f(e^\alpha x) d\alpha = \int_0^\theta e^\alpha x \cdot (\nabla f)(e^\alpha x) d\alpha \quad (2.6)$$

for any $x \in \mathbb{R}^n$ and $\theta > 0$. Let $\theta_0 = \ln(R/\varepsilon)$ and let $|x| \geq \varepsilon$. Since $|e^{\theta_0} x| \geq R$, we see that $f(e^{\theta_0} x) = 0$. By (2.6), we have

$$f(x) = - \int_0^{\theta_0} e^\alpha x \cdot (\nabla f)(e^\alpha x) d\alpha$$

for any $|x| \geq \varepsilon$. Thus (1.7) holds. $\square$

3. APPLICATION

In this section, we show some corollaries of Proposition 1.3.

Corollary 3.1. *Let $n \geq 1$ and $1 \leq p < \infty$. Assume that $f \in L_{\text{loc}}^p(\mathbb{R}^n)$ satisfies $x \cdot \nabla f \in L^p$. If there exist sequences $\{\rho_l\}_{l=1}^\infty \subset (0, \infty)$ and $\{R_l\}_{l=1}^\infty \subset (0, \infty)$ such that*

- (i) $R_l \leq R_k$ if $1 \leq l \leq k$,
- (ii) $\lim_{l \rightarrow \infty} R_l = \infty$,
- (iii)

$$\liminf_{l \rightarrow \infty} \left(\frac{R_l + \rho_l}{\rho_l} \right)^p \int_{R_l < |x| < R_l + \rho_l} |f(x)|^p dx = 0, \quad (3.1)$$

then $f \in L^p$.

Remark 7. Taking $\rho_l = R_l$, we see from the above Corollary that if we have $x \cdot \nabla f \in L^p$ and

$$\liminf_{R \rightarrow \infty} \int_{R < |x| < 2R} |f(x)|^p dx = 0, \quad (3.2)$$

then $f \in L^p$. If the left hand side of (3.2) is positive, then $f \notin L^p$ (even if $x \cdot \nabla f \in L^p$).

Proof of Corollary 3.1. Set

$$\phi_l(x) = \begin{cases} 1 & \text{for } |x| \leq R_l, \\ \zeta \left(\frac{|x| - R_l + \rho_l}{\rho_l} \right) & \text{for } |x| > R_l. \end{cases}$$

We immediately see that

$$\sup_{l \geq 1} \|\phi_l\|_\infty = 1 \quad \text{and} \quad \lim_{l \rightarrow \infty} \phi_l(x) = 1 \text{ for a.e. } x.$$

Since

$$x \cdot \nabla \phi_l(x) = \begin{cases} \frac{|x|}{\rho_l} \zeta' \left(\frac{|x| - R_l + \rho_l}{\rho_l} \right) & \text{if } R_l < |x| < R_l + \rho_l, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\liminf_{l \rightarrow \infty} \left\| (x \cdot \nabla \phi_l) f \right\|_p^p \leq \liminf_{l \rightarrow \infty} \left(\frac{R_l + \rho_l}{\rho_l} \right)^p \int_{R_l < |x| < R_l + \rho_l} |f(x)|^p dx = 0.$$

Thus, we see from Proposition 1.3 that $f \in L^p$. $\square$

Corollary 3.2. *Let $n \geq 1$, $1 \leq p < \infty$. Assume that $f \in L_{\text{loc}}^p(\mathbb{R}^n)$ satisfies $x \cdot \nabla f \in L^p$. If we have either of the following three conditions, then $f \in L^p$:*

(i) *It follows that*

$$\lim_{R \rightarrow \infty} \sup_{R < |x| < 2R} |x|^{n/p} |f(x)| = 0. \quad (3.3)$$

(ii) $p = 1$ and there exists some spherically symmetric, unbounded open set A such that

$$\lim_{|x| \rightarrow \infty, x \in A} |x|^n |f(x)| = 0. \quad (3.4)$$

(iii) $n = 1$, $p = 1$ and there exists some unbounded open set A such that $-\inf A = \sup A = \infty$ and (3.4) holds.

Remark 8. If

$$\liminf_{|x| \rightarrow \infty} |x|^{n/p} |f(x)| > 0,$$

then $f \notin L^p$ even if $x \cdot \nabla f \in L^p$.

Proof of Corollary 3.2. We first prove (i). It follows that

$$\begin{aligned} \int_{R < |x| < 2R} |f(x)|^p dx &= \int_{R < |x| < 2R} (|x|^{n/p} |f(x)|)^p |x|^{-n} dx \\ &\leq \left(\sup_{R < |x| < 2R} |x|^{n/p} |f(x)| \right)^p \int_{R < |x| < 2R} |x|^{-n} dx \\ &\leq \ln 2 \left(\sup_{R < |x| < 2R} |x|^{n/p} |f(x)| \right)^p \left(\int_{\mathbb{S}^{n-1}} d\omega \right) \\ &\rightarrow 0 \quad \text{as } R \rightarrow \infty. \end{aligned}$$

From Remark 7, we have $f \in L^p$.

We next show (ii). There exist $\{\rho_l\}_{l=1}^\infty \subset (0, 1)$ and $\{R_l\}_{l=1}^\infty \subset (0, \infty)$ such that $\lim_{l \rightarrow \infty} \rho_l = 0$, $R_l \leq R_k$ if $1 \leq l \leq k$, $\lim_{l \rightarrow \infty} R_l = \infty$ and

$$A_l \equiv \{x \in \mathbb{R}^n; R_l < |x| < R_l + \rho_l\} \subset A.$$

Then we obtain

$$\begin{aligned}
& \frac{R_l + \rho_l}{\rho_l} \int_{R_l < |x| < R_l + \rho_l} |f(x)| dx \\
& \leq \frac{R_l + \rho_l}{\rho_l} \left(\sup_{R_l < |x| < R_l + \rho_l} |x|^n |f(x)| \right) \left(\int_{\mathbb{S}^{n-1}} d\omega \right) \int_{R_l}^{R_l + \rho_l} r^{-1} dr \\
& \leq \left(\sup_{|x| > R_l, x \in A} |x|^n |f(x)| \right) \left(\int_{\mathbb{S}^{n-1}} d\omega \right) \frac{R_l + \rho_l}{\rho_l} \cdot \rho_l \cdot \frac{1}{R_l} \\
& \rightarrow 0 \quad \text{as } l \rightarrow \infty.
\end{aligned}$$

Applying Corollary 3.1, we see that $f \in L^1$.

Finally, we prove (iii). There exist $\{x_l\}_{l=1}^\infty \subset \mathbb{R}$ and $\{r_l\}_{l=1}^\infty \subset (0, 1)$ such that

$$\begin{aligned}
1 &< x_{2k-1} < x_{2k+1} && \text{for any } k \geq 1, \\
-1 &> x_{2k} > x_{2k+2} && \text{for any } k \geq 1, \\
|r_l| &\rightarrow 0 && \text{as } l \rightarrow \infty, \\
\{x \in \mathbb{R}; |x - x_l| < r_l\} &\subset A && \text{for any } l \geq 1.
\end{aligned}$$

Set

$$\begin{aligned}
\phi_1(x) &= \begin{cases} \zeta(x) & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq x_1, \\ \zeta\left(\frac{x - x_1 + r_1}{r_1}\right) & \text{if } x > x_1, \end{cases} \\
\phi_{2k}(x) &= \begin{cases} \zeta\left(\frac{x + x_{2k} - r_{2k}}{r_{2k}}\right) & \text{if } x < x_{2k}, \\ 1 & \text{if } x_{2k} \leq x \leq 0, \\ \phi_{2k-1}(x) & \text{if } x > 0, \end{cases} \\
\phi_{2k+1}(x) &= \begin{cases} \phi_{2k}(x) & \text{if } x < 0, \\ 1 & \text{if } 0 \leq x \leq x_{2k+1}, \\ \zeta\left(\frac{x - x_{2k+1} + r_{2k+1}}{r_{2k+1}}\right) & \text{if } x > x_{2k+1}, \end{cases}
\end{aligned}$$

where $k = 1, 2, \dots$. Then we have

$$\begin{aligned}
& \sup_{l \geq 1} \|\phi_l\|_\infty = 1, \\
& \phi_l f \in L^p \quad \text{for any } l \geq 1, \\
& \lim_{l \rightarrow \infty} \phi_l(x) = 1 \quad \text{for all } x \in \mathbb{R}.
\end{aligned}$$

Furthermore, we see that

$$x \cdot \nabla \phi_{2k+1} = \begin{cases} 0 & \text{if } x < x_{2k} - r_{2k}, \\ \frac{x}{r_{2k}} \zeta' \left(\frac{x + x_{2k} - r_{2k}}{r_{2k}} \right) & \text{if } x_{2k} - r_{2k} \leq x \leq x_{2k}, \\ 0 & \text{if } x_{2k} < x < x_{2k+1}, \\ \frac{x}{r_{2k+1}} \zeta' \left(\frac{x - x_{2k+1} + r_{2k+1}}{r_{2k+1}} \right) & \text{if } x_{2k+1} \leq x \leq x_{2k+1} + r_{2k+1}, \\ 0 & \text{if } x > x_{2k+1} + r_{2k+1}, \end{cases}$$

for any $k \geq 1$. Thus, we obtain

$$\begin{aligned} & \| (x \cdot \nabla \phi_{2k+1}) f \|_1 \\ & \leq \frac{-x_{2k} + r_{2k}}{r_{2k}} \int_{x_{2k}-r_{2k}}^{x_{2k}} |f(x)| dx + \frac{x_{2k+1} + r_{2k+1}}{r_{2k+1}} \int_{x_{2k+1}}^{x_{2k+1}+r_{2k+1}} |f(x)| dx \\ & \leq \frac{-x_{2k} + r_{2k}}{r_{2k}} \cdot r_{2k} \cdot \frac{1}{-x_{2k}} \left(\sup_{x_{2k}-r_{2k} \leq x \leq x_{2k}} |x| |f(x)| dx \right) \\ & \quad + \frac{x_{2k+1} + r_{2k+1}}{r_{2k+1}} \cdot r_{2k+1} \cdot \frac{1}{x_{2k+1}} \left(\sup_{x_{2k+1} \leq x \leq x_{2k+1}+r_{2k+1}} |x| |f(x)| dx \right). \end{aligned}$$

Hence we observe that

$$\liminf_{l \rightarrow \infty} \| (x \cdot \nabla \phi_l) f \|_1 = 0.$$

From Proposition 1.3, we have $f \in L^1$. □

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(T. Ozawa) DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY, 060-0810, JAPAN.
E-mail address: ozawa@math.sci.hokudai.ac.jp

(H.Sasaki) DEPARTMENT OF MATHEMATICS, OSAKA UNIVERSITY, 563-0043, JAPAN.
E-mail address: hisasaki@cr.math.sci.osaka-u.ac.jp