HOKKAIDO UNIVERSITY

Title	A simple proof of the Allen-Cahn equation to Brakke's motion
Author(s)	Sato, Norifumi
Citation	Hokkaido University Preprint Series in Mathematics, 862, 1-14
Issue Date	2007
DOI	https://doi.org/10.14943/84012
Doc URL	https://hdl.handle.net/2115/69671
Type	departmental bulletin paper
File Information	re862.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

A SIMPLE PROOF OF THE ALLEN-CAHN EQUATION TO BRAKKE'S MOTION

NORIFUMI SATO

Abstract. We give a simple proof that the solution of the Allen-Cahn equation converges to Brakke's motion as a parameter tends to zero by utilizing the recent results of Röger and Schätzle. The proof avoids some of the technicalities in Ilmanen's proof.

1. Introduction

The Allen-Cahn equation

$$\frac{\partial}{\partial t}u^\varepsilon = \Delta u^\varepsilon - \frac{1}{\varepsilon^2} f(u^\varepsilon) \tag{1.1}$$

was introduced by Allen and Cahn [2] to describe the macroscopic motion of phase boundaries driven by surface tension. Here f is the derivative of a potential F with two wells of equal depth at ± 1 and u^ε indicates the phase state at each point. Several authors studied the equation to the conclusion that the zero level set of u^ε approaches a hypersurface with its normal velocity determined by the mean curvature as $\varepsilon \to 0$. The phase boundaries should have the thickness of order ε.

The purpose of this paper is to prove when the space dimension $n = 2$ or 3 that u^ε converges to a mean curvature flow in the sense of Brakke without using the so-called monotonicity formula. Namely we prove μ_t, the limit of the Radon measure

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon(\cdot, t)|^2 + \frac{1}{\varepsilon} F(u^\varepsilon(\cdot, t)) \right) dx$$

constructed from u^ε, satisfies Brakke's inequality

$$\limsup_{s \to t} \frac{\mu_s(\phi) - \mu_t(\phi)}{s - t} \le \int -\phi H^2 + \nabla \phi \cdot (T_x \mu_t)^\perp \cdot \vec{H} \, d\mu_t$$

for all $\phi \in C_c^2(\Omega, \mathbb{R}^+)$ and all $t \ge 0$. Convergence result of u^ε to Brakke's flow is proved by Ilmanen [15]. In his paper, some conditions for initial data u_0^ε are assumed such as boundedness of μ_0^ε, upper density ratio bound of μ_0^ε and approximability of u_0^ε to derive the monotonicity formula. The monotonicity formula is then used to prove the clearing-out lemma, the lower density bound of μ_t, and the equipartition of the

2000 *Mathematics Subject Classification.* Primary 35K57, Secondary 49Q20, 53C44.

limit of discrepancy Radon measure

$$d\xi_t^\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon} F(u^\varepsilon) \right) dx,$$

which in the end prove convergence of μ_t^ε to Brakke's flow. In this paper we prove convergence of μ_t^ε without the monotonicity formula by applying the results in Röger and Schätzle [18]. (We introduce their results in §2.5.) We are able to simplify the proof of Ilmanen, even though the dimension is restricted to $n = 2$ and 3. In addition, we may assume less conditions on initial data u_0^ε. The main reason why the proof of convergence of μ_t^ε becomes shorter is that the varifold corresponding to the limit μ_t is integral from their results. (We introduce the notion of varifolds in §2.2.) In this case, the right-hand side of Brakke's inequality becomes

$$\int -\phi H^2 + \nabla\phi \cdot \vec{H} \, d\mu_t,$$

from which we can prove the main results directly. We remark that the integrality of varifold corresponding to μ_t is proved in Tonegawa [23].

We mention the results related to the Allen-Cahn equation (1.1). The formal derivation was given by Fife [13], Rubinstein, Sternberg and Keller [19], and others. De Mottoni and Schatzman [9], Chen [6], Chen and Elliott [7] and others proved the existence of the limit of u^ε in the general case with the assumption that there exists a classical solution of (1.1). Bronsard and Kohn [5] proved the convergence result for radially symmetric u^ε. Evans, Soner and Souganidis [10] showed that the limit of the level-set solution of the Allen-Cahn equation is contained in the viscosity solution for the mean curvature flow studied by Evans and Spruck [12], Chen, Giga, and Goto [8] and others. Ilmanen [15] showed that the limit is a mean curvature flow in the sense of Brakke [4] by using geometric measure theory. Subsequently Soner [21] gave proofs that more general initial data may be admitted in Ilmanen's result. There are many results related to the general subject of various Allen-Cahn type equation with modifications and those coupled with other field variables.

Finally we mention the organization of this paper. In §2, we set forth the notation and setting of our convergence theorem and define Brakke's varifolds moving by mean curvature and introduce the results in Röger and Schätzle [18]. In §3, we state our main result and lemmas which are needed to prove the main result and we show that there exists a subsequence $\{\mu_t^{\varepsilon_i}\}_{i\geq 1}$ of $\{\mu_t^\varepsilon\}_{\varepsilon>0}$ and a limit μ_t such that $\mu_t^{\varepsilon_i}$ converges μ_t for all $t \geq 0$. In §4, we prove the main lemma by using results in [18] and, in §5, we prove the main theorem.

2. Preliminaries

2.1. **Equation and measure.** Let Ω be an open set of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and u^ε be a smooth unique solution of

$$\begin{aligned} \frac{\partial}{\partial t}u^\varepsilon &= \Delta u^\varepsilon - \frac{1}{\varepsilon^2}f(u^\varepsilon) \quad \text{in } \Omega\times(0,\ \infty) \\ u^\varepsilon(\cdot,\ 0) &= u_0^\varepsilon(\cdot) \quad \text{on } \Omega\times\{0\} \\ \frac{\partial u^\varepsilon}{\partial \nu} &= 0 \quad \text{on } \partial\Omega\times(0,\ \infty), \end{aligned} \tag{2.1}$$

where ν denotes the unit outer normal vector of $\partial\Omega$ and we assume that

$$F(t) := \frac{1}{2}(1-t^2)^2, \quad f(t) := F'(t) = 2t(t^2-1).$$

Define the Radon measures μ_t^ε, ξ_t^ε and α_t^ε for $t \geq 0$ by

$$\begin{aligned} d\mu_t^\varepsilon &:= \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(\cdot,t)|^2 + \frac{1}{\varepsilon}F(u^\varepsilon(\cdot,t))\right)dx, \\ d\xi_t^\varepsilon &:= \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(\cdot,t)|^2 - \frac{1}{\varepsilon}F(u^\varepsilon(\cdot,t))\right)dx, \\ d\alpha_t^\varepsilon &:= \varepsilon\left(-\Delta u^\varepsilon(\cdot,t) + \frac{1}{\varepsilon^2}f(u^\varepsilon(\cdot,t))\right)^2 dx. \end{aligned}$$

And we suppose that there exists a constant E_0 such that

$$\mu_0^\varepsilon(\Omega) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon(\cdot,0)|^2 + \frac{1}{\varepsilon}F(u^\varepsilon(\cdot,0))\right)dx \leq E_0.$$

Since u^ε is a gradient flow of the energy functional

$$M(u) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u(x)|^2 + \frac{1}{\varepsilon}F(u(x))\right)dx,$$

it follows that

$$\mu_t^\varepsilon(\Omega) \leq E_0 \quad \text{for all } t \geq 0. \tag{2.2}$$

Namely, μ_t^ε is uniformly bounded with respect to ε.

2.2. **Varifold notations.** We introduce the notion of varifolds. For details, we refer to [1], [4], [20]. A *general k-varifolds* in Ω is a Radon measure in $\Omega\times G_k(\mathbb{R}^n)$, where $G_k(\mathbb{R}^n)$ is the Grassman manifold of unoriented k-planes in $\mathbb{R}^n$. We denote the set of all general k-varifolds in Ω by $\mathbf{V}_k(\Omega)$. When S is a k-plane, we also use S to denote the orthogonal projection $\mathbb{R}^n \to S$. We write $A : B$ for the inner product $\sum A_{ij}B_{ij}$ of matrices, and $A\cdot V$ for the application of a matrix to a

vector. In this notation we write the first variation formula

$$\begin{aligned}\delta V(X) &= \int DX(x) : S \, dV(x,S) \\ &:= \int \sum_{e_i \| S} < D_{e_i} X(x), e_i > \, dV(x,S) \\ &= \int -X(x) \cdot \vec{H}(x) \, d\|V\|(x) \quad \text{if } |\delta V| \ll \|V\|.\end{aligned}$$

Here V is a general k-varifold, δV is the first variation of V, $\|V\|$ is the mass measure of V, $X \in C^1_c(\Omega, \mathbb{R}^n)$, $\{e_1, \cdots, e_k\}$ is an orthogonal basis of S, $\vec{H} = \vec{H}_V$ is the generalized mean curvature vector if it exists and $|\delta V| \ll \|V\|$ denotes that $|\delta V|$ is absolutely continuous with respect to $\|V\|$. The quantity $DX : S$ is also written $\mathrm{div}_V X$.

2.3. **Integral Radon measures.** We call a Radon measure μ in Ω *k-rectifiable* if either of the following equivalent conditions hold:
(a) $\mu = \theta \mathcal{H}^k \lfloor_X$, where X is a locally k-rectifiable, $\mathcal{H}^k$-measurable set, and $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k \lfloor_X, (0, \infty))$.
(b) The measure theoretic tangent plane $T_x\mu$ exists μ-a.e., where we define $T_x\mu$ by

$$T_x\mu := \lim_{\lambda \to 0} \mu_{x,\lambda} \quad \text{in the sense of Radon measures}$$

provided the limit exists and is a positive multiple of $\mathcal{H}^k$ restricted to some k-plane. Here $\mu_{x,\lambda}(A) := \lambda^{-k}\mu(x + \lambda A)$, for $A \subseteq \mathbb{R}^n$.
We denote the set of all k-rectifiable Radon measures in Ω by $\mathcal{M}_k(\Omega)$. If θ in (a) takes integer values, we call μ *integral* and write $\mu \in \mathcal{IM}_k(\Omega)$.

In this circumstance there is a corresponding *rectifable k-varifolds* $V = V_\mu$ defined by

$$\int \psi(x,S) \, dV(x,S) = \int \psi(x, T_x\mu) \, d\mu(x) \quad \text{for } \psi \in C_c(\Omega \times G_k(\mathbb{R}^n), \mathbb{R}).$$

Note that $\mu = \|V\|$. We denote the set of all rectifiable k-varifolds in Ω by $\mathbf{RV}_k(\Omega)$. This is in one-to-one correspondance with $\mathcal{M}_k$ by its definition. We write $\vec{H} = \vec{H}_\mu = \vec{H}_V$.

We call varifold $V = V_\mu$ is *integral* if μ is integral and denote the set of all integral k-varifolds in Ω by $\mathbf{IV}_k(\Omega)$. Note that $\vec{H} \perp T_x\mu$ μ-a.e. provided V is integral (see [4, 5.8]).

2.4. **Brakke's motion.** We define $\mathcal{B}(\mu, \phi)$ for any Radon measures μ and any test functions $\phi = \phi(x)$ on Ω as follows.

If $\mu \lfloor_{\{\phi > 0\}} \notin \mathcal{M}_k(\Omega)$ we define

$$(*) \qquad \mathcal{B}(\mu, \phi) := -\infty.$$

If $\mu \lfloor_{\{\phi>0\}} \in \mathcal{M}_k(\Omega)$ and $|\delta\mu| \lfloor_{\{\phi>0\}} \not\ll \mu \lfloor_{\{\phi>0\}}$, we define $\mathcal{B}(\mu, \phi)$ as in $(*)$. If $\mu \lfloor_{\{\phi>0\}} \in \mathcal{M}_k(\Omega)$ and $|\delta\mu| \lfloor_{\{\phi>0\}} \ll \mu \lfloor_{\{\phi>0\}}$, we can define the

generalized mean curvature vector $\vec{H}$, and if $\int \phi|\vec{H}|^2\,d\mu = \infty$ we define $\mathcal{B}(\mu,\phi)$ as in $(*)$. Otherwise we define

$$\mathcal{B}(\mu,\phi) := \int_\Omega -\phi H^2 + \nabla\phi\cdot(T_x\mu)^\perp\cdot\vec{H}\,d\mu.$$

Note that if μ is integral we have

$$\mathcal{B}(\mu,\phi) = \int -\phi H^2 + \nabla\phi\cdot\vec{H}\,d\mu.$$

by the perpendicularity of $\vec{H}$.

A family $\{\mu_t\}_{t\geq 0}$ of Radon measures is called *Brakke's motion* provided

$$\overline{D}_t\mu_t(\phi) \leq \mathcal{B}(\mu_t,\phi) \tag{2.3}$$

for all $\phi \in C_c^2(\Omega,\mathbb{R}^+)$ and all $t \geq 0$. Here $\mu_t(\phi) = \int \phi\,d\mu_t$ and $\overline{D}_t f(t)$ is the upper derivative

$$\limsup_{s\to t}\frac{f(s)-f(t)}{s-t}.$$

2.5. **New results by Röger and Schätzle.** We introduce the results proved by Röger and Schätzle in [18]. We require these to prove our main results. In their paper,

$$\mu^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon)\right)\mathcal{L}^n,$$
$$\xi^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{1}{\varepsilon}F(u^\varepsilon)\right)\mathcal{L}^n,$$
$$\alpha^\varepsilon := \varepsilon\left(-\Delta u^\varepsilon + \frac{1}{\varepsilon^2}f(u^\varepsilon)\right)^2\mathcal{L}^n,$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

Theorem 2.1 (**[18] Theorem 4.1, 5.1**). *Let n=2, 3 and $\Omega \subset \mathbb{R}^n$ an open set. Suppose that there exists a constant C such that*

$$\mu_\varepsilon(\Omega) + \alpha_\varepsilon(\Omega) \leq C \tag{2.4}$$

and

$$\mu^\varepsilon \to \mu \quad \text{as measures.} \tag{2.5}$$

Then $\sigma^{-1}\mu$ is an integral $(n-1)$-Radon measure. Here $\sigma := \int_{-1}^1 \sqrt{2F(t)}\,dt$.

Theorem 2.2 (**[18] Theorem 4.9**). *Let n=2, 3 and suppose (2.4) and*

$$\xi^\varepsilon \to \xi \quad \text{as measures.}$$

Then

$$|\xi^\varepsilon| \to 0 \quad \text{as measures}$$

and

$$\xi = 0.$$

These results are proved in the study of the modified conjecture of De Giorgi. For related results, see Bellettini and Mugnai [3], Moser [16], Nagase and Tonegawa [17], and Tonegawa [22].

3. Main theorem and main lemma

In this section, we state the main result and the main lemma, and we prove the existence of the subsequence of measure $\{\mu_t^{\varepsilon_i}\}_{i\geq 1} \subset \{\mu_t^{\varepsilon}\}_{\varepsilon>0}$ such that $\mu_t^{\varepsilon_i}$ converges to the Radon measure μ_t for all $t \geq 0$. The key point of the proof is the semi-decreasing property of μ_t^{ε}.

Main theorem is the following.

Theorem 3.1. *Let n=2 or 3, u^{ε} be the solution of (2.1) and μ_t^{ε} be as in §2.1. Then there exists a Radon measure μ_t such that μ_t^{ε} subsequentially converges to μ_t as ε tends to 0 and μ_t satisfies (2.3) for all $\phi \in C_c^2(\Omega, \mathbb{R}^+)$ and all $t \geq 0$, namely,*

$$\overline{D}_t \mu_t(\phi) \leq \mathcal{B}(\mu_t, \phi) \quad for\ \phi \in C_c^2(\Omega, \mathbb{R}^+),\ t \geq 0.$$

Moreover, whenever $\overline{D}_t \mu_t(\phi) > -\infty$, $\sigma^{-1}\mu_t \lfloor_{\{\phi>0\}}$ is integral.

We prove the main theorem by the following lemma.

Lemma 3.2. *Let n=2, 3 and $\{u^i\}_{i\geq 1}$ be a sequence of smooth functions on Ω with $\mathcal{L}^n(\{|\nabla u^i| = 0\}) = 0$ for each $i \geq 1$. Let $\{\varepsilon_i\}_{i\geq 1}$ be a sequence converging to 0. Define μ^i, ξ^i, V^i as in §2, namely,*

$$d\mu^i = \left(\frac{\varepsilon_i}{2}|\nabla u^i|^2 + \frac{1}{\varepsilon_i}F(u^i)\right) dx,$$

$$d\xi^i = \left(\frac{\varepsilon_i}{2}|\nabla u^i|^2 - \frac{1}{\varepsilon_i}F(u^i)\right) dx,$$

$V^i \in \mathbf{V}_{n-1}(\Omega)$, $\|V^i\| = \mu^i$, $V^i(x)$ supported at $(\nabla u^i(x))^{\perp}$ for each $x \in \Omega$. Let $\phi \in C_c^2(\Omega, \mathbb{R}^+)$ and define

$$\mathcal{B}^{\varepsilon_i}(u^i, \phi) := \int_{\Omega} -\varepsilon_i \phi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 + \varepsilon_i \nabla\phi \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx.$$

Assume

(i) $\mu^i \to \mu$ as Radon measures on Ω

(ii) There exists a constant C such that $\mathcal{B}^{\varepsilon_i}(u^i, \phi) \geq -C$ for $i \geq 1$.

Then the following hold:

(iii) $\sigma^{-1}\mu\lfloor_{\{\phi>0\}} \in \mathcal{IM}_{n-1}(\Omega)$.

(iv) There are $V \in \mathbf{RV}_{n-1}(\Omega)$ and $\tilde{V} \in \mathbf{IV}_{n-1}(\Omega)$ such that

$$V = \sigma\tilde{V}, \quad V^i\lfloor_{\{\phi>0\}} \to V, \quad \|V\| = \mu\lfloor_{\{\phi>0\}}.$$

(v) For all $Y \in C_c^1(\{\phi > 0\}, \mathbb{R}^n)$,

$$\delta V(Y) = \lim_{i\to\infty} \int -\varepsilon_i Y \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx.$$

(vi) $\mathcal{B}(\mu,\phi) \geq \limsup_{i\to\infty} \mathcal{B}^{\varepsilon_i}(u^i,\phi)$.

Remark 3.3. *Our lemma is similar to Ilmanen [15]. The difference between our lemma and the lemma in Ilmanen [15, 9.3] is whether the limit μ_t of Radon measure μ_t^ε is integral or rectifiable. If μ_t is integral, as we have seen in §2.4, the right-hand side of Brakke's inequality becomes a simpler form. Then we can derive the upper semicontinuity of $\mathcal{B}^{\varepsilon_i}$ (vi) with relative ease.*

We postpone the proof of these results to §4 and §5. Now we prove the existence of a subsequence $\{\mu_t^{\varepsilon_i}\}_{i\geq 1} \subset \{\mu_t^\varepsilon\}_{\varepsilon\geq 0}$ converging to a Radon measure μ_t as $i\to\infty$ for all $t\geq 0$. The proof is similar to Ilmanen [15]. But we include the proof for the reader's convenience.

Proposition 3.4 (**[15] Proposition 5.3**). *For $\phi \in C_c^2(\Omega,\mathbb{R}^+)$, the function $\mu_t^\varepsilon(\phi) - C(\phi)E_0 t$ of t is nonincreasing.*

Proof. For $\phi \in C_c^2(\Omega,\mathbb{R}^+)$ we derive by integration by parts

$$\begin{aligned}
\frac{d}{dt}\int \phi\, d\mu_t^\varepsilon &= \int \phi \frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\right) dx \\
&= \int \phi\left(\varepsilon \nabla u\cdot\nabla\frac{\partial}{\partial t}u + \frac{1}{\varepsilon}f(u)\frac{\partial}{\partial t}u\right) dx \\
&= \int -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 + \varepsilon\nabla\phi\cdot\nabla u\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) dx.
\end{aligned}$$

And we have by using Schwarz's inequality and (2.2)

$$\begin{aligned}
&\int -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right)^2 + \varepsilon\nabla\phi\cdot\nabla u\left(-\Delta u + \frac{1}{\varepsilon^2}f(u)\right) dx \\
&\leq \int -\varepsilon\phi\left(-\Delta u + \frac{1}{\varepsilon^2}f(u) + \frac{\nabla\phi\cdot\nabla u}{2\phi}\right)^2 + \varepsilon|\nabla u|^2\frac{|\nabla\phi|^2}{4\phi}\, dx \\
&\leq C_1(\phi)\mu_t^\varepsilon(\{\phi>0\}) \\
&\leq C_1(\phi)E_0,
\end{aligned}$$

where

$$C_1(\phi) := \sup_{\{\phi>0\}} \frac{|\nabla\phi|^2}{2\phi} \leq \sup|\nabla^2\phi| \tag{3.1}$$

(see [14, 6.6]). Therefore for $\phi \in C_c^2(\Omega,\mathbb{R}^+)$ the function $\mu_t^\varepsilon(\phi) - C_1(\phi)E_0 t$ is nonincreasing.

Proposition 3.5 (**[15] Propositon 5.5**). *There is a subsequence $\{\mu_t^{\varepsilon_i}\}_{i\geq 1} \subset \{\mu_t^\varepsilon\}_{\varepsilon>0}$ such that*

$$\mu_t^{\varepsilon_i} \to \mu \quad \text{as Radon measures on } \Omega \text{ for all } t\geq 0$$

Proof. 1. Choose a countable dense set $B_1 \subset [0,\infty)$. Then by (2.2), the weak convergence of Radon measures and diagonal argument, we can select a subsequence $\{\mu_t^{\varepsilon_i}\}_{i\geq 1}$ and $\{\mu_t\}_{t\in B_1}$ such that

$$\mu_t^{\varepsilon_i} \to \mu_t \quad \text{as Radon measures for } t \in B_1.$$

and by proposition 3.4 $\mu_t(\phi) - C(\phi)E_0 t$ is nonincreasing for $t \in B_1$.

2. Now let $\{\phi_i\}_{i\geq 1}$ be a countable dense set in $C_c^2(\Omega, \mathbb{R}^+)$. Then by proposition 3.4 there is a co-countable set $B_2 \subset [0,\infty)$ such that for any $t \in B_2$ and $i \geq 1$, $\mu_s(\phi_i)$ is continuous at t as a function of $s \in B_1$.

3. For any fixed $t \in B_2$, we can find a further subsequence $\{\mu_t^{i_j}\}_{j\geq 1} \subset \{\mu_t^i\}_{i\geq 1}$ and a limit μ_t such that

$$\mu_t^{i_j} \to \mu_t \quad \text{as Radon measures.}$$

Then by proposition 3.4 $\{\mu_s(\phi_i)\}_{s\in B_1\cup\{t\}}$ is continuous at t, for each i. Since $\{\phi_i\}_{i\geq 1}$ is dense, it follows that μ_t is uniquely determined by $\{\mu_s\}_{s\in B_1}$. Therefore the full subsequence converges. In this way we define μ_t for each $t \in B_2$.

4. On the countable set $[0,\infty) \setminus B_2$, by diagonal argument we can choose further subsequence $\{\mu_t^{\varepsilon_{i_j}}\}_{j\geq 1}$ such that

$$\mu_t^{\varepsilon_{i_j}} \to \mu_t \quad \text{as Radon measures for } t \in [0,\infty) \setminus B_2$$

Thus we proved

$$\mu_t^{\varepsilon_{i_j}} \to \mu_t \quad \text{as Radon measures for all } t \geq 0.$$

4. Proof of main lemma

In this section, we prove the main lemma 3.2. Though the proof of (v) and (vi) is similar to Ilmanen [15], we can derive (vi) with less work owing to (iii) and (iv) proved by Röger and Schätzle [18].

1. First we prove (iii). By using Cauchy's inequality and (3.1), we derive from (ii)

$$\begin{aligned}
-C &\leq \int_\Omega -\varepsilon_i\phi\left(-\Delta u^i + \frac{1}{\varepsilon_i^2}f(u)\right)^2 + \varepsilon_i\nabla\phi\cdot\nabla u^i\left(-\Delta u^i + \frac{1}{\varepsilon_i^2}f(u^i)\right)dx \\
&\leq \int_{\{\phi>0\}} -\varepsilon_i\phi\left(-\Delta u^i + \frac{1}{\varepsilon_i^2}f(u)\right)^2 + \frac{\varepsilon_i|\nabla\phi|^2|\nabla u^i|^2}{2\phi} \\
&\qquad\qquad + \frac{\varepsilon_i}{2}\phi\left(-\Delta u^i + \frac{1}{\varepsilon_i^2}f(u^i)\right)^2 dx \\
&\leq -\frac{1}{2}\int \varepsilon_i\phi\left(-\Delta u^i + \frac{1}{\varepsilon_i^2}f(u^i)\right)^2 dx + C(\phi)\mu^i(\Omega).
\end{aligned}$$

So we obtain

$$\int_\Omega \varepsilon_i\phi\left(-\Delta u^i + \frac{1}{\varepsilon_i^2}f(u^i)\right)^2 dx \leq \tilde{C}.$$

Here we fix an open set $\tilde{\Omega} \subset\subset \{\phi > 0\}$. Then we derive

$$\begin{aligned}\tilde{C} &\geq \inf_{x\in\tilde{\Omega}} \phi(x) \int_{\tilde{\Omega}} \varepsilon_i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx \\ &= \alpha^i(\tilde{\Omega}) \inf_{x\in\tilde{\Omega}} \phi(x),\end{aligned}$$

which shows

$$\alpha^i(\tilde{\Omega}) \leq \left(\inf_{x\in\tilde{\Omega}} \phi(x)\right)^{-1} \tilde{C}.$$

Therefore $\mu^i(\tilde{\Omega}) + \alpha^i(\tilde{\Omega})$ is bounded. Then theorem 2.1 implies that $\sigma^{-1}\mu \in \mathcal{IM}_{n-1}(\tilde{\Omega})$ for any $\tilde{\Omega} \subset\subset \{\phi > 0\}$. Consequently $\sigma^{-1}\mu \in \mathcal{IM}_{n-1}(\{\phi > 0\})$.

2. Next we prove (iv). Let $\tilde{V}$ be the varifold corresponding to $\sigma^{-1}\mu$. Then $\tilde{V} \in \mathbf{IV}_{n-1}(\Omega)$ and $||\tilde{V}|| = \sigma^{-1}\mu$. We denote the varifold corresponding to μ by V. Then by one-to-one correspondance between $\mathcal{M}_{n-1}(\Omega)$ and $\mathbf{RV}_{n-1}(\Omega)$

$$V = \sigma\tilde{V}, \quad \|V\| = \mu.$$

By the compactness of Radon measures and (i), there is a subsequence $\{V^{i_j}\}_{j\geq 1} \subset \{V^i\}_{i\geq 1}$ and a limit $\bar{V}$ such that

$$V^{i_j} \to \bar{V} \quad \text{as varifolds.}$$

Then by one-to-one correspondance between $\mathcal{M}_{n-1}(\Omega)$ and $\mathbf{RV}_{n-1}(\Omega)$ $\bar{V}$ is determined by μ, independent of the subsequence. Therefore

$$V^i \to V \quad \text{as varifolds,}$$

which proved (iv).

3. Next we prove (v). Define the *stress tensor* T_{kl} by

$$T = \frac{\varepsilon_i}{2}|\nabla u^i|^2 I - \varepsilon_i \nabla u^i \otimes \nabla u^i + \frac{1}{\varepsilon^i} F(u^i) I,$$

where I denotes the identity matrix of $\mathbb{R}^n$. Note that

$$\frac{\partial}{\partial x_k} T_{kl} = \varepsilon_i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) \frac{\partial}{\partial x_l} u^i.$$

Then for any $Y \in C_c^1(\Omega, \mathbb{R}^n)$ we derive

$$
\begin{aligned}
&\int \varepsilon_i Y \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i) \right) dx \\
&= \int D_k T_{kl} Y^l \, dx \\
&= \int \left(\varepsilon_i \nabla u^i \otimes \nabla u^i - \frac{\varepsilon_i}{2} |\nabla u^i|^2 I - \frac{1}{\varepsilon_i} F(u^i) I \right) : DY \, dx \\
&= -\int \left(\frac{\varepsilon_i}{2} |\nabla u^i|^2 + \frac{1}{\varepsilon_i} F(u^i) \right) (I - \nu^i \otimes \nu^i) : DY \, dx \\
&\qquad\qquad + \int \left(\frac{\varepsilon_i}{2} |\nabla u^i|^2 - \frac{1}{\varepsilon_i} F(u^i) \right) \nu^i \otimes \nu^i : DY \, dx \\
&= -\int (I - \nu^i \otimes \nu^i) : DY \, d\mu^i + \int \nu^i \otimes \nu^i : DY \, d\xi^i,
\end{aligned}
\tag{4.1}
$$

where $\nu^i := \nabla u^i / |\nabla u^i|$. Fix $U \subset\subset \{\phi > 0\}$. Then for $Y \in C_c^1(U, \mathbb{R}^n)$ we obtain

$$
\begin{aligned}
\delta V^i(Y) &= \int DY : S \, dV^i(x, S) \\
&= \int DY : (I - \nu^i \otimes \nu^i) \, d\mu^i \\
&= \int -\varepsilon_i Y \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i) \right) dx - \int \nu^i \otimes \nu^i : DY \, d\xi^i.
\end{aligned}
\tag{4.2}
$$

Passing to limits and using theorem 2.2, we have

$$
\delta V(Y) = \lim_{i \to \infty} \int -\varepsilon_i Y \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i) \right) dx.
$$

and (v) is proved.

4. Next we prove (vi). The proof is similar to [15]. Let $\psi \in C_c^2(\{\phi > 0\}, \mathbb{R}^+)$ with $\psi^{\frac{1}{2}} \in C^1$. By the rectifiability of μ_t, we can approximate by smooth functions (see [14, 7.4]) to obtain

$$
\left(\int \psi H^2 \, d\mu \right)^{\frac{1}{2}} = \sup \left\{ \int \psi^{\frac{1}{2}} \vec{H} \cdot Y \, d\mu \ ; \ Y \in C_c^\infty(\Omega), \ \|Y\|_{L^2(\mu)} \leq 1 \right\}.
$$

Now, using (4.2), theorem 2.2 and Schwarz's inequality,

$$\begin{aligned}
&\int \psi^{\frac{1}{2}} Y \cdot \vec{H}\, d\mu \\
&= -\delta V(\psi^{\frac{1}{2}} Y) \\
&= -\lim_{i\to\infty} \delta V^i(\psi^{\frac{1}{2}} Y) \\
&= \lim_{i\to\infty} \int \varepsilon_i \psi^{\frac{1}{2}} Y \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx \\
&\qquad\qquad + \lim_{i\to\infty} \int \nu^i \otimes \nu^i : D(\psi^{\frac{1}{2}} Y)\, d\xi^i \\
&\le \liminf_{i\to\infty} \left(\int \varepsilon_i |\nabla u^i|^2 |Y|^2\, dx\right)^{\frac{1}{2}} \left(\int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx\right)^{\frac{1}{2}} \\
&\le \lim_{i\to\infty} \left(\int |Y|^2\, d\mu^i\right)^{\frac{1}{2}} \liminf_{i\to\infty} \left(\int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx\right)^{\frac{1}{2}} \\
&= \|Y\|_{L^2(\mu)} \liminf_{i\to\infty} \left(\int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx\right)^{\frac{1}{2}},
\end{aligned}$$

which yields

$$\int \psi H^2\, d\mu \le \liminf_{i\to\infty} \int \varepsilon_i \psi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx.$$

Passing ψ to ϕ by the monotone convergence theorem, we have

$$\int \phi H^2\, d\mu \le \liminf_{i\to\infty} \int \varepsilon_i \phi \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right)^2 dx. \tag{4.3}$$

5. (Compare to "seven-epsilon proof" of [15].) By (iii), (iv) and (v), we obtain for $\psi \in C_c^2(\{\phi > 0\}, \mathbb{R}^+)$

$$\begin{aligned}
\int \nabla\psi \cdot \vec{H}\, d\mu &= \delta V(\nabla\psi) \\
&= \lim_{i\to\infty} \int -\varepsilon_i \nabla\psi \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon_i^2} f(u^i)\right) dx.
\end{aligned}$$

Now we pass ψ to ϕ and get

$$\int \nabla\phi \cdot \vec{H}\, d\mu = \lim_{i\to\infty} \int -\varepsilon_i \nabla\phi \cdot \nabla u^i \left(-\Delta u^i + \frac{1}{\varepsilon^2} f(u^i)\right) dx. \tag{4.4}$$

By (4.3) and (4.4), (vi) is proved and we complete the proof of lemma 3.2.

5. Proof of main theorem

Finally we prove the main theorem 3.1 by using the main lemma 3.2. The proof is also similar to [15]. The key point of the proof is the semi-decreasing property of $\{\mu_t^\varepsilon\}$.

Let $t_0 \geq 0$, $\phi \in C_c^2(\Omega, \mathbb{R}^+)$, and assume without loss of generality

$$-\infty < D_0 := \overline{D}_{t_0}\mu_t(\phi). \tag{5.1}$$

Then there is a sequence $\{\delta_q\}_{q\geq 1}$ with $\delta_q \to 0$ and $\{t_q\}_{q\geq 1}$ with $t_q \to t_0$ such that

$$D_0 - \delta_q \leq \frac{\mu_{t_q}(\phi) - \mu_{t_0}(\phi)}{t_q - t_0}.$$

We may assume that $t_q > t_0$ for all q. (The other case is similar.)

By the convergence $\mu_t^{\varepsilon_i} \to \mu_t$ there is a sequence $\{r_q\}_{q\geq 1}$ with $r_q \to \infty$ such that

$$D_0 - 2\delta_q \leq \frac{\mu_{t_q}^{\varepsilon_{r_q}}(\phi) - \mu_{t_0}^{\varepsilon_{r_q}}(\phi)}{t_q - t_0} = \frac{1}{t_q - t_0}\int_{t_0}^{t_q} \frac{d}{dt}\mu_t^{\varepsilon_{r_q}}(\phi)\, dt. \tag{5.2}$$

Now by proposition 3.4 there is a constant $D_1 = D_1(\phi)$ such that

$$\frac{d}{dt}\mu_t^{\varepsilon_i}(\phi) \leq D_1 \quad \text{for } i \geq 1,\ t \geq 0.$$

If

$$T := \left\{ t \in [t_0,\ t_q] : \frac{d}{dt}\mu_t^{\varepsilon_{r_q}}(\phi) \geq D_0 - 3\delta_q \right\},$$

then by (5.2) there exists $s_q \in T \subset [t_0, t_q]$ such that

$$D_0 - 3\delta_q \leq \left.\frac{d}{dt}\mu_t^{\varepsilon_{r_q}}(\phi)\right|_{t=s_q} = \mathcal{B}^{\varepsilon_{r_q}}(u^{\varepsilon_{r_q}}(\cdot, s_q),\ \phi). \tag{5.3}$$

Now we suppose that the subsequence $\{\mu_{s_q}^{\varepsilon_{r_q}}\}_{q\geq 1}$ converges to a Radon measure $\tilde{\mu}$ as q tends to ∞. By applying proposition 3.4 and (5.1), it is possible to prove (see [14, 7.1]) that

$$\tilde{\mu}\lfloor_{\{\phi>0\}} = \mu_{t_0}\lfloor_{\{\phi>0\}}, \tag{5.4}$$

so that $\mathcal{B}(\tilde{\mu}, \phi) = \mathcal{B}(\mu_{t_0}, \phi)$.

By (5.3) and (5.4) we have verified the hypotheses of lemma 3.2 for the sequences $\{u^{\varepsilon_{r_q}}(\cdot, s_q)\}_{q\geq 1}$ and $\{\mu_{s_q}^{\varepsilon_{r_q}}\}_{q\geq 1}$ on $\{\phi > 0\}$. Therefore from (5.1), (5.3) and (5.4), as $q \to \infty$, we obtain

$$\overline{D}_{t_0}\mu_t(\phi) \leq \mathcal{B}(\mu_{t_0}, \phi). \tag{5.5}$$

Hence we have proven Brakke's inequality (2.3).

References

[1] W. Allard, *On the first variation of a varifold*, Ann. of Math. **95** (1972), no. 2, 417–491.

[2] S. M. Allen, J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening*, Acta. Metal. **27** (1979), 1084–1095.

[3] G. Bellettini, L. Mugnai, *On the approximation of the elastica functional in radial symmetry*, Calculus of Var. P.D.E. **24** (2005), no. 1, 1–20.

[4] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, NJ, (1978).

[5] L. Bronsard, R. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Diff. Eqns. **90** (1991), no. 2, 211–237.

[6] X. Chen, *Generation and propagation of interfaces in reaction-diffusion equations*, J. Diff. Eqns. **96** (1992), no. 1, 116–141.

[7] X. Chen, C. M. Elliott, *Asymptotics for a parabolic double obstacle problem*, Proc. Royal. Soc. London Ser. A **444** (1994), No. 1992, 429–445.

[8] Y. G. Chen, Y. Giga, S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Diff. Geom. **33** (1991), no. 3, 749–786.

[9] P. de Mottoni, M. Schatzman, *Evolution géométric d'interfaces*, C. R. Acad. Paris Sci. Sér. I Math. **309** (1989), no. 7, 453–458.

[10] L. C. Evans, H. M. Soner, P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure. Appl. Math. **45** (1992), no. 9, 1097–1123.

[11] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Math. CRC Press, (1992).

[12] L. C. Evans, J. Spruck, *Motion by mean curvature I*, J. Diff. Geom. **33** (1991), no. 3, 635–681.

[13] P. C. Fife, *Dynamics of internal layers and diffusive interfaces*, SIAM, CBMS (1988).

[14] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520

[15] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Diff. Geom. **38** (1993), 417–461.

[16] R. Moser, *A higher order asymptotic problem related to phase transitions*, SIAM J. Math. Anal. **37** (2005), no. 3, 712–736.

[17] Y. Nagase, Y. Tonegawa, *A singular perturbation problem with integral curvature bound*, to appear in Hiroshima Math. J.

[18] M. Röger, R. Schätzle, *On a modified conjecture of De Giorgi*, Math. Z. **254** (2006), no. 4, 675–714.

[19] J. Rubinstein, P. Steinberg, J. B. Keller, *Fast reaction, slow diffusion and curve shotening*, SIAM J. Appl. Math. **49** (1989), no. 1, 116–133.

[20] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[21] H. M. Soner, *Ginzburg-Landau equation and motion by mean curvature, I: convergence*, J. Geom. Anal. **7** (1997), no. 3, 437–475.

[22] Y. Tonegawa, *Phase field model with a variable chemical potential*, Proc. of the Royal Soc. of Edinburgh, Sec A. **132** (2002), no. 4, 993–1019.

[23] Y. Tonegawa, *Integrality of varifolds in the singular limit of reaction-diffusion equations*, Hiroshima Math. J. **33** (2003), no. 3, 323–341.

Division of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan
E-mail address: sato@math.hokudai.ac.jp