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ELLIPTIC DEDEKIND-RADEMACHER SUMS AND TRANSFORMATION FORMULAE OF CERTAIN INFINITE SERIES

TOMOYA MACHIDE

Abstract. We give a transformation formula for certain infinite series in which some elliptic Dedekind-Rademacher sums arise. In the course of its proof, we also obtain a transformation formula for elliptic Dedekind-Rademacher sums. When a complex parameter τ tends to $i\infty$, these represent some classical results which include the reciprocity formula for Apostol-Dedekind sums.

1. Introduction

Let $\tilde{B}_1(x)$ be the first Bernoulli function defined by

$$\tilde{B}_1(x) := \begin{cases} \{x\} - 1/2 & (\text{if } x \text{ is not an integer}), \\ 0 & (\text{if } x \text{ is an integer}). \end{cases}$$

If p and q are relatively prime integers with $q \neq 0$, the Dedekind sum $s(p,q)$ is

$$s(p,q) := \operatorname{sign} q \sum_{j(|q|)} \tilde{B}_1(\frac{j}{q})\tilde{B}_1(\frac{pj}{q}). \tag{1.1}$$

Here $\{x\}$ means the fractional part of a real number x, sign q equals $q/|q|$, and the summation runs through a complete residue system modulo $|q|$. R. Dedekind [De] introduced this sum in connection with the transformation formula for the Dedekind η-function under the group $\mathrm{SL}_2(\mathbb{Z})$ of the two by two matrices with integer entries and determinant one, and deduced from this his reciprocity formula which is a special case of the following (transformation) formula studied in [Ca2, Eq. (4,5)], [Ha, Theorem 2], [HH, Eq. (26)] and [RG]. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$ with $c \neq 0$ and $cp + dq \neq 0$,

$$\begin{aligned} &s(p,q) - s(ap+bq, cp+dq) + s(d,c) \\ &\qquad = -\frac{1}{4}\operatorname{sign} cq(cp+dq) + \frac{1}{12}\Big(\frac{c}{q(cp+dq)} + \frac{cp+dq}{cq} + \frac{q}{(cp+dq)c}\Big). \end{aligned} \tag{1.2}$$

Dedekind's reciprocity formula is the case that $a = d = 0$ and $-b = c = 1$.

The Dedekind sum has been generalized by many authors (see [Bec] and [RG] for good pictures of their generalizations). For any nonnegative integer m with $m \neq 1$, let $\tilde{B}_m(x) := B_m(\{x\})$ denote the m-th Bernoulli function where $B_m(x)$ is the m-th Bernoulli polynomial defined by $\sum_{n=1}^{\infty} \frac{B_n(x)}{n!}X^n = X\mathrm{e}^{xX}/(\mathrm{e}^X - 1)$. If a, b and c are integers with $c \geq 1$, the generalized Dedekind-Rademacher sum is

$$S_{m,n}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} := \sum_{j(c)} \tilde{B}_m\Big(a\frac{j+z}{c} - x\Big)\tilde{B}_n\Big(b\frac{j+z}{c} - y\Big). \tag{1.3}$$

T.M. Apostol [Ap1] showed that the sums $S_{m,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & 0 & 0\end{smallmatrix}\right)$ arise in the transformation formula of certain Lambert series, and L. Carlitz [Ca1] deduced from this a reciprocity relation for the sums $S_{m,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & 0 & 0\end{smallmatrix}\right)$ which includes (1.2) and the reciprocity formula for so-called Apostol-Dedekind sums $S_{1,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & 0 & 0\end{smallmatrix}\right)$. U. Halbritter [Ha] gave a transformation formula for the sums $S_{m,n}\left(\begin{smallmatrix}1 & p & q\\ 0 & x & y\end{smallmatrix}\right)$ which includes them too. It is noted that there are many proofs of the reciprocity formulas for Dedekind sums and Apostol-Dedekind sums (for example, see [RG] for those of Dedekind sums, and see [Ap2, Bec, Ber3, Fu, Ha, HWZ, Mi, NOS, Ta] for those of Apostol-Dedekind sums), and that a special case of Apostol's transformation formula is Ramanujan's formula (cf. [Ber4, p.275, Entry 21(i)]). The monograph [RG] of H. Rademacher and E. Grosswald is the standard reference for Dedekind sums.

T. Arakawa [Ar1, Ar2], B.C. Berndt [Ber1, Ber2, Ber3], S. Iseki [Is] and J. Lewittes [Le] also gave transformation formulas which are considered as generalizations or analogues of Apostol's transformation formula. Arakawa's [Ar1, Ar2] which is deduced from Berndt's [Ber1] uses the infinite series

$$\eta(\alpha, s, y, x) := \sum_{m=1}^{\infty} \frac{\mathrm{e}(mx)}{m^{1-s}} \frac{\mathrm{e}(ym\alpha)}{1-\mathrm{e}(m\alpha)}. \tag{1.4}$$

In their transformation formulas, Berndt let α to be a complex number with positive imaginary part, but Arakawa to be an irrational real algebraic number. If s is an integer with $s \leq -1$, then Arakawa's transformation formula with $x = y = 0$ becomes the same form of Apostol's and consequently yields Carlitz's reciprocity relation.

On the other hand, various elliptic analogues of Dedekind sums have been studied by many people ([As, Ba1, Ba2, Eg, FY, It1, It2, Sc]). Recently the author [Ma] has introduced elliptic analogue of generalized Dedekind-Rademacher sums which we call *elliptic Dedekind-Rademacher sums*, and has obtained their reciprocity formula. His elliptic analogue and the others have had no transformation formula in which themselves appear.

The purpose of this paper is to give a transformation formula for certain infinite series in which not all but some elliptic Dedekind-Rademacher sums arise. Our transformation formula is considered as an elliptic analogue of Arakawa's with $s \in \mathbb{Z}$ and $s \leq -2$, because it reproduces Arakawa's when a parameter τ in the upper half plane tends to $i\infty$, and the infinite series in it satisfy a modular property on the parameter τ under $\mathrm{SL}_2(\mathbb{Z})$. In the course of its proof, we also obtain elliptic generalizations of Carlitz's reciprocity relation and a part of Halbritter's transformation formula.

The paper is organized as follows: We introduce elliptic Dedekind-Rademacher sums and certain infinite series in Section 2 and 3 respectively. In Section 4, we give our transformation formula. Section 5, 6 and 7 devote the proofs of propositions in Section 2, 3 and 4. Section 6 also includes a transformation formula for elliptic Dedekind-Rademacher sums which reproduce a part of Halbritter's transformation formula.

2. Elliptic Dedekind-Rademacher sums

In order to define the elliptic Dedekind-Rademacher sums which arise in our transformation formula, we introduce Kronecker's double series and their generating

function. Let τ be in the upper half plane, and $\mathrm{e}(x) := \mathrm{e}^{2\pi i x}$. If $q = \mathrm{e}(\tau)$, Jacobi's theta function $\theta(x;\tau)$ is defined by

$$\begin{aligned}(2.1)\qquad \theta(x;\tau) &:= \sum_{m\in\mathbb{Z}} \mathrm{e}(\frac{1}{2}(m+\frac{1}{2})^2\tau + (m+\frac{1}{2})(x+\frac{1}{2})) \\ &= iq^{1/8}(\mathrm{e}(\frac{x}{2}) - \mathrm{e}(-\frac{x}{2})) \prod_{m=1}^{\infty}(1-\mathrm{e}(-x)q^m)(1-\mathrm{e}(x)q^m)(1-q^m)\end{aligned}$$

which is an odd and quasi periodic entire function:

(2.2)

$$\theta(-x;\tau) = -\theta(x;\tau), \quad \theta(x+1;\tau) = -\theta(x;\tau), \quad \theta(x+\tau;\tau) = -\mathrm{e}(-\frac{\tau}{2} - x)\theta(x;\tau).$$

Let $\theta'(x;\tau)$ denote the derivative of $\theta(x;\tau)$ with respect to x. For any vector $\vec{x} = (x', x)$ in $\mathbb{R}^2 \setminus \mathbb{Z}^2$, Kronecker's double series $B_m(\vec{x};\tau)$ and their generating function $\underline{F}(\vec{x}; X; \tau)$ are defined by

(2.3)

$$\underline{F}(\vec{x}; X; \tau) := \mathrm{e}(xX)\frac{\theta'(0;\tau)\theta(-x'+x\tau+X;\tau)}{\theta(-x'+x\tau;\tau)\theta(X;\tau)} = \sum_{m=0}^{\infty}\frac{B_m(\vec{x};\tau)}{m!}(2\pi i)^m X^{m-1}.$$

Let $\vec{a}$ be a vector in $\mathbb{Z}^2$. We find from (2.1) that the function $\underline{F}(\vec{x}; X; \tau)$ with respect to X is meromorphic with only simple poles on the lattice $\mathbb{Z}+\tau\mathbb{Z}$, and from (2.2) that

$$(2.4)\qquad \underline{F}(\vec{x}; X+1; \tau) = \mathrm{e}(x)\underline{F}(\vec{x}; X; \tau), \qquad \underline{F}(\vec{x}; X+\tau; \tau) = \mathrm{e}(x')\underline{F}(\vec{x}; X; \tau),$$

$$(2.5)\qquad \underline{F}(-\vec{x}; -X; \tau) = -\underline{F}(\vec{x}; X; \tau), \qquad \underline{F}(\vec{x}+\vec{a}; X; \tau) = \underline{F}(\vec{x}; X; \tau).$$

We deduce from (2.3) and (2.5)

$$(2.6)\qquad B_m(-\vec{x};\tau) = (-1)^m B_m(\vec{x};\tau), \quad B_m(\vec{x}+\vec{a};\tau) = B_m(\vec{x};\tau).$$

Put $q = \mathrm{e}(\tau)$. We also have explicit expressions of Kronecker's double series (see [Ma, Eq. (9)]):

$$\begin{aligned}(2.7)\quad B_m(\vec{x};\tau) = m\Bigg(&\sum_{j=1}^{\infty}(x-j)^{m-1}\frac{\mathrm{e}(-x\tau)q^j}{\mathrm{e}(-x') - \mathrm{e}(-x\tau)q^j} \\ &- \sum_{j=1}^{\infty}(x+j)^{m-1}\frac{\mathrm{e}(x\tau)q^j}{\mathrm{e}(x') - \mathrm{e}(x\tau)q^j} + x^{m-1}\frac{\mathrm{e}(-x'+x\tau)}{\mathrm{e}(-x'+x\tau)-1}\Bigg) + B_m(x).\end{aligned}$$

So $B_1(\vec{x};\tau)$ and $B_2(\vec{x};\tau)$ are continuous at $\vec{x} \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ and discontinuous at $\vec{x} \in \mathbb{Z}^2$, but the others are continuous at every $\vec{x} \in \mathbb{R}^2$.

Let us define the elliptic Dedekind-Rademacher sums. Let r be a rational number. There is a unique pare $(n(r), d(r))$ of integers with $\gcd(n(r), d(r)) = 1, d(r) \geq 1$ and $r = n(r)/d(r)$. Let $\mathrm{M}_2(\mathbb{R})$ be the set of the two by two matrices with real number entries, and $\mathrm{SM}_2(r)$ be its subset defined as follows:

$$(2.8)\qquad \mathrm{SM}_2(r) := \left\{ \begin{pmatrix} x' & x \\ y' & y \end{pmatrix} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \middle| \vec{y}, n(r)\vec{y} - d(r)\vec{x} \notin \mathbb{Z}^2 \right\},$$

where $\vec{x}$ and $\vec{y}$ denote the vectors (x', x) and (y', y) respectively. If $\Xi = \left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right)$ is in $\mathrm{SM}_2(r)$, then the elliptic Dedekind-Rademacher sum is defined by

$$S^{\tau}_{m,n}(\Xi; r) := \frac{1}{d(r)} \sum_{j', j(d(r))} B_m\Big(\frac{\vec{j}+\vec{y}}{d(r)}; \tau\Big) B_n\Big(n(r)\frac{\vec{j}+\vec{y}}{d(r)} - \vec{x}; \tau\Big), \tag{2.9}$$

where $\vec{j}$ denotes the vector (j', j). By continuity of Kronecker's double series, the elliptic Dedekind-Rademacher sum $S^{\tau}_{m,n}(\Xi; r)$ is continuous at $\Xi \in \mathrm{SM}_2(r)$. In particular, if $m, n \geq 3$, $S^{\tau}_{m,n}(\Xi; r)$ is continuous at every $\Xi \in \mathrm{M}_2(\mathbb{R})$. The relation between this sum and the sum $S^{\tau}_{m,n}\left(\begin{smallmatrix}(a',a) & (b',b) & (c',c)\\ (x',x) & (y',y) & (z',z)\end{smallmatrix}\right)$ defined in [Ma] is

$$S^{\tau}_{m,n}(\Xi; r) = S^{\tau}_{m,n}\begin{pmatrix}(1,1) & (n(r),n(r)) & (d(r),d(r))\\ (0,0) & \vec{x} & \vec{y}\end{pmatrix}. \tag{2.10}$$

It is noted that the sums $S^{\tau}_{m,n}\left(\begin{smallmatrix}(1,1) & (b',b) & (c',c)\\ (0,0) & \vec{x} & \vec{y}\end{smallmatrix}\right)$ with $b' \neq b$ or $c' \neq c$ can not be written in terms of $S^{\tau}_{m,n}(\Xi; r)$, or do not appear in our transformation formula.

To give some properties of the elliptic Dedekind-Rademacher sums, the cotangent sum studied by U. Dieter [Di] is introduced:

$$\mathfrak{c}(a, b, c; x, y, z) := \frac{1}{c} {\sum_{j(c)}}' \cot\Big(\pi(a\frac{j+z}{c} - x)\Big) \cot\Big(\pi(b\frac{j+z}{c} - y)\Big),$$

where the prime of the summation means excluding j such that $a\frac{j+z}{c} - x \in \mathbb{Z}$ or $b\frac{j+z}{c} - y \in \mathbb{Z}$. For convenience, we shall use

$$S_{m,n}\left(\begin{smallmatrix}x\\ y\end{smallmatrix}; r\right) = S_{m,n}\left(\begin{smallmatrix}1 & n(r) & d(r)\\ 0 & x & y\end{smallmatrix}\right), \quad \mathfrak{c}\left(\begin{smallmatrix}x\\ y\end{smallmatrix}; r\right) = \mathfrak{c}(1, n(r), d(r); 0, x, y).$$

If $V = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ is a two by two matrix, let $V\tau$ and $j(V;\tau)$ denote $(a\tau+b)/(c\tau+d)$ and $c\tau+d$ respectively. The elliptic Dedekind-Rademacher sum $S^{\tau}_{m,n}(\Xi; r)$ has a modular property, and reproduces the sum $S_{m,n}\left(\begin{smallmatrix}x\\ y\end{smallmatrix}; r\right)$ when τ tends to $i\infty$. We shall prove these properties in Section 7.

PROPOSITION 2.1. *Let r be a rational number, and $\Xi = \begin{pmatrix}x' & x\\ y' & y\end{pmatrix}$ be a matrix in $SM_2(r)$.*
(i) *If V is in $SL_2(\mathbb{Z})$, then*

$$S^{\tau}_{m,n}(\Xi; r) = \frac{1}{j(V;\tau)^{m+n}} S^{V\tau}_{m,n}(\Xi\ {}^tV; r), \tag{2.11}$$

where ${}^tV$ means the transpose of the matrix V. In particular, $S^{\tau}_{m,n}\left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}; r\right)$ is a modular form of weight $m+n$ if $m, n \geq 3$.
(ii) *Let* Re z *denote the real part of a complex number z. Then we have*

$$\mathrm{Re}\lim_{\tau\to i\infty} S^{\tau}_{m,n}(\Xi; r) = \begin{cases} S_{m,n}\left(\begin{smallmatrix}x\\ y\end{smallmatrix}; r\right) - \frac{1}{4}\mathfrak{c}\left(\begin{smallmatrix}x'\\ y'\end{smallmatrix}; r\right) & (m = n = 1, x, y \in \mathbb{Z}),\\ S_{m,n}\left(\begin{smallmatrix}x\\ y\end{smallmatrix}; r\right) & (\textit{otherwise}).\end{cases} \tag{2.12}$$

3. Certain infinite series

Let α be an irrational real algebraic number and l an integer with $l \geq 3$. If $\Xi = \left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right) = \left(\begin{smallmatrix}x' & x\\ y' & y\end{smallmatrix}\right)$ is in $\mathrm{M}_2(\mathbb{R})$ with $\vec{y} \notin \mathbb{Z}^2$, the certain infinite series which appears

in our transformation formula is defined by

$$H_l^{\tau}(\Xi;\alpha) := (-1)^l \frac{l!}{(2\pi i)^{l+1}} \sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \underline{F}(-\vec{y};\alpha(\tau m'+m);\tau). \tag{3.1}$$

Here the summation ranges over all elements in $\mathbb{Z}^2$ with $(m',m)\neq(0,0)$, $\vec{m}$ denotes the vector (m',m), and $\vec{m}\cdot\vec{x}$ means the inner product of $\vec{m}$ and $\vec{x}$, or $m'x'+mx$. By the following lemma, the series in (3.1) absolutely converges, that is, $H_l^{\tau}(\Xi;\alpha)$ is a continuous function at $\vec{x}\in\mathbb{R}^2$ and $\vec{y}\in\mathbb{R}^2\setminus\mathbb{Z}^2$.

LEMMA 3.1. *Let s be a complex number, and λ',λ nonnegative real numbers. The series $\sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})m'^{\lambda'}m^{\lambda}}{(\tau m'+m)^s}\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)$ absolutely converges if $\mathrm{Re}\ s > \lambda'+\lambda+2$.*

Proof. Let $I = \sum_{m',m}{}' \Big|\frac{\mathrm{e}(\vec{m}\cdot\vec{x})m'^{\lambda'}m^{\lambda}}{(\tau m'+m)^s}\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)\Big|$, and $\vec{x},\vec{y},\tau$ be fixed. Since the function $\underline{F}(\vec{x};X;\tau)$ with respect to X is meromorphic with only simple poles on the lattice $\mathbb{Z}+\tau\mathbb{Z}$, there is a positive real number $C=C(\vec{y},\tau)$ such that if $X=\tau\xi'+\xi$ is a complex number with $-1/2\le\xi',\xi\le 1/2$, then

$$|X\underline{F}(\vec{y};X;\tau)|\le C.$$

For any real number x, let $[[x]]$ and $\langle\langle x\rangle\rangle$ respectively denote the integer and the real number with $-1/2<\langle\langle x\rangle\rangle\le 1/2$, $x=[[x]]+\langle\langle x\rangle\rangle$. By (2.4),

$$\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau) = \mathrm{e}(-y'[[\alpha m']]-y[[\alpha m]])\underline{F}(-\vec{y};\tau\langle\langle\alpha m'\rangle\rangle+\langle\langle\alpha m\rangle\rangle;\tau),$$

thus we have

$$I \le C\sum_{m',m}{}' \frac{|m'|^{\lambda'}|m|^{\lambda}}{|\tau m'+m|^s}\frac{1}{|\tau\langle\langle\alpha m'\rangle\rangle+\langle\langle\alpha m\rangle\rangle|}.$$

If X' and X are real numbers, some calculations show that

$$\begin{aligned} &|\tau X'+X| \ge \frac{\mathrm{Im}\ \tau}{|\tau|}|X|, \quad |\tau X'+X| \ge |\tau|^2 \mathrm{Im}\ (-\frac{1}{\tau})\ |X'|, \\ &|\tau X'+X|^2 \ge 2(|\tau|-|\mathrm{Re}\ \tau|)|X'X|. \end{aligned} \tag{3.2}$$

Hence, there is a positive real number $D=D(\vec{y},\tau)$ such that the following inequality holds.

$$\begin{aligned} I \le D\Big(&\sum_{m'}{}'\sum_{m}{}' \frac{1}{|m'|^{(s-\lambda'-\lambda)/2}|m|^{(s-\lambda'-\lambda)/2}|\langle\langle\alpha m'\rangle\rangle|^{1/2}|\langle\langle\alpha m\rangle\rangle|^{1/2}} \\ &+\frac{1}{|\tau|^{s-\lambda'-\lambda+1}}\sum_{m'}{}'\frac{1}{|m'|^{s-\lambda'-\lambda}|\langle\langle\alpha m'\rangle\rangle|}+\sum_{m}{}'\frac{1}{|m|^{s-\lambda'-\lambda}|\langle\langle\alpha m\rangle\rangle|}\Big), \end{aligned}$$

where the summation $\sum_m{}'$ ranges over all integers except zero. Since it was obtained in the proof of [Ar1, Lemma 1] that the series $\sum_{m=1}^{\infty}\frac{1}{m^{1+\epsilon}|\langle\langle\alpha m\rangle\rangle|}$ converges if $\epsilon>0$, we complete the proof. □

Let us give some properties of the infinite series $H_l^{\tau}(\vec{x},\vec{y};\alpha)$. For any real number x, let $\langle x\rangle$ be $\{x\}$ if x is not an integer, and 1 otherwise. By using Berndt's results

[Ber1, Ber3], Arakawa [Ar1, Ar2] gave his transformation formula for the infinite series defined by

$$H(\alpha,s,y,x):=\eta(\alpha,s,\langle y\rangle,x)+\mathrm{e}(s/2)\eta(\alpha,s,\langle -y\rangle,-x). \tag{3.3}$$

In particular, if l is an integer with $l\geq 2$,

$$H(\alpha,-l+1,y,x)=\begin{cases}\dfrac{i}{2}\Big(\xi_l(x;\alpha)+i\sum\limits_{m=1}^{\infty}\dfrac{\mathrm{e}(mx)+(-1)^{l-1}\mathrm{e}(-mx)}{m^l}\Big) & (y\in\mathbb{Z}),\\ \dfrac{i}{2}\Big(\xi_l(x+\{y\}\alpha;\alpha)-i{\sum_m}'\dfrac{\mathrm{e}(m(x+\{y\}\alpha))}{m^l}\Big) & (y\notin\mathbb{Z}),\end{cases} \tag{3.4}$$

where $\xi_l(x;\alpha):={\sum_m}'\dfrac{\mathrm{e}(mx)}{m^l}\cot\pi m\alpha$. In analogy to Proposition 2.1, the infinite series $H_l^{\tau}(\Xi;\alpha)$ has a modular property, and reproduces $H(\alpha,-l+1,-y,x)$ when τ tends to $i\infty$. We shall prove these properties in Section 7 too.

PROPOSITION 3.2. *Let α be an irrational real algebraic number, l an integer with $l\geq 3$, and $\Xi=\begin{pmatrix}x' & x\\ y' & y\end{pmatrix}$ be a matrix in $M_2(\mathbb{R})$ with $(y',y)\notin\mathbb{Z}^2$.*

(i) *If V is in $SL_2(\mathbb{Z})$, then*

$$H_l^{\tau}(\Xi;\alpha)=\frac{1}{j(V;\tau)^{l+1}}H_l^{V\tau}(\Xi\ {}^tV;\alpha). \tag{3.5}$$

(ii) *Set $CL(x)=\dfrac{1}{2}\sum\limits_{m=1}^{\infty}\dfrac{\mathrm{e}(mx)+(-1)^{l-1}\mathrm{e}(-mx)}{m^l}$. Then we have*

$$\operatorname*{Re}\lim_{\tau\to i\infty}H_l^{\tau}(\Xi;\alpha)=\begin{cases}(-1)^{l+1}\dfrac{l!}{(2\pi i)^l}\Big(H(\alpha,-l+1,0,x)+CL(x)\Big) & (y\in\mathbb{Z}),\\ (-1)^{l+1}\dfrac{l!}{(2\pi i)^l}H(\alpha,-l+1,-y,x) & (y\notin\mathbb{Z}).\end{cases} \tag{3.6}$$

4. Transformation formula

In this section, we give a transformation formula for the infinite series $H_l^{\tau}(\Xi;\alpha)$ in which the elliptic Dedekind-Rademacher sums $S_{m,n}^{\tau}(\Xi;\alpha)$ arise. For any matrix $V=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$, let $\mathrm{SM}_2(V)$ be the subset of $\mathrm{M}_2(\mathbb{R})$ such that

$$\mathrm{SM}_2(V):=\left\{\begin{pmatrix}x' & x\\ y' & y\end{pmatrix}=\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\in\mathrm{M}_2(\mathbb{R})\,\middle|\,\vec{y},d\vec{y}+c\vec{x}\notin\mathbb{Z}^2\right\}. \tag{4.1}$$

It is noted that we use the same symbol SM_2 for a rational number r and for a matrix V in $\mathrm{SL}_2(\mathbb{Z})$ because $\mathrm{SM}_2(r)$ and $\mathrm{SM}_2(V)$ have a close relation, i.e., $\mathrm{SM}_2(V)=\mathrm{SM}_2(-d/c)$ when $c\neq 0$. If $\Xi=\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right)$ is in $\mathrm{SM}_2(V)$, c is positive, and z is a complex number, then a sum of elliptic Dedekind-Rademacher sums is defined by

$$R_l^{\tau}(\Xi;z;V):=\begin{cases}\dfrac{(-1)^l}{l+1}\sum\limits_{k=-1}^{l}\binom{l+1}{k+1}\big(-j(V,z)\big)^k S_{k+1,l-k}^{\tau}\left(\begin{matrix}-\vec{x}\\ \vec{y}\end{matrix};\frac{d}{c}\right) & (c>0),\\ 0 & (c=0).\end{cases} \tag{4.2}$$

When c is negative, $R_l^\tau(\Xi; z; V)$ also denotes $R_l^\tau(\Xi; z; -V)$. It is a continuous function at $\Xi \in \mathrm{SM}_2(V)$ by virtue of continuity of elliptic Dedekind-Rademacher sums. The transformation formula is

THEOREM 4.1. *Let α be an irrational real algebraic number, l an integer with $l \geq 3$, and V a matrix in $SL_2(\mathbb{Z})$. If Ξ is a matrix in $SM_2(V)$, then*

$$H_l^\tau(\Xi; \alpha) - j(V; \alpha)^{l-1} H_l^\tau(V\Xi; V\alpha) = R_l^\tau(\Xi; \alpha; V). \tag{4.3}$$

To give its proof, we need the following two lemmas which shall be proved in Section 5 and 6.

LEMMA 4.2. *Let T and S denote the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively. If $V \in \{T^{\pm}, S\}$ and $\Xi \in SM_2(V)$, then (4.3) holds.*

LEMMA 4.3. *Let z be a complex number, l a positive integer, and V, V_1, V_2 matrices in $SL_2(\mathbb{Z})$ with $V = V_2V_1$. If Ξ is in $SM_2(V) \bigcap SM_2(V_1)$, then*

$$R_l^\tau(\Xi; z; V_1) + j(V_1, z)^{l-1} R_l^\tau(V_1\Xi; V_1 z; V_2) = R_l^\tau(\Xi; z; V). \tag{4.4}$$

Proof of Theorem 4.1. Put $G = \{T^{\pm}, S\}$. For any positive integer n, let U_n be a subset of $\mathrm{SL}_2(\mathbb{Z})$ such that if $V \in U_n$, there are n matrices $V_1, \ldots, V_n$ in G with $V = V_nV_{n-1}\cdots V_1$. If V is a matrix in $\mathrm{SL}_2(\mathbb{Z})$, there is a positive integer n with $V \in U_n$ by [Ap3, Theorem 2.1]. We will prove (4.3) by induction on n. If n is one, the claim has been shown in Lemma 4.2. Suppose that it is true when $V \in U_{n-1}$. Let V be a matrix in U_n. Then there are two matrices V_1 and V_2 such that $V_1 \in G, V_2 \in U_{n-1}$ and $V = V_2V_1$. Since $j(V; \alpha) = j(V_1; \alpha)j(V_2; V_1\alpha)$, the left hand side of (4.3) equals

$$H_l^\tau(\Xi; \alpha) - j(V_1; \alpha)^{l-1} H_l^\tau(V_1\Xi; V_1\alpha) + \\ j(V_1; \alpha)^{l-1}\Big(H_l^\tau(V_1\Xi; V_1\alpha) - j(V_2; V_1\alpha)^{l-1} H_l^\tau(V_2(V_1\Xi); V_2(V_1\alpha))\Big).$$

Thus we obtain (4.3) with $\Xi \in \mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1)$ by virtue of Lemma 4.3 and the induction hypothesis. The set $\mathrm{SM}_2(V) \bigcap \mathrm{SM}_2(V_1)$ is dense in $\mathrm{M}_2(\mathbb{R})$, so we also get (4.3) with $\Xi \in \mathrm{SM}_2(V)$ by continuity of $H_l^\tau(\Xi; \alpha)$ and $R_l^\tau(\Xi; \alpha; V)$, which completes the proof. □

REMARK 4.4. When τ tends to $i\infty$, $R_l^\tau(\Xi; z; V)$ represents $f_{l+1}(-d, c; z)$ in [Ca2, Eq. (1.2)], that is, (4.4) is an elliptic generalization of Carlitz's reciprocity relation [Ca2, Eq. (1.11)].

By (2.12) and (3.6), we can reproduce a part of Arakawa's transformation formula [Ar1, Ar2]. Since it is not a new result, we do not give the formula here.

Motivated by Knopp's identity [Kn], J.A. Parson and K.H. Rosen [PR] gave a formula for the sums $S_{m,n}\left(\begin{smallmatrix} 1 & p & q \\ 0 & 0 & 0 \end{smallmatrix}\right)$ by looking at the action of the Hecke operators on certain Lambert series studied by Apostol [Ap1] together with the transformation formula for these series. Aarakawa [Ar2] also obtained an identity of Knopp type in a different way from theirs. For the identity, he not only considered the Hecke operators on the Dirichlet series $\xi(s, \alpha) = \sum_{m=1}^{\infty} \dfrac{\cot \pi m\alpha}{m^s}$ with the transformation formula for this, but also used the fact that the function $\xi(s, \alpha)$ for a real quadratic number α can be analytically continued to a meromorphic function of s in the whole

complex plane. It is noted that the Dirichlet series $\xi(s,\alpha)$ relates to the infinite series $H(\alpha,1-s,1,0)$ (see [Ar2, Eq. (2.2)]).

$$\xi(s;\alpha)=-2i\Big(\frac{1}{1+\mathrm{e}((1-s)/2)}H(\alpha,1-s,1,0)+\frac{1}{2}\zeta(s)\Big), \tag{4.5}$$

where $\zeta(s)$ denotes the Riemann zeta function. We will end the section by posing three questions.

Questions (i) Can we construct Hecke operators on the series $H_l^{\tau}(\Xi;\alpha)$, and give elliptic analogues of identities of Knopp, Parson and Rosen by using the action of the Hecke operators together with the transformation formula for the series?

(ii) If α is a real quadratic number, can we analytically continue the series

$$\sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^s}\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)$$

to a meromorphic function of s in the whole complex plane, and produce an elliptic analogue of Arakawa's identity in a similar way of his?

(iii) Not all elliptic Dedekind-Rademacher sums appear in our transformation formula. Is there a transformation formula in which the others arise?

5. Proof of Lemma 4.2

We give a proof of Lemma 4.2. Its method is based on Siegel's idea [Si] (see [Ka]). Set

$$\sum_{m',m}{}_e=\lim_{M\to\infty}\sum_{m'=-M}^{M}\sum_{m=-M}^{M},\qquad \sum_{m',m}{}_e'=\lim_{M\to\infty}\sum_{m'=-M}^{M}\sum_{m=-M}^{M}{}_{(m',m)\neq(0,0)}.$$

In order to prove Lemma 4.2, we need the following equation.

PROPOSITION 5.1. *Let α be an irrational real algebraic number, and s be a complex number with* Re $s>3$. *If $x',x,y',y\in\mathbb{R}\setminus\mathbb{Z}$, then*

$$\Big(\sum_{m',m}{}_e'\sum_{n',n}{}_e'-\sum_{n',n}{}_e'\sum_{m',m}{}_e'\Big)\frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^s}\frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{\alpha(\tau m'+m)+\tau n'+n}=0, \tag{5.1}$$

or the order of (m',m) and (n',n) in the sum can be changed. Here $\vec{x},\vec{y},\vec{m}$ and $\vec{n}$ denote the vectors $(x',x),(y',y),(m',m)$ and (n',n) respectively.

We give the proof of Lemma 4.2 before showing (5.1).

Proof of Lemma 4.2. Let V be in $\{T^{\pm},S\}$ and $\left(\begin{smallmatrix}x' & x\\ y' & y\end{smallmatrix}\right)=\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right)$ in $\mathrm{SM}_2(V)$. It follows from (2.4) that

$$H_l^{\tau}\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix};\alpha\right)=H_l^{\tau}\left(\begin{smallmatrix}\vec{x}+\vec{y}\\ \vec{y}\end{smallmatrix};\alpha+1\right),$$

thus we obtain (4.3) when $V=T^{\pm}$. We will show (4.3) with $V=S$. Let l be an integer with $l\geq 4$, and $x',x,y',y\in\mathbb{R}\setminus\mathbb{Z}$. Since $Y^l-X^l=(Y-X)\sum_{k=0}^{l-1}X^kY^{l-1-k}$, we have

$$\frac{1}{X^l}\frac{1}{\alpha X+Y}-\alpha^{l-1}\frac{1}{(-Y)^l}\frac{1}{\frac{1}{\alpha}Y+X}=\sum_{k=0}^{l-1}(-\alpha)^k\frac{1}{Y^{k+1}}\frac{1}{X^{l-k}}.$$

By substituting $\tau m'+m$ and $\tau n'+n$ for X and Y respectively, we get

$$\sum_{n',n}{}'_e \sum_{m',m}{}'_e \Big(\frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{\alpha(\tau m'+m)+\tau n'+n} - \alpha^{l-1}\frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{(-\tau n'-n)^l} \times \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{\frac{1}{\alpha}(\tau n'+n)+\tau m'+m}\Big) = \sum_{k=0}^{l-1}(-\alpha)^k \sum_{n',n}{}'_e \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{(\tau n'+n)^{k+1}} \sum_{m',m}{}'_e \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^{l-k}}. \tag{5.2}$$

On the other hand, L. Kronecker [Kr] showed that

$$\underline{F}(\vec{x};X;\tau) = \sum_{m',m}{}_e \frac{\mathrm{e}(-\vec{m}\cdot\vec{x})}{X+\tau m'+m}, \quad B_k(\vec{x};\tau) = -\frac{k!}{(2\pi i)^k}\sum_{m',m}{}'_e \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^k}.$$

Thus, by (5.1), the left hand side of (5.2) equals

$$\sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l}\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau) + \frac{1}{\alpha}\frac{(2\pi i)^{l+1}}{(l+1)!}B_{l+1}(\vec{x};\tau)$$
$$-\alpha^{l-1}\Big(\sum_{n',n}{}'_e \frac{\mathrm{e}(-\vec{n}\cdot\vec{y})}{(\tau n'+n)^l}\underline{F}(-\vec{x};-\frac{1}{\alpha}(\tau n'+n);\tau) + (-1)^l\alpha\frac{(2\pi i)^{l+1}}{(l+1)!}B_{l+1}(\vec{y};\tau)\Big),$$

and the right hand side of (5.2) equals

$$\frac{(2\pi i)^{l+1}}{(l+1)!}\sum_{k=0}^{l-1}(-\alpha)^k\binom{l+1}{k+1}B_{k+1}(\vec{y};\tau)B_{l-k}(\vec{x};\tau).$$

These yield (4.3) with $l\geq 4$ and $V=S$ since the set $\big\{\left(\begin{smallmatrix}x' & x\\ y' & y\end{smallmatrix}\right)\big| x',x,y',y\in\mathbb{R}\setminus\mathbb{Z}\big\}$ is dense in $\mathrm{M}_2(\mathbb{R})$. In order to get (4.3) with $l=3$ and $V=S$, we introduce differential equations for Kronecker's double series and their generating function.

$$(\tau\frac{\partial}{\partial x'}+\frac{\partial}{\partial x})\underline{F}(\vec{x};X;\tau) = 2\pi i X\underline{F}(\vec{x};X;\tau),$$
$$(\tau\frac{\partial}{\partial x'}+\frac{\partial}{\partial x})B_m(\vec{x};\tau) = mB_{m-1}(\vec{x};\tau).$$

If $\mathrm{Re}\ s>3$ and $z\in\{x',x,y',y\}$, we see from Lemma 3.1 that the series

$$\sum_{m',m}{}'\frac{\partial}{\partial z}\Big(\frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^s}\underline{F}(-\vec{y};\alpha(\tau m'+m);\tau)\Big)$$

absolutely converges and its term wise differentiation is possible. So, by applying the differential operator $\tau\dfrac{\partial}{\partial x'}+\dfrac{\partial}{\partial x}$ to (4.3) with $l=4$, we obtain (4.3) with $l=3$. □

To give a proof of Proposition 5.1, the following lemma is introduced.

LEMMA 5.2. *Let y',y,λ be real numbers, and N_1',N_1,N_2',N_2,m',m integers. If $y',y\notin\mathbb{Z},\lambda>0,N_1'\leq N_2',N_1\leq N_2$ and $(m',m)\neq(0,0)$, then there is a positive real number $D=D(y',y,\alpha,\lambda,\tau)$ such that*

$$\Big|\sum_{n'=N_1'}^{N_2'}\sum_{n=N_1}^{N_2}\frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{\alpha(\tau m'+m)+\tau n'+n}\Big| \leq D\Big(|\tau m'+m|^{\lambda+1}+1\Big). \tag{5.3}$$

Proof. First, we verify (5.3) when $-\alpha m' \notin [N_1', N_2']$ and $-\alpha m \notin [N_1, N_2]$. Set $z = \alpha(\tau m' + m)$ and

$$a'_{n'} = \frac{\mathrm{e}(n'y')}{\mathrm{e}(y') - 1}, \quad a_n = \frac{\mathrm{e}(ny)}{\mathrm{e}(y) - 1}, \quad f_{n',n} = z + \tau n' + n.$$

Since $\mathrm{e}(n'y') = a'_{n'+1} - a'_{n'}$ and $\mathrm{e}(ny) = a_{n+1} - a_n$, we find that

$$\begin{aligned}(5.4)\quad \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1}^{N_2} \frac{\mathrm{e}(\vec{n}\cdot\vec{y})}{z+\tau n'+n} &= \sum_{n'=N_1'+1}^{N_2'+1} \sum_{n=N_1+1}^{N_2+1} \frac{a'_{n'}a_n}{f_{n'-1,n-1}} \\ &- \sum_{n'=N_1'+1}^{N_2'+1} \sum_{n=N_1}^{N_2} \frac{a'_{n'}a_n}{f_{n'-1,n}} - \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1+1}^{N_2+1} \frac{a'_{n'}a_n}{f_{n',n-1}} + \sum_{n'=N_1'}^{N_2'} \sum_{n=N_1}^{N_2} \frac{a'_{n'}a_n}{f_{n',n}}.\end{aligned}$$

If $\xi', \xi, \xi_1', \xi_1, \xi_2'$ and ξ_2 are real numbers with $\xi_1' \le \xi_2', \xi_1 \le \xi_2, -\alpha m' \notin [\xi_1', \xi_2']$ and $-\alpha m \notin [\xi_1, \xi_2]$, then we have

(5.5)

$$\begin{aligned}&\frac{1}{f_{\xi_1',\xi}} - \frac{1}{f_{\xi_2',\xi}} = \tau \int_{\xi_1'}^{\xi_2'} \frac{d\mu'}{(z+\tau\mu'+\xi)^2}, \quad \frac{1}{f_{\xi',\xi_1}} - \frac{1}{f_{\xi',\xi_2}} = \int_{\xi_1}^{\xi_2} \frac{d\mu}{(z+\tau\xi'+\mu)^2}, \\ &\frac{1}{f_{\xi_1',\xi_1}} - \frac{1}{f_{\xi_1',\xi_2}} - \frac{1}{f_{\xi_2',\xi_1}} + \frac{1}{f_{\xi_2',\xi_2}} = 2\tau \int_{\xi_1'}^{\xi_2'} \int_{\xi_1}^{\xi_2} \frac{d\mu' d\mu}{(z+\tau\mu'+\mu)^3}.\end{aligned}$$

Let $A = A(\vec{y}, \tau)$ denote $(1+2|\tau|)/(|\mathrm{e}(y')-1||\mathrm{e}(y)-1|)$, and I be the left hand side of (5.3). It follows from (5.4) and (5.5) that

$$\begin{aligned}(5.6)\quad I \le A\Big(&\int_{N_1'}^{N_2'} \int_{N_1}^{N_2} \frac{d\mu' d\mu}{|z+\tau\mu'+\mu|^3} + \int_{N_1'}^{N_2'} \frac{d\mu'}{|z+\tau\mu'+N_1|^2} \\ &+ \int_{N_1'}^{N_2'} \frac{d\mu'}{|z+\tau\mu'+N_2|^2} + \int_{N_1}^{N_2} \frac{d\mu}{|z+\tau N_1'+\mu|^2} + \int_{N_1}^{N_2} \frac{d\mu}{|z+\tau N_2'+\mu|^2} \\ &+ \frac{1}{|f_{N_1',N_1}|} + \frac{1}{|f_{N_1',N_2}|} + \frac{1}{|f_{N_2',N_1}|} + \frac{1}{|f_{N_2',N_2}|}\Big).\end{aligned}$$

Put $t' = \alpha m'$ and $t = \alpha m$. Since

$$\begin{aligned}&\int_{N_1'}^{N_2'} \frac{d\mu'}{|z+\tau\mu'+N|^2} = \frac{1}{|\tau|^2 |\mathrm{Im}\ \frac{1}{\tau}||t+N|} \left[\arctan \frac{t'+\mu'+\mathrm{Re}\ (\frac{1}{\tau})(t+N)}{|\mathrm{Im}\ \frac{1}{\tau}||t+N|}\right]_{\mu'=N_1'}^{\mu'=N_2'}, \\ &\int_{N_1}^{N_2} \frac{d\mu}{|z+\tau N'+\mu|^2} = \frac{1}{|\mathrm{Im}\ \tau||t'+N'|} \left[\arctan \frac{t+\mu+\mathrm{Re}\ (\tau)(t'+N')}{|\mathrm{Im}\ \tau||t'+N'|}\right]_{\mu=N_1}^{\mu=N_2},\end{aligned}$$

we can find from (3.2) and (5.6) that there is a positive real number $B = B(\vec{y}, \tau)$ with

$$\begin{aligned}I \le B\Big(&\Big(\frac{1}{|\alpha m'+N_1'|^{1/2}} + \frac{1}{|\alpha m'+N_2'|^{1/2}}\Big)\Big(\frac{1}{|\alpha m+N_1|^{1/2}} + \frac{1}{|\alpha m+N_2|^{1/2}}\Big) \\ &+ \frac{1}{|\alpha m'+N_1'|} + \frac{1}{|\alpha m'+N_2'|} + \frac{1}{|\alpha m+N_1|} + \frac{1}{|\alpha m+N_2|}\Big).\end{aligned}$$

On the other hand, by the theorem of Thue-Siegel-Roth in the diophantine approximation theory, there is a positive real number $C = C(\alpha, \lambda)$ such that

$$\Big|\alpha - \frac{k}{l}\Big| > \frac{1}{Cl^{\lambda+2}} \quad (k, l \in \mathbb{Z},\ l > 0).$$

It can be assumed that C is more than one. We deduce from this

$$\frac{1}{|\alpha l + k|} < C(|l|^{\lambda+1} + 1) \le C(|l|^{(\lambda+1)/2} + 1)^2 \quad (k, l \in \mathbb{Z},\ (k, l) \ne (0, 0)).$$

Thus we have

$$I \le 8BC\Big(|m'|^{(\lambda+1)/2}|m|^{(\lambda+1)/2} + |m'|^{\lambda+1} + |m|^{\lambda+1} + 1\Big)$$

which together with (3.2) implies (5.3) with $-\alpha m' \notin [N_1', N_2']$ and $-\alpha m \notin [N_1, N_2]$. It is remarked that if $m' = N_1' = N_2' = 0$ or $m = N_1 = N_2 = 0$, one can also prove (5.3) in a similar way.

In order to complete the proof, we consider the case when $-\alpha m' \in [N_1', N_2']$ and $-\alpha m \notin [N_1, N_2]$. Let $[x]$ denote the integer part of a real number x, i.e., $x = [x] + \{x\}$. If $m' \ne 0$, then $-\alpha m' \notin [N_1', [-\alpha m']]$ and $-\alpha m' \notin [[-\alpha m'] + 1, N_2']$, and if $m' = 0$, then $-\alpha m' \notin [N_1', -1], -\alpha m' \notin [1, N_2']$ and $-\alpha m' \in [0, 0]$. Thus this case is reduced to the case studied above, and verified. The other cases are also proved the same way. □

We are in a position to prove Proposition 5.1 now.

Proof of Proposition 5.1. Since $\mathrm{Re}\ s > 3$, there is positive real numbers λ, s_0 with $\mathrm{Re}\ s - \lambda - 1 > s_0 > 2$. Let ϵ be an arbitrary positive real number. Since the series $\sum_{m',m}{}' \frac{1}{|\tau m' + m|^{s_0}}$ and $\sum_{n',n}{}_e' \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\xi + \tau n' + n}$ converge, there are positive integers L and M such that if $|m_0'|, |m_0| \le M, (m_0', m_0) \ne (0, 0)$ and $N \ge L$, then

$$\sum_{\substack{m' \ge M \\ or\ \ m \ge M}} \frac{1}{|\tau m' + m|^{s_0}} < \epsilon, \quad \Big|\sum_{\substack{|n'| \ge N \\ or\ \ |n| \ge N}} \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\alpha(\tau m_0' + m_0) + \tau n' + n}\Big| < \epsilon.$$

A calculation shows that

$$\begin{aligned}
&\Big|\Big(\sum_{m',m}{}_e' \sum_{n',n}{}_e' - \sum_{\substack{|n'|<N \\ and\ \ |n|<N}}{}' \sum_{m',m}{}_e'\Big) \frac{\mathrm{e}(\vec{m} \cdot \vec{x})}{(\tau m' + m)^s} \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\alpha(\tau m' + m) + \tau n' + n}\Big| \\
&\qquad = \Big|\Big(\sum_{\substack{|m'| \ge M \\ or\ \ |m| \ge M}} + \sum_{\substack{|m'|<M \\ and\ \ |m|<M}}{}'\Big) \sum_{\substack{|n'| \ge N \\ or\ \ |n| \ge N}} \frac{\mathrm{e}(\vec{m} \cdot \vec{x})}{(\tau m' + m)^s} \frac{\mathrm{e}(\vec{n} \cdot \vec{y})}{\alpha(\tau m' + m) + \tau n' + n}\Big|.
\end{aligned}$$

Let I be the left hand side of the above equation. By (5.3), we get

$$\begin{aligned}
I &\le D \sum_{\substack{|m'| \ge M \\ or\ \ |m| \ge M}} \frac{1}{|\tau m' + m|^s}(|\tau m' + m|^{\lambda+1} + 1) + \epsilon \sum_{\substack{|m'|<M \\ and\ \ |m|<M}}{}' \frac{1}{|\tau m' + m|^s} \\
&\le \epsilon\Big(2D + \sum_{m',m}{}' \frac{1}{|\tau m' + m|^s}\Big),
\end{aligned}$$

which completes the proof. □

6. Proof of Lemma 4.3 and transformation formulae of elliptic Dedekind-Rademacher sums

In order to prove Lemma 4.3, we give the following transformation formula for the elliptic Dedekind-Rademacher sums $S^{\tau}_{1,l}(\Xi;r)$ which reproduces a part of Halbritter's result [Ha, Theorem 2].

THEOREM 6.1. *Let $r=n(r)/d(r)$ be a rational number, and l a positive integer. If $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $SL_2(\mathbb{Z})$ with $j(V;r)\neq 0$ and Ξ is in $SM_2(V)\bigcap SM_2(r)$, then we have*

$$S^{\tau}_{1,l}(\Xi;r)-j(V;r)^{l-1}S^{\tau}_{1,l}(V\Xi;Vr)=R^{\tau}_{l}(\Xi;r;V)+\frac{lcB_{l+1}(n(r)\vec{y}-d(r)\vec{x};\tau)}{(l+1)d(r)^{l+1}j(V;r)}, \tag{6.1}$$

For any two by two matrix $V=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, put $\tilde{j}(V;r)=\dfrac{c}{j(V;r)}$. First we prove Lemma 4.3 by using (6.1).

Proof of Lemma 4.3. Suppose that $\Xi\in \mathrm{SM}_2(V)\bigcap \mathrm{SM}_2(V_1)\bigcap \mathrm{SM}_2(r)$. It is easily seen that $\Xi\in \mathrm{SM}_2(V)\bigcap \mathrm{SM}_2(V_1)$ yields $V_1\Xi\in \mathrm{SM}_2(V_2)$. Since the determinant of V is one, we get $\gcd(an(r)+bd(r),cn(r)+dd(r))=1$ and

$$(n(Vr),d(Vr))=\begin{cases} (an(r)+bd(r),cn(r)+dd(r)) & (j(V;r)>0),\\ -(an(r)+bd(r),cn(r)+dd(r)) & (j(V;r)<0). \end{cases} \tag{6.2}$$

If ξ and η are variables, a calculation shows that

$$(an(r)+bd(r))(c\xi+d\eta)-(cn(r)+dd(r))(a\xi+b\eta)=n(r)\eta-d(r)\xi, \tag{6.3}$$

which together with $\Xi\in \mathrm{SM}_2(r)\bigcap \mathrm{SM}_2(V_1)$ gives $V_1\Xi\in \mathrm{SM}_2(V_1r)$. Thus we find from (6.1) that

$$\begin{aligned} &R^{\tau}_{l}(\Xi;r;V)-R^{\tau}_{l}(\Xi;r;V_1)-j(V_1,r)^{l-1}R^{\tau}_{l}(V_1\Xi;V_1r;V_2)\\ &\quad=-\frac{lB_{l+1}(n(r)\vec{y}-d(r)\vec{x};\tau)}{(l+1)d(r)^{l+1}}\Big(\tilde{j}(V;r)-\tilde{j}(V_1;r)-\frac{1}{j(V_1;r)^2}\tilde{j}(V_2;V_1r)\Big)=0. \end{aligned}$$

Since the set $\mathrm{SM}_2(r)$ is dense in $\mathrm{M}_2(\mathbb{R})$, (4.4) holds if $z=r$ is a rational number and $\Xi\in \mathrm{SM}_2(V)\bigcap \mathrm{SM}_2(V_1)$. The function $R^{\tau}_{l}(\Xi;z;V)$ with respect to z is meromorphic, thus we can remove the condition that z is rational. □

To verify (6.1), we introduce a further transformation formula. Let $r=n(r)/d(r)$ be a rational number, l a positive integer, $\Xi=\left(\begin{smallmatrix} \vec{x} \\ \vec{y} \end{smallmatrix}\right)$ be a matrix in $\mathrm{M}_2(\mathbb{R})$, and X,Y be complex variables. If Ξ is in $\mathrm{SM}_2(r)$, then a function appeared in the transformation formula is defined by

$$\begin{aligned} S^{\tau}_{l}(\Xi;\tbinom{X}{Y};r):=\frac{1}{d(r)}\sum_{j',j(d(r))}&\underline{F}\Big(\frac{\vec{j}+\vec{y}}{d(r)};n(r)Y-d(r)X;\tau\Big)\\ &\times\underline{F}^{(l-1)}\Big(n(r)\frac{\vec{j}+\vec{y}}{d(r)}-\vec{x};-Y;\tau\Big), \end{aligned} \tag{6.4}$$

where $\underline{F}^{(m)}(\vec{x};X;\tau)$ denotes $\dfrac{1}{(2\pi i)^m}\Big(\dfrac{\partial}{\partial X}\Big)^m\underline{F}(\vec{x};X;\tau)$. This corresponds to the elliptic Dedekind-Rademacher sum $S^{\tau}_{1,l}(\Xi;r)$ in (6.1). If $\vec{x}=(x',x)$ and $\vec{y}=(y',y)$,

we see from (2.4) that

$$\begin{aligned} &S_l^\tau(\Xi;\,{}^{X+1}_{Y};r)=\mathrm{e}(-y)S_l^\tau(\Xi;\,{}^{X}_{Y};r),\ \ S_l^\tau(\Xi;\,{}^{X}_{Y+1};r)=\mathrm{e}(x)S_l^\tau(\Xi;\,{}^{X}_{Y};r),\\ &S_l^\tau(\Xi;\,{}^{X+\tau}_{Y};r)=\mathrm{e}(-y')S_l^\tau(\Xi;\,{}^{X}_{Y};r),\ \ S_l^\tau(\Xi;\,{}^{X}_{Y+\tau};r)=\mathrm{e}(x')S_l^\tau(\Xi;\,{}^{X}_{Y};r). \end{aligned} \tag{6.5}$$

We also introduce a function which corresponds to $R_l^\tau(\Xi;r;V)$ in (6.1). If $V=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$ and Ξ is in $\mathrm{SM}_2(V)$, then the function is defined by

(6.6)

$$R_l^\tau(\Xi;\,{}^{X}_{Y};r;V):=\begin{cases}(-1)^l\displaystyle\sum_{k=0}^{l-1}\binom{l-1}{k}\big(-j(V,r)\big)^k S^\tau_{k+1,l-k}\left(\begin{matrix}-\vec{x}\\ \vec{y}\end{matrix};\begin{matrix}X\\Y\end{matrix};\frac{d}{c}\right) & (c\neq 0),\\ 0 & (c=0),\end{cases}$$

where $S^\tau_{k+1,l-k}\left(\begin{smallmatrix}-\vec{x}\\ \vec{y}\end{smallmatrix};\begin{smallmatrix}X\\Y\end{smallmatrix};\frac{d}{c}\right)$ denotes

$$\frac{1}{c}\sum_{j',j(|c|)}\underline{F}^{(k)}(\frac{\vec{j}+\vec{y}}{c};-cX-dY;\tau)\underline{F}^{(l-1-k)}(d\frac{\vec{j}+\vec{y}}{c}+\vec{x};Y;\tau). \tag{6.7}$$

We shall need the following two lemmas in a proof of the transformation formula for the functions $S_l^\tau(\Xi;\,{}^{X}_{Y};r)$ on the parameter r under $\mathrm{SL}_2(\mathbb{Z})$.

LEMMA 6.2. *Let $r=n(r)/d(r)$ be a rational number, and $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ a matrix in $SL_2(\mathbb{Z})$. If m',m,n',n,M',M,N',N are integers with $\left(\begin{smallmatrix}m'&n'\\m&n\end{smallmatrix}\right)=\left(\begin{smallmatrix}M'&N'\\M&N\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, then we have*

$$r(\tau m'+m)+\tau n'+n=(ar+b)(\tau M'+M)+(cr+d)(\tau N'+N). \tag{6.8}$$

In particular,

$$\frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})=r(\tau\mathbb{Z}+\mathbb{Z})+\tau\mathbb{Z}+\mathbb{Z}=(ar+b)(\tau\mathbb{Z}+\mathbb{Z})+(cr+d)(\tau\mathbb{Z}+\mathbb{Z}). \tag{6.9}$$

Proof. A direct calculation verifies (6.8). The first equal sign in (6.9) follows from the fact that $\gcd(n(r),d(r))=1$, and the second from (6.8) and the fact that the determinant of the matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is one. □

LEMMA 6.3. *Let c be a nonzero integer, l a nonnegative integer, and $\vec{i}=(i',i)$ a vector in $\mathbb{Z}^2$. Then we have*

$$\underline{F}^{(l)}(\vec{x};X+\frac{\tau i'+i}{c};\tau)=c^{l-1}\sum_{j',j(|c|)}\mathrm{e}(\vec{i}\cdot\frac{\vec{j}+\vec{x}}{c})\underline{F}^{(l)}(\frac{\vec{j}+\vec{x}}{c};cX;\tau), \tag{6.10}$$

$$B_m(\vec{x};\tau)=c^{m-2}\sum_{j',j(|c|)}B_m(\frac{\vec{j}+\vec{x}}{c};\tau). \tag{6.11}$$

Proof. Let $\vec{j}$ be the vector $(j',j)\in\mathbb{Z}^2$ and $X\notin\frac{\tau}{c}\mathbb{Z}+\frac{1}{c}\mathbb{Z}$. Set

$$f(z)=\underline{F}(\vec{x};-z+X;\tau)\underline{F}(\frac{\vec{j}+\vec{x}}{c};cz;\tau).$$

We find from (2.4) that the function $f(z)$ is a doubly periodic function with respect to 1 and τ, and from the pole situation of the function $\underline{F}(\vec{x};X;\tau)$ that $f(z)$ has the

only simple poles on the lattices $\frac{\tau}{c}\mathbb{Z}+\frac{1}{c}\mathbb{Z}$ and $X+\tau\mathbb{Z}+\mathbb{Z}$. Since the sum of the residues of $f(z)$ at its poles in any period parallelogram equals zero, we have

$$\frac{1}{c}\sum_{k',k(|c|)}\underline{F}(\vec{x};X+\frac{\tau k'+k}{c};\tau)\mathrm{e}(-k'\frac{j'+x'}{c}-k\frac{j+x}{c})=\underline{F}(\frac{\vec{j}+\vec{x}}{c};cX;\tau).$$

If we add each side of the above equation for $j',j=0,1,\dots,|c|-1$, then

$$c\underline{F}(\vec{x};X;\tau)=\sum_{j',j(|c|)}\underline{F}(\frac{\vec{j}+\vec{x}}{c};cX;\tau).$$

Replacing X by $X+\dfrac{\tau i'+i}{c}$ and (2.4) imply (6.10). (6.11) follows from (6.10) with $i'=i=0$ and (2.3). □

The transformation formula is the following. The method of the proof is to apply Liouville's theorem.

PROPOSITION 6.4. *Let $r=n(r)/d(r)$ be a rational number, l a positive integer, and X,Y complex variables. If V is in $SL_2(\mathbb{Z})$ with $j(V;r)>0$ and Ξ is in $SM_2(V)\bigcap SM_2(r)$, then we have*

$$S_l^\tau(\Xi;\tbinom{X}{Y};r)-j(V;r)^{l-1}S_l^\tau(V\Xi;V\tbinom{X}{Y};Vr)=R_l^\tau(\Xi;\tbinom{X}{Y};r;V). \tag{6.12}$$

Proof. Put $\Xi=\left(\begin{smallmatrix}\vec{x}\\ \vec{y}\end{smallmatrix}\right)=\left(\begin{smallmatrix}x' & x\\ y' & y\end{smallmatrix}\right)$ and $V=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. Let Ξ and Y be fixed, and let $L(X)$ and $R(X)$ be the left hand and the right hand side of (6.12) respectively. The function $LR(X)$ is defined by $L(X)-R(X)$. We find from (2.4) and (6.5) that

$$LR(X+1)=\mathrm{e}(-y)LR(X),\quad LR(X+\tau)=\mathrm{e}(-y')LR(X). \tag{6.13}$$

Suppose that $Y\notin\frac{1}{d(Vr)}(\tau\mathbb{Z}+\mathbb{Z})$. In order to obtain $LR(X)=0$, we will show that the function $LR(X)$ is an entire function. Since $j(V;r)$ is positive, we see from (6.2) that $n(Vr)=an(r)+bd(r)$ and $d(Vr)=cn(r)+dd(r)$. By (6.3), we obtain

$$\begin{aligned}S_l^\tau(V\Xi;V\tbinom{X}{Y};Vr)=\frac{1}{d(Vr)}\sum_{j',j(d(Vr))}&\underline{F}\Big(\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)};n(r)Y-d(r)X;\tau\Big)\\ &\times\underline{F}^{(l-1)}\Big(\frac{n(Vr)\vec{j}+n(r)\vec{y}-d(r)\vec{x}}{d(Vr)};-cX-dY;\tau\Big).\end{aligned} \tag{6.14}$$

It is seen from (6.4), (6.6) and (6.14) that every possible pole of the function $LR(X)$ is on the lattices

$$\begin{cases}rY+\dfrac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z}),\quad -\dfrac{d}{c}Y+\dfrac{1}{c}(\tau\mathbb{Z}+\mathbb{Z}) & (c\neq 0),\\ rY+\dfrac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z}) & (c=0).\end{cases}$$

Note that $\big(rY+\frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})\big)\bigcap\big(-\frac{d}{c}Y+\frac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})\big)=\phi$ since $Y\notin\frac{1}{d(Vr)}(\tau\mathbb{Z}+\mathbb{Z})$.

Let us examine behaviors of the functions $L(X)$ and $R(X)$ at the lattices. Let z be a complex number on $\frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})$. By (6.8) and (6.9), there are integers m',m, n',n, M',M, N',N such that $\left(\begin{smallmatrix}m' & n'\\ m & n\end{smallmatrix}\right)=\left(\begin{smallmatrix}M' & N'\\ M & N\end{smallmatrix}\right)\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and

$$z=r(\tau m'+m)+\tau n'+n=(ar+b)(\tau M'+M)+(cr+d)(\tau N'+N).$$

Put $w=-\vec{m}\cdot\vec{x}-\vec{n}\cdot\vec{y}$ and $W=-\vec{M}\cdot(a\vec{x}+b\vec{y})-\vec{N}\cdot(c\vec{x}+d\vec{y})$, where $\vec{m},\vec{n},\vec{M}$ and $\vec{N}$ denote the vectors $(m',m),(n',n),(M',M)$ and (N',N) respectively. Set $\vec{\xi}=d(r)\vec{x}-n(r)\vec{y}$. Then we see from (2.3) and (6.4) that

$$S_l^\tau(\Xi;\,{}^{X+rY+z}_{\ Y};r)=\frac{(-1)^{l-1}\mathrm{e}(w)}{d(r)^2X}\sum_{j',j(d(r))}\mathrm{e}(\vec{m}\cdot\frac{-n(r)\vec{j}+\vec{\xi}}{d(r)})\\ \times\underline{F}^{(l-1)}\Big(\frac{-n(r)\vec{j}+\vec{\xi}}{d(r)};Y;\tau\Big)+O(1),$$

where $O(1)$ means a holomorphic function at $X=0$. Since it follows from (6.3) that

$$n(Vr)\vec{M}\cdot(\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)})=\vec{M}\cdot(a\vec{x}+b\vec{y})+\vec{M}\cdot(\frac{n(Vr)\vec{j}-\vec{\xi}}{d(Vr)}),$$

we also see from (2.3) and (6.14) that

$$S_l^\tau(V\Xi;V\big({}^{X+rY+z}_{\ Y}\big);Vr)=\frac{(-1)^{l-1}\mathrm{e}(W)}{d(r)d(Vr)X}\sum_{j',j(d(Vr))}\mathrm{e}(\vec{M}\cdot\frac{-n(Vr)\vec{j}+\vec{\xi}}{d(Vr)})\\ \times\underline{F}^{(l-1)}\Big(\frac{-n(Vr)\vec{j}+\vec{\xi}}{d(Vr)};\frac{d(Vr)}{d(r)}Y+cz;\tau\Big)+O(1).$$

Because some calculations show that $w=W$ and

$$\frac{\tau m'+m}{d(r)}=\frac{cz}{d(Vr)}+\frac{\tau M'+M}{d(Vr)},$$

we find from (6.10) that $L(X+rY+z)=O(1)$, i.e., the function $LR(X)$ is holomorphic at $X\in rY+\frac{1}{d(r)}(\tau\mathbb{Z}+\mathbb{Z})$.

Suppose that c is not zero. Let $z=\frac{1}{c}(\tau m'+m)$ be in $\frac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})$. Then one has

$$S_l^\tau(V\Xi;V\big({}^{X-(d/c)Y+z}_{\ Y}\big);Vr)=\frac{1}{d(Vr)}\sum_{j',j(d(Vr))}\mathrm{e}(-\vec{m}\cdot\frac{n(Vr)\vec{j}-\vec{\xi}}{d(Vr)})\\ \times\underline{F}\Big(\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)};-d(r)(X+z)+\frac{d(Vr)Y}{c};\tau\Big)\underline{F}^{(l-1)}\Big(\frac{n(Vr)\vec{j}-\vec{\xi}}{d(Vr)};-cX;\tau\Big).\tag{6.15}$$

Since

$$\underline{F}^{(n)}(\vec{x};X;\tau)=\frac{(-1)^n n!}{(2\pi i)^nX^{n+1}}+\sum_{m=0}^{\infty}\frac{B_{m+n+1}(\vec{x};\tau)}{(m+n+1)m!}(2\pi i)^{m+1}X^m,\tag{6.16}$$

the right hand side of (6.15) equals

$$\frac{(l-1)!}{d(Vr)(2\pi i)^{l-1}c^l}\mathrm{e}(\frac{\vec{m}\cdot\vec{\xi}+n(Vr)\vec{m}\cdot(c\vec{x}+d\vec{y})}{d(Vr)})\sum_{k=0}^{l-1}\frac{(2\pi id(r))^k}{k!X^{l-k}}\sum_{j',j(d(Vr))}\\ \mathrm{e}(-n(Vr)\frac{\vec{m}\cdot(\vec{j}+c\vec{x}+d\vec{y})}{d(Vr)})\underline{F}^{(k)}\Big(-\frac{\vec{j}+c\vec{x}+d\vec{y}}{d(Vr)};-\frac{d(Vr)Y}{c}+d(r)z;\tau\Big)+O(1).$$

By (6.10) and (6.3), this is equal to

$$\frac{(l-1)!}{(2\pi i)^{l-1}c^l}\mathrm{e}(\vec{m}\cdot(a\vec{x}+b\vec{y}))\sum_{k=0}^{l-1}\frac{(2\pi id(r))^k}{k!d(Vr)^kX^{l-k}}\underline{F}^{(k)}\Big(-c\vec{x}-d\vec{y};-\frac{Y}{c}+az;\tau\Big)+O(1).$$

So we conclude that

$$L(X-\frac{d}{c}Y+z)=-\frac{(l-1)!}{(2\pi i)^{l-1}c^l}\mathrm{e}(\vec{m}\cdot(a\vec{x}+b\vec{y}))\times\sum_{k=0}^{l-1}\frac{(2\pi i)^k j(V;r)^{l-1-k}}{k!X^{l-k}}\underline{F}^{(k)}\Big(-c\vec{x}-d\vec{y};-\frac{Y}{c}+az;\tau\Big)+O(1). \tag{6.17}$$

On the other hand, we have

$$R(X-\frac{d}{c}Y+z)=-\frac{1}{c}\sum_{k=0}^{l-1}\binom{l-1}{k}j(V,r)^k$$
$$\times\sum_{j',j(|c|)}\frac{k!}{(2\pi i)^k(cX)^{k+1}}\mathrm{e}(-\vec{m}\cdot\frac{\vec{j}+\vec{y}}{c})\underline{F}^{(l-1-k)}(-\frac{d\vec{j}+c\vec{x}+d\vec{y}}{c};-Y;\tau)+O(1).$$

Since $\vec{m}\cdot\vec{y}=a\vec{m}\cdot(c\vec{x}+d\vec{y})-c\vec{m}\cdot(a\vec{x}+b\vec{y})$, we get

$$R(X-\frac{d}{c}Y+z)=-\frac{\mathrm{e}(\vec{m}\cdot(a\vec{x}+b\vec{y}))}{c^l}\sum_{k=0}^{l-1}\binom{l-1}{k}j(V,r)^k\times\frac{k!}{(2\pi i)^kX^{k+1}}\underline{F}^{(l-1-k)}(-c\vec{x}-d\vec{y};-\frac{Y}{c}+az;\tau)+O(1).$$

This together with (6.17) implies that the function $LR(X)$ is holomorphic at $X\in-\frac{d}{c}Y+\frac{1}{c}(\tau\mathbb{Z}+\mathbb{Z})$, and it is an entire function.

We see from (6.13) that $LR(X)$ is a bounded function, and from Liouville's theorem that $LR(X)$ is a constant function. Since $\vec{y}\notin\mathbb{Z}^2$ by $\Xi\in\mathrm{SM}_2(r)$, we obtain $LR(X)=0$ by (6.13), which completes the proof because the set $\{Y|Y\notin\frac{1}{d(Vr)}(\tau\mathbb{Z}+\mathbb{Z})\}$ is dense in $\mathbb{C}$. □

In order to obtain the transformation formula for elliptic Dedekind-Rademacher Sums $S^\tau_{1,l}(\Xi;r)$, we prepare a lemma which states that the functions $S^\tau_l(\Xi;\left(\begin{smallmatrix}X\\Y\end{smallmatrix}\right);r)$ and $R^\tau_l(\Xi;\left(\begin{smallmatrix}X\\Y\end{smallmatrix}\right);r;V)$ yield $S^\tau_{1,l}(\Xi;r)$ and $R^\tau_l(\Xi;r;V)$ respectively.

LEMMA 6.5. *For any meromorphic function $f(X,Y)$, we denote by $C_Y(f(X,Y))$ and $C_{Y,X}(f(X,Y))$ the coefficients of $Y^0(=1)$ in $f(X,Y)$ and X^0 in $C_Y(f(X,Y))$ respectively. If $V=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is in $SL_2(\mathbb{Z})$ with $c\neq0$ and $j(V;r)>0$, the following three equations hold.*

$$C_{Y,X}(S^\tau_l(\Xi;\left(\begin{smallmatrix}X\\Y\end{smallmatrix}\right);r))=\frac{(2\pi i)^2}{l}\Big[-\frac{r^l}{l+1}B_{l+1}(\vec{y};\tau)+S^\tau_{1,l}(\Xi;r)\Big], \tag{6.18}$$

$$C_{Y,X}(S^\tau_l(V\Xi;V\left(\begin{smallmatrix}X\\Y\end{smallmatrix}\right);Vr))=\frac{(2\pi i)^2}{l}\Big[\frac{(-1)^{l-1}}{(l+1)c^lj(V;r)^l}B_{l+1}(c\vec{x}+d\vec{y};\tau)+\frac{lc}{(l+1)d(r)d(Vr)^l}B_{l+1}(n(r)\vec{y}-d(r)\vec{x};\tau)+S^\tau_{1,l}(V\Xi;Vr)\Big], \tag{6.19}$$

$$C_{Y,X}(R_l^\tau(\Xi;\tbinom{X}{Y};r;V)) = \frac{(2\pi i)^2}{l}\Big[\frac{-r^l}{l+1}B_{l+1}(\vec{y};\tau) + \frac{(-1)^l}{(l+1)c^l j(V;r)}B_{l+1}(c\vec{x}+d\vec{y};\tau) + R_l^\tau(\Xi;r;V)\Big]. \tag{6.20}$$

Proof. We find from (6.16) and Lemma 6.3 that

$$C_Y(S_l^\tau(\Xi;\tbinom{X}{Y};r)) = \frac{2\pi i}{l}\Big[\frac{1}{d(r)}\sum_{j',j(d(r))}\underline{F}\Big(\frac{\vec{j}+\vec{y}}{d(r)};-d(r)X;\tau\Big) \times B_l\Big(n(r)\frac{\vec{j}+\vec{y}}{d(r)}-\vec{x};;\tau\Big) - r^l\underline{F}^{(l)}(\vec{y};-X;\tau)\Big].$$

By using (6.16) again, one gets (6.18). We can also obtain (6.19) and (6.20) in a similar way, so we omit the proof. □

We are in a position to prove Theorem 6.1 now.

Proof of Theorem 6.1. The case that c is zero is trivial, so suppose that c is not zero. If $j(V;r)$ is positive, (6.1) follows from Proposition 6.4 and Lemma 6.5. It is seen that $j(-V;r) = -j(V;r)$, $S_{1,l}^\tau(-V\Xi;(-V)r) = (-1)^{l-1}S_{1,l}^\tau(V\Xi;Vr)$, and $R_l^\tau(\Xi;r;-V) = R_l^\tau(\Xi;r;V)$, thus (6.1) with $j(V;r) < 0$ is reduced to the case that $j(V;r) > 0$. □

As a corollary of Theorem 6.1, we shall reproduce a part of Halbritter's result, i.e., the transformation formula for the sums $S_{1,n}(\tbinom{x}{y};r)$. In analogy to the function $R_l^\tau(\Xi;z;V)$, we define a sum of generalized Dedekind-Rademacher sums as follows:

$$R_l\begin{pmatrix}x\\y\end{pmatrix};z;V\Big) := \begin{cases}\dfrac{(-1)^l}{l+1}\displaystyle\sum_{k=-1}^{l}\binom{l+1}{k+1}\big(-j(V,z)\big)^k S_{k+1,l-k}\begin{pmatrix}-x\\y\end{pmatrix};\frac{d}{c}\Big) & (c>0),\\ 0 & (c=0).\end{cases} \tag{6.21}$$

When c is negative, $R_l(\tbinom{x}{y};z;V)$ also denotes $R_l(\tbinom{x}{y};z;-V)$. By (2.12), we have

$$\mathrm{Re}\lim_{\tau\to i\infty}R_l^\tau(\Xi;z;V) = \begin{cases}R_l(\tbinom{x}{y};z;V) + \frac{1}{4}\mathfrak{c}\big(\tbinom{-x'}{y'};d/c\big) & (l=1,x,y\in\mathbb{Z}),\\ R_l(\tbinom{x}{y};z;V) & (otherwise),\end{cases} \tag{6.22}$$

where $\mathfrak{c}\big(\tbinom{-x'}{y'};(\pm1)/0\big)$ means zero.

THEOREM 6.6 (cf. [Ha] Theorem 2)**.** *Let $r = n(r)/d(r)$ be a rational number, x, y real numbers, and V a matrix in $SL_2(\mathbb{Z})$ with $j(V;r) \neq 0$.*
(i) *If l is an integer more than one, then we have*

$$S_{1,l}(\tbinom{x}{y};r) - j(V;r)^{l-1}S_{1,l}(V\tbinom{x}{y};Vr) = R_l(\tbinom{x}{y};r;V) + \frac{lc\tilde{B}_{l+1}(n(r)y-d(r)x)}{(l+1)d(r)^{l+1}j(V;r)}. \tag{6.23}$$

(ii) *Put* $p = n(r)$, $q = d(r)$ *and* $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *If* c *is not zero, then we have*

$$S_{1,1}(\begin{smallmatrix} x \\ y \end{smallmatrix}; r) - S_{1,1}(V\begin{pmatrix} x \\ y \end{pmatrix}; Vr) + S_{1,1}(\begin{smallmatrix} -x \\ y \end{smallmatrix}; d/c) = \frac{1}{2}\Big(\frac{c}{q(cp+dq)}\tilde{B}_2(py - qx) + \frac{cp+dq}{cq}\tilde{B}_2(y) + \frac{q}{(cp+dq)c}\tilde{B}_2(cx+dy)\Big) + \begin{cases} -\dfrac{\text{sign } c(cp+dq)}{4} & (x, y \in \mathbb{Z}), \\ 0 & (otherwise). \end{cases} \tag{6.24}$$

Proof. Suppose that $x, y \in \mathbb{Z}$. By (2.12), (6.1) and (6.22), one has

$$S_{1,1}(\begin{smallmatrix} x \\ y \end{smallmatrix}; r) - S_{1,1}(V\begin{pmatrix} x \\ y \end{pmatrix}; Vr) + S_{1,1}(\begin{smallmatrix} -x \\ y \end{smallmatrix}; d/c) = \frac{1}{2}\Big(\frac{c\tilde{B}_2(n(r)y - d(r)x)}{d(r)^2 j(V;r)} + \frac{j(V;r)}{c}\tilde{B}_2(y) + \frac{1}{cj(V;r)}\tilde{B}_2(cx+dy)\Big) + \frac{1}{4}\Big(\mathfrak{c}(\begin{smallmatrix} x' \\ y' \end{smallmatrix}; r) - \mathfrak{c}\big(V\begin{pmatrix} x' \\ y' \end{pmatrix}; Vr\big) + \mathfrak{c}(\begin{smallmatrix} -x' \\ y' \end{smallmatrix}; d/c)\Big).$$

Since a calculation shows that

$$\lim_{y' \to i\infty} \lim_{x' \to i\infty} \Big(\mathfrak{c}(\begin{smallmatrix} x' \\ y' \end{smallmatrix}; r) - \mathfrak{c}\big(V\begin{pmatrix} x' \\ y' \end{pmatrix}; Vr\big) + \mathfrak{c}(\begin{smallmatrix} -x' \\ y' \end{smallmatrix}; d/c)\Big) = -\text{sign } c(cp+dq),$$

we obtain (6.24). The other cases immediately follow from (2.12), (6.1) and (6.22). □

REMARK 6.7. If p and q are relatively prime integers with $q \neq 0$,

$$S_{1,l}(\begin{smallmatrix} x \\ y \end{smallmatrix}; p/q) = \text{sign } q \sum_{j(|q|)} \tilde{B}_1\Big(\frac{j+y}{q}\Big)\tilde{B}_l\Big(p\frac{j+y}{q} - x\Big). \tag{6.25}$$

Thus (6.23) and (6.24) imply a part of Halbritter's result [Ha, Theorem 2]. In particular, (6.24) with $x = y = 0$ yields (1.2), and (6.24) with $x = y = a = d = 0$ and $-b = c = 1$ gives the reciprocity formula for Apostol-Dedekind sums.

7. Proofs of Proposition 2.1 and 3.2

We prove Proposition 2.1 and 3.2. Throughout this section, let $\Xi = \begin{pmatrix} x' & x \\ y' & y \end{pmatrix}$ be in $\mathrm{M}_2(\mathbb{R})$, and $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$. We denote the vectors (x', x) and (y', y) by $\vec{x}$ and $\vec{y}$ respectively.

For a proof of Proposition 2.1, suppose that $r = n(r)/d(r)$ is a rational number and Ξ is in $\mathrm{SM}_2(r)$.

Proof of Proposition 2.1. Set $F(x, X; \tau) := \dfrac{\theta'(0;\tau)\theta(x+X;\tau)}{\theta(x;\tau)\theta(X;\tau)}$. This function has the modular property (see the theorem in [Za, Section 3]):

$$F(x, X; \tau) = \frac{1}{c\tau + d}\mathrm{e}(-\frac{cxX}{c\tau+d})F(\frac{x}{c\tau+d}, \frac{X}{c\tau+d}; V\tau).$$

We note that $\theta(x;\tau)$ is equal to $i\theta_\tau(2\pi i x)$ in [Za]. Since the function $\underline{F}(\vec{x}; X; \tau)$ equals $\mathrm{e}(xX)F(-x' + x\tau, X; \tau)$,

$$\underline{F}(\vec{x}; X; \tau) = \frac{1}{j(V;\tau)}\underline{F}(\vec{x}\ {}^tV; \frac{X}{j(V;\tau)}; V\tau) \tag{7.1}$$

from which we deduce $B_m(\vec{x};\tau) = \dfrac{1}{j(V;\tau)^m} B_m(\vec{x}\ {}^tV; V\tau)$. Thus we have

$$S^{\tau}_{m,n}(\Xi; r)$$
$$= \frac{1}{d(r) j(V;\tau)^{m+n}} \sum_{j', j(d(r))} B_m\Big(\frac{\vec{j}+\vec{y}}{d(r)}\ {}^tV; V\tau\Big) B_n\Big(\big(n(r)\frac{\vec{j}+\vec{y}}{d(r)} - \vec{x}\big)\ {}^tV; V\tau\Big).$$

From the fact that the determinant of V is one, it follows that $\vec{j}\ {}^tV$ runs over all elements in $(\mathbb{Z}/d(r)\mathbb{Z})^2$ when $\vec{j}$ does too. This together with the above equation implies (2.11). (2.12) is a special case of [Ma, Eq. (25)]. □

In order to give a proof of Proposition 3.2, let α be an irrational real algebraic number, l an integer with $l \geq 3$, and $\vec{y} \notin \mathbb{Z}^2$.

Proof of Proposition 3.2. Let $I^{\tau} = \sum_{m',m}{}' \dfrac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m' + m)^l} \underline{F}(-\vec{y}; \alpha(\tau m' + m); \tau)$, and $\vec{x}, \vec{y}$ be fixed. For any vector $\vec{m} = (m', m) \in \mathbb{Z}^2$, let $\vec{n} = (n', n)$ be the vector with $\vec{m} = \vec{n}\ V$. We see from (7.1) that

$$I^{\tau} = \frac{1}{j(V;\tau)^{l+1}} \sum_{m',m}{}' \frac{\mathrm{e}(\vec{n}\cdot(\vec{x}\ {}^tV))}{(V\tau n' + n)^l} \underline{F}(-\vec{y}\ {}^tV; \alpha(V\tau n' + n); V\tau).$$

Since the determinant of V is one, (n', n) runs over all elements in $\mathbb{Z}^2$ except $(0,0)$ when (m', m) does too. Thus we obtain (3.5).

In order to prove (3.6), we introduce the Fourier expansion for the function $F(\xi, X; \tau)$ (see [We, p. 70], [Za, p. 456]). If $|\mathrm{Im}\ \xi|, |\mathrm{Im}\ X| < |\mathrm{Im}\ \tau|$, then

$$F(\xi, X; \tau) = \pi\big(\cot \pi\xi + \cot \pi X\big) - 2\pi i \sum_{i,j=1}^{\infty} (\mathrm{e}(i\xi + jX) - \mathrm{e}(-i\xi - jX))\mathrm{e}(ij\tau).$$

Let ξ', ξ, X' and X be a real number with $|X'| \leq 1/2$ and $|\xi| < 1$. We find from this that

$$\underline{F}(\xi', \xi; X'\tau + X; \tau) = \pi \mathrm{e}(\xi(X'\tau + X))\Big(\cot \pi(-\xi' + \xi\tau) + \cot \pi(X + X'\tau)\Big) \tag{7.2}$$
$$-2\pi i \mathrm{e}(\xi X) \sum_{i,j=1}^{\infty} \Big(\mathrm{e}(-i\xi' + jX)\mathrm{e}((i+X')(j+\xi)\tau) - \mathrm{e}(i\xi' - jX)\mathrm{e}((i-X')(j-\xi)\tau)\Big).$$

Let S be the last term in the right hand side of (7.2). If $c = \min\{1 \pm \xi, 1/2\}$, then

$$|S| \leq 4\pi \sum_{i,j=1}^{\infty} \mathrm{e}(c^2 ij\tau). \tag{7.3}$$

For any non-zero integer m', let $y_{m'}$ be the real number with $y_{m'} \equiv -y \pmod 1$, $|y_{m'}| < 1$ and $y_{m'}\langle\langle \alpha m' \rangle\rangle \geq 0$, and let y_0 be $\{-y\}$. Since it follows from (2.4) and (2.5) that

$$\underline{F}(-\vec{y}; \alpha(\tau m' + m); \tau) = \mathrm{e}(-[[\alpha m']] y')\underline{F}(-y', y_{m'}; \langle\langle \alpha m' \rangle\rangle \tau + \alpha m; \tau),$$

we find from (7.2) and (7.3) that

$$\lim_{\tau\to i\infty} I^\tau = \lim_{\tau\to i\infty} \pi \sum_{m',m}{}' \frac{\mathrm{e}(\vec{m}\cdot\vec{x})}{(\tau m'+m)^l} \mathrm{e}(-[[\alpha m']]y') \times \mathrm{e}(y_{m'}(\langle\langle \alpha m'\rangle\rangle\tau+\alpha m))\Big(\cot\pi(y'+y_{m'}\tau)+\cot\pi(\alpha m+\langle\langle\alpha m'\rangle\rangle\tau)\Big). \tag{7.4}$$

Let $S_{m',m}$'s be the summands of the summation in right hand side of (7.4). If

$$\lim_{\tau\to i\infty} \sum_{\substack{m',m\\(m'\neq 0)}}{}' S_{m',m} = 0, \tag{7.5}$$

then we have

$$\lim_{\tau\to i\infty} I^\tau = \pi \sum_m{}' \frac{\mathrm{e}(mx)}{m^l} \lim_{\tau\to i\infty} \mathrm{e}(\{-y\}\alpha m)\Big(\cot\pi(y'+\{-y\}\tau)+\cot\pi\alpha m\Big)$$
$$= \begin{cases} \pi\Big(\xi_l(x;\alpha)+\cot\pi y' \sum_m{}' \dfrac{\mathrm{e}(mx)}{m^l}\Big) & (y\in\mathbb{Z}),\\ \pi\Big(\xi_l(x+\{-y\}\alpha;\alpha)-i\sum_m{}' \dfrac{\mathrm{e}(m(x+\{-y\}\alpha))}{m^l}\Big) & (y\notin\mathbb{Z}) \end{cases}$$

from which we deduce (3.6). So we may prove (7.5). Let c be a positive real number such that if $|\mathrm{Re}\ z|, |\mathrm{Im}\ z| \le 1/2$, then $|z\cot\pi z|\le c$. Since $\cot\pi z$ converges at $\pm i$ when z tends to $\mp i\infty$, there is a positive real number c' such that if $|\mathrm{Re}\ z|\le 1/2$ and z is not zero, then

$$|\cot\pi z| \le \frac{c}{|z|}+c'.$$

Set $C=\max\{c,c'\}$. Since $y_{m'}\langle\langle\alpha m'\rangle\rangle \ge 0$, one obtains

$$\sum_{\substack{m',m\\(m'\neq 0)}}{}' |S_{m',m}| \le \sum_{\substack{m',m\\(m'\neq 0)}}{}' \frac{C}{|\tau m'+m|^l}\Big(\frac{1}{|\langle\langle y'\rangle\rangle+y_{m'}\tau|}+\frac{1}{|\langle\langle\alpha m\rangle\rangle+\langle\langle\alpha m'\rangle\rangle\tau|}+2\Big).$$

This together with (3.2) implies (7.5), which completes the proof. □

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Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan

E-mail address: machide@math.sci.hokudai.ac.jp