Title	Global properties on spacelike submanifolds of codimension two in Minkowski space
Author(s)	Izumiya, Shyuichi; Nuno Ballesteros, Juan Jose; Romero Fuster, Maria del Carmen
Citation	Hokkaido University Preprint Series in Mathematics, 878, 1-25
Issue Date	2007-10-10
DOI	https://doi.org/10.14943/84028
Doc URL	https://hdl.handle.net/2115/69687
Type	departmental bulletin paper
File Information	re878.pdf

Global properties on spacelike submanifolds of codimension two in Minkowski space

Shyuichi IZUMIYA [*], Juan José NUNO BALLESTEROS
and
María del Carmen ROMERO FUSTER [†]

October 10, 2007

Abstract

We consider codimension two spacelike submanifolds with a parallel normal field (i.e. vanishing normal curvature) in Minkowski space. We use the analysis of their contacts with hyperplane and hyperquadrics in order to get some global informations on them. As a consequence we obtain new versions of Caratèodory's and Loewner's conjectures on spacelike surfaces in 4-dimensional Minkowski space and 4-flattenings therorems for closed spacelike curves in 3-dimensional Minkowski space.

1 Introduction

The study of the contacts of submanifolds with hyperplanes and hyperspheres (i.e., totally umbilical hypersurfaces) in Euclidean space by means of the analysis of the singularities of appropriate functions has been useful in order to obtain global results concerning their geometry and topology. For instance, a classical consequence of Morse Theory establishes that *a closed (compact without boundary) surface is a* 2*-sphere if and only if it admits some Morse function with exactly two critical points.* Also, from the Extrinsic Geometry viewpoint, there is the following result due to Nomizu and Rodriguez ([44]): *Every distance squared Morse function on a closed connected Riemannian* n*-manifold* M *in the Euclidean space has index* 0 *or* n *at all of its critical points if and only if* M *is emmbedded as a Euclidean* n*-sphere.*

On the other hand, the study of the degenerate contacts of curves with hyperplanes in Euclidean n-space has lead to several results on the existence of flattenings (zeroes of the $(n-1)$th curvature function) for closed curves with appropriate convexity conditions ([2], [50], [51]).

---

[*] Work partially supported by by Grant-in-Aid for formation of COE "Mathematics of Nonlinear Structure via Singularities"

[†] Work partially supported by DGCYT grant no. MTM2006-06027

2000 Mathematics Subject classification:53C40, 53A35

Key Words and Phrases: lightlike principal curvature directions, lightlike asymptotic directions, lightlike umbilics.

In the case of surfaces immersed in 3-space, there is a conjecture, classically known as the Carathèodory conjecture, that asserts that any 2-sphere immersed in $\mathbb{R}^3$ has at least two umbilical points (critical points of its principal configuration). Such points can also be characterized as corank 2 singularities of distance squared functions on the surfaces. A generic proof of this result is due to E. A. Feldman ([7]), who showed that generically immersed 2-spheres must have at least 4 of them. The general case remains as a conjecture so far. An attempt to prove it has lead to the following,

Loewner's conjecture: *The index of an umbilic point of any surface immersed in* $\mathbb{R}^3$ *is at most one.*

Several works have been devoted to the proof of this conjecture in the real analytic case ([3], [17], [38],[53],[55]). More recently, V. V. Ivanov has given a more complete version in [19]. Some other works, intended to prove the conjecture in the smooth case, have been done by R. Garcia, C. Gutierrez, F. Mercuri and Sánchez-Bringas ([9], [12], [14], [15]) and by B. Smyth and F. Xavier ([52]).

A generalization of Feldman's result for convex surfaces generically immersed in Euclidean 4-space was obtained in [10] as a consequence of the study of the generic behaviour of height functions on them.

This paper is the sequel of a recently appeared work of the first and third authors [36], concerning the geometrical properties related to the contacts of codimension 2 spacelike submanifolds with lightlike hyperplanes in Minkowski space. It was there proven that an analogous of the Gauss-Bonnet theorem holds in this context. There were obtained some consequences for the particular case of spacelike submanifolds with a parallel normal field. Our aim here consists in obtaining further global results for these submanifolds. For this purpose we use the following basic idea, which is based in the method introduced by the second author in [45] in order to obtain a proof for the Carathèodory's conjecture for analytic surfaces with vanishing normal curvature in 4-dimensional Euclidean space. We show that the properties related to the contacts with hyperplanes of spacelike $(n-1)$-submanifolds with vanishing normal curvature in Minkowski $(n+1)$-space can be put in terms of the corresponding properties for hypersurfaces in Hyperbolic n-space. And then, by means of the conformal map which is given by the composition of the stereographic projections, in terms of properties concerning the contacts of hypersurfaces with hyperspheres in Euclidean n-space. In this way, under appropriate assumptions, we can "transport" to the first known results on the last ones.

The paper is organized as follows: Sections 2-5 contain the notation and some preliminary results concerning spacelike submanifolds of codimension 2 in Minkowski space. In section 3, we consider the shape operators associated to different normal fields on the submanifolds together with their corresponding principal configurations. A spacelike submanifold of codimension 2 in Minkowski space has a well defined lightcone Gauss map whose associated shape operator is known as the normalized lightcone shape operator [36], and the corresponding principal configuration is known as the lightcone principal configuration. In the particular case of hypersurfaces in Hyperbolic space, these coincide respectively with the horospherical shape operator and the horospherical principal configuration [21]. This is discussed in section 4. Sections 5 is devoted to the description of Lorentzian distance squared functions and their connections with the contacts of the submanifolds with hyperquadrics in Minkowski space. An important observation arising in this context is the fact that the principal directions can be considered as contact directions of the submanifold with hyperquadrics and, in particular, the lightcone principal directions are contact directions of the submanifold with its focal lightcones. In section 6 we consider three naturally defined height function families, respectively called timelike, spacelike and lightlike. Associated to the degenerate singularities of such functions we have the concepts of osculating hyperplanes and asymptotic directions. We see here, that in the case hypersurfaces of Hyperbolic space, their contacts with hyperspheres, equidistant hypersurfaces and hyperhorospheres can be described in terms of height functions. Then as a consequence of the characterization of metric spheres in hyperbolic space due to Cecil and Ryan in [6], we can assert:

a) Suppose that M is a compact connected smooth $(n-1)$- manifold immersed in Hyperbolic n-space. Then every non degenerate timelike height function has exactly two critical points if and only if M is embedded as a metric $(n-1)$-sphere.

b) Suppose that M is a connected complete smooth hypersurface in Hyperbolic n-space. Then every non degenerate timelike or spacelike height function on M has index 0 or $(n-1)$ if and only if M is embedded as a hypersphere, hyperhorosphere, or equidistant hypersurface.

In Section 7 we concentrate our attention in the codimension 2 submanifolds with a parallel normal field and show, that analogously to what happens with codimension 2 submanifolds of Euclidean space, the following property, that shall be fundamental in the obtention of global results of section 8, also holds for spacelike codimension 2 submanifolds in Minkowski space:

If M admits some globally defined parallel vector field then there exists an orthonormal frame of common eigenvectors for the shape operators associated to al normal fields over M.

Moreover, we show that provided M admits a non degenerate unit timelike normal field whose image $\bar{M}$ in Hyperbolic n-space has no self-intersections (i.e. is embedded) then M and $\bar{M}$ have the "same kind of contacts" with hyperplanes.

This allows us to conclude that, although in the general case of a spacelike $(n-1)$-submanifold of Minkowski $(n+1)$-space we cannot ensure the existence of any asymptotic direction at every point, those admitting a parallel normal field have exactly $(n-1)$ orthogonal asymptotic directions at every (non critical) point.

Finally, in Section 8 we use the above properties in order to transport known global results concerning contacts of hypersurfaces with hyoperspheres in Euclidean space to new results concerning the flat geometry of spacelike codimension 2 submanifolds with hyperplanes in Minkowski space, such as:

1. Suppose that M is a compact connected smooth $(n-1)$- manifold immersed in Minkowski $(n+1)$-space. Then M is a metric $(n-1)$-sphere contained in a spacelike hyperplane if and only if M has a globally defined non-degenerate parallel normal field and every non degenerate timelike height function has exactly two critical points on M.

2. Loewner's and Caratheodory's conjectures on umbilic points of surfaces in Euclidean 3-space hold if and only if they hold for lightcone umbilics of semiumbilical spacelike surfaces in Minkowski 4-space. So, relying on the analytic version of Loewner's conjecture for surfaces in Euclidean 3-space ([19]), we can assert that analytic semiumbilical spacelike 2-sphere immersed in Minkowski 4-space have at least two lightcone umbilics.

3. Any closed curve that admits a globally defined non-degenerated parallel timelike normal field ν has at least two flattening points. If ν satisfies that $\nu(s) \neq \nu(s'), \forall s \neq s'$, then the curve has at least 4 flattening points.

Which leads to the following 4-vertex theorems for closed spacelike curves in the de Sitter 2-space and the 2-dimensional lightcone:

3.a) Any regular closed spacelike curve immersed in de Sitter 2-space with non vanishing *curvature and geodesic curvature functions has at least* 4 *geodesic vertices (flattening points).*

3.b) Any regular closed spacelike curve immersed in the 2*-dimensional lightcone with non vanishing Gauss curvature function has at least* 4 *flattening points.*

2 Basic facts and notations on Minkowski space

We introduce in this section some basic notations on Minkowski $n+1$-space and spacelike submanifolds. For basic concepts and properties, see [46].

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\ \}$ be an $n+1$-dimensional cartesian space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n),\ \boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ *Minkowski* $n+1$*-space.* We denote $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the canonical projection $\pi : \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Here we identify $\{\boldsymbol{0}\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and it is considered as Euclidean n-space whose scalar product is induced from the pseudo scalar product $\langle , \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

The *Hyperbolic n-space* is given by

$$H^n_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}.$$

Any non empty hypersurface of $H^n_+(-1)$ determined by the intersection of $H^n_+(-1)$ with either a spacelike, a timelike or a lightlike hyperplane is respectively called *hypersphere, equidistant hyperplane* or *hyperhorosphere.*

Other well known pseudo-spheres in Minkoswki space are the *de Sitter n-space*, given by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}.$$

And the (*open*) *lightcone*:

$$LC^* = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}_1\ | x_0 \neq 0,\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

The subset

$$LC^*_+ = \{\boldsymbol{x} \in LC^*\ | x_0 > 0, \}$$

is called *future lightcone.* We denote the *n-dimensional lightcone with vertex* $\boldsymbol{\lambda}$ *in*$I\!\!R^{n+1}_1$ by

$$LC^*_{\boldsymbol{\lambda}} = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x} - \boldsymbol{\lambda}, \boldsymbol{x} - \boldsymbol{\lambda} \rangle = 0\}.$$

If $\boldsymbol{x}=(x_0,x_1,\dots,x_2)$ is a non-zero lightlike vector, then $x_0\neq 0$. Therefore we have

$$\widetilde{\boldsymbol{x}}=\left(1,\frac{x_1}{x_0},\dots,\frac{x_2}{x_0}\right)\in S^{n-1}_{+}=\{\boldsymbol{x}=(x_0,x_1,\dots,x_n)\mid\langle\boldsymbol{x},\boldsymbol{x}\rangle=0,\ x_0=1\}.$$

We call S^{n-1}_{+} the *lightcone* (or*, spacelike*) *unit* $n-1$*-sphere.*

For any $\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_n\in\mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\cdots\wedge\boldsymbol{x}_n$ by

$$\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\cdots\wedge\boldsymbol{x}_n=\begin{vmatrix}-\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n\\ x^1_0 & x^1_1 & \cdots & x^1_n\\ x^2_0 & x^2_1 & \cdots & x^2_n\\ \vdots & \vdots & \cdots & \vdots\\ x^n_0 & x^n_1 & \cdots & x^n_n\end{vmatrix},$$

where $\boldsymbol{e}_0,\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{x}_i=(x^i_0,x^i_1,\dots,x^i_n)$. We can easily check that

$$\langle\boldsymbol{x},\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\cdots\wedge\boldsymbol{x}_n\rangle=\det(\boldsymbol{x},\boldsymbol{x}_1,\dots,\boldsymbol{x}_n),$$

so that $\boldsymbol{x}_1\wedge\boldsymbol{x}_2\wedge\cdots\wedge\boldsymbol{x}_n$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i=1,\dots,n)$.

3 Extrinsic dynamics on spacelike submanifolds of codimension two

Let $\mathbb{R}^{n+1}_1$ be an oriented and timelike oriented space. We choose $\boldsymbol{e}_0=(1,0,\dots,0)$ as the future timelike vector field. We consider a spacelilke embedding $\boldsymbol{X}:U\longrightarrow\mathbb{R}^{n+1}_1$ from an open subset $U\subset\mathbb{R}^{n-1}$. We write $M=\boldsymbol{X}(U)$ and identify M and U through the embedding $\boldsymbol{X}$. We say that $\boldsymbol{X}$ is *spacelike* if X_{u_i} $i=1,\dots,n-1$ are always spacelike vectors. Therefore, the tangent space T_pM of M is a spacelike subspace (i.e., consists of spacelike vectors) for any point $p\in M$. In this case, the pseudo-normal space N_pM is a timelike plane (i.e., Lorentz plane) (cf.,[46]). We denote by $N(M)$ the pseudo-normal bundle over M. Let $\boldsymbol{n}$ be a section of this bundle, that is, a normal field on M. Under the identification of M and U through $\boldsymbol{X}$, we have a linear mapping provided by its derivative, $d_p(\boldsymbol{n}):T_pM\longrightarrow T_p\mathbb{R}^{n+1}_1=T_pM\oplus N_p(M)$ at each point $p\in M$. By composing this with the orthogonal projections, $\pi^t:T_pM\oplus N_p(M)\to T_p(M)$ and $\pi^n:T_p(M)\oplus N_p(M)\to N_p(M)$, we obtain the $\boldsymbol{n}$*-shape operator*

$$S_p(\boldsymbol{n})=d_p(\boldsymbol{n})^t=\pi^t\circ d_p(\boldsymbol{n})$$

and the *normal connection with respect to* $\boldsymbol{n}$,

$$d_p(\boldsymbol{n})^n=\pi^n\circ d_p(\boldsymbol{n}),$$

evaluated at the point p. Its eigenvectors are called $\boldsymbol{n}$*-principal directions* and the corresponding eigenvalues are the $\boldsymbol{n}$*-principal curvatures* $\{\kappa(\boldsymbol{n})\}_{i=1}^{n-1}$. A $\boldsymbol{n}$-principal direction whose corresponding $\boldsymbol{n}$-principal curvature vanishes at a point $p\in M$ is said to be *asymptotic direction (associated to* $\boldsymbol{n}$*)* of M at p. The function $K(\boldsymbol{n})(p)=detS_p(\boldsymbol{n})$ is called *Gauss-Kronnecker* $\boldsymbol{n}$*-curvature.* The points at which $K(\boldsymbol{n})$ vanishes are called $\boldsymbol{n}$*-parabolic points.* We say that $\boldsymbol{n}$ is *non-degenerate* at p provided $K(\boldsymbol{n})(p)\neq 0$, that is, the subset of $\boldsymbol{n}$-parabolic points is empty. Clearly, if there exists an asymptotic direction of M at p then p is $\boldsymbol{n}$-parabolic, for some $\boldsymbol{n}$. The points at which two or more of the $\boldsymbol{n}$-principal curvatures coincide are said to be *$\boldsymbol{n}$-preumbilical* points. A point at which all the $\boldsymbol{n}$-principal curvatures coincide is said to be *$\boldsymbol{n}$-umbilical.* We say that a normal field $\boldsymbol{n}$ is *umbilical* over M, or alternatively, that manifold M is *$\boldsymbol{n}$-umbilical* if all its points are $\boldsymbol{n}$-umbilical.

Since $N(M)$ is a trivial bundle, we can arbitrarily choose a future directed unit timelike normal section $\boldsymbol{n}^T(u) \in N_p(M)$, where $p = \boldsymbol{X}(u)$. Here, we say that $\boldsymbol{n}^T$ is *future directed* if $\langle \boldsymbol{n}^T, \boldsymbol{e}_0\rangle < 0$. Therefore we can construct a spacelike unit normal section $\boldsymbol{n}^S(u) \in N_p(M)$ by

$$\boldsymbol{n}^S(u) = \frac{\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|},$$

and we have $\langle \boldsymbol{n}^T, \boldsymbol{n}^T\rangle = -1$, $\langle \boldsymbol{n}^T, \boldsymbol{n}^S\rangle = 0$, $\langle \boldsymbol{n}^S, \boldsymbol{n}^S\rangle = 1$. Although we could also choose $-\boldsymbol{n}^S(u)$ as a spacelike unit normal section with the above properties, we fix the direction $\boldsymbol{n}^S(u)$ throughout this paper. We call $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ a *future directed normal frame* along $M = \boldsymbol{X}(U)$. Clearly, the vector $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ is lightlike. Here we choose $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along M. Since $\{\boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_{n-1}}(u)\}$ is a basis of T_pM, the system $\{\boldsymbol{n}^T(u), \boldsymbol{n}^S(u), \boldsymbol{X}_{u_1}(u), \ldots, \boldsymbol{X}_{u_{n-1}}(u)\}$ provides a basis for $T_p\mathbb{R}^{n+1}_1$.

It has been shown in [36] that given two future directed unit timelike normal sections $\boldsymbol{n}^T(u), \bar{\boldsymbol{n}}^T(u) \in N_p(M)$, the corresponding lightlike normal sections $\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u), \bar{\boldsymbol{n}}^T(u) + \bar{\boldsymbol{n}}^S(u)$ are parallel.

By applying the above procedure to the lightlike vector field $\boldsymbol{n}^T + \boldsymbol{n}^S$ as in [36], we obtain *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-shape operator* of $M = \boldsymbol{X}(U)$ at $p = \boldsymbol{X}(u)$. Its eigenvectors are called *lightcone principal directions* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p, and the corresponding eigenvalues, denoted by $\{\kappa_i(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)\}_{i=1}^{n-1}$, are the *lightcone principal curvatures* with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ at p. A point $p = \boldsymbol{X}(u)$ is a *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic point* if all the principal curvatures coincide at p and thus $S_p(\boldsymbol{n}^T, \boldsymbol{n}^S) = \kappa(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)1_{T_{p_0}M}$, for some function κ. This gives rise to the *$(\boldsymbol{n}^T, \boldsymbol{n}^S)$-lightcone principal configuration* on M, composed by the foliations determined by the integral lines of the lightcone principal directions fields with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ and the sets of $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-preumbilics and $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilics. We observe that this configuration does not depend on the choice of the pair $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ and it is preserved by the Lorentz transformations. Lightcone principal directions whose associated lightcone principal curvature vanishes are called *lightcone asymptotic directions* on M. We say that $M = \boldsymbol{x}(U)$ is *totally $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic* if all points on M are $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic.

Since $\boldsymbol{X}_{u_i}$ $(i = 1, \ldots n-1)$ are spacelike vectors, we have a Riemannian metric (the *hyperbolic first fundamental form*) on $M = \boldsymbol{X}(U)$ defined by $ds^2 = \sum_{i=1}^{n-1} g_{ij}du_idu_j$, where $g_{ij}(u) = \langle \boldsymbol{X}_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. We also have a *lightcone second fundamental invariant with respect to the normal vector field* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ defined by $h_{ij}(\boldsymbol{n}^T, \boldsymbol{n}^S)(u) = \langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i}(u), \boldsymbol{X}_{u_j}(u)\rangle$ for any $u \in U$. The following result was proven in [36].

Proposition 3.1 *Under the above notations, we have the following lightcone Weingarten formula with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$:*

(a) $(\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^S, \boldsymbol{n}^T_{u_i}\rangle(\boldsymbol{n}^T + \boldsymbol{n}^S) - \sum_{j=1}^{n-1} h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t \circ (\boldsymbol{n}^T + \boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^{n-1} h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here $\left(h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\right) = \left(h_{ik}(\boldsymbol{n}^T, \boldsymbol{n}^S)\right)\left(g^{kj}\right)$ *and* $\left(g^{kj}\right) = (g_{kj})^{-1}$.

4 Lightcone Gauss map and principal configurations

Given a spacelike embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^{n+1}_1$ from an open subset $U \subset \mathbb{R}^{n-1}$, and a point $p = \boldsymbol{X}(u)$, consider a future directed unit timelike normal section $\boldsymbol{n}^T(u) \in N_p(M)$ and the corresponding spacelike unit normal section $\boldsymbol{n}^S(u) \in N_p(M)$ constructed in the previous section. Since given any other future directed unit timelike nomal section $\bar{\boldsymbol{n}}^T(u)$, we have $(\widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S})(u) = (\widetilde{\bar{\boldsymbol{n}}^T + \bar{\boldsymbol{n}}^S})(u) \in S^{n-1}_+$, it is possible to define a *lightcone Gauss map* of $M = \boldsymbol{X}(U)$ as

$$\begin{array}{rccl} \widetilde{\mathbb{L}}: & U & \longrightarrow & S^{n-1}_+ \\ & u & \longmapsto & (\widetilde{\boldsymbol{n}^T + \boldsymbol{n}^S})(u). \end{array}$$

This induces a linear mapping $d\widetilde{\mathbb{L}}_p : T_pM \longrightarrow T_p\mathbb{R}^{n+1}_1$ under the identification of U and M, where $p = \boldsymbol{X}(u)$. The following normalized lightcone Weingarten formula was proven in [36]:

$$\pi^t \circ \widetilde{\mathbb{L}}_{u_i} = -\sum_{j=1}^{n-1} \frac{1}{\ell_0(u)} h^j_i(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j},$$

where $\mathbb{L}(u) = (\ell_0(u), \ell_1(u), \dots, \ell_n(u))$.

We call the linear transformation $\widetilde{S}_p = -\pi^t \circ d\widetilde{\mathbb{L}}_p$ the *normalized lightcone shape operator* of $M = \boldsymbol{X}(U)$ at p. The *normalized lightcone Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ is defined to be $\widetilde{K}_\ell(u) = \det \widetilde{S}_p$. We say that $p = \boldsymbol{X}(u)$ is a *lightlike parabolic point* if $\widetilde{K}_\ell(u) = 0$.

The eigenvalues $\{\widetilde{\kappa}_i(p)\}_{i=1}^{n-1}$ of $\widetilde{S}_p$ are called *normalized lightcone principal curvatures.* It follows from the above formula that $\widetilde{\kappa}_i(p) = (1/\ell_0)\kappa_i(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$. Clearly, the eigenvectors of $\widetilde{S}_p$ coincide with the lightcone principal directions with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$, for any future directed frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ on M, therefore, we can refer to the $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-lightcone principal configuration, simply as the *lightcone principal configuration* on M. The $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilics and preumbilics shall be called *lightlike umbilics* and *preumbilics.* We say that $M = \boldsymbol{X}(U)$ is *totally lightlike umbilic* if all points on M are lightlike umbilic, as usual. The point p is called a *lightlike flat point* if p is both lightlike umbilic and parabolic. The spacelike submanifold $M = \boldsymbol{X}(U)$ is called *lightlike flat* provided every point of M is lightlike flat. As observed in the previous section, the lightcone principal configuration is preserved by Lorentz transformations, although the lightcone principal curvatures are not. Nevertheless, we shall see below that the Lorentz transformations preserve the lightcone asymptotic directions and hence the lightlike parabolic points and the lightlike flat points. Therefore, the lightlike flatness is a Lorentzian property.

In the particular case of a $(n-1)$-submanifold $M = \boldsymbol{X}(U)$ contained in hyperbolic n-space $H^n_+(-1)$, we can take $\boldsymbol{n}^T = \boldsymbol{X}$, then $\boldsymbol{n}^S = \boldsymbol{e} \in S^n_1$ is univocally defined, and we have that the lightcone Gauss map on M coincides with the *hyperbolic Gauss map* ([32]), given by

$$\widetilde{\mathbb{L}}(u_0) = \boldsymbol{X}\widetilde{(u_0) + \boldsymbol{e}}(u_0).$$

In this case, we call $\widetilde{S}_p$, the *horospherical shape operator*, the corresponding eigenvectors, eigenvalues and umbilics are respectively called *horospherical principal directions*, *horospherical principal curvatures* and *horoumbilics* and determine the *horospherical principal configuration* on M. Consequently, we have that the horospherical principal directions of M at a point $\boldsymbol{p}_0$, considered as a hypersurface of $H^n_+(-1)$, are the lightcone principal directions of M at $\boldsymbol{p}_0$, considered as a codimension 2 submanifold of $\mathbb{R}^{n+1}_1$.

5 Lorentzian distance squared functions and principal configurations

We can also use the theory of contact developed by Montaldi [41, 42] in order to characterize the lightcone principal directions.

Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of* X_1 and Y_1 at y_1 is of the same type as the *contact of* X_2 and Y_2 at y_2 if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. In his paper [41], Montaldi gives the following characterization of the notion of contact by using the terminology of singularity theory:

Theorem 5.1 *Let* M_i, N_i $(i = 1, 2)$ *be submanifolds of* $\mathbb{R}^n$ *with* $\dim M_1 = \dim M_2$ *and* $\dim N_1 = \dim N_2$. *Let* $g_i : (M_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ *be immersion germs and* $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^r, 0)$ *be submersion germs with* $(N_i, y_i) = (f_i^{-1}(0), y_i)$. *Then* $K(M_1, N_1; y_1) = K(M_2, N_2; y_2)$ *if and only if* $f_1 \circ g_1$ *and* $f_2 \circ g_2$ *are* $\mathcal{K}$*-equivalent.*

So, given two submanifolds M and N of $\mathbb{R}^n$, with a common point p, and an immersion germ $g : (M, x) \longrightarrow (\mathbb{R}^n, p)$ and a submersion germ $f : (\mathbb{R}^n, p) \longrightarrow (\mathbb{R}^r, 0)$, such that $N = f^{-1}(0)$, we have that the contact of $M \equiv g(M)$ and N at p is completely determined by the singularity type of the germ $(f \circ g, x)$. When N is a hypersurface, we have $r = 1$, and the function germ $(f \circ g, x)$ has a degenerate singularity if and only if its Hessian, $\mathcal{H}(f \circ g)(x)$, is a degenerate quadratic form. In such case, the tangent directions lying in the kernel of this quadratic form are called *contact directions* for M and N at p.

We consider next some functions that describe the contacts of a spacelike $(n-1)$-submanifold $M = \boldsymbol{X}(U)$ of $\mathbb{R}^{n+1}_1$ with lightcones.

The *Lorentzian distance-squared functions family* on a spacelike $(n-1)$-submanifold $M = \boldsymbol{X}(U)$ of $\mathbb{R}^{n+1}_1$ was introduced in [30] for the case $n = 3$. It is defined as

$$\begin{array}{rlcl} G: & U \times \mathbb{R}^{n+1}_1 & \longrightarrow & \mathbb{R} \\ & (u, \boldsymbol{\lambda}) & \longmapsto & \langle \boldsymbol{X}(u) - \boldsymbol{\lambda}, \boldsymbol{X}(u) - \boldsymbol{\lambda} \rangle. \end{array}$$

Given $p = \boldsymbol{X}(u)$, for any fixed $\lambda_0 \in \mathbb{R}^{n+1}_1$, we write $g(p) = G_{\lambda_0}(p) = G(p, \boldsymbol{\lambda}_0)$. The following proposition was proven in [30] for the case $n = 3$. Its proof in the general case is analogous.

Proposition 5.2 *Let* M *be a spacelike* $(n-1)$*-submanifold and* $G : M \times \mathbb{R}^{n+1}_1 \to \mathbb{R}$ *the Lorentzian distance-squared function on* M. *Let* $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ *be a future directed frame. Suppose that* $p_0 \neq \boldsymbol{\lambda}_0$. *Then we have the following:*

(1) $g(p_0) = \partial g/\partial u_1(p_0) = \cdots = \partial g/\partial u_{n-1}(p_0) = 0$ *if and only if* $p_0 - \boldsymbol{\lambda}_0 = \mu(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(p_0)$ *for some* $\mu \in \mathbb{R} \setminus \{0\}$.

(2) $g(p_0) = \partial g/\partial u_1(p_0) = \cdots = \partial g/\partial u_{n-1}(p_0) = \det\mathcal{H}(g)(p_0) = 0$ (*where* $\det\mathcal{H}(g)(p_0)$ *is the determinant of the Hessian matrix*) *if and only if*

$$p_0 - \boldsymbol{\lambda}_0 = \mu(\widetilde{\boldsymbol{n}^T \pm \boldsymbol{n}^S})(p_0), \ \mu = \frac{1}{\kappa_i^{\pm}(p_0)}, i = 1, \cdots, n-1.$$

As a consequence, we see that the lightcone principal directions are contact directions associated to the above family. In other words, these are the tangent directions along which M has a degenerate (= non Morse) contact with some lightcone (*focal lightcone*) at p_0.

Analogously, for any $\lambda \in \mathbb{R}^{n+1}_1$, and $r \in \mathbb{R}^+$, the function $g^+_{\lambda,r}(p) = G(p, \boldsymbol{\lambda}) + r^2$ measures the contacts of M at p_0 with $H^n_+(-r) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x} - \lambda, \boldsymbol{x} - \lambda \rangle = -r^2, x_0 > 0\}$ and the function $g^-_{\lambda,r}(p) = G(p, \boldsymbol{\lambda}) - r^2$ measures the contacts of M at p_0 with $S^n_r = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \langle \boldsymbol{x} - \lambda, \boldsymbol{x} - \lambda \rangle = r^2\ \}$. The principal directions associated to any normal field $\boldsymbol{n}$ on M can also be characterized as contact directions associated to these functions.

Then straightforward calculations show that $\boldsymbol{n}$-umbilic points and, in particular the lightlike umbilics, can also be characterized as corank $(n-1)$ singularities of the Lorentzian distance squared functions.

6 Height functions and contacts with hyperplanes

We consider now three families of functions that describe respectively the contacts of $M = \boldsymbol{X}(U)$ with spacelike, timelike and lightlike hyperplanes in $\mathbb{R}^{n+1}_1$.

1. The *timelike height functions family* , given by

$$\begin{array}{rccc} H^t: & U \times H^n_+(-1) & \longrightarrow & \mathbb{R} \\ & (u, \boldsymbol{v}) & \longmapsto & \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle. \end{array}$$

2. The *spacelike height functions family* , given by

$$\begin{array}{rccc} H^s: & U \times S^n_1 & \longrightarrow & \mathbb{R} \\ & (u, \boldsymbol{v}) & \longmapsto & \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle. \end{array}$$

3. The *lightcone height functions family* , given by

$$\begin{array}{rccc} H^\ell: & U \times S^{n-1}_+ & \longrightarrow & \mathbb{R} \\ & (u, \boldsymbol{v}) & \longmapsto & \langle \boldsymbol{X}(u), \boldsymbol{v} \rangle. \end{array}$$

We denote the *Hessian matrix* of the timelike height function $h^t_{v_0}(u) = H^t(u, \boldsymbol{v}_0)$ at u_0 as $\mathrm{Hess}(h^t_{v_0})(u_0)$. Analogously, we denote by $\mathrm{Hess}(h^s_{v_0})(u_0)$ and $\mathrm{Hess}(h^\ell_{v_0})(u_0)$ the hessians of the spacelike and lightcone height functions at u_0. A normal direction $v \in N_pM$ is said to be *binormal* provided it induces a degenerate height function on M at p. A normal field $\boldsymbol{n}$ is said to be a *binormal field* on M if and only if $\boldsymbol{n}(p)$ is a binormal direction at p, for all $p \in M$. It is not difficult to check that for an appropriate local coordinates system the matrix of the shape operator $S_p(\boldsymbol{n})$ coincides with that of $\mathrm{Hess}(h^t_{\boldsymbol{n}(u)})(u)$. As a consequence, we have:

A tangent direction w of M at $p = \boldsymbol{x}(u)$ is an asymptotic direction for some timelike (resp. spacelike, lightlike) normal field $\boldsymbol{n}$ at p if and only if u is a degenerate singularity of the height function $h^t_{\boldsymbol{n}(u)}$ (resp. $h^s_{\boldsymbol{n}(u)}$, $h^\ell_{\boldsymbol{n}(u)}$) and w lies in the kernel of $\mathrm{Hess}(h^t_{\boldsymbol{n}(u)})(u)$ *(resp.* $\mathrm{Hess}(h^s_{\boldsymbol{n}(u)})(u)$*,* $\mathrm{Hess}(h^\ell_{\boldsymbol{n}(u)})(u)$ *).*

In other words, w is a contact direction of M with some spacelike (resp. timelike, lightlike) hyperplane at p. This hyperplane is said to be an *osculating hyperplane.* In a similar way to the case of codimension 2 submanifolds of Euclidean space (studied in [40]) we can see that the number of asymptotic directions (or of binormal directions, or of osculating hyperplanes) at any point of M is at most $(n-1)$.

We observe that given some globally defined binormal field $\boldsymbol{b}$ on M, we have an associated foliation of asymptotic curves (with possible critical points) on M. This foliation coincides with one of the principal foliations (with vanishing principal curvature) associated to the normal field $\boldsymbol{b}$ on M.

The following result on lightcone height functions was shown in ([36, Proposition 4.2]):

(1) $\partial H^\ell/\partial u_i(u_0, \boldsymbol{v}_0) = 0$ $(i = 1, \dots n-1)$ if and only if $\boldsymbol{v}_0 = \widetilde{\mathbb{L}^{\pm}}(u_0)$, where $\mathbb{L}^{\pm}(u) = \boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$.

Moreover, since in an appropriate coordinate system, $\mathrm{Hess}(h^\ell_{v_0})(u_0) = \widetilde{S}_{p_0}$, we have that for $\boldsymbol{v}_0 = \widetilde{\mathbb{L}}(u_0)$,

(2) p_0 is a lightlike parabolic point if and only if $\det \mathrm{Hess}(h^\ell_{v_0})(u_0) = 0$.

(3) p_0 is a lightlike flat point if and only if $\mathrm{rank}\, \mathrm{Hess}(h^\ell_{v_0})(u_0) = 0$.

Consider now the particular case of a $(n-1)$-submanifold $M = \boldsymbol{X}(U)$ contained in hyperbolic n-space $H^n_+(-1)$. Given a vector $\boldsymbol{v} \in H^n_+(-1)$ (resp. S^n_1, S^{n-1}_+) and a real number c, denote by $S(\boldsymbol{v}, c)$ the hypersphere (resp. equidistant hyperplane, hyperhorosphere) determined by the intersection of the hyperplane $HP(\boldsymbol{v}, c)$ with $H^n_+(-1)$. Given $p = \boldsymbol{X}(u) \in M$, suppose that $\boldsymbol{v} \in N_pM$.

Lemma 6.1 *The germ of the height function* $h^t_{\boldsymbol{v}}$ *(resp.* $h^s_{\boldsymbol{v}}$, $h^\ell_{\boldsymbol{v}}$*) at* p *describes the contact of* $M = \boldsymbol{X}(U)$ *and the hypersphere (resp. equidistant hypersurface, hyperhorosphere)* $S(\boldsymbol{v}, c) = \{\boldsymbol{x} \in H^n_+(-1) \mid \langle \boldsymbol{v}, \boldsymbol{x}\rangle = c\}$ *at* p*, where* $\langle \boldsymbol{v}, p\rangle = c$.

Proof: For any $\boldsymbol{v} \in \mathbb{R}^{n+1}$, consider the function $\lambda_{\boldsymbol{v},c} : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $\lambda_{\boldsymbol{v},c}(x) = \langle \boldsymbol{x}, \boldsymbol{v}\rangle - c$. Denote by $\bar{\lambda}_{\boldsymbol{v},c}$ the restriction of $\lambda_{\boldsymbol{v},c}$ to $H^n_+(-1)$. So $\bar{\lambda}^{-1}_{\boldsymbol{v},c}(0)$ is respectively a hypersphere, equidistant hypersurface, or hyperhorosphere in $H^n_+(-1)$, according to $\boldsymbol{v}$ is timelike, spacelike, or lightlike. Clearly, $\lambda_{\boldsymbol{v},c} \cdot \boldsymbol{X} = \bar{\lambda}_{\boldsymbol{v},c} \cdot \boldsymbol{X}$ coincides respectively with $h^t_{\boldsymbol{v}}$, $h^s_{\boldsymbol{v}}$ or $h^\ell_{\boldsymbol{v}}$ and we have the required result. $\square$

Therefore, we have that for $M \subset H^n_+(-1)$ the singularities of the lightcone height functions family measure the contacts of M with hyperhorospheres in $H^n_+(-1)$ and *the horospherical principal directions of* M (i.e. the lightcone principal directions of M, considered as a codimension 2 submanifold of $\mathbb{R}^{n+1}_1$) *coincide with the asymptotic directions determined by the family* H^ℓ *of lightlike height functions on* M (i.e. the contact directions corresponding to degenerate singularities of the functions $h^\ell_{\boldsymbol{v}}$, $\boldsymbol{v} \in S^{n-1}_+$ on M).

The following provide alternative contact function germs for M with hyperspheres, equidistant hypersurfaces and hyperhorospheres in $H^n_+(-1)$ respectively ([6]):

a) Given $\boldsymbol{v} \in H^n_+(-1)$, the distance squared function from $\boldsymbol{v}$, $L_{\boldsymbol{v}} : H^n_+(-1) \to \mathbb{R}$, is given by $L_{\boldsymbol{v}}(\boldsymbol{x}) = (cosh^{-1}(-\langle \boldsymbol{x}, \boldsymbol{v}\rangle)^2$. Now, given $p_0 = \boldsymbol{X}(u_0) \in M = \boldsymbol{X}(U)$, let $r \in \mathbb{R}$ be such that $L_{\boldsymbol{v}}(p_0) = r^2$. Then the germ $(L_{\boldsymbol{v}} \cdot \boldsymbol{X}, u_0)$ is a contact function germ for the pair $(M, S(\boldsymbol{v}, r))$ at p_0, where $S(\boldsymbol{v}, r)$ denotes the hypersphere $S(\boldsymbol{v}, r) = HP(\boldsymbol{v}, r^2) \cap H^n_+(-1)$.

b) For $\boldsymbol{v} \in S^n_1$ and $r \in \mathbb{R}$ such that $L_{\boldsymbol{v}}(p_0) = r$, we take $L_{\boldsymbol{v}} : H^n_+(-1) \to \mathbb{R}$, given by $L_{\boldsymbol{v},r}(\boldsymbol{x}) = (sinh^{-1}(-\langle \boldsymbol{x}, \boldsymbol{v}\rangle)$. Then, analogously, $(L_{\boldsymbol{v}} \cdot \boldsymbol{X}, u_0)$ is a contact function germ for M and the equidistant hypersurface given by the intersection $HP(\boldsymbol{v}, r) \cap H^n_+(-1)$.

c) For $\boldsymbol{v} \in LC^*_+$ and $r \in \mathbb{R}$ such that $L_{\boldsymbol{v}}(p_0) = r$, we put $L_{\boldsymbol{v}}(\boldsymbol{x}) = log(-\langle \boldsymbol{x}, \boldsymbol{v}\rangle)$ and again $L_{\boldsymbol{v}} \cdot \boldsymbol{X}$ is a contact function for M and the hyperhorosphere $H(\boldsymbol{v}) = \{\boldsymbol{x} \in H^n_+(-1) \mid log(-\langle \boldsymbol{x}, \boldsymbol{v}\rangle) = r\}$.

It follows from the Theorem 5.1 that the functions $h^t_{\boldsymbol{v}}, h^s_{\boldsymbol{v}}$ and $h^\ell_{\boldsymbol{v}}$ must be respectively $\mathcal{K}$-equivalent to the distance functions $L_{\boldsymbol{v}}$ in a), b) and c).

It was shown in [5] that if M is a compact connected smooth $(n-1)$- manifold immersed in $H^n_+(-1)$, then every Morse function $L_{\boldsymbol{v}}$ in a) has exactly two critical points if and only if M is embedded as a metric $(n-1)$-sphere. In view of the above considerations we can now state this in terms of hyperbolic height functions as follows:

Corollary 6.2 *Suppose that M is a compact connected smooth $(n-1)$- manifold immersed in $H^n_+(-1)$. Then every non degenerate timelike height function has exactly two critical points if and only if M is embedded as a metric $(n-1)$-sphere.*

On the other hand, a characterization for complete totally umbilic submanifolds of $H^n_+(-1)$ in terms of the above distance functions was obtained in [6]. In the case of connected, complete hypersurface M in $H^n_+(-1)$, it tells us that M is embedded as a hypersphere, hyperhorosphere, or equidistant hypersurface if and only if every non degenerate function $L_{\boldsymbol{v}}$ of the types a) and b) above has index 0 or $(n-1)$. So we can rephrase this result in terms of height functions in Minkowski space as follows.

Corollary 6.3 *Suppose that M is a connected complete smooth hypersurface in $H^n_+(-1)$. Then every non degenerate timelike or spacelike height function on M has index 0 or $(n-1)$ if and only if M is embedded as a hypersphere, hyperhorosphere, or equidistant hypersurface.*

7 Spacelike submanifolds with a parallel normal frame

We say that a normal vector field $\boldsymbol{n}$ is *parallel* if $D_X\boldsymbol{n}$, for any $X \in T_pM$ and any $p \in M$, where $D_X\boldsymbol{n}$ denotes the normal component of the vector $d\boldsymbol{n}(X) \in T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_pM$.

It can be seen [36] that a manifold M admits a parallel normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ made of a timelike and a spacelike vector fields if and only if it admits some parallel normal field $\boldsymbol{n}$ (which may be either lightlike, timelike or spacelike).

The *normal curvature* of M at p is defined by

$$\begin{array}{rcccl} R^\perp_p & : & T_pM \times T_pM \times N_pM & \longrightarrow & N_pM \\ & & \big(X, Y, \boldsymbol{n}\big) & \longmapsto & D_X(D_Y\boldsymbol{n}) - D_Y(D_X\boldsymbol{n}) - D_{[X,Y]}\boldsymbol{n}. \end{array}$$

We remind that in case that the normal curvature vanishes identically, then M is said to have *flat normal bundle.*

Lemma 7.1 *A spacelike submanifold M admits locally some parallel normal frame in $\mathbb{R}^{n+1}_1$ if and only if its normal curvature vanishes identically.*

Proof. This is a well known property for any connection on a fibre bundle that it is flat if and only if it is locally parallelizable (see for instance [37]). □

Corollary 7.2 *Suppose that $\boldsymbol{n}^T$ is a parallel timelike field over M and let $p \in M$. Then there exists an orthonormal frame $\{\boldsymbol{X}_1, \cdots \boldsymbol{X}_{n-1}\}$ of T_pM of common eigenvectors for the shape operators associated to the fields $\boldsymbol{n}^T$, $\boldsymbol{n}^S$ and $\boldsymbol{n}^T + \boldsymbol{n}^S$, where $\boldsymbol{n}^S$ is obtained from $\boldsymbol{n}^T$ as in §3.*

Proof. Since M has a parallel normal vector field, the above lemma implies that its normal curvature vanishes identically. We denote by $S_p(\boldsymbol{n}^T)$ and $S_p(\boldsymbol{n}^S)$ the shape operators of the normal vectors $\boldsymbol{n}^T$, $\boldsymbol{n}^S$. The Ricci equation (see [46]) implies that

$$0 = \langle R_p^{\perp}(X,Y)\boldsymbol{n}^T, \boldsymbol{n}^S\rangle = \langle S_p(\boldsymbol{n}^T)(X), S_p(\boldsymbol{n}^S)(Y)\rangle - \langle S_p(\boldsymbol{n}^T)(Y), S_p(\boldsymbol{n}^S)(X)\rangle,$$

for any $X, Y \in T_pM$ and $p \in M$. But this can be written as

$$\langle S_p(\boldsymbol{n}^S) \circ S_p(\boldsymbol{n}^T)(X), Y\rangle = \langle Y, S_p(\boldsymbol{n}^T) \circ S_p(\boldsymbol{n}^S)(X)\rangle,$$

for any $X, Y \in T_pM$ and $p \in M$. This is equivalent to the fact that the two self-adjoint operators $S_p(\boldsymbol{n}^T)$ and $S_p(\boldsymbol{n}^S)$ commute, which is also equivalent to that they can be diagonalized simultaneously. That is, there is an orthonormal frame $\{\boldsymbol{X}_1, \cdots \boldsymbol{X}_{n-1}\}$ of T_pM of common eigenvectors for $S_p(\boldsymbol{n}^T)$ and $S_p(\boldsymbol{n}^S)$. Obviously, the vectors of the frame are also eigenvectors of $S_p(\boldsymbol{n}^T + \boldsymbol{n}^S)$. □

Remark 7.3 *It follows that the orthonormal frame* $\{\boldsymbol{X}_1, \cdots \boldsymbol{X}_{n-1}\}$ *provides a basis of principal directions for any normal field* $\boldsymbol{n}$ *on* M.

Given a unit timelike normal field $\boldsymbol{n}^T$ on a spacelike $(n-1)$-submanifold $M = \boldsymbol{X}(U)$ of $\mathbb{R}_1^{n+1}$, we can consider the map $\mathbb{L}_{\boldsymbol{n}^T} : U \to H_+^n(-1)$, given by $\mathbb{L}_{\boldsymbol{n}^T}(u) = \boldsymbol{n}^T(u)$. If $\boldsymbol{n}^T$ is non-degenerate, then $\mathbb{L}_{\boldsymbol{n}^T}$ is an immersion. We denote $\bar{M} = \mathbb{L}_{\boldsymbol{n}^T}(M)$ and $\bar{p}_0 = \mathbb{L}_{\boldsymbol{n}^T}(u_0)$, where $p_0 = \boldsymbol{X}(u_0)$. We observe that, $\widetilde{\mathbb{L}}(p) = \widetilde{\mathbb{L}}(\mathbb{L}_{\boldsymbol{n}^T}(p)), \forall p \in M$, where the left-hand-side refers to the lightcone Gauss map on $M \subset \mathbb{R}_1^{n+1}$ and the right-hand-side to the horospherical Gauss map on $\bar{M} \subset H_+^n(-1)$.

Proposition 7.4 *Let* $\boldsymbol{n}^T$ *be a unit parallel timelike non-degenerate normal field on* M. *Then we have:*

a) $T_{\bar{p}}\bar{M} = T_pM$, $N_{\bar{p}}\bar{M} = N_pM$ *and* $d_p\mathbb{L}_{\boldsymbol{n}^T} : T_pM \to T_{\bar{p}}\bar{M}$ *is an isomorphism.*

b) The linear map $d_p\mathbb{L}_{\boldsymbol{n}^T}$ *takes lightcone principal directions of* M *at* p *into horospherical principal directions of* $\bar{M}$ *at* $\bar{p}$, $\forall p \in M$.

Proof. a) Given $X \in T_pM$, it follows from Weingarten equation that

$$d_p\mathbb{L}_{\boldsymbol{n}^T}(X) = d_p\boldsymbol{n}^T(X) = D_X\boldsymbol{n}^T - S_p(\boldsymbol{n}^T)(X).$$

If $\boldsymbol{n}^T$ is parallel, we have that $d_p\mathbb{L}_{\boldsymbol{n}^T}(X) = -S_p(\boldsymbol{n}^T)(X) \in T_pM$. This shows that $T_{\bar{p}}\bar{M} \subset T_pM$ and the equality follows from the fact that $\boldsymbol{n}^T$ is non-degenerate. Obviously, we also have that $N_{\bar{p}}\bar{M} = N_pM$ and $d_p\mathbb{L}_{\boldsymbol{n}^T} : T_pM \to T_{\bar{p}}\bar{M}$ is an isomorphism.

b) We denote by II and $\bar{I}I$ the shape tensors of M and $\bar{M}$ respectively. That is, given $X, Y \in T_pM$, we have

$$II(X,Y) = (d_p\tilde{Y}(X))^{\perp}, \quad \bar{I}I(X,Y) = (d_p\bar{Y}(X))^{\perp},$$

where $\tilde{Y}, \bar{Y}$ denote local extensions of Y in $M, \bar{M}$ respectively. Assume that $\bar{Y}$ is give in local coordinates by $\bar{Y} = \sum_i f_j \frac{\partial}{\partial x_i}$ for some functions f_i locally defined in a neighbourhood of $\bar{p}$ in

$\bar{M}$. Then we can also consider the induced extension $\tilde{Y} = \sum_i (f_j \circ \mathbb{L}_{\boldsymbol{n}^T})\frac{\partial}{\partial x_i}$ in M. By using these extensions, we obtain

$$\begin{aligned} II(X,Y) &= (d_p\tilde{Y}(X))^{\perp} \\ &= \left(\sum_i X(f_j \circ \mathbb{L}_{\boldsymbol{n}^T})\frac{\partial}{\partial x_i}\right)^{\perp} \\ &= \left(\sum_i d_p\mathbb{L}_{\boldsymbol{n}^T}(X)(f_i)\frac{\partial}{\partial x_i}\right)^{\perp} \\ &= (d_p\bar{Y}(d_p\mathbb{L}_{\boldsymbol{n}^T}(X)))^{\perp} \\ &= \bar{I}I(d_p\mathbb{L}_{\boldsymbol{n}^T}(X),Y) \\ &= -\bar{I}I(S_p(\boldsymbol{n}^T)(X),Y). \end{aligned}$$

We denote now by $S_p(\boldsymbol{n})$ and $\bar{S}_p(\boldsymbol{n})$ the shape operators associated to a normal vector $\boldsymbol{n}$ in M and $\bar{M}$ respectively. The above computation give us the relationship between both operators:

$$\bar{S}_p(\boldsymbol{n}) = -S_p(\boldsymbol{n}) \circ S_p(\boldsymbol{n}^T)^{-1}. \tag{1}$$

By corollary 7.2 there is an orthonormal frame $\{\boldsymbol{X}_1, \cdots \boldsymbol{X}_{n-1}\}$ of T_pM of principal directions for the fields $\boldsymbol{n}^T$, $\boldsymbol{n}^S$ and $\boldsymbol{n}^T + \boldsymbol{n}^S$ in M. This means that

$$S_p(\boldsymbol{n}^T)(\boldsymbol{X}_i) = \lambda_i\boldsymbol{X}_i, \quad S_p(\boldsymbol{n}^S)(\boldsymbol{X}_i) = \mu_i\boldsymbol{X}_i,$$

for some $\lambda_i \neq 0$ and μ_i, $i = 1, \ldots, n-1$. By using (1), this gives in $\bar{M}$ that

$$\bar{S}_p(\boldsymbol{n}^T)(\boldsymbol{X}_i) = -\boldsymbol{X}_i, \quad \bar{S}_p(\boldsymbol{n}^S)(\boldsymbol{X}_i) = -\frac{\mu_i}{\lambda_i}\boldsymbol{X}_i.$$

In particular, $\{\boldsymbol{X}_1, \cdots, \boldsymbol{X}_{n-1}\}$ are also principal directions for $\boldsymbol{n}^T$, $\boldsymbol{n}^S$ and $\boldsymbol{n}^T + \boldsymbol{n}^S$ in $\bar{M}$. Note that since $\bar{M}$ is contained in $H^n_+(-1)$, the lightcone and horospherical principal directions coincide. □

Proposition 7.5 *Let $\boldsymbol{n}^T$ be a non degenerate timelike parallel normal field on M and let $HP(\boldsymbol{v}, p_0)$ represent the hyperplane orthogonal to $\boldsymbol{v}$ through the point $p_0 \in M$. Then the contact function of M with $HP(\boldsymbol{v}, p_0)$ at p_0 has the same corank and codimension than the contact function of $\bar{M}$ with $HP(\boldsymbol{v}, \bar{p}_0)$ at $\bar{p}_0$. In particular, M has non degenerate contact with $HP(\boldsymbol{v}, p_0)$ at p_0 if and only if $\bar{M}$ has non degenerate contact with $HP(\boldsymbol{v}, \bar{p}_0)$ at $\bar{p}_0$.*

Proof. Assume that M is parametrized locally as $M = \boldsymbol{X}(U)$, where $u_0 \in U$ and $p_0 = \boldsymbol{X}(u_0)$. The contact function of M with $HP(\boldsymbol{v}, p_0)$ at p_0 is denoted by $h_{\boldsymbol{v}} : U \to \mathbb{R}$ and is given by $h_{\boldsymbol{v}}(u) = \langle \boldsymbol{v}, \boldsymbol{X}(u)\rangle$.

Analogously, $\bar{M}$ is parametrized locally as $\bar{M} = \mathbb{L}_{\boldsymbol{n}^T} \circ \boldsymbol{X}(U)$ with $\bar{p}_0 = \mathbb{L}_{\boldsymbol{n}^T} \circ \boldsymbol{X}(u_0)$. The contact function of $\bar{M}$ with $HP(\boldsymbol{v}, \bar{p}_0)$ at $\bar{p}_0$ is $\bar{h}_{\boldsymbol{v}} : U \to \mathbb{R}$, defined by $\bar{h}_{\boldsymbol{v}}(u) = \langle \boldsymbol{v}, \mathbb{L}_{\boldsymbol{n}^T} \circ \boldsymbol{X}(u)\rangle$.

Because of proposition 7.4, part a), we deduce

$$d_p\mathbb{L}_{\boldsymbol{n}^T}\left(\frac{\partial X}{\partial u_i}\right) = \sum_{i=1}^{n-1} a_{ij}\frac{\partial X}{\partial u_j},$$

for some smooth functions a_{ij} such that $\det(a_{ij}) \neq 0$. From this we obtain that

$$\frac{\partial \bar{h}_{\boldsymbol{v}}}{\partial u_i} = \sum_{i=1}^{n-1} a_{ij} \frac{\partial h_{\boldsymbol{v}}}{\partial u_j}.$$

In particular, $\bar{h}_{\boldsymbol{v}}$ and $h_{\boldsymbol{v}}$ have the same jacobian ideals (i.e., the ideals generated by partial derivatives) in the local algebra $C^\infty(U, u_0)$. Since both the corank and the codimension are computed from the jacobian ideals, we deduce that they have the same corank and codimension at u_0. $\square$

We observe that in general $\mathcal{K}(M, HP(\boldsymbol{v}, p_0); p_0) \neq \mathcal{K}(\bar{M}, HP(\boldsymbol{v}, \bar{p}_0); \bar{p}_0)$, although both contact classes must have the same Thom-Boardman symbol [11].

Corollary 7.6 *Spacelike $(n-1)$-submanifolds with vanishing normal curvature in $\mathbb{R}^{n+1}_1$ admit exactly $n-1$ orthogonal asymptotic directions at one of its non critical points (for which all the directions are asymptotic).*

Proof. Let M be a $(n-1)$ submanifold with vanishing normal curvature and let $\boldsymbol{n}$ be a parallel unit timelike field defined in a neighbourhood of a point $p \in M$. Let $\bar{M} = \boldsymbol{n}(M) \subset H^n_+(-1)$. It follows from proposition 7.5 that a direction $\theta \in T_pM$ is an asymptotic direction for M at p if and only if $\bar{\theta} = d_pL_{\boldsymbol{n}^T}(\theta) \in T_{\bar{p}}\bar{M}$, where $\bar{p} = \boldsymbol{n}(p)$, is an asymptotic direction for $\bar{M}$ at $\bar{p}$. This means that there is some (osculating) hyperplane $H(\boldsymbol{v}, c)$ having a degenerate contact with $\bar{M}$ at $\bar{p}$ along the direction $\bar{\theta}$. But then it follows from Lemma 6.1 that the hypersphere $S(\boldsymbol{v}, c) = H(\boldsymbol{v}, c) \cap H^n_+(-1)$ of $H^n_+(-1)$ has degenerate contact with $\bar{M}$ at $\bar{p}$ along the direction $\bar{\theta}$ too. So $\bar{\theta}$ can be seen as a principal direction associated to some normal field on $\bar{M}$.

We consider now the conformal map $\varphi : H^n_+(-1) \to \mathbb{R}^n_0 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 | \, x_0 = 0\}$ which is given by the composition of the stereographic projections. This a diffeomorphism taking hyperspheres (where we include the equivariant hypersurfaces and hyperhorospheres as degenerate ones) in $H^n_+(-1)$ to hyperspheres (including hyperplanes) in $\mathbb{R}^n_0$ preserving their respective contacts with $\bar{M}$ and $\varphi(\bar{M})$. Consequently, it determines a bijection between their corresponding contact directions. But these are the principal directions of $\varphi(\bar{M})$ and the asymptotic directions of $\bar{M}$. Since $\varphi(\bar{M})$ has exactly $(n-1)$ orthogonal principal directions at each non umbilic point, so must have $\bar{M}$. Therefore, M also has exactly $(n-1)$ orthogonal asymptotic directions at each non critical point. $\square$

Remark 7.7 *Since each foliation of the asymptotic configuration on M can be seen as a principal foliation associated to a binormal field on M it follows that when M has vanishing normal curvature the asymptotic foliations grid must coincide with the lightcone principal configuration and its critical points coincide with the lightcone umbilics.*

A particular case of codimension 2 spacelike submanifolds with a parallel normal field is given by those that admit some umbilic normal field. First of all we observe that, as a consequence of the Ricci equation ([46], p. 125), it can be shown (in a similar manner to the riemannian case) that if p is an umbilic point for some normal field $\boldsymbol{n}$ then $R^\perp_p = 0$. Moreover, having vanishing normal curvature on M is equivalent to having flat normal bundle, and thus we have that spacelike submanifolds that admit some umbilic field also admit some parallel normal frame.

It was shown in ([25], Theorem 4.3) that given a spacelike $(n-1)$-submanifold M in $\mathbb{R}^{n+1}_1$, which is totally umbilic for some lightlike parallel normal field $\boldsymbol{n}$ with curvature κ, then either M is contained in some light cone (if $\kappa \neq 0$), or M lies in a lightlike hyperplane ($\kappa = 0$). Moreover,

i) If a spacelike $(n-1)$-submanifold M is contained in hyperbolic n-space, then the position vector field $\boldsymbol{X}$ is a parallel timelike normal field along M which is umbilic with constant (non vanishing) curvature on M.

ii) If a spacelike $(n-1)$-submanifold M is contained in de Sitter n-space, then the position vector field $\boldsymbol{X}$ is a parallel spacelike normal field along M which is umbilic with constant (non vanishing) curvature on M. Then we have that the vector field

$$\nu(u) = \frac{\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)}{\|\boldsymbol{X}(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{X}_{u_{n-1}}(u)\|},$$

is a timelike parallel field globally defined on M. We call the map $\nu : M \to H^n_+(-1)$ *timelike Gauss map* on M. We observe that ν is non-degenerate if and only if the ν-parabolic set is empty. In other words, if and only if it defines an immersion of M in $H^n_+(-1)$.

iii) If a spacelike $(n-1)$-submanifold M is contained in the lightcone of $\mathbb{R}^{n+1}_1$, then the position vector field $\boldsymbol{X}$ is a parallel lightlike normal field along M which is umbilic with constant (non vanishing) curvature on M.

iv) If a spacelike $(n-1)$-submanifold M is contained in a (spacelike, timelike or lightlike) hyperplane of $\mathbb{R}^{n+1}_1$, then the normal vector $\boldsymbol{v}$ to the hyperplane determines a constant (timelike, spacelike or lightlike) normal field along M which is umbilic with vanishing curvature on M.

Proposition 7.8 *A spacelike $(n-1)$-submanifold M of $\mathbb{R}^{n+1}_1$ is umbilic for some lightlike normal field $\boldsymbol{n}^L$ if and only if M is contained in some lightcone $LC_{\boldsymbol{\lambda}}$ (provided $\boldsymbol{n}^L$ is not constant) or in a lightlike hyperplane pseudo-orthogonal to $\boldsymbol{n}^L$ (in case that $\boldsymbol{n}^L$ is a constant field).*

Proof. Suppose that $\boldsymbol{n}^L$ is a lightlike normal field which is umbilic over M. Then there exists some parallel field $\boldsymbol{n}^T$, that we can assume timelike over M. We can construct as in §3 another field $\boldsymbol{n}^S$ which is also parallel over M. Then the lightlike field $\boldsymbol{n}^T + \boldsymbol{n}^S$ is also parallel and satisfies that $\boldsymbol{n}^T + \boldsymbol{n}^S = \mu \boldsymbol{n}^L$. Since $\boldsymbol{n}^L$ is umbilic, we have that $\boldsymbol{n}^T + \boldsymbol{n}^S$ is also umbilic. Therefore M admits an umbilic lightlike parallel field, which implies the required result. □

8 Some global consequences

8.1 Characterization of metric spheres

We use the results obtained in §7 in order to transport some known results concerning hypersurfaces in Euclidean space to new results on spacelike codimension 2 submanifolds with vanishing normal curvature in Minkowski space. We consider first the non degenerate (Morse) contacts of hypersurfaces with hyperspheres and we get the following characterization of total umbilicity in terms of timelike height functions.

Theorem 8.1 *Suppose that M is a compact connected smooth $(n-1)$- manifold immersed in $\mathbb{R}^{n+1}_1$. Then M is a metric $(n-1)$-sphere contained in a spacelike hyperplane if and only if M*

has a globally defined non-degenerate parallel normal field and every non degenerate timelike height function has exactly two critical points on M.

Proof: We first observe that a metric $(n-1)$-sphere contained in a spacelike hyperplane in $\mathbb{R}^{n+1}_1$ necessarily has a globally defined non-degenerate parallel normal field. Then the result follows as a consequence of Corollary 6.2 and Proposition 7.5. $\square$

Observe that the existence of some globally defined non-degenerate parallel normal field on M implies that M has vanishing normal curvature and never vanishing Gaussian curvature. We have the following corollary of the above theorem.

Corollary 8.2 *Suppose that M is a compact connected $n-1$-manifold spacelike immersed in S^n_1. Then M is a spacelike $(n-1)$-sphere in S^n_1 if and only if the timelike normal is non-degenerate and every non-degenerate timelike height function has exactly two critical points on M. Here a spacelike $(n-1)$-sphere in S^n_1 is defined to be the intersection of a spacelike hyperplane with S^n_1.*

8.2 Lightcone configurations and Caratèodory's type Conjectures on surfaces

We shall look next to the degenerate contacts in order to relate the lightcone configurations of spacelike codimension 2 submanifolds with vanishing normal curvature in Minkowski space with the principal configurations of submanifolds of codimension 2 in Euclidean space.

Consider the conformal map $\varphi : H^n_+(-1) \to \mathbb{R}^n_0 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 |\ x_0 = 0\}$ which is given by the composition of the stereographic projections. Since φ is a conformal map, it maps the hyperspheres of $H^n_+(-1)$ into hyperspheres of $\mathbb{R}^n_0$. On the other hand φ is a diffeomorphism and thus preserves contacts and contact directions. Therefore, given any $(n-1)$-submanifold $M \subset H^n_+(-1)$ we have that $d\varphi$ takes the contact direction of M with any hypersphere S of $H^n_+(-1)$ at a point $p \in M$ to the contact directions of $\varphi(M)$ with the hypersphere $\varphi(S)$ at $\varphi(p)$ in $\mathbb{R}^n$.

Theorem 8.3 *Given $M \subset H^n_+(-1)$, the conformal map $\varphi : H^n_+(-1) \to \mathbb{R}^n_0$ takes the horospherical configuration of M into the principal configuration of $\varphi(M)$ in $\mathbb{R}^n_0$.*

Proof: This follows from the following facts: a) The map φ preserves contacts with hyperspheres; b) the horospherical principal directions at each point are principal directions for any normal field on M. $\square$

We consider next the particular case of surfaces in Minkowski 4-space. A surface $M \subset H^3_+(-1)$ shall be called *generic* provided the height functions families H^t, H^s and H^ℓ are generic families of functions in the sense that the germ of $\lambda(\boldsymbol{X})$ at (u,v) is a versal unfolding of the germ of $\boldsymbol{x}_v = \lambda(\boldsymbol{X})(-,v)$ at u, for all $u \in U$ and for all $v \in H^3_+(-1), S^3_1$ or S^2_+. We observe that this means that the family of squared-distance functions on the surface $\varphi(M) \subset \mathbb{R}^3$, $\lambda(\varphi \cdot \boldsymbol{X}) : U \times S^2 \to \mathbb{R}$, is a generic family too. It follows from Looijenga's genericity theorem [39], or equivalently from Montaldi's genericity theorem [41] that the subset of generic immersions of a given surface M in $\mathbb{R}^3$ is residual in the Whitney C^∞-topology on the total set of immersions of M in $\mathbb{R}^3$. Consequently, the subset of generic immersions of a given surface in $H^3_+(-1)$ is residual in the Whitney C^∞-topology on the total set of immersions of M in $H^3_+(-1)$ too.

It is a well known fact that the umbilic points of generically immersed surfaces in $\mathbb{R}^3$ are of Darbouxian type, and thus they have index $\pm\frac{1}{2}$ ([13], [54]). Then it follows:

Theorem 8.4 *The horospherical configurations in a neighbourhood of a horoumbilical point in a generic surface M in $H^3_+(-1)$ are of Darbouxian type $D_i, i = 1, 2, 3$. Therefore, the index of the lightcone principal direction fields at a lightlike umbilic point of a generic surface generically immersed in $H^3_+(-1)$ is $\pm\frac{1}{2}$.*

As a consequence of Poincaré-Hopf formula we have the horospherical analogous of Feldman's result [7] on the number of umbilic points of generic closed surfaces in Euclidean 3-space:

Corollary 8.5 *The number of horoumbilical points of any closed (compact without boundary) surface M generically immersed in $H^3_+(-1)$ is greater or equal than $2|\chi(M)|$, where $\chi(M)$ denotes the Euler number of M.*

Consequently any 2-sphere generically immersed in $H^3_+(-1)$ has at least 4 horoumbilical points.

In a general (non necessarily generic) situation, there is a Carathéodory's conjecture that states that any 2-sphere immersed in euclidean 3-space has at least 2 umbilics.

We can also use the above arguments in order to assert that Loewner's and Caratheodory's conjectures on umbilic points of surfaces in Euclidean 3-space hold if and only if they hold for surfaces in Hyperbolic 3-space. Therefore, as a consequence of the proof of the analytic version of Loewner's conjecture for surfaces in $\mathbb{R}^3$, we obtain

Theorem 8.6 *The index of the horospherical principal direction fields at a horoumbilical point of an analytic surface in $H^3_+(-1)$ is at most 1.*

From which the following Carathéodory's type result follows,

Corollary 8.7 *Any 2-sphere analytically immersed in $H^3_+(-1)$ has at least two horoumbilical points.*

We can now use the considerations made in §7 in order to transport these results to the lightcone configurations and lightlike umbilic points of spacelike surfaces with vanishing normal curvature in Minkowski 4-space.

A spacelike surface in $\mathbb{R}^4_1$ is said to be *semiumbilical* if it admits some umbilic field locally defined at each one of its points. This is equivalent to asking that the curvature ellipse degenerates into a segment at every point, moreover the umbilic normal field is pseudo-normal to this segment (see [25]). On the other hand, in the particular case of spacelike surfaces immersed in 4-dimensional Minkowski space, the semiumbilicity condition is equivalent to the existence of a globally defined parallel normal field. In particular we can take this normal field to be timelike. Again, as a consequence of the above results on surfaces in $H^3_+(-1)$ together with Proposition 7.4 we have that *Loewner's and Caratheodory's conjectures on umbilic points of surfaces in Euclidean 3-space hold if and only if they hold for semiumbilical spacelike surfaces in Minkowski 4-space.* So, relying on the analytic version of Loewner's conjecture for surfaces in $\mathbb{R}^3$, we can state

Theorem 8.8 *The index of the lightcone principal direction fields at a lightcone umbilic of a spacelike analytic surface with vanishing normal curvature in* $\mathbb{R}^4_1$ *is at most* 1.

From which the following Carathéodory's type result would follow,

Corollary 8.9 *Analytic semiumbilical spacelike* 2*-sphere immersed with vanishing normal curvature in* $\mathbb{R}^4_1$ *have at least two lightcone umbilics.*

We observe that these results have been obtained by through the analysis of the contacts of the submanifolds with hyperplanes in Minkowski space. Moreover, we have that affine transformations, preserve these contacts and thus they take asymptotic configurations into asymptotic configurations (but do not respect their orthogonality). So we can conclude that the critical points of the asymptotic configuration (inflection points) of any spacelike surface which is affinely equivalent to semiumbilic analytic spacelike surface also satisfy the above properties. On the other hand, we cannot say the same with respect to the lightcone principal configurations, for they are preserved by Lorentz transformations (that also preserve the semiumbilicity property) but not by affine transformations. In view of this, we think that it is relevant to push forward the following more general:

Carathéodory's type conjecture for spacelike surfaces in $\mathbb{R}^4_1$: *Any spacelike* 2*-sphere immersed in* $\mathbb{R}^4_1$ *whose asymptotic foliations are globally defined has at least two inflection points.*

8.3 4-Flattenings theorems for closed spacelike curves in Minkowski 3-space

A vertex of a curve α in the Euclidean plane is an extremum of its curvature function. These points can also be characterized as:

a) Singular points of the evolute (locus of centers of curvature) of α;

b) Points at which the contact of α with its osculating circle is of order at least three.

Observe that the osculating circle at a point $\alpha(s)$ is characterized by having contact of order at least two with the curve at $\alpha(s)$. This condition can be paraphrased in terms of singularities of distance functions (as in section 5) by saying that the distance squared function from the center of the circle has a singularity of type $A_{k\geq 2}$ at the point s. Moreover, the condition b) the above is equivalent to saying that a vertex is a singularity of type $A_{k\geq 3}$ of the distance squared function from the corresponding curvature center of α ([4]).

Given a spacelike curve $\boldsymbol{\gamma} : S^1 \to \mathbb{R}^3_1$ in Minkowski 3-space parameterized by the arc-length parameter s, we can take $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and define the *curvature* of $\boldsymbol{\gamma}$ as $\kappa(s) = ||\boldsymbol{\gamma}''(s)||$. If $\kappa(s) \neq 0$, then the unit principal normal vector $\boldsymbol{n}(s)$ of $\boldsymbol{\gamma}$ at s is given by $\boldsymbol{\gamma}''(s) = \kappa(s)\boldsymbol{n}(s)$. Provided $\boldsymbol{\gamma}''(s)$ neither vanish, nor is a lightlike vector, we have that $\kappa(s) \neq 0$, in such case we define the *binormal vector* of $\boldsymbol{\gamma}$ at s as $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$. If we put $\delta(s) = \langle \boldsymbol{n}(s), \boldsymbol{n}(s)\rangle$, we have that $\langle \boldsymbol{b}(s), \boldsymbol{b}(s)\rangle = -\delta(s)$. Therefore, $\boldsymbol{b}(s)$ is spacelike (timelike resp.) if and only if $\boldsymbol{n}(s)$ is timelike (spacelike resp.). Then the following Frenet-Serre type formulae hold:

$\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s),$

$$\begin{aligned}
&\boldsymbol{n}'(s) = -\delta(s)\kappa(s)\boldsymbol{t}(s) - \tau(s)\boldsymbol{b}(s),\\
&\boldsymbol{b}'(s) = \tau(s)\boldsymbol{n}(s),
\end{aligned}$$

where $\tau(s)$ is the torsion of $\boldsymbol{\gamma}$ at s ([20]). Analogously to the case of curves in Euclidean 3-space, we say that a point $\boldsymbol{\gamma}(s)$ is a *flattening* of $\boldsymbol{\gamma}$ provided $\tau(s) = 0$. It is not difficult to see that, analogously to what happens in the Euclidean case ([4]), a point $\boldsymbol{\gamma}(s)$ is a flattening of $\boldsymbol{\gamma}$ if and only if the height function in the direction $\boldsymbol{b}(s)$ has a singularity of type $A_{k\geq 3}$. In other words, $\boldsymbol{\gamma}$ has contact of order at least three with its osculating plane.

It follows from lemma 6.1 that in the particular case of a curve immersed in $H^2_+(-1)$, a flattening is a point at which the curve has contact of order at least three with some circle, equidistant line, or horocycle according to the vector $\boldsymbol{b}(s)$ is either timelike, spacelike or lightlike. Such points are also known as *geodesic vertices* of $\boldsymbol{\gamma}$ as a curve in $H^2_+(-1)$, that is, zeroes of the geodesic curvature. In fact, for a unit speed curve $\boldsymbol{\gamma} : I \to H^2_+(-1)$, we can take $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ as above and define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s)$, so we get a pseudo-orthonormal frame $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{e}\}$ along $\boldsymbol{\gamma}$, for which the following Frenet-Serre type equations hold ([28]),

$$\begin{aligned}
&\boldsymbol{\gamma}'(s) = \boldsymbol{t}(s),\\
&\boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{e}(s),\\
&\boldsymbol{e}'(s) = -\kappa_g(s)\boldsymbol{t}(s),
\end{aligned}$$

where $\kappa_g(s) = det(\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{t}'(s))$ is the *geodesic curvature function* on $\boldsymbol{\gamma}$.

Now, we can write $\boldsymbol{n}$ and $\boldsymbol{b}$ in terms of $\boldsymbol{\gamma}$ and $\boldsymbol{e}$:

$$\boldsymbol{n} = \frac{1}{\sqrt{|\kappa_g^2 - 1|}}(\boldsymbol{\gamma} + \kappa_g\boldsymbol{e}), \quad \boldsymbol{b} = \frac{1}{\sqrt{|\kappa_g^2 - 1|}}(\kappa_g\boldsymbol{\gamma} + \boldsymbol{e}).$$

Provided $\kappa_g(s) \neq 1$ (or equivalently, $\kappa(s) = 0$), we can distinguish two cases:

a) $\kappa_g^2(s) > 1$, which implies that $\delta(s) = -1$ and thus $\boldsymbol{b} \in H^2_+(-1)$,

b) $\kappa_g^2(s) < 1$, which implies that $\delta(s) = 1$ and thus $\boldsymbol{b} \in S^2_1$.

By derivation in the above expression of $\boldsymbol{n}$, we obtain the following relations between τ, κ and κ_g:

$$\tau(s) = \frac{\kappa_g'}{1 - \kappa_g^2}, \quad \kappa(s) = \sqrt{|\kappa_g^2 - 1|}.$$

And thus it follows that provided $\kappa(s_0) \neq 0$, then $\boldsymbol{\gamma}(s_0)$ is a flatenning if and only if it is a geodesic vertex.

By considering the stereographic projection $\phi : H^2_+(-1) \to \mathbb{R}^2$ we see, as in the previous section, that since ϕ is a conformal map and preserves contacts with circles (equidistant lines and horocycles considered as a particular case), it must take the vertices of a curve in $H^2_+(-1)$ onto the vertices of its plane image. It now follows from the 4-vertex theorem for curves in the Euclidean plane ([43]) that *any closed regular simple curve in* $H^2_+(-1)$ *has at least* 4 *vertices (i.e., flattening points). If we allow the curve to have self-intersections then we can only ensure the existence of two flattening points* (for any closed plane curve has a minimum and a maximum of its curvature function). We observe that this is a well known result. A first proof (that uses different tools) was obtained by C. M. Fulton [8] (some related results can be seen in [18] and [56]).

We now use the techniques developed in Section 7 in order to generalize this to a wider class of spacelike curves in Minkowski 3-space. Suppose that $\boldsymbol{\gamma}$ is a closed spacelike curve that admits some globally defined non-degenerate parallel timelike normal field ν. Then the composition $\nu \cdot \boldsymbol{\gamma} = \bar{\boldsymbol{\gamma}}$ is a closed regular curve in the hyperbolic plane $H^2_+(-1)$. The curve $\bar{\boldsymbol{\gamma}}$ is simple provided $\nu(s) \neq \nu(s'), \forall s \neq s'$. So as a consequence of the Proposition 7.5 we can state,

Theorem 8.10 *Any closed curve that admits a globally defined non-degenerated parallel timelike normal field* ν *has at least two flattening points. If* ν *satisfies that* $\nu(s) \neq \nu(s'), \forall s \neq s'$, *then the curve has at least* 4 *flattening points.*

We now investigate under which conditions we can ensure the existence of some globally defined non-degenerate timelike parallel normal field along the closed spacelike curve $\boldsymbol{\gamma}$ in $\mathbb{R}^3_1$. We observe first that any parallel field along $\boldsymbol{\gamma}$ must have constant norm, therefore it is either globally timelike, spacelike, or lightlike. Moreover, the existence of a spacelike parallel normal field implies the existence of a timelike one (just rotate this field a right angle in the normal plane of the curve). Let ν be a unit normal field along $\boldsymbol{\gamma}$. Then we can write $\nu(s) = cosh\theta(s)\boldsymbol{n}(s) - \delta sinh\theta(s)\boldsymbol{b}(s)$. We observe that $\langle \boldsymbol{n}, \boldsymbol{n} \rangle = \langle \nu, \nu \rangle$.

By derivating and applying the Frenet-Serre equations we get,

$$\begin{aligned} \nu'(s) &= -\theta' sinh\theta \boldsymbol{n} + cosh\theta(-\delta\kappa \boldsymbol{t} - \tau \boldsymbol{b}) - \delta(\theta' cosh\theta \boldsymbol{b} + sinh\theta\tau \boldsymbol{n}) \\ &= -\delta\kappa cosh\theta \boldsymbol{t} - (\delta sinh\theta\tau + \theta' sinh\theta)\boldsymbol{n} - (\delta\theta' cosh\theta + \tau cosh\theta)\boldsymbol{b}. \end{aligned}$$

Therefore we have that

a) ν is parallel if and only if $\theta' = -\delta\tau$;

b) ν is non-degenerate at s if and only if $\nu(s) \neq \boldsymbol{b}(s)$ and $\kappa(s) \neq 0$.

So, the existence of a globally defined parallel timelike field ν taking $\boldsymbol{\gamma}$ into a closed spacelike curve $\boldsymbol{\gamma}^\nu \subset H^2_+(-1)$ is equivalent to the vanishing of the total torsion $\int \tau$ of $\boldsymbol{\gamma}$. On the other hand, $\boldsymbol{\gamma}^\nu$ is a regular curve if and only if ν is non-degenerate, which implies that $\boldsymbol{\gamma}$ must have non vanishing curvature.

In the particular case of a regular simple spacelike curve $\boldsymbol{\gamma} : S^1 \to S^2_1$ in de Sitter 2-space we have a natural parallel timelike normal field globally defined. In fact, if $\boldsymbol{\gamma}$ has unit speed, we have that the position vector $\boldsymbol{\gamma}(s)$ determines a parallel spacelike normal field along $\boldsymbol{\gamma}$ and then $\boldsymbol{\gamma}(s) \wedge \boldsymbol{\gamma}'(s)$, is also a parallel timelike normal field globally defined on $\boldsymbol{\gamma}$. Similarly to the case of curves in Hyperbolic plane, we can put $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s)$. Then we get $\boldsymbol{\gamma}'(s) = \boldsymbol{t}(s)$,

$$\begin{aligned} &\boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{e}(s), \\ &\boldsymbol{e}'(s) = -\kappa_g(s)\boldsymbol{t}(s). \end{aligned}$$

with $\kappa_g(s) = det(\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{t}'(s))$ is the *geodesic curvature function* on $\boldsymbol{\gamma}$. We call the map $\boldsymbol{e} : S^1 \to H^2_+(-1)$, *timelike Gauss map* on $\boldsymbol{\gamma}$.

Proceeding as above, we obtain now:

$$\tau(s) = \frac{\kappa_g'}{\kappa_g^2 - 1}, \quad \kappa(s) = \sqrt{|1 - \kappa_g^2|}.$$

And hence, we again have that $\tau(s) = 0$ if and only if $\kappa_g'(s) = 0$ and $\kappa_g(s) \neq \pm 1$. Points satisfying $\kappa_g'(s) = 0$ are called *geodesic vertices* of $\boldsymbol{\gamma}$. We observe that $\kappa_g(s) = \pm 1$ if and only if $\kappa(s) = 0$.

Lemma 8.11 *The image $\bar{\gamma}$ of the timelike Gauss map $\boldsymbol{e}$ on $\boldsymbol{\gamma}$ is an embedded curve in $H_+^2(-1)$ if and only if $\kappa_g \neq 0$.*

Proof: Since $\boldsymbol{e}'(s) = -\kappa_g(s)\boldsymbol{t}(s)$, we have that $\boldsymbol{e}$ has a singular point at s if and only if $\kappa_g(s) = 0$. On the other hand, we can write $\kappa_g(s) = \langle \boldsymbol{\gamma}'', \boldsymbol{e} \rangle$. So, provided there are s_1 and s_2 such that $\boldsymbol{e}(s_1) = \boldsymbol{e}(s_2) = \boldsymbol{v}$, the points s_1 and s_2 are both critical points of the height function $h_{\boldsymbol{v}}^t$. Since ${h_{\boldsymbol{v}}^t}''(s_i) = \kappa_g(s_i) \neq 0$, we have that either one of them is a local maximum and the other a local minimum, or both points are local maxima (or minima). If one of them, say s_1, is a maximum and the other is a minimum then $\kappa_g(s_1) < 0$ and $\kappa_g(s_2) > 0$, so there must exist some s_0 between s_1 and s_2 such that $\kappa_g(s_0) = 0$. In the other case, there must necessarily be a local minimum (or maximum) s_3 of $h_{\boldsymbol{v}}^t$ between s_1 and s_2, but in this case we would have that $\kappa_g(s_i) < 0 (> 0), i = 1, 2$ and $\kappa_g(s_3) > 0 (< 0)$, so again there must exist some s_0 such that $\kappa_g(s_0) = 0$ and the proof is completed.$\square$

We can thus state the following 4-vertex theorem for closed curves in de Sitter 2-space:

Corollary 8.12 *Any regular closed spacelike curve immersed in de Sitter 2-space with non vanishing curvature as a spacelike curve in $\mathbb{R}_1^3$ and geodesic curvature functions has at least 4 geodesic vertices (flattening points).*

We consider now closed spacelike curves in the 2-dimensional lightcone. Let $\boldsymbol{\gamma} : S^1 \to LC^*$ be such a curve, that we can assume has unit speed. We denote by $\mathbb{R}_0^2 = \{\boldsymbol{x} \in \mathbb{R}_1^3 | x_0 = 0\}$ the Euclidean plane in $\mathbb{R}_1^3$ and by $\boldsymbol{r} : S^1 \to \mathbb{R}_0^2$ the orthogonal projection of $\boldsymbol{\gamma}$ onto $\mathbb{R}_0^2$. Let $\boldsymbol{N} : S^1 \to \{\boldsymbol{x} \in \mathbb{R}_0^2 | x_1^2 + x_2^2 = 1\}$ be the (Euclidean) Gauss map of the curve $\boldsymbol{r}$ and denote by $\boldsymbol{\gamma}^\ell : S^1 \to LC^*$ the lifting of $\boldsymbol{N}$ to LC^* (so $\pi \circ \boldsymbol{\gamma}^\ell = \boldsymbol{N}$, where $\pi : \mathbb{R}_1^3 \to \mathbb{R}_0^2$ is the orthogonal projection). We can explicitely write that

$$\boldsymbol{\gamma}^\ell(s) = \left(\frac{\|\boldsymbol{r}(s)\|}{(\boldsymbol{r}(s) \cdot \boldsymbol{N}(s))^2}, \frac{\boldsymbol{r}(s) - 2(\boldsymbol{r}(s) \cdot \boldsymbol{N}(s))\boldsymbol{N}(s)}{(\boldsymbol{r}(s) \cdot \boldsymbol{N}(s))^2} \right),$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ is the canonical Euclidean scalar product (i.e., $\boldsymbol{a} \cdot \boldsymbol{b} = \langle \boldsymbol{a}, \boldsymbol{b} \rangle | \mathbb{R}_0^2$). It can be shown that $\langle \boldsymbol{\gamma}, \boldsymbol{\gamma}^\ell \rangle = -2$ (see [34], or [35] for details on this calculation). Since $\boldsymbol{\gamma}$ lies in LC^*, we have that the position vector $\boldsymbol{\gamma}$ is a parallel lightlike normal field along $\boldsymbol{\gamma}$ and so is $\boldsymbol{\gamma}^\ell$. The *timelike Gauss map* of $\boldsymbol{\gamma}$ is defined as the map $\boldsymbol{e} : S^1 \to H_+^2(-1)$ given by

$$\boldsymbol{e}(s) = \frac{\boldsymbol{\gamma}(s) + \boldsymbol{\gamma}^\ell(s)}{2}.$$

It follows that $\boldsymbol{e}$ is a globally defined parallel timelike field on $\boldsymbol{\gamma}$. We take $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ and define the *lightcone curvature* on $\boldsymbol{\gamma}$ as the function

$$\kappa_\ell = -\langle {\boldsymbol{\gamma}^\ell}', \boldsymbol{t} \rangle.$$

Then we have that

$$\boldsymbol{e}'(s) = \frac{\boldsymbol{\gamma}'(s) + {\boldsymbol{\gamma}^\ell}'(s)}{2} = \frac{1 + \kappa_\ell}{2} \boldsymbol{t}.$$

So the *timelike Gauss curvature* on $\boldsymbol{\gamma}$ is given by $\kappa_{\boldsymbol{e}} = (1+\kappa_\ell)/2$ and we get that $\kappa_{\boldsymbol{e}} = 0$ if and only if $\kappa_\ell = -1/2$. An analogous argument to that of Lemma 8.3 shows that the image of $\boldsymbol{e}$ is an embedded closed curve if and only if κ_ℓ never vanishes. Therefore, we get the following 4-flattenings theorem for closed spacelike curves in the 2-dimensional lightcone:

Corollary 8.13 *Any regular closed spacelike curve immersed in the* 2*-dimensional lightcone with non vanishing timelike Gauss curvature function has at least* 4 *flattening points.*

In [35] we have defined the notion of the total evolute TE_γ of $\boldsymbol{\gamma} : S^1 \longrightarrow LC^*$ which is decomposed into $TE_\gamma = HE_\gamma \cup DE_\gamma$, where $HE_\gamma \subset H^2(-1)$ and $DE_\gamma \subset S^2_1$. We have shown that the singularities of the total evolute is corresponding to the flattening points of γ. Especially these points are the ordinary cusps for generic spacelike curve $\boldsymbol{\gamma}$. By Corollary 8.13, there are at least 4 cusps on the total evolute for generic spacelike curve $\boldsymbol{\gamma}$, some of them are located in $H^2(-1)$ and others are in S^2_1. (See also [28]).

We finally observe that having a flattening point is a stable property in the sense that it is preserved by small enough local perturbations of the curve. So we can say that closed spacelike curves which are close enough in the Whitney C^3-topology to some of the above ones also have at least 4 flattening points.

References

[1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I.* Birkhäuser (1986).

[2] V. I. Arnol'd, *On the Legendrian Sturm theory of space curves.* (Russian) Funktsional. Anal. i Prilozhen. 32 (1998), 1–7, 95; translation in Funct. Anal. Appl. 32 (1998), 75–80

[3] G. Bol, *Uber Nabelpunkte auf einer Eifläche.* Math Z., 49 (1943/1944), 399–410.

[4] J.W. Bruce and P.J. Giblin, *Curves and singularities. A geometrical introduction to singularity theory.* Cambridge University Press, Cambridge (1984).

[5] T. E. Cecil, *A characterization of metric spheres in hyperbolic space by Morse theory.* Tohoku Math. J. 55 (1974) 5–31.

[6] T. E. Cecil and P. J. Ryan, *Distance functions and umbilic submanifolds of Hyperbolic space.* Nagoya Math. J. 74 (1979), 67–75.

[7] E. A. Feldman, *On parabolic and umbilic points of immersed surfaces.* Trans. Amer. Math. Soc. 127 (1967), 1–28.

[8] C.M. Fulton, *The four-vertex theorem in hyperbolic space.* J. Differential Geometry 4 (1970), 255–256.

[9] R. Garcia and C. Gutierrez, *Ovaloids of* R^3 *and their umbilics: a differential equation approach.* Special issue in celebration of Jack K. Hale's 70th birthday, Part 1 (Atlanta, GA/Lisbon, 1998). J. Differential Equations 168 (2000), no. 1, 200–211.

[10] R. Garcia, D. Mochida, M.C. Romero Fuster and M.A.S. Ruas, *Inflection points and topology of surfaces in* 4*-space.* Trans. Amer. Math. Soc. 352 (2000), no. 7, 3029–3043.

[11] M. Golubitsky and V. Guillemin, *Stable mappings and Their Singularities.* Graduate Texts in Mathematics, Vol. 14. Springer-Verlag, New York-Heidelberg, 1973.

[12] C. Gutierrez, F. Mercuri and F. Sanchez-Bringas, *On a conjecture of Carathéodory: analyticity versus smoothness.* Experiment. Math. 5 (1996), no. 1, 33–37.

[13] C. Gutierrez and J. Sotomayor, *Lines of curvature, umbilic points and Carathéodory conjecture.* Resenhas 3 (1998), no. 3, 291–322.

[14] C. Gutierrez and F. Sánchez-Bringas, *On a Carathéodory's conjecture on umbilics: representing ovaloids.* Rend. Sem. Mat. Univ. Padova 98 (1997), 213–219.

[15] C. Gutierrez and F. Sánchez-Bringas, *Planar vector field versions of Carathéodory's and Loewner's conjectures.* Proceedings of the Symposium on Planar Vector Fields (Lleida, 1996). Publ. Mat. 41 (1997), no. 1, 169–179.

[16] C. Gutierrez and F. Sánchez-Bringas, *On a Loewner umbilic-index conjecture for surfaces immersed in* R^4. J. Dynam. Control Systems 4 (1998) 127–136.

[17] H. Hamburguer, *Beweis einer Carathéodoryschen Vermütung.* Ann. of Math. 41 (1940), 63-68, II, III, and Acta Math. 73 (1941), 174-332.

[18] O. Haupt, *Vierscheitelsätze in der ebenen hyperbolischen Geometrie.* Geometriae Dedicata 1 (1973), 399–414.

[19] V. V. Ivanov, *The Analytic Caratheodory Conjecture.* Siberian Math. J. 43 (2002), 251–322.

[20] S. Izumiya and A. Takiyama, *A time-like surface in Minkowski* 3*-space which contains pseudocircles.* Proc. Edinburgh Math. Soc. (2) 40 (1997), 127–136.

[21] S.Izumiya, D-H. Pei and T. Sano, *Singularities of hyperbolic Gauss maps.* Proc. London Math. Soc. 86 (2003), 485–512.

[22] S. Izumiya, D-H. Pei and M. Takahashi, *Singularitites of evolutes of hypersufaces in hyperbolic space.* Proceedings of the Edinburgh Mathematical Society. 47 (2004), 131–153.

[23] S.Izumiya, D-H. Pei and T. Sano, *Horospherical surface of curve in Hyperbolic space.* Publ. Math. Debrecen 64 (2004), 1-13.

[24] S. Izumiya, D. Pei and M.C. Romero Fuster, *The lightcone Gauss map of a spacelike surface in Minkowski* 4*-space*, Asian J. Math., vol. 8 (2004), 511–530.

[25] S. Izumiya, D. Pei and M.C. Romero Fuster, *Umbilicity of spacelike submanifolds of Minkowski space*, Proceedings of the Royal Society of Edinburgh, 134A (2004), 375–387.

[26] S. Izumiya, D. Pei, M.C. Romero-Fuster and M. Takahashi, *On the horospherical ridges of submanifolds of codimension 2 in hyperbolic* n*-space.* Bull. Braz. Math. Soc. (N.S.) 35 (2004), no. 2, 177–198.

[27] S. Izumiya, D. Pei and M. Takahashi, *Curves and surfaces in hyperbolic space.* Geometric Singularity Theory, 107–123, Banach Center Publ., 65, Polish Acad. Sci., Warsaw, 2004.

[28] S. Izumiya, D. Pei, T. Sano and E. Torii,*Evolutes of hyperbolic plane curves.* Acta Math. Sin. (Engl. Ser.) 20 (2004), no. 3, 543–550.

[29] S. Izumiya, D. Pei, M.C. Romero Fuster and M. Takahashi, *The horospherical geometry of submanifolds in hyperbolic space.* J. London Math. Soc. (2) 71 (2005), 779–800.

[30] S. Izumiya, M. Kossowski, D. Pei and M.C. Romero Fuster, *Singularities of lightlike hypersurfaces in Minkowski* 4*-space.* Tohoku Math. J. (2) 58 (2006), 71–88.

[31] S. Izumiya, D-H. Pei and M.C. Romero-Fuster, *The horospherical geometry of surfaces in Hyperbolic* 4*-space.* Israel J. Maths. 154 (2006), 361–379.

[32] S. Izumiya and M.C. Romero-Fuster, *The horospherical Gauss-Bonnet type theorem in hyperbolic space.* J. Math. Soc. Japan 58 (2006), 965–984.

[33] S. Izumiya, *Differential geometry from the viewpoint of Lagrangian and legendrian singularity theory.* Singularity Theory, Proceedings of the 2005 Marseille Singularity School and Conference. Ed. by D. Chéniot, N. Dutertre, C. Murolo, D. Trotman and A. Pichon. World Sci (2007), 241–275.

[34] S. Izumiya, *Legedrian dualities and spacelike hypersurfaces in the lightcone.* Preprint.

[35] S. Izumiya and M. Takahashi, *Spacelike parallels and evolutes in Minkowski pseudospheres.* J. Geom. and Physics 57 (2007), 1569–1600.

[36] S. Izumiya and M.C. Romero-Fuster, *The lightlike flat geometry on spacelike submanifolds of codimension two in Minkowski space.* Selecta Mathematica (NS) 13 (2007), 23–55.

[37] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I.* Interscience Publishers, New York-London 1963.

[38] T. Klotz, *On Bol's proof of Carathéodory's Conjecture.* Comm. Pure Appl. Math., 12 (1959), 277–311.

[39] E.J.N. Looijenga, *Structural stability of smooth families of C^∞-functions.* Doctoral Thesis, University of Amsterdam, 1974.

[40] D.K.H. Mochida, M.C. Romero-Fuster and M.A.S. Ruas, *Osculating hyperplanes and asymptotic directions of codimension two submanifolds of Euclidean spaces.* Geom. Dedicata 77 (1999), no. 3, 305–315.

[41] J. A. Montaldi, *On contact between submanifolds*, Michigan Math. J., **33** (1986), 81–85.

[42] J. A. Montaldi, *On generic composites of maps*, Bull. London Math. Soc., 23 (1991), 81–85.

[43] S. Mukhopadhyaya, *New Methods in the geometry of a plane arc.* Bull. Calcutta Math. Soc. 1 (1909), 31–37.

[44] K. Nomizu and L. Rodriguez, *Umbilical submanifolds and Morse functions.* Nagoya Math. J. 48 (1972), 197–201.

[45] J.J. Nunõ-Ballesteros, *Submanifolds with a non-degenerate parallel normal vector field in euclidean spaces.* Singularity Theory and Its Applications. Adv. Stud. in Pure Maths. 43 (2006), 311–332.

[46] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[47] I.R. Porteous, *The normal singularities of a submanifold.* J. Differential Geometry 5 (1971), 543–564.

[48] I.R. Porteous, *Geometric differentiation. For the intelligence of curves and surfaces.* Second edition. Cambridge University Press, Cambridge, 2001.

[49] R. S. Palais and C-L. Terng, *Critical Point Theory and Submanifolds Geometry*, Lecture Notes in Math 1353, Springer-Verlag 1988.

[50] M.C. Romero Fuster, *Convexly-generic curves in* $\mathbb{R}^3$. Geom. Dedicata 28 (1988), 7–29.

[51] V. D. Sedykh, *Four vertices of a convex space curve.* Bull. London Math. Soc. 26 (1994), 177–180.

[52] B. Smyth and F. Xavier, *A sharp geometric estimate for the index of an umbilic point on a smooth surface.* Bull. London Math. Soc. 24 (1992), 176–180.

[53] H. Scherbel, *A new proof of Hamburger's Index Theorem on umbilical points.* Dissertation ETH No. 10281.

[54] J. Sotomayor, J and C. Gutierrez, *Structurally stable configurations of lines of principal curvature.* Bifurcation, ergodic theory and applications (Dijon, 1981). Astérisque, 98-99, Soc. Math. France, Paris (1982), 195–215.

[55] Titus, *A proof of a conjecture of Loewner and of the conjecture of Carathéodory on umbilic points.* Acta Math. 131 (1973), 43–77.

[56] B. Wegner, *A cyclographic approach to the vertices of plane curves.* Journal of geometry 50 (1944), 186–201.

Shyuichi Izumiya, Department of Mathematics, Hokkaido University, Sapporo 060-0810,Japan
e-mail:`izumiya@math.sci.hokudai.ac.jp`

Juan José Nuno Ballesteros and María del Carmen Romero Fuster, Departament de Geometría i Topología, Universitat de València, 46100 Burjassot (València), Espanya
e-mail:`nuno@uv.es`
e-mail:`carmen.romero@post.uv.es`