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Title	Optimal intertemporal risk allocation applied to insurance pricing
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Citation	Hokkaido University Preprint Series in Mathematics, 882, 1-20
Issue Date	2007
DOI	https://doi.org/10.14943/84032
Doc URL	https://hdl.handle.net/2115/69691
Type	departmental bulletin paper
File Information	re882.pdf



OPTIMAL INTERTEMPORAL RISK ALLOCATION APPLIED TO INSURANCE PRICING

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ABSTRACT. We present a general approach to the pricing of products in finance and insurance in the multi-period setting. It is a combination of the utility indifference pricing and optimal intertemporal risk allocation. We give a characterization of the optimal intertemporal risk allocation by a first order condition. Applying this result to the exponential utility function, we obtain an essentially new type of premium calculation method for a popular type of multi-period insurance contract. This method is simple and can be easily implemented numerically. We see that the results of numerical calculations are well coincident with the risk loading level determined by traditional practices. The results also suggest a possible implied utility approach to insurance pricing.

1. INTRODUCTION

The insurer of an insurance contract needs to ensure that the premium contains a necessary conservative margin — the so called risk loading or safety loading — to put up the risk capital. When determining this margin in a multi-period insurance contract, the insurer faces two types of risks to evaluate. The first one comes from unfavorable fluctuations in the level of investment funded by accumulated premiums. The second risk comes from the uncertainty of (life) time, i.e., the risk of the unfavorable event occurring at an inopportune time, e.g., before the funding target is reached. It is desirable to determine the margin that reflects both types of risks adequately. However, there seems to be no theoretically established solution to this challenging problem. The main difficulty is in the inseparable nature of the two types of risks themselves; the insurance contract guarantees a defined payment at an uncertain time of the insured event occurring by uncertain funding.

In this paper, toward a solution to the problem above, we present a fairly general approach to the multi-period pricing problem. It is a combination of the *utility indifference pricing* and *optimal intertemporal risk allocation*. Though both are quite general concepts, their combination leads us to an interesting new premium calculation method in a multi-period setting.

The general setting of the utility indifference pricing is as follows: we define the *indifference price* $H(Z)$ of a risk Z by

$$(IP) \quad U(w + H(Z) - Z) = U(w),$$

where $U(W)$ denotes the utility of a risk W and the constant w is the initial wealth of the seller of Z . The price $H(Z)$ is the so-called *selling* indifference price: $H(Z)$ is the amount that leaves the seller of the risk Z indifferent between selling and

2000 Mathematics Subject Classification Primary 62P05; Secondary 91B28.

Date: November 8, 2007.

Key words and phrases. Indifference pricing, optimal intertemporal risk allocation, Pareto optimality, exponential utility, insurance, premium calculation method.

being paid for Z , and neither selling nor being paid for Z . In mathematical finance, the indifference pricing approach is becoming one of the major pricing methods in incomplete markets (see, e.g., Hodges and Neuberger [22], Rouge and El Karoui [26], Musiela and Zariphopoulou [24], Bielecki et al. [4], and Møller and Steffensen [25]). The indifference pricing also fits the pricing of insurance well. For example, in the single-period pricing, we can show that many known premium principles are obtained by this method. The expectation, variance and exponential premium principles are among them. Thus, the utility indifference pricing approach has the potential advantage of pricing products in finance and insurance coherently.

We write $\mathcal{A}(W)$ for the class of admissible intertemporal risk allocations $(Y_t)_{t \in \mathbb{T}}$ of W over the multi-period interval $\mathbb{T} := \{1, 2, \dots, T\}$ (see Definition 2.1 below): $(Y_t)_{t \in \mathbb{T}}$ is an essentially bounded adapted process satisfying the risk allocation condition

$$(RA) \quad \sum_{t \in \mathbb{T}} \tilde{Y}_t = W \quad \text{a.s.},$$

where $\tilde{Y}_t$ denotes the discounted value of Y_t . In this paper, we adopt the following utility $U(\cdot)$ in (IP):

$$(U) \quad U(W) := \sup \left\{ \sum_{t \in \mathbb{T}} E[u_t(\tilde{Y}_t)] : (Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(W) \right\}.$$

Here $u_t(x)$ is a time-dependent utility function describing the intertemporal preferences of an economic agent such as an insurance company. This definition says that if an allocation $(X_t) \in \mathcal{A}(W)$ attains the supremum in (U), then the utility of W is based on the choice of (X_t) . Thus, to precisely investigate $U(\cdot)$, whence $H(\cdot)$, we are led to the problem of finding $(X_t) \in \mathcal{A}(W)$ that attains the supremum in (U), which we call the *optimal intertemporal risk allocation* of W .

The optimal risk allocation problems date back to the classical work of Borch [5, 6, 7], where Pareto optimality in uncertain circumstances is studied extensively, motivated mainly by reinsurance. Since then, various types of optimal risk allocation problems have been considered by Bühlmann [8, 9], Gerber [19], Bühlmann and Jewell [10], and many others. See also Gerber and Pafumi [20], Duffie [16], Dana and Jeanblanc [13] and Dana and Scarsini [14]. Recently, many authors consider the problems based on the preferences defined by coherent or convex risk measures introduced by Artzner et al. [2], Delbaen [15], and Föllmer and Schied [17] (see also [18]). See, e.g., Heath and Ku [21], Barrieu and El Karoui [3], Burgert and Rüschendorf [11], Acciaio [1], and Jouini et al. [23].

Unlike most of these references where the problems of optimal risk allocation among several economic agents are discussed, we consider a single agent in the multi-period framework who seeks to find the optimal intertemporal allocation of her/his risk. As the definition itself suggests, this optimality is closely related to Pareto optimality. Note, however, that classical Pareto optimality is concerned with allocations of risk among economic agents in single-period models, while the Pareto optimality we consider in this paper is concerned with intertemporal allocations of the aggregate risk of a single agent in the multi-period setting, whence it may be called *time* Pareto optimality.

Our key finding about the optimal intertemporal risk allocation (Theorem 2.8) is that an allocation $(X_t) \in \mathcal{A}(W)$ is optimal if and only if the following first order condition is satisfied:

$$(FO) \quad (u'_t(\tilde{X}_t))_{t \in \mathbb{T}} \text{ is an } (\mathcal{F}_t)\text{-martingale,}$$

where $u'_t(x) := (du_t/dx)(x)$ and $(\mathcal{F}_t)_{t \in \mathbb{T}}$ is the underlying information structure. It is perhaps interesting that this first order condition involves a martingale property. By applying this characterization to the exponential utility, we can derive an algorithm to compute the optimal intertemporal risk allocation and indifference price $H(\cdot)$ for it (Theorem 3.4). We illustrate the usefulness of this algorithm by applying it to a popular type of multi-period insurance contract, whereby obtaining an essentially new type of premium calculation method in the multi-period setting (Theorem 4.3). This method is simple and can be easily implemented numerically. We see that the results of numerical calculations are well coincident with the risk loading level determined by traditional practices. The results also suggest a possible *implied utility approach* to insurance pricing.

In §2, we give basic results on the optimal intertemporal risk allocation, including its characterization by (FO) and its relationship to Pareto optimality. In §3, we apply the results in §2 to the exponential utility function and derive the optimal intertemporal risk allocation and indifference price for it. Section 4 is devoted to the applications of the results in §3 to insurance pricing. We also discuss properties of the indifference prices and some results of numerical calculations.

2. OPTIMAL INTERTEMPORAL RISK ALLOCATION

Let $\mathbb{T} := \{1, 2, \dots, T\}$. Throughout the paper, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \{0\} \cup \mathbb{T}}, P)$. We write $L^\infty := L^\infty(\Omega, \mathcal{F}_T, P)$ for the space of all essentially bounded, real-valued $\mathcal{F}_T$ -measurable random variables. Let $(r_t)_{t \in \mathbb{T}}$ be a spot rate process. We assume that the process $(r_t)_{t \in \mathbb{T}}$ is bounded, nonnegative and predictable, i.e., r_t is bounded, nonnegative and $\mathcal{F}_{t-1}$ -measurable for all $t \in \mathbb{T}$. Let B_t be the price of the riskless bond:

$$B_0 = 1, \quad B_t = \prod_{k=1}^t (1 + r_k) \quad \text{for } t = 1, \dots, T.$$

Throughout the paper, we use $(B_t)_{t \in \mathbb{T}}$ as the numéraire, and for each price process $(X_t)_{t \in \mathbb{T}}$, we denote by $(\tilde{X}_t)_{t \in \mathbb{T}}$ its discounted price process:

$$\tilde{X}_t := X_t / B_t, \quad t \in \mathbb{T}.$$

2.1. Optimality. We consider an economic agent such as an insurance company who wishes to allocate her/his aggregate risk W over the multi-period interval $\mathbb{T}$. In the next definition, we define the collection of all such possible intertemporal allocations of W .

Definition 2.1. For $W \in L^\infty$, we write $\mathcal{A}(W)$ for the following set of *admissible intertemporal allocations* $(Y_t)_{t \in \mathbb{T}}$ of W :

$$\mathcal{A}(W) := \left\{ (Y_t)_{t \in \mathbb{T}} : \begin{array}{l} (Y_t)_{t \in \mathbb{T}} \text{ is an } (\mathcal{F}_t)\text{-adapted process satisfying} \\ \text{(RA) and } Y_t \in L^\infty \text{ for all } t \in \mathbb{T}. \end{array} \right\}.$$

Example 2.2. We consider the aggregate risk W of a life insurance contract with duration T in which the insured receives c_t dollars at time $t \in \mathbb{T}$ if she/he dies in the period $(t-1, t]$. Then, we have $W = \sum_{t \in \mathbb{T}} \tilde{Y}_t$ with $Y_t := c(t)1_{(t-1 < \tau \leq t)}$, where τ is the stopping time representing the lifetime of the insured. Notice that $(Y_t)_{t \in \mathbb{T}}$

itself is in $\mathcal{A}(W)$. If we define $(X_t)_{t \in \mathbb{T}}$ by

$$X_t = \begin{cases} 0, & t = 1, \\ (1 + r_t)Y_{t-1}, & t = 2, \dots, T-1, \\ Y_T + (1 + r_T)Y_{T-1}, & t = T, \end{cases}$$

then $(X_t)_{t \in \mathbb{T}}$ is also in $\mathcal{A}(W)$. Insurance companies which have many contracts with policyholders will be able to regard W as the aggregate risk of $(X_t)_{t \in \mathbb{T}}$, rather than that of $(Y_t)_{t \in \mathbb{T}}$, at a negligible cost.

We assume that the intertemporal preferences of the agent is described by the time-dependent utility function $u_t(x)$. This means that a rational choice of the agent's allocation $(Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ is based on the integrated expected utility $\sum_{t \in \mathbb{T}} E[u_t(\tilde{Y}_t)]$. Throughout §2, we assume that the utility function $u_t(x)$ satisfies the following condition:

$$(2.1) \quad \begin{cases} \text{for } t \in \mathbb{T}, \mathbb{R} \ni x \mapsto u_t(x) \in \mathbb{R} \text{ is a strictly concave, } C^1\text{-class function} \\ \text{such that } u'_t(x) := (du_t/dx)(x) > 0 \text{ for } x \in \mathbb{R}. \end{cases}$$

Using $u_t(x)$, we define the utility $U(W) \in \mathbb{R} \cup \{+\infty\}$ of the risk $W \in L^\infty$ by (U).

Definition 2.3. An intertemporal risk allocation $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ of the risk $W \in L^\infty$ is *optimal* if it attains the supremum in (U).

In other words, $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ is optimal if it solves the following problem:

$$(P) \quad \text{Maximize } \sum_{t \in \mathbb{T}} E[u_t(\tilde{Y}_t)] \quad \text{among all } (Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(W).$$

Proposition 2.4. *The optimal intertemporal risk allocation $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ of $W \in L^\infty$ is unique if it exists.*

Proof. Suppose that there are two distinct optimal intertemporal allocations (X_t) and (Y_t) of W . If we put $Z_t := (1/2)X_t + (1/2)Y_t$ for $t \in \mathbb{T}$, then (Z_t) is also in $\mathcal{A}(W)$. However, concavity of $u_t(\cdot)$ yields

$$\sum_{t \in \mathbb{T}} E[u_t(\tilde{Z}_t)] > \sum_{t \in \mathbb{T}} E[(1/2)u_t(\tilde{X}_t) + (1/2)u_t(\tilde{Y}_t)] = U(W),$$

which is a contradiction. Thus the optimal allocation of W is unique. $\square$

2.2. Indifference pricing. In this section, we assume that $U(W) < \infty$ for all $W \in L^\infty$. This condition holds, for example, if $u_t(x)$ is bounded from above. This also holds if the optimal intertemporal risk allocation exists for all $W \in L^\infty$. We thus have the utility functional $U : L^\infty \rightarrow \mathbb{R}$. We write $w \in \mathbb{R}$ for the initial wealth of the agent.

Proposition 2.5. *The functional U has the following properties for $W, Z \in L^\infty$.*

- (a) *Strict Monotonicity:* If $W \geq Z$ a.s. and $P(W > Z) > 0$, then $U(W) > U(Z)$.
- (b) *Concavity:* If $a \in [0, 1]$, then $U(aW + (1-a)Z) \geq aU(W) + (1-a)U(Z)$.

Proof. (a) For $(Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(Z)$, we define $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ by

$$X_t = \begin{cases} Y_t, & t \neq T, \\ Y_T + B_T(W - Z), & t = T. \end{cases}$$

Choosing $m > 0$ so that $\max(|W|, |Z|) \leq m$ a.s., we define $c := \inf_{|y| \leq m} u'_T(y)$. Then, by (2.1), $c > 0$. Since $u_T(\tilde{X}_T) \geq u_T(\tilde{Y}_T) + c(W - Z)$, we have

$$U(W) \geq \sum_{t \in \mathbb{T}} E[u_t(\tilde{X}_t)] \geq \sum_{t \in \mathbb{T}} E[u_t(\tilde{Y}_t)] + cE[W - Z].$$

The property (a) follows from this.

(b) The property (b) follows easily from the concavity of u_t , $t \in \mathbb{T}$. $\square$

From Proposition 2.5, we see that for $Z \in L^\infty$, the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) := U(w + x - Z)$ is concave (whence continuous) and strictly increasing. Moreover, since Z is bounded, we have $U(w + x - Z) < U(w)$ for x small enough and $U(w + x - Z) > U(w)$ for x large enough. We are thus led to the following definition.

Definition 2.6. We define the *indifference price* $H(Z) = H(Z; w) \in \mathbb{R}$ of $Z \in L^\infty$ by $U(w + H(Z) - Z) = U(w)$.

From Proposition 2.5, we immediately obtain the next proposition.

Proposition 2.7. *The indifference price functional $H : L^\infty \rightarrow \mathbb{R}$ has the following properties for $W, Z \in L^\infty$.*

- (a) *Strict Monotonicity:* If $W \geq Z$ a.s. and $P(W > Z) > 0$, then $H(W) > H(Z)$.
- (b) *Convexity:* If $a \in [0, 1]$, then $H(aW + (1 - a)Z) \leq aH(W) + (1 - a)H(Z)$.

2.3. Characterization by the first order condition. It should be noticed that, in general, the optimal intertemporal risk allocation may not exist. However, to precisely investigate the utility $U(\cdot)$, whence the indifference price $H(\cdot)$, it seems indispensable to find and describe the optimal intertemporal risk allocation. In this section, we show that the condition (FO) is necessary and sufficient for $(X_t) \in \mathcal{A}(W)$ to be optimal. This characterization plays a key role in this paper. In the proof below, and throughout the paper, we write

$$E_t[Y] := E[Y | \mathcal{F}_t], \quad Y \in L^1(\Omega, \mathcal{F}, P), \quad t \in \mathbb{T}.$$

Here is the characterization of the optimality.

Theorem 2.8. *For $W \in L^\infty$ and $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$, the following conditions are equivalent:*

- (a) *$(X_t)_{t \in \mathbb{T}}$ is optimal.*
- (b) *The condition (FO) is satisfied.*

Proof. First, we prove (a) $\Rightarrow$ (b). Let $(X_t) \in \mathcal{A}(W)$ be the optimal allocation. Choose $k, m \in \mathbb{T}$ so that $k < m$, and put, for $t \in \mathbb{T}$, $y \in \mathbb{R}$ and $A \in \mathcal{F}_k$,

$$X_t(y) = \begin{cases} X_m + yB_m 1_A, & t = m, \\ X_k - yB_k 1_A, & t = k, \\ X_t, & \text{otherwise.} \end{cases}$$

Then, $\sum_{t \in \mathbb{T}} \tilde{X}_t(y) = W$, so that $(X_t(y))_{t \in \mathbb{T}} \in \mathcal{A}(W)$. Since $(X_t(0)) = (X_t)$ is optimal, the function f defined by $f(y) := \sum_{t \in \mathbb{T}} E[u_t(\tilde{X}_t(y))]$ takes the maximal value at $y = 0$. Thus $f'(0) = 0$ or $E[\{u'_m(\tilde{X}_m) - u'_k(\tilde{X}_k)\} 1_A] = 0$, which implies that $(u'_t(\tilde{X}_t))$ is an $(\mathcal{F}_t)$ -martingale.

Next, we prove (b) $\Rightarrow$ (a). Assume that $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ and that $(u'_t(\tilde{X}_t))_{t \in \mathbb{T}}$ is an $(\mathcal{F}_t)$ -martingale. By concavity of $u_t(\cdot)$, we have $u_t(y) \leq u_t(x) + u'_t(x)(y - x)$ for $x, y \in \mathbb{R}$, so that for any $Y = (Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$,

$$\sum_{t \in \mathbb{T}} u_t(\tilde{Y}_t) \leq \sum_{t \in \mathbb{T}} \lambda_t u_t(\tilde{X}_t) + \sum_{t \in \mathbb{T}} u'_t(\tilde{X}_t)(\tilde{Y}_t - \tilde{X}_t).$$

Since $(u'_t(\tilde{X}_t))$ is an $(\mathcal{F}_t)$ -martingale and both (X_t) and (Y_t) are in $\mathcal{A}(W)$, we see that

$$\begin{aligned} & E \left[\sum_{t \in \mathbb{T}} u'_t(\tilde{X}_t)(\tilde{Y}_t - \tilde{X}_t) \right] \\ &= \sum_{t \in \mathbb{T}} E \left[E_t[u'_t(\tilde{X}_t)](\tilde{Y}_t - \tilde{X}_t) \right] = \sum_{t \in \mathbb{T}} E \left[u'_T(\tilde{X}_T)(\tilde{Y}_t - \tilde{X}_t) \right] \\ &= E \left[u'_T(\tilde{X}_T) \sum_{t \in \mathbb{T}} (\tilde{Y}_t - \tilde{X}_t) \right] = E \left[u'_T(\tilde{X}_T)(W - W) \right] = 0. \end{aligned}$$

Combining, $\sum_{t \in \mathbb{T}} E[u_t(\tilde{Y}_t)] \leq \sum_{t \in \mathbb{T}} E[u_t(\tilde{X}_t)]$. Thus, (X_t) is optimal. $\square$

Remark 2.9. We clearly find similarity between the theorem above and Borch's theorem which characterizes (classical) Pareto optimality by a first order condition (see Borch [5, 6, 7]; see also Gerber and Pafumi [20]).

2.4. Pareto optima. In this section, we introduce Pareto optimality of intertemporal risk allocations. It is closely related to the optimality introduced above.

Definition 2.10. For $W \in L^\infty$, the allocation $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ is *Pareto optimal* if there does not exist $(Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ satisfying the following two conditions:

- (a) $E[u_t(\tilde{Y}_t)] \geq E[u_t(\tilde{X}_t)]$ for all $t \in \mathbb{T}$.
- (b) $E[u_{t_0}(\tilde{Y}_{t_0})] > E[u_{t_0}(\tilde{X}_{t_0})]$ for at least one $t_0 \in \mathbb{T}$.

For $\lambda = (\lambda_1, \dots, \lambda_T) \in \mathbb{R}_+^T \setminus \{0\}$, we consider the following problem:

$$(P_\lambda) \quad \text{Maximize} \quad \sum_{t \in \mathbb{T}} \lambda_t E[u_t(\tilde{Y}_t)] \quad \text{among all} \quad (Y_t)_{t \in \mathbb{T}} \in \mathcal{A}(W).$$

Lemma 2.11. Let $\lambda = (\lambda_1, \dots, \lambda_T) \in \mathbb{R}_+^T \setminus \{0\}$.

- (a) If $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ is the solution to Problem P_λ , then $(\lambda_t u'_t(\tilde{X}_t))_{t \in \mathbb{T}}$ is an $(\mathcal{F}_t)$ -martingale.
- (b) If Problem P_λ has a solution, then $\lambda \in (0, \infty)^T$.

Proof. The proof of (a) is almost the same as that of the implication (a) $\Rightarrow$ (b) in Theorem 2.8, whence we omit it.

We prove (b). Assume that $\lambda_k = 0$ for $k \in \mathbb{T}$, and choose m so that $\lambda_m > 0$. If Problem P_λ has a solution $(X_t) \in \mathcal{A}(W)$, then, by (a), $(\lambda_t u'_t(\tilde{X}_t))$ is an $(\mathcal{F}_t)$ -martingale. However, since $\lambda_k u'_k(\tilde{X}_k) = 0$ and $\lambda_m u'_m(\tilde{X}_m) > 0$, this can never be the case. Thus, (b) follows. $\square$

Proposition 2.12. Let $\lambda \in (0, \infty)^T$. Then the solution $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ to Problem P_λ is unique if exists.

The proof is almost the same as that of Proposition 2.4, whence we omit it.

The next theorem is an analogue of the *second fundamental theorem of welfare economics*.

Theorem 2.13. For $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$, the following conditions are equivalent:

- (a) $(X_t)_{t \in \mathbb{T}}$ is Pareto optimal.

(b) *There exists $\lambda \in (0, \infty)^T$ such that $(X_t)_{t \in \mathbb{T}}$ solves Problem P_λ .*

Proof. (b) $\Rightarrow$ (a). If $(X_t)_{t \in \mathbb{T}}$ is not Pareto optimal, then clearly it is not the solution to Problem P_λ for any $\lambda \in (0, \infty)^T$.

(a) $\Rightarrow$ (b). We define $f(Y) := \phi(X) - \phi(Y)$ for $Y \in \mathcal{A}(W)$, where

$$\phi(Y) := \left(E[u_1(\tilde{Y}_1)], \dots, E[u_T(\tilde{Y}_T)] \right).$$

Then $f : \mathcal{A}(W) \rightarrow \mathbb{R}^T$ is $\mathbb{R}_+^T$ -convex: for $p \in (0, 1)$ and $Y, Y' \in \mathcal{A}(W)$,

$$pf(Y) + (1-p)f(Y') - f(pY + (1-p)Y') \in \mathbb{R}_+^T.$$

If $X \in \mathcal{A}(W)$ is Pareto optimal, then $-f(Y) \notin (0, \infty)^T$ for $Y \in \mathcal{A}(W)$. Hence, by Gordan's Alternative Theorem (see, e.g., Craven [12], Chapter 2), there exists $\lambda \in \mathbb{R}_+^T$, $\lambda \neq 0$, such that

$$\lambda \cdot f(Y) = \lambda \cdot [\phi(X) - \phi(Y)] \geq 0, \quad Y \in \mathcal{A}(W),$$

which implies that X is the solution to Problem P_λ . Finally, Lemma 2.11 gives $\lambda \in (0, \infty)^T$. $\square$

By Theorem 2.13, we see that the set of Pareto optimal intertemporal risk allocations in $\mathcal{A}(W)$ is parametrized by the $T-1$ parameters $(\lambda_2/\lambda_1, \dots, \lambda_T/\lambda_1) \in (0, \infty)^{T-1}$. We also see that the Pareto optimal allocation $(X_t) \in \mathcal{A}(W)$ corresponding to Problem (P_λ) with $\lambda = (\lambda_1, \dots, \lambda_T)$ is optimal with respect to the intertemporal preferences described by the utility function $v_t(x) := \lambda_t u_t(x)$. Therefore, from Theorem 2.8, we immediately obtain the next characterization of Pareto optimality.

Theorem 2.14. *For $W \in L^\infty$ and $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$, the following conditions are equivalent:*

- (a) *$(X_t)_{t \in \mathbb{T}}$ is Pareto optimal.*
- (b) *There exists $(\lambda_1, \dots, \lambda_T) \in (0, \infty)^T$ such that the process $(\lambda_t u'_t(\tilde{X}_t))_{t \in \mathbb{T}}$ is an $(\mathcal{F}_t)$ -martingale.*

3. EXPONENTIAL UTILITY

Let $(r_t)_{t \in \mathbb{T}}$ and $(B_t)_{t \in \mathbb{T}}$ be as in Section 2. In this section, we adopt the following time-dependent exponential utility function:

$$(EU) \quad \begin{cases} u_t(x) = \frac{1}{\alpha_t} [1 - \exp(-\alpha_t x)], & t \in \mathbb{T}, x \in \mathbb{R} \\ \text{with } \alpha := (\alpha_1, \dots, \alpha_T) \in (0, \infty)^T. \end{cases}$$

In what follows, we may also write $\alpha(t) = \alpha_t$. We have

$$(3.1) \quad u'_t(x) = \exp(-\alpha_t x), \quad u_t(0) = 0, \quad u'_t(0) = 1.$$

3.1. The optimal allocation for the exponential utility. In this section, we describe the optimal intertemporal risk allocation for the exponential utility function $u_t(x)$ in (EU). Thus, the problem that we consider here is Problem (P) for $u_t(x)$ in (EU).

To derive the optimal allocation $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ or the solution to (P), we consider the transform $M_t = \exp(-\alpha_t \tilde{X}_t)$ for $t \in \mathbb{T}$. Then, by Theorem 2.8, Problem (P) reduces to

Problem M. For $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0, \infty)^T$, derive a positive $(\mathcal{F}_t)$ -martingale $(M_t)_{t \in \mathbb{T}}$ satisfying

$$(3.2) \quad \prod_{t \in \mathbb{T}} M_t^{1/\alpha(t)} = \exp(-W) \quad \text{a.s.}$$

For $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0, \infty)^T$, we define the adapted process $(L_t(\alpha, W))_{t \in \mathbb{T}}$ by the following backward iteration:

$$(L1) \quad \begin{cases} L_T(\alpha, W) := \exp(-\alpha_T W), \\ L_{t-1}(\alpha, W) := E_{t-1}[L_t(\alpha, W)]^{\beta(t-1)/\beta(t)}, \quad t = 2, \dots, T, \end{cases}$$

where $E_t[Y] := E[Y|\mathcal{F}_t]$ as before, and we define β_t , or $\beta(t)$, in $(0, \infty)$ by

$$(\beta) \quad \frac{1}{\beta_t} = \sum_{k=t}^T \frac{1}{\alpha_k}, \quad t \in \mathbb{T}.$$

Notice that for all $t \in \mathbb{T}$, $L_t(\alpha, W)$ is bounded away from 0 and ∞ . We also define the adapted process $(M_t(\alpha, W))_{t \in \mathbb{T}}$ by

$$(M) \quad \begin{cases} M_t(\alpha, W) = L_t(\alpha, W) \cdot \prod_{k=1}^{t-1} L_k(\alpha, W)^{-\beta(k+1)/\alpha(k)}, \quad t = 2, \dots, T, \\ M_1(\alpha, W) = L_1(\alpha, W). \end{cases}$$

Here is the solution to the martingale problem M above.

Theorem 3.1. For $W \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0, \infty)^T$, the solution $(M_t)_{t \in \mathbb{T}}$ to Problem M is unique and given by $M_t = M_t(\alpha, W)$ for $t \in \mathbb{T}$.

Proof. For simplicity, we write $L_t = L_t(\alpha, W)$ for $t \in \mathbb{T}$.

Step 1. Let $t \geq 3$. Since $\prod_{k=1}^{t-1} L_k^{-\beta(k+1)/\alpha(k)}$ is $\mathcal{F}_{t-1}$ -measurable, the process $(M_t)_{t \in \mathbb{T}}$ defined by $M_t = M_t(\alpha, W)$ satisfies

$$E_{t-1}[M_t] = E_{t-1}[L_t] \cdot \prod_{k=1}^{t-1} L_k^{-\beta(k+1)/\alpha(k)}.$$

However, since $E_{t-1}[L_t] = L_{t-1}^{\beta(t)/\beta(t-1)}$, we get

$$\begin{aligned} E_{t-1}[M_t] &= L_{t-1}^{\beta(t)/\beta(t-1)} \cdot L_{t-1}^{-\beta(t)/\alpha(t-1)} \cdot \prod_{k=1}^{t-2} L_k^{-\beta(k+1)/\alpha(k)} \\ &= L_{t-1} \cdot \prod_{k=1}^{t-2} L_k^{-\beta(k+1)/\alpha(k)} = M_t. \end{aligned}$$

Treating the case $t = 2$ similarly, we see that (M_t) is an $(\mathcal{F}_t)$ -martingale. Also,

$$\begin{aligned} \prod_{t \in \mathbb{T}} M_t^{1/\alpha(t)} &= L_1^{1/\alpha(1)} \cdot \prod_{t=2}^T L_t^{1/\alpha(t)} \left(\prod_{k=1}^{t-1} L_k^{-\beta(k+1)/\{\alpha(k)\alpha(t)\}} \right) \\ &= \left[\prod_{t \in \mathbb{T}} L_t^{1/\alpha(t)} \right] \cdot \left[\prod_{t=2}^T \prod_{k=2}^t L_{k-1}^{-\beta(k)/\{\alpha(k-1)\alpha(t)\}} \right] \\ &= \left[\prod_{t \in \mathbb{T}} L_t^{1/\alpha(t)} \right] \cdot \left[\prod_{k=2}^T \left(\prod_{t=k}^T L_{k-1}^{-1/\alpha(t)} \right)^{-\beta(k)/\alpha(k-1)} \right] \\ &= \left[\prod_{t \in \mathbb{T}} L_t^{1/\alpha(t)} \right] \cdot \left[\prod_{k=2}^T L_{k-1}^{-1/\alpha(k-1)} \right] = L_T^{1/\alpha(T)}, \end{aligned}$$

yielding (3.2). Thus (M_t) is a solution to Problem M.

Step 2. We show the uniqueness. Assume that $(M_t)_{t \in \mathbb{T}}$ is a solution to Problem M. Then,

$$(3.3) \quad \left[\prod_{k=1}^{T-2} M_k^{1/\alpha(k)} \right] \cdot M_{T-1}^{1/\beta(T-1)} = E_{T-1}[L_T]^{1/\alpha(t)}.$$

From this, we have the decomposition

$$(3.4) \quad M_{T-1} = L_{T-1} \cdot N_{T-2},$$

where N_{T-2} is an $\mathcal{F}_{T-2}$ -measurable random variable. We see that N_{T-2} satisfies

$$\left[\prod_{k=1}^{T-2} M_k^{1/\alpha(k)} \right] \cdot N_{T-2}^{1/\beta(T-1)} = 1.$$

However,

$$M_{T-2} = E_{T-2}[M_{T-1}] = E_{T-2}[L_{T-1}] \cdot N_{T-2} = L_{T-2}^{\beta(T-1)/\beta(T-2)} \cdot N_{T-2},$$

so that

$$\left[\prod_{k=1}^{T-3} M_k^{1/\alpha(k)} \right] \cdot N_{T-2}^{1/\beta(T-2)} = L_{T-2}^{-\beta(T-1)/\{\alpha(T-2)\beta(T-2)\}}.$$

Thus, N_{T-2} also has the decomposition

$$N_{T-2} = L_{T-2}^{-\beta(T-1)/\alpha(T-2)} \cdot N_{T-3},$$

where N_{T-3} is $\mathcal{F}_{T-3}$ -measurable. Moreover, this and (3.4) give

$$(3.5) \quad M_{T-1} = L_{T-1} \cdot L_{T-2}^{-\beta(T-1)/\alpha(T-2)} \cdot N_{T-3}.$$

The random variable N_{T-3} satisfies

$$\left[\prod_{k=1}^{T-3} M_k^{1/\alpha(k)} \right] \cdot N_{T-3}^{1/\beta(T-2)} = 1.$$

However, from

$$\begin{aligned} E_{T-2}[L_{T-1}] &= L_{T-2}^{\beta(T-1)/\beta(T-2)}, \\ E_{T-3}[L_{T-2}] &= L_{T-3}^{\beta(T-2)/\beta(T-3)}, \end{aligned}$$

we find that

$$\begin{aligned} M_{T-3} &= E_{T-3}[M_{T-1}] = E_{T-3}[L_{T-1} \cdot L_{T-2}^{-\beta(T-1)/\alpha(T-2)}] \cdot N_{T-3} \\ &= E_{T-3}[E_{T-2}[L_{T-1}] \cdot L_{T-2}^{-\beta(T-1)/\alpha(T-2)}] \cdot N_{T-3} \\ &= E_{T-3}[L_{T-2}] \cdot N_{T-3} = L_{T-3}^{\beta(T-2)/\beta(T-3)} \cdot N_{T-3}. \end{aligned}$$

Therefore,

$$\left[\prod_{k=1}^{T-4} M_k^{1/\alpha(k)} \right] \cdot N_{T-3}^{1/\beta(T-3)} = L_{T-3}^{-\beta(T-2)/\{\alpha(T-3)\beta(T-3)\}},$$

so that N_{T-3} has the decomposition

$$N_{T-3} = L_{T-3}^{-\beta(T-2)/\alpha(T-3)} \cdot N_{T-4},$$

where N_{T-4} is $\mathcal{F}_{T-4}$ -measurable. Moreover, from this and (3.5), we get

$$M_{T-1} = L_{T-1} \cdot L_{T-2}^{-\beta(T-1)/\alpha(T-2)} \cdot L_{T-3}^{-\beta(T-2)/\alpha(T-3)} \cdot N_{T-4}.$$

Repeating the arguments above, we finally obtain

$$M_{T-1} = L_{T-1} \cdot \prod_{k=1}^{T-2} L_k^{-\beta(k+1)/\alpha(k)}.$$

On the other hand, we find from (3.2) and (3.3) that

$$M_T = \frac{M_{T-1} \cdot L_T}{E_{T-1}[L_T]}.$$

Moreover, $E_{T-1}[L_T] = L_{T-1}^{\alpha(T)/\beta(T-1)}$. Combining,

$$\begin{aligned} M_T &= L_T \cdot L_{T-1}^{-\alpha(T)/\beta(T-1)} \cdot M_{T-1} \\ &= L_T \cdot L_{T-1}^{-\alpha(T)/\beta(T-1)} \cdot L_{T-1} \cdot \prod_{k=1}^{T-2} L_k^{-\beta(k+1)/\alpha(k)} \\ &= L_T \cdot \prod_{k=1}^{T-1} L_k^{-\beta(k+1)/\alpha(k)}. \end{aligned}$$

Thus M_T coincides with $M_T(\alpha, W)$. However, since both (M_t) and $(M_t(\alpha, W))$ are $(\mathcal{F}_t)$ -martingales, this implies that the two processes are identical. Thus the solution to Problem M is unique. $\square$

The next theorem follows immediately from Theorems 2.8 and 3.1.

Theorem 3.2. *The optimal intertemporal risk allocation $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(W)$ of $W \in L^\infty$ for the exponential utility function $u_t(x)$ in (EU) is unique and given by*

$$(3.6) \quad \exp(-\alpha_t \tilde{X}_t) = M_t(\alpha, W), \quad t \in \mathbb{T}.$$

We need the next proposition later.

Proposition 3.3. *Let $x \in \mathbb{R}$, $Z \in L^\infty$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0, \infty)^T$. Then, the following assertions hold:*

- (a) $L_t(\alpha, x) = \exp(-\beta_t x)$ for $t \in \mathbb{T}$.
- (b) $L_t(\alpha, x - Z) = \exp(-\beta_t x) L_t(\alpha, -Z)$ for $t \in \mathbb{T}$.

Proof. The assertion (a) follows immediately from the definition of $(L_t(\alpha, x))$. If we put $L'_t := \exp(-\beta_t x) L_t(\alpha, -Z)$ for $t \in \mathbb{T}$, then $(L'_t)_{t \in \mathbb{T}}$ satisfies

$$\begin{cases} L'_T = \exp[-\alpha_T(x - Z)], \\ L'_{t-1} = E_{t-1}[L'_t]^{\beta(t-1)/\beta(t)}, \quad t = 2, \dots, T, \end{cases}$$

whence $L'_t = L_t(\alpha, x - Z)$ for $t \in \mathbb{T}$ or (b). $\square$

3.2. The indifference prices for the exponential utility. In this section, we derive the indifference prices for the exponential utility $u_t(x)$ in (EU). Let $U, H : L^\infty \rightarrow \mathbb{R}$ be the utility and indifference price functionals defined from $u_t(x)$ as above, respectively. Recall β_t , $L_t(\alpha, Z)$ and $M_t(\alpha, Z)$ from Section 3.1.

For the exponential utility, the next theorem reduces the computation of the indifference price $H(Z)$ to that of $L_1(\alpha, -Z)$.

Theorem 3.4. *We assume (EU). Then, for $x \in \mathbb{R}$ and $Z \in L^\infty$, the following assertions hold:*

- (a) $U(Z) = \frac{1}{\beta_1} \{1 - E[L_1(\alpha, Z)]\}.$
- (b) $U(x - Z) = \frac{1}{\beta_1} \{1 - \exp(-\beta_1 x) \cdot E[L_1(\alpha, -Z)]\}.$
- (c) $H(Z) = \frac{1}{\beta_1} \log E[L_1(\alpha, -Z)].$

Proof. Define $(X_t)_{t \in \mathbb{T}} \in \mathcal{A}(Z)$ by (3.6). Then, by Theorem 3.2, the supremum in (U) is attained by (X_t) . Since $(M_t(\alpha, Z))_{t \in \mathbb{T}}$ is an $(\mathcal{F}_t)$ -martingale and $M_1(\alpha, Z) = L_1(\alpha, Z)$, we have

$$\begin{aligned} U(Z) &= \sum_{t \in \mathbb{T}} \frac{1}{\alpha_t} E[1 - \exp(-\alpha_t \tilde{X}_t)] = \sum_{t \in \mathbb{T}} \frac{1}{\alpha_t} E[1 - M_t(\alpha, Z)] \\ &= \{1 - E[M_1(\alpha, Z)]\} \sum_{t \in \mathbb{T}} \frac{1}{\alpha_t} = \frac{1}{\beta_1} \{1 - E[L_1(\alpha, Z)]\}. \end{aligned}$$

Thus (a) follows. The assertion (b) follows from (a) and Proposition 3.3 (b). Finally, (c) follows from (a), (b) and Proposition 3.3 (a). $\square$

From Theorem 3.4 (c), we see that the indifference price $H(Z)$ does not depend on the level w of the initial wealth for the exponential utility function.

The next proposition describes the optimal intertemporal allocation of the selling position $w + H(Z) - Z$ for the exponential utility.

Proposition 3.5. *We assume (EU). For $x \in \mathbb{R}$ and $Z \in L^\infty$, let $(X_t) \in \mathcal{A}(x - Z)$ be the optimal intertemporal allocation of $x - Z$: $\sum_{t \in \mathbb{T}} E[u_t(\tilde{X}_t)] = U(x - Z)$. Then, $(X_t)_{t \in \mathbb{T}}$ is given by*

$$\begin{aligned} X_1 &= \frac{B_1}{\alpha_1} [\beta_1 x - \log L_1(\alpha, -Z)], \\ X_t &= \frac{B_t}{\alpha_t} \left[\beta_1 x - \log L_t(\alpha, -Z) + \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \log L_k(\alpha, -Z) \right], \quad t = 2, \dots, T. \end{aligned}$$

Proof. Let $t \geq 2$ (the case $t = 1$ can be treated similarly). By Theorem 3.2 and Proposition 3.3, the optimal intertemporal allocation (X_t) of $x - Z$ satisfies

$$\begin{aligned} e^{-\alpha_t X_t / B_t} &= M_t(\alpha, x - Z) = L_t(\alpha, x - Z) \cdot \prod_{k=1}^{t-1} L_k(\alpha, x - Z)^{-\beta(k+1)/\alpha(k)} \\ &= e^{-\beta(t)x} \prod_{k=1}^{t-1} \left(e^{-\beta(k)x} \right)^{-\beta(k+1)/\alpha(k)} \\ &\quad \times L_t(\alpha, -Z) \cdot \prod_{k=1}^{t-1} L_k(\alpha, -Z)^{-\beta(k+1)/\alpha(k)}, \end{aligned}$$

whence

$$\begin{aligned} \frac{\alpha_t}{B_t} X_t &= \left\{ \beta_t - \sum_{k=1}^{t-1} \frac{\beta_k \beta_{k+1}}{\alpha_k} \right\} x \\ &\quad - \log L_t(\alpha, -Z) + \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \log L_k(\alpha, -Z). \end{aligned}$$

However, by simple calculation, we see that

$$\beta_t - \sum_{k=1}^{t-1} \frac{\beta_k \beta_{k+1}}{\alpha_k} = \beta_1.$$

Thus, the proposition follows. $\square$

4. INSURANCE PRICING

In this section, we apply the approach above to the computation of insurance premiums.

4.1. Life insurance contract. We consider a life insurance contract with duration T in which the insurer pays the insured c_t dollars at time $t \in \mathbb{T}$ if the insured dies in the interval $(t-1, t]$. Here c_t 's are deterministic. The insured pays the insurer a one-time premium at time $t = 0$.

We denote by τ the future life time of the insured, i.e., she/he dies at time τ . We assume that τ is a random variable on $(\Omega, \mathcal{F}, P)$ satisfying $\tau(\omega) > 0$ for all $\omega \in \Omega$ and $P(\tau = t) = 0$ for all $t \in [0, \infty)$.

If the insured pays the insurer H dollars as one time premium at time $t = 0$, then the present value of the cashflow of the insurer is given by $H - Z$ with

$$Z = \sum_{t \in \mathbb{T}} \tilde{c}_t 1_{(t-1 < \tau \leq t)}, \quad \tilde{c}_t := c_t/B_t \quad \text{for } t \in \mathbb{T}.$$

In the traditional pricing, the premium $H_0(Z)$ based on the *principle of equivalence* is often used: $H_0(Z)$ is defined by $E[H_0(Z) - Z] = 0$ or $H_0(Z) = E[Z]$. If the interest rates are deterministic, $H_0(Z)$ is given by

$$H_0(Z) = \sum_{t \in \mathbb{T}} \tilde{c}_t P(t-1 < \tau \leq t).$$

Notice that this price lacks the *safety loading* if the real mortality table is used. Usually, insurance companies use modified mortality tables to ensure the necessary safety loading (see §4.4 below).

We define a discrete-time process $(D_t)_{t \in \mathbb{T}}$ by

$$D_t := 1_{(\tau \leq t)}, \quad t = 0, 1, \dots, T.$$

Then, $(D_t)_{t \in \mathbb{T}}$ is a $\{0, 1\}$ -valued nondecreasing process with $D_0 = 0$. Notice that for $t \in \mathbb{T}$, $D_t = 0$ (resp., $D_t = 1$) if and only if the insurer is alive (resp., dead) at time t . We denote by $(\mathcal{H}_t)_{t \in \mathbb{T}}$ the filtration associated with the process $(D_t)_{t \in \{0\} \cup \mathbb{T}}$:

$$(4.1) \quad \mathcal{H}_t := \sigma(D_s : s = 0, \dots, t), \quad t = 0, 1, \dots, T.$$

We consider the following conditional probabilities:

$$\begin{aligned} q_t &:= P(\tau \leq t+1 \mid \tau > t), \quad t = 0, \dots, T-1, \\ p_t &:= 1 - q_t = P(\tau > t+1 \mid \tau > t), \quad t = 0, \dots, T-1. \end{aligned}$$

We have the following equalities:

$$q_t + p_t = 1 \quad (t = 0, \dots, T-1), \quad q_0 = P(\tau \leq 1), \quad p_0 = P(1 < \tau).$$

We use the following well-known result.

Lemma 4.1. *The following assertions hold:*

- (a) $E[D_t \mid \mathcal{H}_{t-1}] = D_{t-1} + (1 - D_{t-1})q_{t-1}$ for $t \in \mathbb{T}$.
- (b) $E[(1 - D_t) \mid \mathcal{H}_{t-1}] = (1 - D_{t-1})p_{t-1}$ for $t \in \mathbb{T}$.

4.2. Algorithm for the premium computation. The aim of this section is to derive an algorithm to compute the indifference premium of the life insurance contract. To this end, in addition to (EU), we assume the following conditions:

- (R) The interest rate process $(r_t)_{t \in \mathbb{T}}$ is deterministic.
- (F) The filtration $(\mathcal{F}_t)_{t \in \{0\} \cup \mathbb{T}}$ is given by $(\mathcal{H}_t)_{t \in \{0\} \cup \mathbb{T}}$ in (4.1).

The condition (R) implies that the riskless bond price process $(B_t)_{t \in \mathbb{T}}$ is also deterministic.

The σ -algebra $\mathcal{F}_T$ is generated by the following decomposition of Ω :

$$\Omega = (0 < \tau \leq 1) \cup (1 < \tau \leq 2) \cup \dots \cup (T-1 < \tau \leq T) \cup (T < \tau).$$

Hence, if $Z \in L^\infty(\Omega, \mathcal{F}_T, P)$, then Z has the decomposition of the form

$$(Z) \quad Z = \sum_{t=1}^T z_t 1_{(t-1 < \tau \leq t)} + z_{T+1} 1_{(T < \tau)}$$

with some real deterministic coefficients z_t , $t = 1, \dots, T+1$. We also write $z(t) = z_t$. For example, in the life insurance contract considered in the previous section, we have $z_t = \tilde{c}_t$ for $t = 1, \dots, T$ and $z_{T+1} = 0$.

Recall β_t from (β). For $Z \in L^\infty$ with representation (Z), we define the real deterministic sequence $(h_t)_{t=0}^T$ by the following backward iteration:

$$(h) \quad \begin{cases} h_t = e^{z(T+1)}, \\ h_{t-1} = \left[e^{\beta(t)z(t)} q_{t-1} + h_t^{\beta(t)} p_{t-1} \right]^{1/\beta(t)}, \quad t = 1, \dots, T. \end{cases}$$

Recall the definition of the process $(L_t(\alpha, -Z))_{t \in \mathbb{T}}$ from Section 2.

Proposition 4.2. *We assume (EU), (R) and (F). Then, for $Z \in L^\infty$ with (Z), the process $(L_t(\alpha, -Z))_{t \in \mathbb{T}}$ is given by*

$$(L2) \quad \begin{cases} L_1(\alpha, -Z) = e^{\beta(1)z(1)} D_1 + h_1^{\beta(1)} (1 - D_1), \\ L_t(\alpha, -Z) = \exp \left[\beta_t \sum_{s=1}^{t-1} \{z_s - z_{s+1}\} D_s \right] \\ \quad \times \left[e^{\beta(t)z(t)} D_t + h_t^{\beta(t)} (1 - D_t) \right], \quad t = 2, \dots, T. \end{cases}$$

Proof. For simplicity, we write $L_t = L_t(\alpha, -Z)$. Since

$$1_{(t-1 < \tau \leq t)} = D_t - D_{t-1} \quad (t = 1, \dots, T), \quad 1_{(T < \tau)} = 1 - D_T,$$

we have

$$(4.2) \quad Z = \sum_{t=1}^{T-1} \{z_t - z_{t+1}\} D_t + z_T D_T + z_{T+1} (1 - D_T).$$

To prove (L2), we use backward mathematical induction with respect to t .

First, if $t = T$, then from (4.2),

$$L_T = \exp \left[\beta_T \sum_{t=1}^{T-1} \{z_t - z_{t+1}\} D_t \right] \cdot \exp [\beta_T \{z_T D_T + z_{T+1} (1 - D_T)\}].$$

However, since D_T is either 1 or 0 and $h_t = \exp(z_{T+1})$, we have

$$\exp [\beta_T \{z_T D_T + z_{T+1} (1 - D_T)\}] = e^{\beta(t)z(t)} D_T + h_t^{\beta(t)} (1 - D_T),$$

which implies (L2) with $t = T$.

Next, we assume that (L2) holds for $t \in \{2, \dots, T\}$. Then,

$$E_{t-1}[L_t] = \exp \left[\beta_t \sum_{s=1}^{t-1} \{z_s - z_{s+1}\} D_s \right] \cdot E_{t-1} \left[e^{\beta(t)z(t)} D_t + h_t^{\beta(t)} (1 - D_t) \right],$$

where, as before, we write $E_t[X]$ for $E[X|\mathcal{F}_t]$. By Lemma 4.1,

$$\begin{aligned} E_{t-1} \left[e^{\beta(t)z(t)} D_t + h_t^{\beta(t)} (1 - D_t) \right] &= e^{\beta(t)z(t)} \{D_{t-1} + (1 - D_{t-1})q_{t-1}\} + h_t^{\beta(t)} (1 - D_{t-1})p_{t-1} \\ &= e^{\beta(t)z(t)} D_{t-1} + \left[e^{\beta(t)z(t)} q_{t-1} + h_t^{\beta(t)} p_{t-1} \right] (1 - D_{t-1}) \\ &= e^{\beta(t)z(t)} D_{t-1} + h_{t-1}^{\beta(t)} (1 - D_{t-1}). \end{aligned}$$

Hence, noting that D_{t-1} is either 1 or 0, we obtain

$$\begin{aligned}
L_{t-1} &= E_{t-1}[L_t]^{\beta(t-1)/\beta(t)} \\
&= \exp \left[\beta_{t-1} \sum_{s=1}^{t-1} \{z_s - z_{s+1}\} D_s \right] \\
&\quad \times \left[e^{\beta(t-1)z(t)} D_{t-1} + h_{t-1}^{\beta(t-1)} (1 - D_{t-1}) \right] \\
&= \exp \left[\beta_{t-1} \sum_{s=1}^{t-2} \{z_s - z_{s+1}\} D_s \right] \\
&\quad \times \left[e^{\beta(t-1)z(t-1)} D_{t-1} + h_{t-1}^{\beta(t-1)} (1 - D_{t-1}) \right],
\end{aligned}$$

which implies (L2) with $t-1$. Thus, (L2) holds for $t \geq 1$. $\square$

We are ready to give the algorithms to compute the indifference premium $H(Z)$ and corresponding optimal allocation of the selling position $w + H(Z) - Z$. We see that the computations are reduced to those of h_t , $t = 0, \dots, T$, in (h).

Theorem 4.3. *We assume (EU), (R) and (F). Let $Z \in L^\infty$ with representation (Z). Then, the following assertions hold.*

- (a) *The indifference price $H(Z)$ is given by $H(Z) = \log h_0$.*
- (b) *Let $(X_t) \in \mathcal{A}(w + H(Z) - Z)$ be the optimal intertemporal allocation of $w + H(Z) - Z$: $\sum_{t \in \mathbb{T}} E[u_t(\tilde{X}_t)] = U(w + H(Z) - Z) = U(w)$. Then, $(X_t)_{t \in \mathbb{T}}$ is given by*

$$\begin{aligned}
X_1 &= \frac{B_1}{\alpha_1} [\beta_1(w + H(Z)) - \beta_1 z_1 \cdot 1_{(0 < \tau \leq 1)} - \beta_1 \log h_1 \cdot 1_{(1 < \tau)}], \\
X_t &= \frac{B_t}{\alpha_t} \left[\beta_1(w + H(Z)) - \sum_{k=1}^t \beta_k z_k \cdot 1_{(k-1 < \tau \leq k)} - \beta_t \log h_t \cdot 1_{(t < \tau)} \right. \\
&\quad \left. + \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \beta_k \log h_k \cdot 1_{(k < \tau)} \right], \quad t = 2, \dots, T.
\end{aligned}$$

Proof. (a) Since $E[D_1] = q_0$ and $E[1 - D_1] = p_0$, it follows from Proposition 4.2 that

$$E[L_1(\alpha, -Z)] = e^{\beta(1)z(1)} q_0 + h_1^{\beta(1)} p_0 = h_0^{\beta(1)}.$$

The assertion (a) follows from this and Theorem 3.4 (c).

(b) From Proposition 4.2,

$$\log L_1(\alpha, -Z) = \beta_1 z_1 \cdot 1_{(0 < \tau \leq 1)} + \beta_1 \log h_1 \cdot 1_{(1 < \tau)}$$

and, for $t = 2, \dots, T$,

$$\begin{aligned}
\log L_t(\alpha, -Z) &= \beta_t \left[\sum_{s=1}^{t-1} \{z_s - z_{s+1}\} \cdot D_s + z_t \cdot D_t \right] \\
&\quad + \beta_t \log h_t \cdot (1 - D_t), \\
&= \beta_t \left[\sum_{s=1}^t z_s \cdot 1_{(s-1 < \tau \leq s)} \right] + \beta_t \log h_t \cdot 1_{(t < \tau)}.
\end{aligned}$$

We see that

$$\beta_t - \sum_{k=s}^{t-1} \frac{\beta_{k+1} \beta_k}{\alpha_k} = \beta_s, \quad 1 \leq s < t \leq T.$$

Hence, we have, for $t = 2, \dots, T$,

$$\begin{aligned}
& \log L_t(\alpha, -Z) - \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \log L_k(\alpha, -Z) \\
&= \beta_t \log h_t \cdot 1_{(t < \tau)} - \sum_{s=1}^{t-1} \frac{\beta_{s+1}}{\alpha_s} \beta_s \log h_s \cdot 1_{(s < \tau)} \\
&\quad + \beta_t \left[\sum_{s=1}^t z_s \cdot 1_{(s-1 < \tau \leq s)} \right] - \sum_{k=1}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \beta_k \left[\sum_{s=1}^k z_s \cdot 1_{(s-1 < \tau \leq s)} \right] \\
&= \beta_t \log h_t - \sum_{s=1}^{t-1} \frac{\beta_{s+1}}{\alpha_s} \beta_s \log h_s \\
&\quad + \beta_t z_t \cdot 1_{(t-1 < \tau \leq t)} + \sum_{s=1}^{t-1} \left[\beta_t - \sum_{k=s}^{t-1} \frac{\beta_{k+1}}{\alpha_k} \beta_k \right] z_s \cdot 1_{(s-1 < \tau \leq s)} \\
&= \beta_t \log h_t \cdot 1_{(t < \tau)} - \sum_{s=1}^{t-1} \frac{\beta_{s+1}}{\alpha_s} \beta_s \log h_s \cdot 1_{(s < \tau)} + \sum_{s=1}^t \beta_s z_s \cdot 1_{(s-1 < \tau \leq s)}.
\end{aligned}$$

This and Proposition 3.5 yield the assertion (b) with $t = 2, \dots, T$. We can prove the case $t = 1$ in the same way. $\square$

Remark 4.4. In the premium calculation method in Theorem 4.3, we have assumed that the interest rate process $(r_t)_{t \in \mathbb{T}}$ is deterministic (the condition (R)). If instead we assume, e.g., that $(r_t)_{t \in \mathbb{T}}$ is a Markovian process that is independent of τ , then we obtain a similar pricing method that involves the transition probabilities of $(r_t)_{t \in \mathbb{T}}$. Such extensions to the case of random-interest-rate will be reported elsewhere.

4.3. Dependence on the risk aversion coefficients. As in the previous section, we assume (EU), (R) and (F). The aim of this section is to investigate the dependence of the indifference price $H(Z)$ on the absolute risk aversion coefficient set $\alpha = (\alpha_1, \dots, \alpha_t) \in (0, \infty)^T$. To emphasize the dependence on α , we write $u_t(x; \alpha)$, $U_\alpha(Z)$, $H_\alpha(Z)$ and $h_t(\alpha)$ for the exponential utility function $u_t(x)$, utility $U(Z)$, indifference price $H(Z)$ and h_t in (h), respectively. In what follows, $\alpha \rightarrow 0+$ (resp., $\alpha \rightarrow \infty$) means that $\alpha_t \rightarrow +0$ (resp., $\alpha_t \rightarrow \infty$) for all $t \in \mathbb{T}$.

To study the asymptotic behavior of $H_\alpha(Z)$ as $\alpha \rightarrow 0+$, we need the next lemma.

Lemma 4.5. *For $z \in \mathbb{R}$, $q \in [0, 1]$, and $g : (0, \infty) \rightarrow (0, \infty)$ with limit $c := \lim_{x \rightarrow 0+} \log g(x) \in \mathbb{R}$, we define $f(x) := [qe^{zx} + (1-q)g(x)^x]^{1/x}$ for $x > 0$. Then,*

$$\lim_{x \rightarrow 0+} \log f(x) = qz + (1-q)c.$$

Proof. Take $\varepsilon > 0$. If x is positive and sufficiently close to 0, then

$$\frac{1}{x} \log [qe^{zx} + (1-q)e^{(c-\varepsilon)x}] \leq \log f(x) \leq \frac{1}{x} \log [qe^{zx} + (1-q)e^{(c+\varepsilon)x}],$$

which yields

$$qz + (1-q)(c - \varepsilon) \leq \liminf_{x \rightarrow 0+} \log f(x) \leq \limsup_{x \rightarrow 0+} \log f(x) \leq qz + (1-q)(c + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, the lemma follows. $\square$

For $Z \in L^\infty$ with representation (Z) , we have

$$E[Z] = \sum_{t=1}^T z_t P(t-1 < \tau \leq t) + z_{T+1} P(T < \tau).$$

We define $H_\infty(Z)$ by

$$H_\infty(Z) := \max\{z_1, \dots, z_{T+1}\}.$$

We can view $E[Z]$ (resp., $H_\infty(Z)$) as a lower (resp., upper) bound for any reasonable price of Z . From the next theorem, we see that $H_\alpha(Z)$ takes any value in $(E[Z], H_\infty(Z))$ by a suitable choice of $\alpha \in (0, \infty)^T$.

Theorem 4.6. *We assume (EU), (R) and (F). We also assume $0 < q_t < 1$ for all $t = 0, \dots, T-1$. Then, for $Z \in L^\infty$, the following assertions hold:*

- (a) $E[Z] \leq H_\alpha(Z) \leq H_\infty(Z)$ for all $\alpha \in (0, \infty)^T$.
- (b) $\lim_{\alpha \rightarrow 0+} H_\alpha(Z) = E[Z]$.
- (c) $\lim_{\alpha \rightarrow \infty} H_\alpha(Z) = H_\infty(Z)$.
- (d) For every $\pi \in (E[Z], H_\infty(Z))$ and $\alpha = (\alpha_1, \dots, \alpha_T) \in (0, \infty)^T$, there exists $p \in (0, \infty)$ such that $\pi = H_{p\alpha}(Z)$, where $p\alpha := (p\alpha_1, \dots, p\alpha_T)$.

Proof. (a) By (3.1), we have $u_t(x; \alpha) \leq x$. Hence, for $W \in L^\infty$,

$$\begin{aligned} U_\alpha(W) &= \sup \left\{ \sum_{t=0}^T E[u_t(\tilde{X}_t, \alpha)] : (X_t) \in \mathcal{A}(W) \right\} \\ &\leq \sup \left\{ E \left[\sum_{t=0}^T \tilde{X}_t \right] : (X_t) \in \mathcal{A}(W) \right\} = E[W], \end{aligned}$$

which implies $0 = U_\alpha(H_\alpha(Z) - Z) \leq E[H_\alpha(Z) - Z]$ or $E[Z] \leq H_\alpha(Z)$.

By (h), we have $h_T(\alpha) \leq \exp[H_\infty(Z)]$. Moreover, if $h_t(\alpha) \leq \exp[H_\infty(Z)]$, then

$$h_{t-1}(\alpha) \leq \left[q_{t-1} e^{\beta(t) H_\infty(Z)} + p_{t-1} e^{\beta(t) H_\infty(Z)} \right]^{1/\beta(t)} = e^{H_\infty(Z)}.$$

Thus we finally see that $h_\alpha(0) \leq \exp[H_\infty(Z)]$. This and Theorem 4.3 (a) give $H_\alpha(Z) \leq H_\infty(Z)$.

(b) We have $\beta \rightarrow 0+$ as $\alpha \rightarrow 0+$. Hence, by applying Lemma 4.5 iterately to

$$h_{t-1}(\alpha) = \left[e^{\beta(t) z(t)} q_{t-1} + h_t(\alpha)^{\beta(t)} p_{t-1} \right]^{1/\beta(t)}, \quad t = 1, \dots, T,$$

with $x = \beta$, $q = q_{t-1}$, $z = z_t$, and $g(x) = h_t(\alpha)$, we see the existence of the limits $h_t(0) := \lim_{\alpha \rightarrow 0+} h_t(\alpha)$, $t = 0, \dots, T$, satisfying

$$\begin{cases} \log h_t(0) = z_{T+1}, \\ \log h_{t-1}(0) = q_{t-1} z_t + p_{t-1} \log h_t(0), \quad t = 1, \dots, T. \end{cases}$$

From this, we get

$$\log h_0(0) = q_0 z_1 + \sum_{t=1}^{T-1} \left(\prod_{s=0}^{t-1} p_s \right) q_t z_{t+1} + \left(\prod_{s=0}^{T-1} p_s \right) z_{T+1}.$$

However, we have $q_0 = P(0 < \tau \leq 1)$,

$$p_0 q_1 = P(\tau > 1) P(\tau \leq 2 | \tau > 1) = P(1 < \tau \leq 2),$$

and more generally,

$$\left(\prod_{s=0}^{t-1} p_s \right) q_t = P(t < \tau \leq t+1), \quad t = 1, \dots, T-1.$$

We also have $\prod_{s=0}^{T-1} p_s = P(T < \tau)$. Thus

$$\log h_0(0) = \sum_{t=1}^T z_t P(t-1 < \tau \leq t) + z_{T+1} P(T < \tau) = E[Z]$$

or

$$\lim_{\alpha \rightarrow 0+} H_\alpha(Z) = \lim_{\alpha \rightarrow 0+} \log h_0(\alpha) = E[Z].$$

(c) Let $H_\infty(Z) = z_{t_0}$ with $t_0 \in \{1, \dots, T+1\}$. If $t_0 \geq 2$, then

$$h_{t_0-1}(\alpha) = \left[q_{t_0-1} e^{\beta(t_0)H_\infty(Z)} + p_{t_0-1} h_{t_0}(\alpha)^{\beta(t_0)} \right]^{1/\beta(t_0)} \geq q_{t_0-1}^{1/\beta(t_0)} e^{H_\infty(Z)},$$

which, together with (h), gives

$$h_{t_0-2}(\alpha) \geq p_{t_0-2}^{1/\beta(t_0-1)} h_{t_0-1}(\alpha) \geq p_{t_0-2}^{1/\beta(t_0-1)} q_{t_0-1}^{1/\beta(t_0)} e^{H_\infty(Z)}.$$

Repeating this argument, we finally obtain

$$h_0(\alpha) \geq \left(\prod_{s=0}^{t_0-2} p_s^{1/\beta(s+1)} \right) q_{t_0-1}^{1/\beta(t_0)} e^{H_\infty(Z)}.$$

Similarly, if $t_0 = 1$, then $h_0(\alpha) \geq q_0^{1/\beta(1)} e^{H_\infty(Z)}$. Therefore, since $\beta \rightarrow \infty$ as $\alpha \rightarrow \infty$, we obtain

$$\liminf_{\alpha \rightarrow \infty} H_\alpha(Z) = \liminf_{\alpha \rightarrow \infty} \log h_0(\alpha) \geq H_\infty(Z).$$

However, $H_\alpha(Z) \leq H_\infty(Z)$ by (a), so that $\lim_{\alpha \rightarrow \infty} H_\alpha(Z) = H_\infty(Z)$.

(d) By the construction in (h), $h_0(\alpha)$, whence $H_\alpha(Z) = \log h_0(\alpha)$, is continuous in $\alpha \in (0, \infty)^T$. Therefore, the assertion (d) follows from (a)–(c). $\square$

4.4. Numerical examples. We compare the indifference pricing method in Theorem 4.3 with traditional ones by applying them to the following same insurance contract:

- Type of insurance: term mortality insurance.
- Age at issue: 30 years old.
- Sex: male.
- Term of contract: from 1 year to 30 years.
- Loading of premium: excluded.
- Mortality rate: Standard Mortality Table 2007 for mortality insurance (made by the Institute of Actuaries of Japan).
- Discount rate: 2%.
- Payment method: annual payment.
- Sum assured: 1 (during the entire contract term).

By using the notation in the previous sections, the aggregate risk Z of this contract becomes

$$Z = \sum_{t=1}^T \frac{1}{(1+0.02)^t} 1_{(t-1 < \tau \leq t)}.$$

The traditional pricing methods that we use here are as follows:

(1) Traditional method without risk loading:

$$\text{The premium TP1}(T) = \sum_{t=1}^T \frac{1}{(1+0.02)^t} P(t-1 < \tau \leq t).$$

(2) Traditional method with risk loading:

$$\text{The premium TP2}(T) = \sum_{t=1}^T \frac{1}{(1 + 0.02)^t} Q'_t,$$

where $Q'_t := Q_t + \{Q_t(1 - Q_t)\}^{1/2}$ with $Q_t := P(t - 1 < \tau \leq t)$.

As above, we write $\text{TP1}(T)$ and $\text{TP2}(T)$ for the premiums of the contract with T years of term obtained by the traditional pricing methods (1) and (2), respectively. For the values $a = 1.0, 1.5, 2.0$ and 2.5 , we denote by $\text{IP}_a(T)$ the premium of the same contract obtained by the indifference pricing method in Theorem 4.3 with $\alpha(t) \equiv a$. We also write $\text{IP}_{\text{fit}}(T)$ for the premium of the same contract calculated by the pricing method in Theorem 4.3 with $\alpha(t) = 0.6 + 0.36\sqrt{t}$, the form of which is chosen to fit the graph of the indifference prices to that of TP2. We used the nonlinear least-squares to determine the form of $\alpha(t)$ for $\text{IP}_{\text{fit}}(T)$.

In Figures 4.1–4.3, we plot the graphs of TP1 , TP2 , IP_a , and IP_{fit} . We see that the fitted premiums $\text{IP}_{\text{fit}}(T)$ simultaneously approximate the corresponding traditional prices $\text{TP2}(T)$ well. We have repeated this procedure for various prices and obtained good fits in most cases. This observation suggests the following *implied utility approach* to coherent pricing: insurance companies estimate their implied utility functions by applying this method to existing products, and then refers to them in pricing other products.

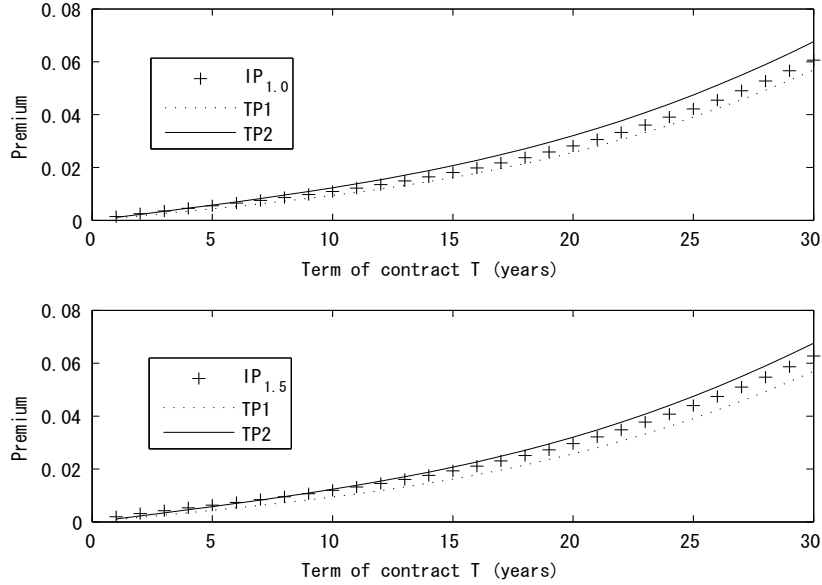


FIGURE 4.1. TP1 and TP2 vs. $\text{IP}_{1.0}$ and $\text{IP}_{1.5}$.

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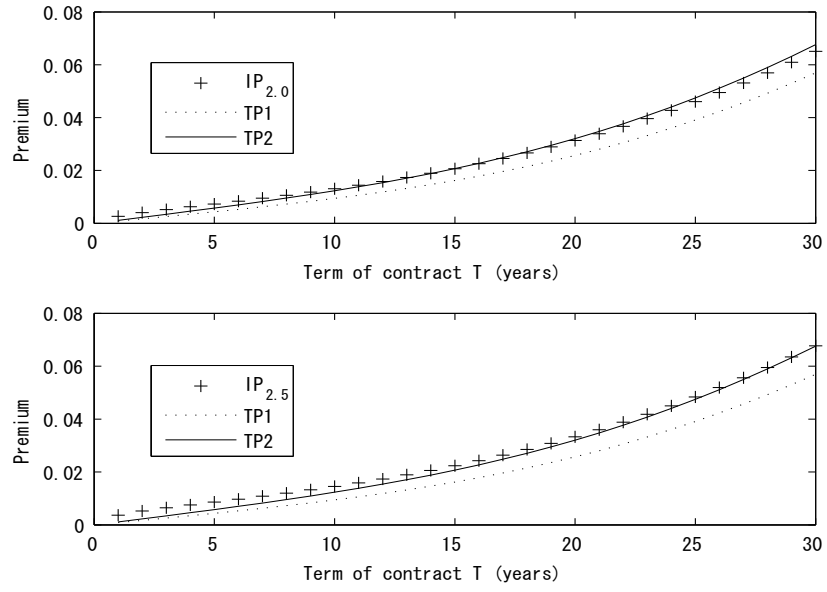


FIGURE 4.2. TP1 and TP2 vs. $IP_{2.0}$ and $IP_{2.5}$.

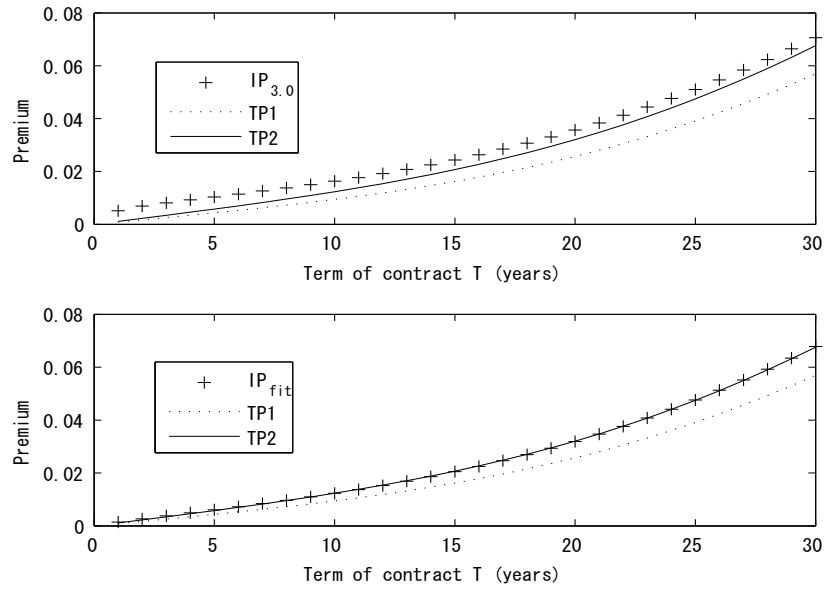


FIGURE 4.3. TP1 and TP2 vs. $IP_{3.0}$ and IP_{fit} .

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