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BAXTER'S INEQUALITY FOR FRACTIONAL BROWNIAN MOTION-TYPE PROCESSES WITH HURST INDEX LESS THAN 1/2

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Abstract. We prove an analogue of Baxter's inequality for fractional Brownian motion-type processes with Hurst index less than 1/2. This inequality is concerned with the norm estimate of the difference between finite- and infinite-past predictor coefficients.

1. Introduction

To explain Baxter's inequality in the classical setup, we consider a centered, weakly stationary process $(X_k : k \in \mathbf{Z})$, and write ϕ_j and $\phi_{j,n}$ for the infinite- and finite-past predictor coefficients, respectively:

$$P_{(-\infty,-1]}X_0 = \sum_{j=1}^{\infty} \phi_j X_{-j}, \qquad P_{[-n,-1]}X_0 = \sum_{j=1}^{n} \phi_{j,n} X_{-j}, \tag{1.1}$$

where $P_{(-\infty,-1]}X_0$ and $P_{[-n,-1]}X_0$ denote the linear least-squares predictors of X_0 based on the observed values $\{X_{-j} : j = 1, 2, \dots\}$ and $\{X_{-j} : j = 1, \dots, n\}$, respectively. There are many models in which $\phi_{j,n}$'s are difficult to compute exactly while the computation of ϕ_j's are relatively easy. In fact, this is usually so for the models with explicit spectral density. It is known that $\lim_{n\to\infty} \phi_{j,n} = \phi_n$ (see, e.g., Pourahmadi, 2001, Theorem 7.14). Therefore, it would be natural to approximate $P_{[-n,-1]}X_0$ replacing the finite-past predictor coefficients $\phi_{j,n}$ by the infinite counterparts ϕ_j. Then the error can be estimated by

$$\left\| P_{[-n,-1]}X_0 - \sum_{j=1}^{n} \phi_j X_{-j} \right\| \le \|X_0\| \sum_{j=1}^{n} |\phi_{j,n} - \phi_j|, \tag{1.2}$$

where $\|Z\| := E[Z^2]^{1/2}$. The question thus arises of estimating the right-hand side of (1.2). Baxter (1962) showed that for short memory processes, there exists a positive constant M such that

$$\sum_{j=1}^{n} |\phi_{j,n} - \phi_j| \le M \sum_{k=n+1}^{\infty} |\phi_k| \qquad \text{for all } n = 1, 2, \dots.$$

This *Baxter's inequality* was extended to long memory processes by Inoue and Kasahara (2006). See also Berk (1974), Cheng and Pourahmadi (1993) and Pourahmadi (2001, Section 7.6.2).

In Inoue and Anh (2007), prediction formulas similar to (1.1) were proved for a class of continuous-time, centered, stationary-increment, Gaussian processes $(X(t) : t \in \mathbf{R})$ that includes fractional Brownian motion $(B_H(t) : t \in \mathbf{R})$ with Hurst index $H \in (0, 1/2)$ (see Section 3 for the definition). For

$$-\infty < t_0 \le 0 \le t_1 < t_2 < \infty, \qquad t_0 < t_1, \qquad T := t_2 - t_1, \qquad t := t_1 - t_0, \tag{1.3}$$

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the prediction formulas take the following forms:

$$P_{(-\infty,t_1]}X(t_2) = \int_0^\infty \psi(s;T)X(t_1-s)ds, \qquad P_{[t_0,t_1]}X(t_2) = \int_0^t \psi(s;T,t)X(t_1-s)ds, \tag{1.4}$$

where $P_{(-\infty,t_1]}X(t_2)$ and $P_{[t_0,t_1]}X(t_2)$ are the linear least-squares predictors of $X(t_2)$ based on the infinite past $\{X(s) : -\infty < s \le t_1\}$ and finite past $\{X(s) : t_0 \le s \le t_1\}$, respectively.

The aim of this paper is to prove an analogue of Baxter's inequality for $(X(t))$. Since $\|X(s)\|$ depends on s, a straightforward analogue of (1.2) is not available. Instead, we have

$$\left\| P_{[t_0,t_1]}X(t_2) - \int_0^t \psi(s;T)X(t_1-s)ds \right\| \le \int_0^t \{\psi(s;T,t) - \psi(s;T)\}\|X(t_1-s)\|ds.$$

Here $\psi(s;T,t) > \psi(s;T) > 0$ (see Section 3 below). We show that there is a positive constant M such that

$$\int_0^t \{\psi(s;T,t) - \psi(s;T)\}\|X(t_1-s)\|ds \le M\int_t^\infty \psi(s;T)\|X(t_1-s)\|ds \quad \text{for all } t \ge t_1, \tag{B}$$

which we call *Baxter's inequality for* $(X(t))$. To the best of our knowledge, this type of inequality has not been demonstrated before. The key ingredient in the proof is the representation of the difference $\psi(s;T,t) - \psi(s;T)$ ((3.2) with Proposition 3.2 below). In fact, we prove a general result that includes (B) (Theorem 4.2 (b)).

2. Fractional Brownian motion

Throughout the paper, we assume $0 < H < 1/2$. We can define the fractional Brownian motion $(B_H(t) : t \in \mathbf{R})$ with Hurst index H by the moving-average representation

$$B_H(t) = \frac{1}{\Gamma(\frac{1}{2}+H)}\int_{-\infty}^\infty \left\{((t-s)_+)^{H-\frac{1}{2}} - ((-s)_+)^{H-\frac{1}{2}}\right\} dW(s) \qquad (t \in \mathbf{R}),$$

where $(x)_+ := \max(x,0)$ and $(W(t) : t \in \mathbf{R})$ is the ordinary Brownian motion. In this section, we study the difference between the finite- and infinite-past predictor coefficients of $(B_H(t))$.

Let t_0, t_1, t_2, t and T be as in (1.3). We define the infinite- and finite-past predictors $P_{(-\infty,t_1]}B_H(t_2)$ and $P_{[t_0,t_1]}B_H(t_2)$ of $(B_H(t))$, respectively, as we defined in Section 1 for $(X(t))$. The following prediction formulas, that is, (1.4) for $(B_H(t))$, were established by Yaglom (1955) and Nuzman and Poor (2000, Theorem 4.4), respectively (see also Anh and Inoue, 2004, Theorem 1):

$$P_{(-\infty,t_1]}B_H(t_2) = \int_0^\infty \psi_0(s;T)B_H(t_1-s)ds, \quad P_{[t_0,t_1]}B_H(t_2) = \int_0^t \psi_0(s;T,t)B_H(t_1-s)ds,$$

where

$$\psi_0(s;T) = \frac{\cos(\pi H)}{\pi}\,\frac{1}{s+T}\left(\frac{T}{s}\right)^{\frac{1}{2}+H} \qquad (0 < s < \infty),$$

$$\psi_0(s;T,t) = \frac{\cos(\pi H)}{\pi}\left[\frac{1}{s+T}\left(\frac{T}{s}\right)^{\frac{1}{2}+H}\left(\frac{t-s}{t+T}\right)^{\frac{1}{2}-H}\right.$$
$$\left. + (\tfrac{1}{2}-H)B_{\frac{t}{t+T}}(H+\tfrac{1}{2}, 1-2H)\,\frac{1}{t}\left\{\left(\frac{t}{s}\right)\left(\frac{t}{t-s}\right)\right\}^{\frac{1}{2}+H}\right] \quad (0 < s < t),$$

with $B_s(p,q) := \int_0^s u^{p-1}(1-u)^{q-1}du$ being the incomplete beta function.

Throughout the paper, $f(t) \sim g(t)$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 1$. A positive measurable function f, defined on some neighbourhood $[M, \infty)$ of ∞, is called *regularly varying with index* $\rho \in \mathbf{R}$, written $f \in R_\rho$, if for all $\lambda \in (0, \infty)$, $\lim_{t\to\infty} f(t\lambda)/f(t) = \lambda^\rho$. When $\rho = 0$, we say that the function is *slowly varying*. A generic slowly varying function is usually denoted by ℓ. See Bingham et al. (1989) for details. The function $\|B_H(t_1 - s)\|$ of s is in R_H since $\|B_H(s)\| = |s|^H \|B_H(1)\|$ for $s \in \mathbf{R}$.

We will use the next lemma in Section 4. For $0 < H < \frac{1}{2}$ and $\rho > -\frac{1}{2} + H$, we put

$$C(H, \rho) := 1 - \rho\, B(\tfrac{1}{2} - H + \rho, \tfrac{1}{2} - H)\frac{1 - 2H}{1 + 2H},$$

where $B(p, q) := \int_0^1 u^{p-1}(1 - u)^{q-1} du$ denotes the beta function.

Lemma 2.1. *Let g be locally bounded in $[0, \infty)$ and $g \in R_\rho$ with $\rho > -\frac{1}{2} + H$. Then, for fixed $T > 0$,*

$$\int_0^t \{\psi_0(s; T, t) - \psi_0(s; T)\} g(s) ds \sim \frac{C(H, \rho)}{\frac{1}{2} + H - \rho} \cdot t\, \psi_0(t; T) g(t) \qquad (t \to \infty).$$

Proof. If t is large enough, then $g(t) > 0$. For such t, we have, by simple computation,

$$\begin{aligned}\frac{1}{t\,\psi_0(t; T) g(t)} \int_0^t \{\psi_0(s; T, t) - \psi_0(s; T)\} g(s) ds &= \int_0^1 \frac{\psi_0(ts; T, t) - \psi_0(ts; T)}{\psi_0(t, T)} \frac{g(ts)}{g(t)} ds \\ &= \int_0^1 \mathrm{I}(s; T, t) \frac{g(ts)}{g(t)} ds + \int_0^1 \mathrm{II}(s; T, t) \frac{g(ts)}{g(t)} ds,\end{aligned}$$

where

$$\mathrm{I}(s; T, t) = s^{-\frac{1}{2} - H} \frac{1 + (T/t)}{s + (T/t)} \left[\left(\frac{1 - s}{1 + (T/t)} \right)^{\frac{1}{2} - H} - 1 \right],$$

$$\mathrm{II}(s; T, t) = (\tfrac{1}{2} - H) B_{\frac{t}{t+T}}(H + \tfrac{1}{2}, 1 - 2H)(t/T)^{\frac{1}{2} + H} \{1 + (T/t)\} \{s(1 - s)\}^{-\frac{1}{2} - H}.$$

Since $B_{t/(t+T)}(H + \frac{1}{2}, 1 - 2H) \sim \left(\frac{1}{2} + H\right)^{-1} (T/t)^{\frac{1}{2} + H}$ as $t \to \infty$, we easily see that, for $0 < s < 1$,

$$|\mathrm{I}(s; T, t)| \le \text{const.} \times s^{-\frac{1}{2} - H}, \qquad |\mathrm{II}(s; T, t)| \le \text{const.} \times \{s(1 - s)\}^{-\frac{1}{2} - H} \qquad (t \text{ large enough}).$$

Put $\delta = \frac{1}{2}(\frac{1}{2} - H + \rho) > 0$. Then, for $0 < s < 1$, also we have

$$|g(ts)/g(t)| \le 2s^{\rho - \delta} \quad (t \text{ large enough}) \tag{2.1}$$

(cf. Bingham et al., 1989, Theorem 1.5.2). Therefore, the dominated convergence theorem yields, as $t \to \infty$,

$$\int_0^1 \mathrm{I}(s; T, t) \frac{g(ts)}{g(t)} ds \to \int_0^1 \frac{(1 - s)^{\frac{1}{2} - H} - 1}{s^{\frac{3}{2} + H - \rho}} ds = \frac{1 - (\frac{1}{2} - H) B(\frac{1}{2} - H + \rho, \frac{1}{2} - H)}{\frac{1}{2} + H - \rho}, \tag{2.2}$$

$$\int_0^1 \mathrm{II}(s; T, t) \frac{g(ts)}{g(t)} ds \to \frac{(\frac{1}{2} - H) B(\frac{1}{2} - H + \rho, \frac{1}{2} - H)}{\frac{1}{2} + H}. \tag{2.3}$$

In (2.2), we have used integration by parts. From (2.2) and (2.3), we obtain the lemma. □

Remark 2.2. From Lemma 2.1 with $g(t) = \|B_H(t_1 - t)\|$, whence $\rho = H$, we see that

$$\int_0^t \{\psi_0(s; T, t) - \psi_0(s; T)\} \|B_H(t_1 - s)\| ds \sim \frac{2}{\pi} \cos(\pi H) C(H, H) T^{\frac{1}{2} + H} \|B_H(1)\| \cdot t^{-\frac{1}{2}} \qquad (t \to \infty).$$

It is interesting that the order of decay here is $t^{-1/2}$, whence does not depend on H.

3. Fractional Brownian motion-type processes

In this and next sections, we consider the predictor coefficients for the fractional Brownian motion-type process $(X(t) : t \in \mathbf{R})$ in Inoue and Anh (2007). It is a stationary-increment Gaussian process defined by

$$X(t) = \int_{-\infty}^{\infty} \{c(t-s) - c(-s)\}\, dW(s), \qquad (t \in \mathbf{R}),$$

where the moving-average coefficient c is a function of the form

$$c(t) = \int_0^\infty e^{-ts}\nu(ds) \quad (t > 0), \qquad = 0 \quad (t \le 0)$$

with ν being a Borel measure on $(0,\infty)$ satisfying $\int_0^\infty (1+s)^{-1}\nu(ds) < \infty$. We also assume

$$\lim_{t\to 0+} c(t) = \infty, \qquad c(t) = O(t^q) \quad (t \to 0+) \quad \text{for some } q > -1/2,$$

$$c(t) \sim \frac{1}{\Gamma(\frac{1}{2}+H)} t^{-(\frac{1}{2}-H)}\ell(t) \qquad (t \to \infty),$$

where $\ell(\cdot)$ is a slowly varying function and $H \in (0, 1/2)$.

The process $(X(t))$ also has the autoregressive coefficient a defined by $a(t) := -(d\alpha/dt)(t)$ for $t > 0$, where α is the unique function on $(0,\infty)$ satisfying

$$-iz\left(\int_0^\infty e^{izt}c(t)dt\right)\left(\int_0^\infty e^{izt}\alpha(t)dt\right) = 1 \qquad (\Im z > 0).$$

We know that $a(t) = \int_0^\infty e^{-ts}s\mu(ds)$ for some Borel measure μ on $(0,\infty)$ (see Inoue and Anh, 2007, Corollary 3.3). In particular, a is also positive and decreasing on $(0,\infty)$. By Inoue and Anh (2007, (3.12)), we have

$$a(t) \sim \frac{t^{-(\frac{3}{2}+H)}}{\ell(t)} \cdot \frac{(\frac{1}{2}+H)}{\Gamma(\frac{1}{2}-H)} \qquad (t \to \infty). \tag{3.1}$$

Example 3.1. If ν is given by $\nu(ds) = \pi^{-1}\cos(\pi H)s^{-(\frac{1}{2}+H)}ds$ on $(0,\infty)$, then $c(t) = t^{-(\frac{1}{2}-H)}/\Gamma(\frac{1}{2}+H)$ for $t > 0$, whence $(X(t))$ reduces to $(B_H(t))$. In this case, $a(t) = t^{-(\frac{3}{2}+H)}(\frac{1}{2}+H)/\Gamma(\frac{1}{2}-H)$.

We refer to Inoue and Anh (2007, Example 2.6) for another example of $(X(t))$ which has two different indexes H_0 and H describing its path properties and long-time behaviour, respectively.

We put

$$b(s,u) := \int_0^u c(u-v)a(s+v)dv \qquad (s,u > 0).$$

For $k = 1, 2\ldots$ and $s, t, T > 0$, we define $b_k(s,t;T)$ iteratively by

$$b_1(s;T,t) := b(s,T), \qquad b_k(s;T,t) := \int_0^\infty b(s,u)b_{k-1}(t+u;T,t)du \qquad (k = 2,3,\ldots).$$

Note that b_k's are positive because both c and a are so. By Inoue and Anh (2007, Theorems 3.7 and 1.1), the infinite- and finite-past predictor coefficients $\psi(s;T)$ and $\psi(s;T,t)$ in (1.4)

are given, respectively, by

$$\psi(s;T) = b(s,T) = b_1(s;T,t) \quad (s>0),$$
$$\psi(s;T,t) = \sum_{k=1}^{\infty} \{b_{2k-1}(s;T,t) + b_{2k}(t-s;T,t)\} \quad (0<s<t).$$

Notice that $\psi(s;T,t)$ here corresponds to $h(t-s;T,t)$ in Inoue and Anh (2007). We have

$$\psi(s;T,t) - \psi(s;T) = \sum_{k=1}^{\infty} \{b_{2k}(t-s;T,t) + b_{2k+1}(s;T,t)\} \qquad (0<s<t), \tag{3.2}$$

which plays a key role in the proof of Baxter's inequality (B) in the next section.

To prove Baxter's inequality (B), we need to discuss the following. Consider

$$\beta(t) := \int_0^{\infty} c(v)a(t+v)dv \qquad (t>0),$$

and define $\delta_k(t,u,v)$ for $k=1,2,3,\ldots$ and $t,u,v>0$, iteratively by

$$\delta_1(t,u,v) := \beta(t+u+v), \qquad \delta_k(t,u,v) := \int_0^{\infty} \beta(t+v+w)\delta_{k-1}(t,u,w)dw \qquad (k=2,3,\ldots).$$

Proposition 3.2. *For $s,t,T>0$ and $k\geq 2$,*

$$b_k(s;T,t) = \int_0^T c(T-v)dv \int_0^{\infty} a(s+u)\delta_{k-1}(t,u,v)du.$$

This can be proved in the same as in Inoue and Kasahara (2006, Theorem 2.8); we omit the proof.

Next, we give some results on the asymptotic behaviour of δ_k's. For $k=1,2,\ldots$ and $u\geq 0$, we define $f_k(u)$ iteratively by

$$f_1(u) := \frac{1}{\pi(1+u)}, \qquad f_k(u) := \int_0^{\infty} \frac{f_{k-1}(u+v)}{\pi(1+v)}dv \qquad (k=2,3,\ldots).$$

Proposition 3.3. (a) *For $r\in(1,\infty)$, there exists $N>0$ such that $0<\delta_k(t,u,v)\leq f_k(0)\{r\cos(\pi H)\}^k t^{-1}$ for $u,v>0$, $k\in\mathbf{N}$, $t\geq N$.*

(b) *For $k\in\mathbf{N}$ and $u,v>0$, $\delta_k(t,tu,v)\sim t^{-1}f_k(u)\cos^k(\pi H)$ as $t\to\infty$.*

This can be proved in the same as in Inoue and Kasahara (2006, Proposition 3.2); we omit the proof.

4. Baxter's inequality

In this section, we prove Baxter's inequality (B). Let $(X(t))$, $\psi(s;T)$ and $\psi(s;T,t)$ be as in Section 3. Since a is decreasing, we have $a(T+t)\int_0^T c(v)dv \leq \psi(t;T) \leq a(t)\int_0^T c(v)dv$, so that (3.1) implies

$$\psi(t;T) \sim a(t)\int_0^T c(v)dv \sim \frac{t^{-(\frac{3}{2}+H)}}{\ell(t)} \cdot \frac{(\frac{1}{2}+H)}{\Gamma(\frac{1}{2}-H)} \int_0^T c(v)dv \qquad (t\to\infty). \tag{4.1}$$

Here is the extension of Lemma 2.1 to $(X(t))$.

Lemma 4.1. *Lemma 2.1 with $\psi_0(s;T,t)$ and $\psi_0(s;T)$ replaced by $\psi(s;T,t)$ and $\psi(s;T)$, respectively, holds.*

Proof. For t large enough, using (3.2), we may write

$$D(t) := \frac{1}{t\,\psi(t;T)g(t)} \int_0^t \{\psi(s;T,t) - \psi(s;T)\} g(s) ds = \int_0^1 \frac{\psi(ts;T,t) - \psi(ts;T)}{\psi(t;T)} \frac{g(ts)}{g(t)} ds$$
$$= \sum_{k=1}^{\infty} \int_0^1 \frac{b_{2k}(ts;T,t)}{\psi(t;T)} \frac{g(t(1-s))}{g(t)} ds + \sum_{k=1}^{\infty} \int_0^1 \frac{b_{2k+1}(ts;T,t)}{\psi(t;T)} \frac{g(ts)}{g(t)} ds$$

and

$$\frac{b_k(ts;T,t)}{\psi(t;T)} = \frac{a(t)}{\psi(t;T)} \int_0^T c(T-v)dv \int_0^{\infty} \frac{a(ts+u)}{a(t)}\, \delta_{k-1}(t,u,v)du$$
$$= \frac{a(t)}{\psi(t;T)} \int_0^T c(T-v)dv \int_0^{\infty} \frac{a(t(s+u))}{a(t)} \cdot t\, \delta_k(t,tu,v)du.$$

Put $\delta = \frac{1}{3}\min\{\frac{1}{2} - H, \frac{1}{2} - H + \rho\} > 0$. By (3.1), we have $a \in R_{-(3/2)-H}$, and

$$a(t\lambda)/a(t) \le 2\lambda^{-\frac{3}{2}-H-\delta} \text{ for } 0 < \lambda < 1, \quad \le 2\lambda^{-\frac{3}{2}} \text{ for } \lambda > 1 \qquad (t \text{ large enough})$$

(cf. Bingham et al., 1989, Theorems 1.5.2 and 1.5.6). Choose $0 < r < 1/\cos(\pi H)$ so that $x := r\cos(\pi k) \in (0,1)$. Then, by Proposition 3.3 (a), we have for $0 < s < 1$ and $v > 0$,

$$\int_0^{\infty} \frac{a(t(s+u))}{a(t)} \cdot t\, \delta_k(t,tu,v)du \le 2f_k(0)x^k \left[\int_0^{1-s} \frac{du}{(s+u)^{\frac{3}{2}+H+\delta}} + \int_{1-s}^{\infty} \frac{du}{(s+u)^{-\frac{3}{2}}}\right]$$
$$\le 2f_k(0)x^k \left[\frac{s^{-\frac{1}{2}-H-\delta}}{\frac{1}{2}+H+\delta} + 2\right] \qquad (t \text{ large enough}).$$

By Inoue and Kasahara (2006, Lemma 3.1), $\sum_{k=0}^{\infty} f_k(0)x^k < \infty$. From these facts as well as (2.1), (4.1), Proposition 3.3 (b) and the dominated convergence theorem, we see that $\lim_{t\to\infty} D(t) = D$, where

$$D := \sum_{k=1}^{\infty} \cos^{2k-1}(\pi H) \int_0^1 \left\{\int_0^{\infty} \frac{f_{2k-1}(u)}{(s+u)^{\frac{3}{2}+H}} du\right\} (1-s)^{\rho} ds$$
$$+ \sum_{k=1}^{\infty} \cos^{2k}(\pi H) \int_0^1 \left\{\int_0^{\infty} \frac{f_{2k}(u)}{(s+u)^{\frac{3}{2}+H}} du\right\} s^{\rho} ds.$$

Since $(B_H(t))$ is a special case of $(X(t))$, this also holds for $\psi_0(t;T)$ and $\psi_0(s;T,t)$. Therefore, from Lemma 2.1, we conclude that $D = C(H,\rho)/(\frac{1}{2} + H - \rho)$. Thus the lemma follows. □

Following theorems are the conclusion of this paper.

Theorem 4.2. *Let g be locally bounded in $[0,\infty)$ and $g \in R_\rho$ with $\rho \in (-\frac{1}{2} + H, \frac{1}{2} + H)$.*

(a) *For fixed $T > 0$, we have*

$$\int_0^t \{\psi(s,T;t) - \psi(s;T)\} g(s) ds \sim C(H,\rho) \int_t^{\infty} \psi(s;T) g(s) ds \qquad (t \to \infty).$$

(b) *There exists a positive constant M such that*

$$\int_0^t \{\psi(s,T;t) - \psi(s;T)\} g(s) ds \le M \int_t^{\infty} \psi(s;T) g(s) ds \qquad (t > 1).$$

Proof. By (4.1), the function $\psi(s;T)g(s)$ in s belongs to $R_{\rho-\frac{3}{2}-H}$. Since $\rho < \frac{1}{2}+H$, we have

$$\int_t^\infty \psi(s;T)g(s)ds \sim \frac{1}{\frac{1}{2}+H-\rho} t\psi(t;T)g(t) \qquad (t\to\infty).$$

The assertion (a) follows from this and Lemma 4.1, while (b) from (a). □

Theorem 4.3. (a) *Baxter's inequality* (B) *holds.*

(b) *For fixed $T>0$, we have, as $t\to\infty$,*

$$\int_0^t \{\psi(s,T;t)-\psi(s;T)\}\|X(t_1-s)\|ds \sim C(H,H)\frac{1+2H}{\Gamma(\frac{1}{2}-H)}\left(\int_0^T c(v)dv\right)\|B_H(1)\|\cdot t^{-\frac{1}{2}}.$$

Proof. By Inoue and Anh (2007, Lemma 2.7), $\|X(t)\| \sim \|B_H(1)\|\, t^H \ell(t)$ as $t\to\infty$. So, (a) follows from Theorem 4.2 (b) if we put $g(s) := \|X(t_1-s)\| = \|X(s-t_1)\|$. Also, (b) follows from Lemma 4.1 and (4.1). □

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