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Proceedings of the 27th Sapporo Symposium on Partial Differential Equations

Edited by T. Ozawa, Y. Giga, S. Jimbo and G. Nakamura

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PREFACE

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on July 30 through August 2 in 2002 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Professor Taira Shirota started the symposium more than 25 years ago. Professor Kôji Kubota and Professor Rentaro Agemi who made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

T. Ozawa, Y. Giga, S. Jimbo and G. Nakamura

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代表者 小澤 徹, 儀我 美一, 神保秀一, 中村 玄
Organizers: T. Ozawa, Y. Giga, S. Jimbo, G. Nakamura

記

1. 日時 2002年7月31日(水) ~ 8月2日(金)
2. 場所 北海道大学大学院 理学研究科 3号館 (数学教室512号室)
および5号館 大講義室 (数学教室南向かい)
3. 講演 プログラム

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- 15:00-16:00 Xian-Gao Liu (Fudan U.)
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Logarithmic Sobolev inequality for infinitely
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A variational characterization of the effective speed of
inhomogeneous travelling waves
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- 14:30-15:00 山田 計幸 (K. Yamada), 北大院 (Hokkaido U.)
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Asymptotic decay toward the rarefaction waves of solutions
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- 17:00-17:30 Free discussion time with speakers in the coffee-tea room*
- 18:00-20:00 懇親会 (Banquet) エンレイソウ

August 2, 2002 (金)

- 9:30-10:30 H. Kang (Seoul National U.)
An asymptotic expansion of solutions to the Lamé system
in the presence of inclusions and applications
- 11:00-12:00 隠居 良行 (Y. Kagei), 九大 (Kyushu U.)
On large time behavior of solutions to the compressible
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- 14:00-15:00 鶉飼 正二 (S. Ukai), 横国大 (Yokohama National U.)
On the half-space problem of the Boltzmann equation
(joint work with T. Yang, S.-H. Yu (City Univ. of Hong Kong))
- 15:00-15:30 Free discussion time with speakers in the coffee-tea room*

* Lecturers in each session are invited to stay in the coffee-tea room during discussion time.

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本年は、最初の2日が数学教室、最終日が5号館大講義室と変則的ですのでご注意ください。

NONLOCAL EVOLUTION PROBLEMS

M. Chipot

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Let Ω be a smooth bounded open subset of \mathbf{R}^n with boundary Γ . Suppose that Γ is divided into Γ_D and $\Gamma_N = \Gamma \setminus \Gamma_D$. Let a_{ij} , a_0 be continuous functions uniformly bounded such that for some positive constant λ it holds that

$$\sum_{i,j=1}^n a_{ij}(\zeta) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbf{R}^n, \forall \zeta \in \mathbf{R}. \quad (1)$$

Set

$$A = A(\zeta) = \sum_{i,j=1}^n \partial_{x_j} (a_{ij}(\zeta) \partial_{x_i}) \quad (2)$$

where ∂_{x_i} denotes the partial derivative in the direction x_i . Denote also by ∂_{ν_A} the usual conormal derivative associated to A . Then, one can easily show that there exists a solution to the problem

$$\begin{cases} u_t - A(\ell(u(\cdot, t)))u + a_0(\ell(u(\cdot, t)))u = f & \text{in } \Omega \times \mathbf{R}^+, \\ \partial_{\nu_{A(\ell(u(\cdot, t)))}} u(x, t) = 0 & \text{for } x \in \Gamma_N, t \in \mathbf{R}^+, \\ u(x, t) = 0 & \text{for } x \in \Gamma_D, t \in \mathbf{R}^+, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \quad (3)$$

where

$$\ell(u(\cdot, t)) = \int_{\Omega} g(x) u(x, t) dx. \quad (4)$$

f , g , u_0 being functions in $L^2(\Omega)$. Then, an interesting issue is to study the asymptotic behaviour of $u(\cdot, t)$ when t goes to $+\infty$. The problem passes of course through the first step of finding the stationary solutions to (3). Let us set

$$A = \mathcal{A}(\zeta) = A(\zeta) - a_0(\zeta) \quad (5)$$

and for $\zeta \in \mathbf{R}$ introduce $\Psi = \Psi_{\mathcal{A}(\zeta)}$ the weak solution to

$$\begin{cases} -\mathcal{A}\Psi = g & \text{in } \Omega, \\ \partial_{\nu_{\mathcal{A}(\zeta)}} \Psi(x) = 0 & \text{for } x \in \Gamma_N, \quad \Psi(x) = 0 & \text{for } x \in \Gamma_D. \end{cases} \quad (6)$$

We can easily see that to every solution of the stationary problem associated to (3) i.e. to every solution of

$$\begin{cases} -\mathcal{A}(\ell(u))u = f & \text{in } \Omega, \\ \partial_{\nu_{\mathcal{A}(\ell(u))}} u(x) = 0 & \text{for } x \in \Gamma_N, \quad u(x) = 0 & \text{for } x \in \Gamma_D, \end{cases} \quad (7)$$

corresponds a solution to the equation in \mathbf{R}

$$\mu = \int_{\Omega} f(x) \Psi_{\mathcal{A}(\mu)}(x) dx, \quad (8)$$

and conversely. Under reasonable assumptions on A and a_0 it is easy to show that the right hand side of (8) is uniformly bounded and thus (7) admits at least one solution. However several solutions might also exist. This complicates the study of the asymptotic behaviour of (3). In the case where

$$A(\zeta) = a(\zeta)\Delta, \quad a_0 = 0, \quad \Gamma_D = \Gamma \quad (9)$$

(Δ is the Laplace operator) the problem has been investigated in [3], [4] and [5]. In this case the problem is a little bit peculiar since the solution to (6) is given by

$$\Psi = \frac{\varphi}{a(\zeta)} \quad (10)$$

where φ is the solution to

$$\begin{cases} -\Delta\varphi = g & \text{in } \Omega, \\ \varphi(x) = 0 & \text{for } x \in \Gamma. \end{cases} \quad (11)$$

(ν denotes the outward unit normal to Γ_N). Thus, in this case (8) becomes

$$a(\mu) = \frac{\int_{\Omega} f\varphi dx}{\mu}. \quad (11)$$

Clearly this equation can have several solutions. Such a simple equation does not arise in the cases

$$A(\zeta) = a(\zeta)\Delta, \quad a_0 = 1 \quad (12)$$

or

$$A(\zeta) = \Delta, \quad a_0 = a(\zeta). \quad (13)$$

Thus in this case the stationary problem (7) is more complicated to solve and the asymptotic behaviour of (3) more involved (see [1], [2]). This are some of these issues that we would like to consider here.

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Analysis of singular sets of Landau-Lifshitz system *

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1 Abstract

The aim of this work is to analyze the singular sets of stationary weak solutions to the Landau-Lifshitz system of Ferromagnetic spin chain from a m -dimensional manifold M into the unit sphere S^2 of R^3 . Landau-Lifshitz system is very similar to the heat flows of harmonic maps into sphere in the form. However the monotonicity inequality, which plays an important role to get partial regularity, does not hold in this case. This becomes a large barrier to regularity. In the present note we get a generalized monotonicity inequality, and prove the partial regularity of the stationary weak solutions with zero m -dimensional parabolic metric Hausdorff measure of singular set. Furthermore suppose that $u_k \rightarrow u$ weakly in $W^{1,2}(M \times R_+, S^2)$ and that Σ^t is the blow up set for fixed t . In here we prove that Σ^t is a \mathcal{H}^{m-2} -rectifiable set for almost all $t \in R_+$. And then we reveal the blow up formulas and expose Σ^t motion by quasi-mean curvature under some assumptions. This investigation is inspired by the study on the heat flow of harmonic maps.

It is well known that the Landau-Lifshitz system describes the evolution of spin fields in continuum ferrimagnetism (see [LL]). It takes the form

$$\partial_t u = -\alpha_1 u \times (u \times F_{eff}) + \alpha_2 u \times F_{eff},$$

where $u = (u^1, u^2, u^3) : M \times R_+ \rightarrow R^3$ is the spin field, “ \times ” denotes the vector cross product in R^3 , $\alpha_1 > 0$ is a Gilbert damping constant, α_2 is a exchange constant, and F_{eff} is the effective field containing contributions from exchange interaction crystalline anisotropy, magneto-static self energy, external magnetic field, etc (see

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[LN]). The magnitude of the spin is finite, i.e., $|u|^2 = \sum_{i=1}^3 (u^i)^2 = 1$. We consider a typical form of F_{eff} corresponding to the pure isotropic case and without external magnetic fields, i.e. $F_{eff} = \Delta u$. In this case the Landau-Lifshitz equation reads

$$\partial_t u = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 u \times \Delta u. \quad (1.1)$$

The system (1.1) was first derived on phenomenological grounds by Landau-Lifshitz [LL]. It plays a fundamental role in the understanding of non-equilibrium magnetism. When $\alpha_1 = 0$, this system is called as Heidelberg system.

We also get the following equivalent equation

$$\lambda_1 \partial_t u - \lambda_2 u \times \partial_t u = \Delta u + |\nabla u|^2 u. \quad (1.2)$$

Where $\lambda_1 = \frac{\alpha_1}{\alpha_1^2 + \alpha_2^2}$, $\lambda_2 = \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2}$.

The initial data is proposed by

$$u(x, 0) = u_0(x), \quad \text{with } |u_0(x)| = 1, x \in M. \quad (1.3)$$

Recently B.Guo, M.Hong [GA] and F.Alouges A.Soyeur [AS] have established the global existence of weak solutions for the Landau-Lifshitz equations in higher dimension.

A natural problem is: how is the regularity of weak solutions to the Landau-Lifshitz system in higher dimension?

One has seen that the Landau-Lifshitz system of the Ferromagnetic spin chain contained the heat flow of harmonic maps from M into S^2 (if $\alpha_2 = 0, \alpha_1 = 1$) and the equation of harmonic maps (if $\partial_t u \equiv 0$). So one cannot expect any regular properties for the general weak solutions of the system without any condition based on the results about the regularity of harmonic maps. Indeed, T.Riviere has constructed a weakly harmonic map $u \in W^{1,2}(B^3, S^2)$ which is discontinuous almost everywhere in B^3 . Thus a more restrictive class of the weak solutions should be considered. Indeed, M.Feldman has proposed a restrictive condition for the weak solutions of the heat flow, as follows the stationary condition of the domain variations for the harmonic maps, i.e. so called "stability hypothesis". An important role of this condition is which implies the monotonicity of the "normalized energy". For heat flow these monotonic inequalities have been proved by M.Struwe. It is unfortunate that the monotonic inequality don't hold in present case. Maybe this is a crucial reason that up to now one could not obtained the regularity of weak solutions of this system in higher dimension.

DEFINITION 1.1 $u(x, t) \in W^{1,2}(R^m \times R_+, S^2)$ is called a **stationary weak solution** of (1.4), if it is a weak solution of (1.4) and satisfies the following two assumptions:

$$\int_{R^m} 2(\lambda_1 u_t - \lambda_2 u \times u_t) \zeta \cdot \nabla u - |\nabla u|^2 \text{div} \zeta + 2\partial_j u \partial_k u \partial_j \zeta^k = 0; \quad (1.4)$$

$$\left(\int_{R^m \times t_2} - \int_{R^m \times t_1} \right) |\nabla u|^2 \vartheta dx \leq - \int_{t_1}^{t_2} \int_{R^m} [2(\lambda_1 u_t^2 \vartheta + \nabla u \nabla \vartheta u_t) - |\nabla u|^2 \vartheta_t] dx dt. \quad (1.5)$$

Where $t_2 > t_1 > 0$, functions ζ, ϑ satisfy the following conditions:

$$\zeta \in C^1(R^m \times R_+, R^m), \vartheta \in C^1(R^m \times R_+), \vartheta \geq 0,$$

$$\zeta, \vartheta, \nabla_{(x,t)} \zeta, \nabla_{(x,t)} \vartheta, \text{ are bounded on } R^m \times R_+,$$

$$\zeta(\cdot, t), \vartheta(\cdot, t) \in C_0^1(R^m).$$

When the equality in (1.5) holds, u is called a **strong stationary weak solution**.

We have

THEOREM 1.1 *Let M be a m -dimensional smooth compact Riemannian manifold. Assume that $u \in W^{1,2}(M \times R_+, S^2)$ is a global stationary weak solution of the Landau-Lifshitz system (1.2) (1.3) with $E(u_0) < \infty, |u_0| = 1$, where $E(u) = \int_M |\nabla u|^2 dV$. Then there exists an open subset U of $M \times R_+$ such that $u \in C^\infty(U, S^2)$ and the singular set $Sing(u)$ of u satisfies $\mathcal{H}_\rho^m(Sing(u)) = 0$ where \mathcal{H}_ρ denotes parabolic metric Hausdorff measure.*

Now we shall consider a sequence u_k of stationary weak solutions of (1.2) with initial data $u_k(x, 0)$ and $\int_{R^m} |\nabla u_k(x, 0)|^2 dx \leq \Lambda$. By the energy inequality we have

$$\int_0^T \int_{R^m} 2\lambda_1 \partial_t u_k^2 dx dt + \int_{R^m} |\nabla u_k|^2(x, T) dx \leq \int_{R^m} |\nabla u_k(x, 0)|^2 dx = E_{k0} \leq \Lambda. \quad (1.6)$$

Therefore we may assume that $u_k \rightarrow u$ weakly in $W^{1,2}(R^m \times R_+, S^2)$.

The small energy regularity and (1.6) imply we may assume that

$$|\nabla u_k|^2(\cdot, t) dx dt \rightarrow |\nabla u|^2(\cdot, t) dx dt + \nu_t dt,$$

$$|\partial_t u_k|^2(\cdot, t) dx dt \rightarrow |\partial_t u|^2(\cdot, t) dx dt + \mu,$$

in the sense of measure as $k \rightarrow \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t , μ is a nonnegative Radon measure in $R^m \times R_+$ supported in Σ .

We have

THEOREM 1.2 *Let u_k be strong stationary weak solutions of (1.2) with initial energy $E_{k0} \leq \Lambda$. Then for almost every $t \in R_+$, ν_t is \mathcal{H}^{m-2} -rectifiable, therefore Σ^t is \mathcal{H}^{m-2} -rectifiable.*

THEOREM 1.3 *Suppose that u_k are strong stationary weak solutions of (1.2) and the limiting map u is also a strong stationary weak solution of (1.2), then the blow up measure $\{\nu_t\}$ is a generalized Brake motion.*

THEOREM 1.4 *Suppose that blow up set Σ^t of Landau-Lifshitz system (1.2) with $\lambda_1 \geq 2|\lambda_2|$ is a smooth family of submaxillas in R^m and assume that it is a generalized Brake' flow in the sense of Theorem above. Then Σ^t is a quasi-mean curvature flow.*

In the following we analyze what happen at blow up points. More generally we assume that $u(x, t)$ is a strong stationary weak solution of (1.2). Let $z_0 = (x_0, t_0)$ be a singular point of u . Set $u_k(z) = u(x_0 + r_k x, t_0 + r_k^2 t)$ where $x \in R^m$ and $t \in R_-$, then u_k satisfies (1.2) and by scaling , for any $z \in R^m \times R_-$,

$$\mathcal{E}(r_k, u, z) = \int_{P_1(z)} |\nabla u_k|^2.$$

We see that for small r_k

$$\int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0),$$

and

$$\int_{P_{1/2}(z)} |\partial_t u_k|^2 \leq c \int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0).$$

Denote , for fixed constant $\delta > 0$,

$$D_k = \left\{ z \in R^m \times R_- \mid x_0 + r_k x \in B_\delta(x_0), t_0 + r_k^2 t \in [t_0 - \delta^2, t_0 + \delta^2] \right\},$$

then $D_k \rightarrow R^m \times R_-$ as $k \rightarrow \infty$, since $r_k \rightarrow 0$. Therefore there is a subsequence (still denoted by r_k) $r_k \rightarrow 0$ such that $u_k(x, t) \rightarrow v(x, t)$ weakly in $W_{loc}^{1,2}(R^m \times R_-, S^2)$,

$$\int |\nabla u_k|^2(\cdot, t) dx dt \rightarrow \int |\nabla v|^2(\cdot, t) dx dt + \nu_t dt$$

and

$$\int |\partial_t u_k|^2(\cdot, t) dx dt \rightarrow \int |\partial_t v|^2(\cdot, t) dx dt + \mu$$

in the sense of measure as $k \rightarrow \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t , μ is a nonnegative Radon measure in $R^m \times R_-$ supported in Σ .

THEOREM 1.5 *If $z_0 = (x_0, t_0)$ is a blow up point, then we have two blow up formulas*

$$\int_{\Sigma^t} \text{div}_{(\Sigma^t)}(\zeta) \nu_t + \int_{R^m} (|\nabla u|^2 \text{div}(\zeta) - 2\partial_j v \partial_k v \partial_j \zeta^k) \quad (1.7)$$

$$\begin{aligned}
&= 2 \int_{R^m} (\lambda_1 \partial_t v - \lambda_2 v \times \partial_t v) \nabla v \zeta + 2 \int_{\Sigma^t} \frac{(\lambda_1 x \cdot \vec{n} - \frac{\lambda_2}{2} a \times (x \cdot \vec{n})) (\zeta \cdot \vec{n})}{2(-t)} \nu_i; \\
&\quad \int_{t_1}^{t_2} \int_{R^m} 2(\lambda_1 v_i^2 \vartheta + \nabla v \nabla \vartheta v_i) dde + \left(\int_{R^m \times t_2} - \int_{R^m \times t_1} \right) |\nabla v|^2 \vartheta dx \\
&= \int_{\Sigma^{t_1}} \vartheta \nu_{t_1} - \int_{\Sigma^{t_2}} \vartheta \nu_{t_2} - \frac{2\lambda_1}{4\lambda_1^2 + \lambda_2^2} \int_{t_1}^{t_2} \int_{\Sigma^t} \vartheta \left| \frac{\lambda_1 x \cdot \vec{n}}{(-t)} \right|^2 \nu_t \\
&\quad - 2 \int_{t_1}^{t_2} \int_{\Sigma^t} \frac{(\lambda_1 x \cdot \vec{n} - \frac{\lambda_2}{2} a \times (x \cdot \vec{n})) (\nabla \vartheta \cdot \vec{n})}{2(-t)} \nu_t, \tag{1.8}
\end{aligned}$$

where $\zeta \in C_0^\infty(R^m, R^m)$ and $\vartheta \in C_0^\infty(R^m, R_+)$.

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Magnetic clusters and fold energies

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This is a joint work with Y.Giga and Y.Tonegawa.

1 Introduction

We are concerned with variational properties for a fold energy in the unit disk $\Omega_0 \subset \mathbb{R}^2$ of the form

$$E^q(\nabla u) = \int_{\Sigma_{\nabla u} \cap \Omega_0} |\nabla u|^q dS$$

for a unit gradient field $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$, $u = u(x, y)$ when ∇u is dailation invariant. Here $q > 0$ is a given positive number. Roughly speaking, $\Sigma_{\nabla u}$ denotes the jump discontinuities of ∇u and $|\nabla u|$ represents the jump on $\Sigma_{\nabla u}$; dS denotes the line element. The precise definition of the energy E^q will be given later.

Our main results are

(1) L^1 -compactness summarized as follows.

If $\{E^q(\nabla u_j)\}_{j \in \mathbb{N}}$ is bounded, then there is an L^1 -convergent subsequence of $\{\nabla u_j\}_{j \in \mathbb{N}}$. (**Theorem 1**)

(2) L^1 -lower semi-continuity of E^q with respect to ∇u . (**Theorem 2**)

(3) Characterization of global minimizers such that $E^q(\nabla u) = 0$. (**Theorem 3**)

(4) Stability of 'saddle point' type stationary configurations is studied.

There is a tendency that more configurations are stable for larger q . (**Theorem 4**)

2 Terminology

To state our results precisely we prepare several functional spaces and notations.

Let Ω_0 be the unit open disk in \mathbb{R}^2 . We call a vector field V on Ω_0 a *cluster* if $V \in L^\infty(\Omega_0)$ satisfies

$$\begin{cases} \text{curl } V = 0 & \text{in } \Omega_0 \setminus \{0\} & \text{(i)} \\ |V| = 1 & \text{in } \Omega_0 & \text{(ii)} \\ V(x) = V(x') & \text{if } \arg x = \arg x' & \text{(iii)} \end{cases}$$

Let K denote the set of clusters. Then we are able to show that $V \in K$ has a potential,

i.e. for any cluster $V \in K$ there exists a function u on Ω_0 to \mathbb{R} such that $\nabla u = V$. We next define the set of finite wall clusters and the fold energy for a finite wall cluster. If a cluster $\nabla u \in K$ is not smooth at $x_0 \in \Omega_0$, then $\nabla u \in K$ is not smooth on the line $l(\theta) = \{x \in \Omega_0 \mid \theta \equiv \arg x_0 = \arg x\}$. We call such $l(\theta)$ a *wall* of $\nabla u \in K$. If a cluster $\nabla u \in K$ has finitely many walls, then ∇u is called a *finite wall cluster*. We denote the set of finite wall clusters by K_f .

For a given $q > 0$ we define the *fold energy* E_f^q on K_f as follows .

For $\nabla u \in K_f$ we set

$$E_f^q(\nabla u) \equiv \int_{\cup l(\theta)} |\nabla u(x)|^q dS$$

where $|\nabla u|(x)$ is the size of the jump of ∇u at $x \in \cup l(\theta)$.

Let

$$K_\infty^q \equiv \{\nabla u \in K \mid \exists \{\nabla u_j\}_{j \in \mathbb{N}} \subset K_f \text{ s.t. } \nabla u_j \rightarrow \nabla u \text{ in } L^1 \\ \sup E_f^q(\nabla u_j) < \infty\}.$$

We call this set K_∞^q the set of *limit clusters*.

We consider the L^1 -lower semi-continuous relaxation E_∞^q of E_f^q on K_∞^q . In fact, for a limit cluster $\nabla u \in K_\infty^q$, the fold energy of ∇u is defined by

$$E_\infty^q(\nabla u) = \inf\{\underline{\lim} E_f^q(\nabla u_j) \mid \{\nabla u_j\}_{j \in \mathbb{N}} \subset K_f \text{ with } \nabla u_j \rightarrow \nabla u \text{ in } L^1 \\ E_f^q(\nabla u_j) < \infty\}.$$

3 Main results

Theorem 1 (L^1 -compactness).

Assume that $\{\nabla u_j\}_{j \in \mathbb{N}} \subset K_\infty^q$ and that $E_\infty^q(\nabla u_j)$ is bounded in j .

Then there exists a subsequence $\{\nabla u_{j(k)}\}_{k \in \mathbb{N}}$ of $\{\nabla u_j\}_{j \in \mathbb{N}}$ and there exists a $\nabla u_0 \in K_\infty^q$ such that $\nabla u_{j(k)}$ converges to ∇u_0 in $L^1(\Omega_0)$.

Sketch of the proof.

It is enough to prove for the set of finite wall clusters.

For any ∇u_j of finite wall cluster we define $g_j : \theta \equiv \arg x \rightarrow \arg \nabla u_j(x)$. Such single-valued function g_j exists since ∇u_j is a unit gradient field (ii) and satisfies dailation invariance (iii).

We next introduce a map M such that $\{M(g_j)\}_{j \in \mathbb{N}}$ is uniformly bounded and equicontinuous. So by the *Ascoli-Arzelà* theorem there exists a subsequence $\{M(g_{j(k)})\}_{k \in \mathbb{N}}$ of $\{M(g_j)\}_{j \in \mathbb{N}}$ and a continuous function h such that $M(g_{j(k)})$ converges uniformly to h . This h has a function g with $M(g) = h$ and this g satisfies the property that $(\cos g_{j(k)}(x), \sin g_{j(k)}(x))$ converges to $(\cos g(x), \sin g(x))$ almost everywhere $x \in \Omega_0$. Thus we can get a subsequence as required.

Theorem 2 (L^1 -lower semi-continuity).

Assume that $\nabla u \in K_\infty^q$, $\{\nabla u_j\}_{j \in \mathbb{N}} \subset K_\infty^q$ and that ∇u_j converges to ∇u in $L^1(\Omega_0)$.

Then $E_\infty^q(\nabla u) \leq \liminf_{j \rightarrow \infty} E_\infty^q(\nabla u_j)$

Moreover, $E_\infty^q = E_f^q$ on K_f .

Sketch of the proof.

It is clear by definition of the fold energy E_∞^q on the set of limit clusters.

Theorem 3 (Global minimizers).

Following two statement are equivalent.

(a) $\nabla u \in K_\infty^q$ is a global minimizer on K_∞^q of E_∞^q .

(b) $\nabla u \in K_\infty^q$ is either $\arg \nabla u(x) = \arg(x)$ or $\arg \nabla u(x) = \arg(-x)$ or $\arg \nabla u \equiv C$.

Sketch of the proof.

Assume (b), then it is obvious that (a) holds.

On the other hand if $\nabla u \in K_\infty^q$ is a global minimizer on K_∞^q of E_∞^q then ∇u has no wall because we know there exists some $\nabla u \in K_\infty^q$ satisfying $E_\infty^q(\nabla u) = 0$. So

we can get the result (b) as required .

We consider a typical series of configurations which is expected to be stationary for E_∞^q .

Definition.

For given $\{\theta_j^*\}_{j=1}^{2n} \subset (0, 2\pi]$ satisfying (I), (II).

(I) $0 < \theta_1^* < \theta_2^* < \dots < \theta_{2n}^* \leq 2\pi$

(II) $\theta_{j+1}^* = \theta_j^* + \frac{1}{n}\pi$ for $\forall j = 1, 2, \dots, 2n - 1$

We define $\nabla u_*^{(n)} \in K_f^q$ as follows.

$\nabla u_*^{(n)} \in K_f^q$ has $2n$ walls $\{l(\theta_j^*)\}_{j=1}^{2n}$ and satisfies for all $x \in \Omega_0$ with $\arg x \in (\theta_1^*, \theta_2^*)$

$$\nabla u_*^{(n)}(x) = \left(\cos\left(\theta_1^* + \frac{n+1}{2n}\right), \sin\left(\theta_1^* + \frac{n+1}{2n}\right) \right)$$

or

$$\nabla u_*^{(n)}(x) = \left(\cos\left(\theta_1^* - \frac{n+1}{2n}\right), \sin\left(\theta_1^* - \frac{n+1}{2n}\right) \right).$$

We denote the set of such $\nabla u_*^{(n)}$'s by Z_{2n} .

Theorem 4(Local minimizers).

Assume that $q > 0$ and $n \in \mathbb{N} \setminus \{1\}$ satisfy $q \sin^2 \frac{\pi}{2n} - 1 > 0$ and that $\nabla u_*^{(n)} \in Z_{2n}$.

Then there exists $\varepsilon_0 > 0$ such that $E_\infty^q(\nabla u_*^{(n)}) \leq E_\infty^q(\nabla u)$ for any $\nabla u \in K_\infty^q$ with $\|\nabla u_*^{(n)} - \nabla u\|_{L^1} < \varepsilon_0$

Sketch of the proof.

It is enough to prove for the set of finite wall clusters.

We first prove the following proposition.

Proposition.

There exists two positive numbers ε_1 and ε_2 such that if $\nabla u \in K_f$ satisfies $\|\nabla u_*^{(n)} - \nabla u\|_{L^1} < \varepsilon_1$ and $E_f^q(\nabla u) < E_f^q(\nabla u_*^{(n)}) + \varepsilon_2$ then ∇u has exactly $2n$ walls and each wall is located near each those of $\nabla u_*^{(n)}$.

By using this proposition we can parameterize such ∇u . So we can parameterize the fold energy of such ∇u , too.

Then the fold energy has a critical point at $\nabla u_*^{(n)}$ and Hessian of the fold energy at $\nabla u_*^{(n)}$ is as follows.

$$\text{Hess } E_f^q \Big|_{\nabla u_*^{(n)}} = C \left\{ q \sin^2 \frac{\pi}{2n} - 1 \right\} A$$

where C is a positive constant and A is a positive matrix.

Thus if $q \sin^2 \frac{\pi}{2n} - 1$ holds then $\nabla u_*^{(n)}$ is a local minimizer.

Corollary.

Assume that

$q > 0$ and $n \in \mathbb{N} \setminus \{1\}$ satisfy $q \sin^2 \frac{\pi}{2n} - 1 < 0$.

Then $\nabla u_*^{(n)} \in Z_{2n}$ are not local minimizers.

4 Related works

H.A.M. van den Berg studies magnetic thin film as follows.[6]

A magnetic field M on a domain Ω of \mathbb{R}^2 is called a *magnetic cluster* if $M \in L^\infty(\Omega)$ satisfies

$$\begin{cases} \operatorname{div} M = 0 & \text{in } \Omega \\ |V| = C & \text{in } \Omega \\ M \cdot \bar{n} = 0 & \text{on } \partial\Omega \end{cases}$$

where C is a given constant and \bar{n} is the unit outer vector on $\partial\Omega$.

He provided many examples of magnetic clusters.

Many people study the problem without dailation invariance. In general we don't know the existence of global minimizers of the fold energy on the set of solutions of the 2-dimensional eikonal equation under suitable boundary conditions. There are several examples that distance function is not the minimizer of the fold energy.[2],[5]

If $q=3$ then the fold energy is lower semi-continuous and compactness holds. If $q>3$ then the fold energy is not lower semi-continuous.

We conjecture that compactness doesn't hold if $q>3$. [1],[3][4]

5 Motivation

We would like to study the stability of stationary state on the set of limit clusters of the fold energy to find and classify local minimizers. We have given examples of stationary state whose stability depends on $q>0$. (**Theorem 4**) We conjecture that there is no local minimizer except examples but we did not pursue this problem here.

In our study it is very important that configurations are restricted as clusters, which satisfy dailation invariance. In general we expect that compactness depends on $q>0$ if we consider clusters without dailation invariance. Originally we thought such compactness depends on q even for clusters. But it turned out that our compactness result does not depend on q . So the compactness property may be different between clusters and general configurations.

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In this work, we consider a system of vector fields $\mathbf{X} = (X_1, \dots, X_n)$ defined on an open domain $\Omega \subseteq \mathbb{R}^n$. We suppose that the system satisfies the following logarithmic growth condition:

$$(1) \quad \|\operatorname{div}(\mathbf{X}^T \mathbf{u})\| \leq C \left(\|\mathbf{X}_i \mathbf{u}\| + \|\mathbf{u}\| \right) \sum_{i=1}^n \|X_i\|, \quad \forall \mathbf{u} \in C_c^\infty(\Omega)$$

where $\mathbf{u} = (u_1, \dots, u_n)^T \in \mathbb{R}^n$. We can find some sufficient conditions for the existence of solutions in $\mathcal{D}'(\Omega)$. The typical example is the system in \mathbb{R}^2 with $\mathbf{X} = (X_1, X_2) = (y, -x)$ with $\Omega = \mathbb{R}^2$. In this case, the condition (1) implies the growth condition of the logarithmic growth condition of second order $\Delta \mathbf{u} = \sum_{i,j=1}^2 \partial_{x_i} \partial_{x_j} u_j$ where \mathbf{X} is the formal adjoint of \mathbf{X}^T .

If Γ is a smooth surface of Ω , we say that Γ is non-characteristic for the system of vector fields \mathbf{X} if the tangential vector fields $\mathbf{X}_T = (X_1^T, \dots, X_n^T)$ are linearly independent in the tangent space $T_x \Gamma$ at each point $x \in \Gamma$. There exists a local coordinate system $(x_1, \dots, x_{n-1}, x_n)$ in a neighborhood of Γ such that $\Gamma = \{x_n = 0\}$. In this case, the second order elliptic equation $\Delta \mathbf{u} = \sum_{i,j=1}^n \partial_{x_i} \partial_{x_j} u_j$ can be written as $\Delta \mathbf{u} = \sum_{i,j=1}^{n-1} \partial_{x_i} \partial_{x_j} u_j + \partial_{x_n}^2 u_n$. Associated with the system of vector fields $\mathbf{X} = (X_1, \dots, X_n)$, we define the following boundary value problem:

$$\begin{cases} \operatorname{div}(\mathbf{X}^T \mathbf{u}) = f & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma \end{cases}$$

where $f \in L^2(\Omega)$ and $\mathbf{g} \in C^\infty$ and non-characteristic for the system of vector fields \mathbf{X} . We define $\mathcal{H}_0^1(\Omega) = \{u \in L^2(\Omega) : \operatorname{div}(\mathbf{X}^T \mathbf{u}) = 0\}$, that this is a Hilbert space. Our first result is the following logarithmic Sobolev inequality:

Theorem 1. Suppose that the system of vector fields $\mathbf{X} = (X_1, \dots, X_n)$ satisfies the logarithmic growth condition (1) for some $C > 0$. Then there exists $C_1 > 0$ and $C_2 > 0$ such that

$$(2) \quad \int_{\Omega} |\operatorname{div}(\mathbf{X}^T \mathbf{u})|^2 \leq C_1 \left(\int_{\Omega} |\mathbf{X}_i \mathbf{u}|^2 + \int_{\Omega} |\mathbf{u}|^2 \right) \sum_{i=1}^n \|X_i\|$$

$$\text{for all } \mathbf{u} \in \mathcal{H}_0^1(\Omega).$$

Concerning this inequality with finite degenerate case of Hörmander's system, for comparison for the system $\mathbf{X} = (X_1, \dots, X_n)$ in \mathbb{R}^n , we have the following

$$\int_{\Omega} |\operatorname{div}(\mathbf{X}^T \mathbf{u})|^2 \leq C \left(\int_{\Omega} |\mathbf{X}_i \mathbf{u}|^2 + \int_{\Omega} |\mathbf{u}|^2 \right) \sum_{i=1}^n \|X_i\|$$

LOGARITHMIC SOBOLEV INEQUALITY FOR INFINITELY DEGENERATE ELLIPTIC OPERATORS

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1. NOTATIONS AND RESULTS

In this work, we consider a system of vector fields $X = (X_1, \dots, X_m)$ defined on an open domain $\tilde{\Omega} \subset \mathbb{R}^d$. We suppose that this system satisfies the following logarithmic regularity estimates,

$$\|(\log \Lambda)^s u\|_{L^2}^2 \leq C \left\{ \sum_{j=1}^m \|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right\}, \quad \forall u \in C_0^\infty(\tilde{\Omega}), \quad (1)$$

where $\Lambda = (e + |D|^2)^{1/2} = \langle D \rangle$. We can find some sufficient conditions for this estimates for example in [?, ?]. The typical example is the system in \mathbb{R}^2 such as $X_1 = \partial_{x_1}, X_2 = e^{-|x_1|^{-1/s}} \partial_{x_2}$ with $s > 0$. Remark that if $s > 1$, the estimate (??) implies the hypoellipticity of the infinitely degenerate elliptic operators of second order $\Delta_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j .

If Γ is a smooth surface of $\tilde{\Omega}$, we say that Γ is non characteristic for the system of vector fields X , if for any point $x_0 \in \Gamma$, there exists at least one vector field of X_1, \dots, X_m which is transversal to Γ at x_0 . Let now $\Gamma = \cup_{j \in J} \Gamma_j$ be the union of a family of smooth surface in $\tilde{\Omega}$. We say that Γ is non characteristic for X , if for any point $x_0 \in \Gamma$, there exists at least one vector field of X_1, \dots, X_m which traverses Γ_j at x_0 for all $j \in J_0 = \{k \in J; x_0 \in \Gamma_k\}$. For this second case, the typical example is $X_1 = \partial_{x_1}, X_2 = \exp(-(x_1^2 \sin^2(\frac{\pi}{x_1}))^{-1/2s}) \partial_{x_2}$, we have $\Gamma_j = \{x_1 = \frac{1}{j}\}, j \in \mathbb{Z} \setminus \{0\}, \Gamma_0 = \{x_1 = 0\}$, and X_1 is transverse to all $\Gamma_j, j \in \mathbb{Z}$.

Associated with the system of vector fields $X = (X_1, \dots, X_m)$, we define the following function spaces :

$$H_X^1(\tilde{\Omega}) = \left\{ u \in L^2(\tilde{\Omega}); X_j u \in L^2(\tilde{\Omega}), j = 1, \dots, m \right\}.$$

Take now $\Omega \subset\subset \tilde{\Omega}$, we suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X . We define $H_{X,0}^1(\Omega) = \{u \in H_X^1(\Omega); u|_{\partial\Omega} = 0\}$, that this is a Hilbert space.

Our first result is the following logarithmic Sobolev inequality.

Theorem 1. *Suppose that the system of vector fields $X = (X_1, \dots, X_m)$ verifies the estimates (??) for some $s > 1/2$. Then there exists $C_0 > 0$ such that*

$$\int_{\Omega} |v|^2 \log^{2s-1} \left(\frac{|v|}{\|v\|_{L^2}} \right) \leq C_0 \left\{ \sum_{j=1}^m \|X_j v\|_{L^2}^2 + \|v\|_{L^2}^2 \right\}, \quad (2)$$

for all $v \in H_{X,0}^1(\Omega)$.

Comparing this inequality with that of finite degenerate case of Hörmander's system, for example, for the system $X_1 = \partial_{x_1}, X_2 = x_1^k \partial_{x_2}$ on \mathbb{R}^2 , we have (see [?])

$$\|v\|_{L^p} \leq C \left(\|\partial_1 v\|_{L^2}^2 + \|x_1^k \partial_2 v\|_{L^2}^2 + \|v\|_{L^2}^2 \right)^{1/2}$$

for all $v \in C_0^\infty(\Omega)$, with $p = 2 + \frac{4}{k}$. Consequently, if k go to infinity, we can only expect to gain the logarithmic estimates as (??). That means that we are not in the elliptic case of [?].

Similarly to the elliptic and subelliptic case (see [?]), by using the Sobolev's inequality, we study the following semi-linear Dirichlet problems

$$\begin{aligned} \Delta_X u &= au \log |u| + bu, \\ u|_{\partial\Omega} &= 0, \end{aligned} \quad (3)$$

where $a, b \in \mathbb{R}$. We have the following theorem.

Theorem 2. *We suppose that the system of vector fields $X = (X_1, \dots, X_m)$ satisfies the following hypotheses :*

H-1) $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X ;

H-2) the system of vector fields X satisfies the finite type of Hörmander's condition on $\tilde{\Omega}$ except an union of smooth surfaces Γ which are non characteristic for X .

H-3) the system of vector fields X verifies the estimates (??) for $s > 3/2$.

Suppose $a \neq 0$ in (??). Then the semi-linear Dirichlet problem (??) posses at least one non trivial weak solution $u \in H_{X,0}^1(\Omega) \cap L^\infty(\Omega)$. Moreover, if $a > 0$, we have $u \in C^\infty(\Omega \setminus \Gamma) \cap C^0(\tilde{\Omega} \setminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \setminus \Gamma$.

As in the elliptic case, we do not know the uniqueness of solutions (see [?]). The regularity of this weak solution near to the infinitely degenerate point of Γ is a more complicate problem, which will be studied in our future works.

2. LOGARITHMIC SOBOLEV'S INEQUALITY

We are following the idea of [?] for the proof of Theorem ???. Take $v \in H_{X,0}^1(\Omega)$, we use the same notation for the extension by 0, As in the classical case, there exists a mollifier family $\{\rho_\varepsilon, \varepsilon > 0\}$ such that $\rho_\varepsilon * v \in C_0^\infty$, $\lim_{\varepsilon \rightarrow 0} \rho_\varepsilon * v = v$ in L^2 and $\|X(\rho_\varepsilon * v)\|_{L^2} \leq C\{\|Xv\|_{L^2} + \|v\|_{L^2}\}$, $\|(\log \Lambda)^s(\rho_\varepsilon * v)\|_{L^2} \leq C\{\|(\log \Lambda)^s v\|_{L^2} + \|v\|_{L^2}\}$ with C independent on ε . We need only to prove the following estimate :

$$\int_{\Omega} |v|^2 \log^{2s-1} \left(\frac{|v|}{\|v\|_{L^2}} \right) \leq C_0 \|(\log \Lambda)^s v\|_{L^2}^2, \quad (4)$$

for all for $v \in C_0^\infty(\Omega)$.

By the homogenization, we prove (??) for $v \in C_0^\infty(\Omega)$ and $\|v\|_{L^2} = 1$. Since $2s - 1 > 0$, we have

$$\begin{aligned} \int_{\Omega} |v|^2 |\log |v||^{2s-1} &\leq C|\Omega| + \int_{|v| \geq \varepsilon} |v|^2 \log^{2s-1} < |v| > \\ &\leq C_0 + \int_{\Omega} |v|^2 \log^{2s-1} < |v| >. \end{aligned}$$

Since Ω is bounded, $v \in L^\infty(\Omega)$ and $2s - 1 > 0$, we have by the definition of Lebesgue integration

$$\begin{aligned} \int_{\Omega} |v|^2 \log^{2s-1} < |v| > &= - \int_0^\infty \lambda^2 \log^{2s-1} < \lambda > d\mu\{|v| > \lambda\} \\ &= \int_0^\infty \left(2\lambda \log^{2s-1} < \lambda > + (2s-1) \frac{\lambda^3}{< \lambda >^2} \log^{2s-2} < \lambda > \right) \mu\{|v| > \lambda\} d\lambda, \end{aligned}$$

where $\mu(\cdot)$ is the Lebesgue measure. Since $\frac{\lambda^3}{< \lambda >^2} \leq \lambda$, $\log < \lambda > \geq 1$, we have that

$$\int_{\Omega} |v|^2 |\log |v||^{2s-1} \leq C_0 + C_s \int_0^\infty \lambda \log^{2s-1} < \lambda > \mu\{|v| > \lambda\} d\lambda. \quad (5)$$

So we need to estimate the second term of right hand side of (??). For $A > 0$ we set $v = v_{1,A} + v_{2,A}$ with $\widehat{v}_{1,A} = \widehat{v}(\xi)\mathbf{1}_{\{|\xi| \leq e^A\}}$. Then $\mu\{|v| > \lambda\} \leq \mu\{|v_{1,A}| > \frac{\lambda}{2}\} + \mu\{|v_{2,A}| > \frac{\lambda}{2}\}$. For the first term we have

$$\|v_{1,A}\|_{L^\infty} \leq \|\widehat{v}_{1,A}\|_{L^1} \leq \|v\|_{L^2} \|\mathbf{1}_{\{|\xi| \leq e^A\}}\|_{L^2} \leq C_d e^{\frac{d}{2}A}.$$

Choose now $A_\lambda = \frac{2}{d} \log\left(\frac{\lambda}{4C_d}\right)$, we have $\mu\{|v_{1,A_\lambda}| > \frac{\lambda}{2}\} = 0$, hence

$$\begin{aligned} & \int_0^\infty \lambda \log^{2s-1} \langle \lambda \rangle \mu(|v| > \lambda) d\lambda \\ & \leq C_0 + C_s \int_e^\infty \lambda \log^{2s-1} \lambda \mu(|v| > \lambda) d\lambda \\ & \leq C_0 + C_s \int_e^\infty \lambda \log^{2s-1} \lambda \mu(|v_{2,A_\lambda}| > \frac{\lambda}{2}) d\lambda \\ & \leq C_0 + 2C_s \int_e^\infty \frac{\log^{2s-1} \lambda}{\lambda} \|v_{2,A_\lambda}\|_{L^2}^2 d\lambda \\ & \leq C_0 + 2C_s \int_e^\infty \frac{\log^{2s-1} \lambda}{\lambda} \int_{\{\xi \in \mathbb{R}^d; |\xi| \geq e^{A_\lambda}\}} |\widehat{v}(\xi)|^2 d\xi d\lambda. \end{aligned}$$

Now $|\xi| \geq e^{A_\lambda}$ implies that $\lambda \leq 4C_d < |\xi| >^{d/2}$. By using Fubini theorem we have

$$\begin{aligned} & \int_0^\infty \lambda \log^{2s-1} \langle \lambda \rangle \mu(|v| > \lambda) d\lambda \\ & \leq C_0 + 2C_s \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 \int_e^{4C_d < |\xi| >^{d/2}} \frac{\log^{2s-1} \lambda}{\lambda} d\lambda d\xi \\ & \leq C_0 + 2C_s \int_{\mathbb{R}^d} \log^{2s} (4C_d < |\xi| >^{d/2}) |\widehat{v}(\xi)|^2 d\xi \\ & \leq C_s \int_{\mathbb{R}^d} \log^{2s} \langle |\xi| \rangle |\widehat{v}(\xi)|^2 d\xi = C_s \|(\log \Lambda)^s v\|_{L^2(\Omega)}^2. \end{aligned}$$

Here we have used the fact $\int_{\mathbb{R}^d} \log^{2s} \langle |\xi| \rangle |\widehat{v}(\xi)|^2 d\xi \geq \int_{\mathbb{R}^d} |\widehat{v}(\xi)|^2 d\xi = 1$. Thus we have proved (??) by using (??).

3. VARIATIONAL PROBLEMS

For $a \in \mathbb{R}$, we study now the following variational problems

$$I_a = \inf_{\|v\|_{L^2} = 1, v \in H_{X,0}^1(\Omega)} \{ \|Xv\|_{L^2(\Omega)}^2 - a \int_{\Omega} |v|^2 \log |v| \}. \quad (6)$$

The Euler-Lagrange equation of this variational problems is the following semilinear Dirichlet problem

$$\begin{aligned} \Delta_X u &= au \log |u| + I_a u, \\ u|_{\partial\Omega} &= 0. \end{aligned} \quad (7)$$

If \widetilde{u} is a weak solution of problem (??), for $c > 0$ we set $u = c\widetilde{u}$, then $\|u\|_{L^2} = c > 0$, $u \geq 0$, $u \in H_{X,0}^1(\Omega)$ and in the weak sense

$$\Delta_X u = au \log |u| + (I_a - \log c)u.$$

Choose $c = e^{I_a - b} > 0$, we get the equation (??). We have

Theorem 3. *Let $a, b \in \mathbb{R}$, $a \neq 0$, under the hypotheses H-1), H-2) and H-3), the Dirichlet problems (??) has at least one non trivial weak solution $u \in H_{X,0}^1(\Omega)$, $u \geq 0$, $\|u\|_{L^2} > 0$.*

The proof for the boundedness and regularity of weak solutions is complicate.

Following this direction, we can study the high order nonlinear eigenvalue problems. Suppose that we have the logarithmic Sobolev inequality

$$\int_{\Omega} |v|^2 \log^{k+1} \left(\frac{|v|}{\|v\|_{L^2}} \right) \leq C_0 \left(\|Xv\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2 \right).$$

For $a_1, \dots, a_k \in \mathbb{R}$, we study the variational problems

$$I_{a_1, \dots, a_k}^k = \inf_{\|v\|_{L^2} = 1, v \in H_{X,0}^1(\Omega)} I_{a_1, \dots, a_k}^k(v),$$

with

$$I_{a_1, \dots, a_k}^k(v) = \|Xv\|_{L^2(\Omega)}^2 - \sum_{j=1}^k a_j \int_{\Omega} |v|^2 \log^j |v|.$$

By similar calculus, we can prove that for any $a_1, \dots, a_k \in \mathbb{R}$, there exists I_{a_1, \dots, a_k}^k such that the following semilinear Dirichlet problems

$$\begin{aligned} \Delta_X u &= \sum_{j=1}^k a_j u \log^j |u| + I_{a_1, \dots, a_k}^k u, \\ u|_{\partial\Omega} &= 0, \end{aligned}$$

has at least one non trivial solution in $H_{X,0}^1(\Omega)$, with $u \geq 0$ and $\|u\|_{L^2} = 1$. Moreover, we have similar regularity results as Theorem ??.

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A Variational Characterization of the Effective Speed of Inhomogeneous Travelling Waves

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Travelling waves in heterogeneous media have gained much attention in the past decade in various fields of science such as ecology, physiology and combustion theory. Previously most of the mathematical studies were focused on spatially periodic cases, and little was known about the nature of traveling waves in spatially aperiodic media. This is in contrast with the case of temporally varying media, for which there is a comprehensive study by Shen (1999).

Recently I have introduced the notion of travelling waves in spatially almost-periodic media, including quasi-periodic ones as special cases. The concept is a natural extension of the classical notion of travelling waves, and I have discussed existence, uniqueness and stability of those travelling waves.

To be more precise, a **travelling wave** is defined to be a solution whose “**current profile**” depends continuously on its “**current landscape**”. Here, roughly speaking, the current profile means the shape of the solution (at each time moment) viewed from the position of the “front”, and the current landscape refers to the spatial environment viewed from that position. For example, in the diffusion equation of the form

$$u_t = u_{xx} + b(x)f(u) \quad (x \in \mathbf{R}, t > 0),$$

the current profile is $u(x + \xi(t), t)$, where $\xi(t)$ denotes the position of the front, and the current landscape is represented by $b(x + \xi(t))$. Thus, travelling waves are characterized by the formula

$$\sigma_{\xi(t)}u(\cdot, t) = W(\sigma_{\xi(t)}b),$$

where σ_ℓ denotes the shift operator $g(x) \mapsto g(x + \ell)$, and W is a continuous map from the hull

$$H_b := \text{closure} \{ \sigma_\ell b \mid \ell \in \mathbf{R} \}$$

into an appropriate function space on \mathbf{R} . By definition, any travelling wave must have a clear “front”, and one can show easily that the front travels at some well-defined average speed.

There is a weaker notion of traveling waves, which I call “pseudo-travelling waves”. In the case of the above one-dimensional equation, a **pseudo-travelling wave** of speed c is defined to be a solution that satisfies

$$\lim_{t \rightarrow \infty} u(x + \tilde{c}t, t) = \begin{cases} \alpha_+ & \text{if } \tilde{c} > c \\ \alpha_- & \text{if } \tilde{c} < c \end{cases}$$

for any $x \in \mathbf{R}$, where α_{\pm} are given zeros of $f(u)$. Unlike travelling waves, pseudo-travelling waves may have only rather a fuzzy front. Clearly any travelling wave is a pseudo-travelling, but the converse is not generally true. However, one can show the following:

- (1) In the case of one-dimensional diffusion equation with a bistable nonlinearity, any pseudo-travelling wave is actually a travelling wave;
- (2) in the case of KPP (i.e. monostable) nonlinearity in any space dimensions but with periodic inhomogeneity, again the same statement as above holds.

In this lecture I will mainly discuss two variational problems associate with travelling waves:

The first is the **mini-max characterization** of propagation speed, which is introduced by Volpert *et al* for homogeneous problems and later extended to periodic problems by Heinze, Papanicolaou and Stevens in the case of bistable nonlinearity. This method enables one to obtain fine rigorous estimates of propagation speed. This method can be extended to quasi-periodic or even almost periodic problems, but it raises a very intriguing question, which I will discuss in my lecture.

The second is concerned with the minimal speed of travelling waves for KPP type equations. As conjectured by Kawasaki-Shigesada (1986), and later proved by Hudson-Zinner (1995 for 1-dim) and Berestycki-Hamel (2002 for higher dim), the minimal speed is characterized by a positive eigenfunction of a certain elliptic eigenvalue problem. In the case of quasi-periodic inhomogeneity, a similar characterization can be formulated, but the corresponding eigenvalue problem may no longer have a positive eigenfunction, because of the degeneracy of the differential operator. This difficulty is related to the small divisor problem, which has bothered celestial mechanists since the 19th century. However, by considering a "generalized" positive eigenfunction, one can partly overcome this difficulty and use this eigenvalue problem to prove the existence of pseudo-travelling waves and to obtain estimates of their minimal speed.

Solvability of viscous Burgers-like equations with linearly growing initial data

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This is a joint work with Y. Giga.

We consider a viscous Burgers-like equation of the form

$$(E) \quad \begin{cases} \partial_t u - \Delta u + \operatorname{div} \mathbf{G}(u) = 0 & \text{in } \mathbf{R}^n \times (0, T), \\ u|_{t=0} = u_0 & \text{in } \mathbf{R}^n, \end{cases}$$

where $\partial_t = \frac{\partial}{\partial t}$. It is well-known that if u_0 is bounded, (E) has a unique global solution. In this paper we consider the case that u_0 is not bounded. This paper specifies the growth of nonlinear term as $\mathbf{G}(r) \sim r^2$ for large r . A typical example is the viscous Burgers equation. Our goal is to solve the initial value problem when the initial data may grow linearly at the space infinity. We shall prove that the problem admits a unique local regular solution. The global existence is not expected in general since $u(x, t) = -\frac{x}{1-t}$ is a solution of the viscous Burgers equation with $u_0(x) = -x$. We also obtain an optimal estimate of the existence time. In fact, the existence time interval $(0, T)$ is estimated from below by a constant multiple over a Lipschitz bound for initial data, $T \geq C_2 \|\nabla u_0\|_\infty$; here the constant is estimated by the structure of \mathbf{G} , and $\|\nabla u_0\|_\infty$ is defined by $\|\nabla u_0\|_\infty = \sum_{i=1}^n \|\partial_i u_0\|_\infty$, where $\partial_i u_0 = \frac{\partial u_0}{\partial x_i}$.

To state our main result precisely we assume the following bounds for $\mathbf{G} = (G_1, \dots, G_n) \in C^{2+\alpha}(\mathbf{R}; \mathbf{R}^n)$ with some $\alpha \in (0, 1)$:

$$(C) \quad \begin{aligned} C_1 &:= \sup_i \sup_{r \in \mathbf{R}} \frac{|G'_i(r)|}{\langle r \rangle} < \infty, \\ C_2 &:= \sup_i \sup_{r \in \mathbf{R}} |G''_i(r)| < \infty, \\ C_3 &:= \sup_i \sup_{r_1, r_2 \in \mathbf{R}} \frac{|G''_i(r_1) - G''_i(r_2)|}{|r_1 - r_2|^\alpha} < \infty. \end{aligned}$$

Here we set $\langle x \rangle = \sqrt{1 + |x|^2}$ for $x \in \mathbf{R}^n$ and G'_i is denotes the derivative of G_i . A typical example satisfying this assumption (C) is $G_i(r) = r^2$ ($1 \leq i \leq n$). We prepare a few function spaces allowing growth at space infinity. Let L^p_m be of the form

$$L^p_m = L^p_m(\mathbf{R}^n) = \left\{ f \in L^p_{loc}(\mathbf{R}^n) \mid \|f\|_{p,m} := \left\| \frac{f(x)}{\langle x \rangle^m} \right\|_p < \infty \right\}.$$

Of course, $L_0^p = L^p$ by definition so that $\|\cdot\|_{p,0} = \|\cdot\|_p$. Let X_B be of the form

$$X_B = \left\{ f \in C^1(\mathbf{R}^n) \mid \|f\|_{X_B} := \|f\|_{\infty,1} + \|\nabla f\|_{\infty} < \infty \right\}.$$

Theorem. Assume that $G \in C^{2+\alpha}(\mathbf{R}; \mathbf{R}^n)$ satisfies bounds (C). Assume that $u_0 \in X_B$. Then there exist $T \geq T_0 := \frac{1}{C_2 \|\nabla u_0\|_{\infty}}$ and $u \in L^\infty(0, T; L_1^\infty(\mathbf{R}^n)) \cap C(\mathbf{R}^n \times [0, T])$ that satisfies (E) in $\mathbf{R}^n \times (0, T)$ with $u|_{t=0} = u_0$. The existence time estimate $T \geq T_0$ is optimal in the sense that a classical solution may not exist in $[0, T_0)$.

Optimality is easily observed by the next example.

Example. We take

$$n = 1, \quad G(r) = \frac{C_2 r^2}{2},$$

so that (E) becomes

$$(E)' \quad \partial_t u - \Delta u + C_2 u \partial_x u = 0.$$

The function

$$u(x, t) = \frac{C_4 x}{1 + C_2 C_4 t}$$

solves (E)' with the initial condition $u(x, 0) = C_4 x$, where C_4 is a constant. If $C_2 C_4 < 0$, the solution of (E)' blows up at $t = \frac{1}{|C_2 C_4|}$.

If (E) is the viscous Burgers equation, we can solve it by using the Hopf-Cole transformation. For example $u_0(x) = x$, we set $v(x, t) = \int_a^x u(y, t) dy + f(t)$ (in this case $f(t) = \frac{a}{1+t}$) and observe that $v(x, t)$ satisfies

$$\partial_t v - \Delta v + \frac{1}{2}(\partial_x v)^2 = 0.$$

We set $w(x, t) = e^{-\frac{1}{2}v(x,t)}$ and observe that w satisfies

$$\partial_t w - \Delta w = 0.$$

This equation is the heat equation. Since we are able to solve the heat equation by using the heat kernel, we can solve the Burgers equation. This method does not apply for our problem for $n > 1$.

A classical result of Tychonov states that the Cauchy problem for the heat equation is uniquely solvable for continuous initial data $u_0(x)$ satisfying growth condition

$$|u_0(x)| \leq C e^{a|x|^2}$$

for some positive constants C, a . Moreover, D. G. Aronson [A] generalized the result of Tychonov for a parabolic operator with variable coefficients

$$Lu = \partial_t u - \partial_i \{ A_{ij}(x, t) \partial_j u + A_j(x, t) u \}$$

with suitable conditions for A_{ij} and A_j for u_0 satisfying

$$\int_{\mathbf{R}^n} |u_0(x)| e^{-b|x|^2} dx < \infty$$

for some positive constant b . He proved that there is a unique solution for $Lu = 0$ with $u|_{t=0} = u_0$.

K. Ishige [I] proved that solvability of Cauchy problem:

$$\begin{cases} \partial_t (|u|^{\beta-1} u) = \operatorname{div} (|\nabla u|^{p-2} \nabla u), \\ |u|^{\beta-1} u(\cdot, 0) = \mu(\cdot), \end{cases}$$

for the initial data μ growing at space infinity when $(p-1)/\beta > 0$. There are some more results for nonlinear equations (see e.g. [I], [BCP]) but these results do not include (E).

Idea of the proof. If u_0 is bounded, (E) can be solved by the following iteration:

$$u_{k+1}(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla u_k(s) \cdot \mathbf{G}'(u_k(s)) ds. \quad (1)$$

But if u_0 is not bounded, it is difficult to solve (E) by the iteration (1). So we use another iteration:

$$u_{k+1}(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \nabla u_{k+1}(s) \cdot \mathbf{G}'(u_k(s)) ds. \quad (2)$$

To use this (2) it is necessary to study the solvability of the linear equation:

$$\partial_t v - \Delta v + \nabla v \cdot \mathbf{p} - vq = 0, \quad (3)$$

for $v \in L^\infty(0, T; L^\infty)$, $\mathbf{p} \in L^\infty(0, T; L_1^\infty)$, $q \in L^\infty(0, T; L^\infty)$. Fortunately, it is not difficult to solve the linear equation (3). Estimating the heat kernel in (2), we get the estimate:

$$\|u_{k+1}(t)\|_{\infty,1} \leq C_T \|u_0\|_{\infty,1} + C_T \int_0^t \|\nabla u_{k+1}(s)\|_\infty \|\mathbf{G}'(u_k(s))\|_{\infty,1} ds. \quad (4)$$

Since u_{n+1} satisfies

$$\partial_t u_{n+1} - \Delta u_{n+1} + \nabla u_{n+1} \cdot \mathbf{G}'(u_n) = 0,$$

$\partial_i u_{n+1}$ satisfies

$$\partial_t (\partial_i u_{n+1}) + \Delta (\partial_i u_{n+1}) + \nabla (\partial_i u_{n+1}) \cdot \mathbf{G}'(u_n) + \nabla u_{n+1} \cdot \mathbf{G}''(u_n) (\partial_i u_n) = 0.$$

The maximum principle for (3) yields

$$\|v\|_\infty \leq \|v_0\|_\infty + \int_0^t \|q(s)\|_\infty \|v(s)\|_\infty ds.$$

Applying the above maximum principle for v , we get

$$\|\nabla u_{k+1}(t)\|_\infty \leq \|\nabla u_0\|_\infty + C_2 \int_0^t \|\nabla u_{k+1}(s)\|_\infty \|\nabla u_k(s)\|_\infty ds.$$

By the Gronwall inequality $\|\nabla u_k(t)\|_\infty$ satisfies

$$\|\nabla u_k(t)\|_\infty \leq \frac{\|\nabla u_0\|_\infty}{1 - C_2 \|\nabla u_0\|_\infty t}, \quad (5)$$

for all k .

By (4) and (5) we see that $\{u_k\}$ is a Cauchy sequence in $L^\infty(0, T_0 - \varepsilon; L^{\infty,1})$ for any $\varepsilon \in (0, T)$ so that $u := \lim_{k \rightarrow \infty} u_k$ is solution of (E). It is easy to prove the uniqueness of solution of (E) by using the maximum principle for equation (3).

Remark. It is natural to consider a linearly growing initial data. We give a formal argument to show that linearly growing initial data is allowed. We postulate that $u(x, t) = x^\alpha f(t)$ is solution of (E). By (E) u must satisfy

$$x^\alpha f'(t) = \alpha(\alpha - 1)x^{\alpha-2}f(t) + \alpha x^{\alpha-1}f(t)G'(x^\alpha f(t)).$$

We observe that the growth of the left hand side is x^α . By the assumption of G the growth of the right hand side is $x^{2\alpha-1}$. Hence α must satisfy $\alpha \leq 2\alpha - 1$ so that $\alpha \leq 1$.

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EXISTENCE OF THE GLOBAL ATTRACTOR FOR WEAKLY DAMPED, FORCED KDV EQUATIONS ON SOBOLEV SPACE OF NEGATIVE INDEX

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1. INTRODUCTION

We consider the global attractor of Korteweg de Vries equations with a weak dissipation and an external forcing term:

$$(1.1) \quad \partial_t u + \gamma u + \partial_x^3 u + \frac{1}{2} \partial_x u^2 = f,$$

$$(1.2) \quad u(x, 0) = u_0(x) \in \dot{H}^s(\mathbb{T})$$

where \mathbb{T} is the one-dimensional torus and the unknown u maps $\mathbb{T} \times [0, \infty)$ into \mathbb{R} , and the damping parameter γ is positive constant, and the external forcing term $f \in \dot{L}^2(\mathbb{T})$ that does not depend on t . Here, we define $\dot{L}^2(\mathbb{T}) = \{u \in L^2(\mathbb{T}); \int_{\mathbb{T}} u(x) dx = 0\}$ and $\dot{H}^s(\mathbb{T}) = \{u \in H^s(\mathbb{T}); \int_{\mathbb{T}} u(x) dx = 0\}$. The existence of the global attractor of (1.1)–(1.2) with $s \geq 0$ has been studied by many authors. In the present paper, we study the lower bound of s to assure of the existence of the global attractor.

We recall the history of the dynamical study of (1.1)–(1.2). For classical smooth solutions to (1.1), i.e. solutions that start from initial data in $H^2(\mathbb{T})$, Ghidaglia (see [4]) proved that the associated KdV semiflow possesses a *weak* global attractor, i.e. a bounded subset of H^2 , that is invariant by the flow and that attracts all the trajectories when t goes to $+\infty$ for the H^2 -weak topology. Moreover, this attractor has finite H^1 -dimension. This result was proved under the assumption that the external force f belongs to H^2 . Actually, it turns out that this weak attractor is a global attractor for the H^2 strong topology (see [5]). The next step is concerned with the issue of the regularity of the attractor. In [13], Moise and Rosa proved that if the external force f belongs to H^3 , then KdV equation provides a dissipative semigroup in H^3 that enjoys the following property: if $f \in H^{3+k}$, then the global attractor for the H^3 topology is a compact subset of H^{3+k} . This result corresponds to the terminology of Haraux (see [9]), where the author proved a regularization at

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$t = +\infty$ for a dissipative wave equation. The asymptotic smoothing effect for dissipative nonlinear Schrödinger equations was first proved in [6].

Next, we consider low regularity solutions. Bourgain (see [1]) proved that the KdV equations (without dissipation and forcing term) is time locally well-posed in $\dot{L}^2(\mathbb{T})$ by introducing Fourier restriction norm method. Because the KdV equations posses L^2 conservation, this time local solutions automatically extend to time global solutions. Kenig, Ponce and Vega improved Bourgain's method and proved that the KdV equations (without dissipation and forcing term) is time locally well-posed in $\dot{H}^s(\mathbb{T})$ with $s > -1/2$ in [10]. By using their method, Goubet proved that when the initial data is in $\dot{L}^2(\mathbb{T})$, equation (1.1) possesses a compact global attractor in L^2 , that is a compact subset of H^3 . These proofs of the time global well-posedness and the existence of global attractor are based on the conservation law. Actually, the KdV equation possesses infinite conservation quantities, each of which is defined in $H^j (j \in \mathbb{Z}, j \geq 0)$. However, because the KdV equations on H^s for $s < 0$ have no conservation law, it seems difficult to consider the long time behavior in Sobolev space of negative index. Moreover, we do not know whether the solutions of (1.1)–(1.2) for $s < 0$ are measurable function or not, because the KdV equations on torus do not have smoothing effects. In [3], Colliander, Keel, Staffilani, Takaoka and Tao overcame this difficulty and proved the time global well-posedness of (1.1)–(1.2) with $s \geq -1/2$ by introducing the operator I and calculating the modified Energy, which is called “ I -method”. Naturally, the following problem arises. If a damping term and a forcing term are added to the KdV equation, does the semiflow corresponding to the weak solution in [3] also has a global attractor? We have the following theorem by using their method in [3].

Theorem 1.1. *We assume $s \geq -1/2$. Then, there exist the semigroup $S(t)$ and maps M_1 and M_2 such that $S(t)u_0$ is the unique solution of (1.1)–(1.2) and*

$$(1.3) \quad S(t)u_0 = M_1(t)u_0 + M_2(t)u_0,$$

$$(1.4) \quad \sup_{t > T_1} \|M_1(t)u_0\|_{L^2} < K,$$

and for $t > T_1$

$$(1.5) \quad \|M_2(t)u_0\|_{H^s} < K \exp(-\gamma(t - T_1))$$

where the constant K depending only on $\|f\|_{L^2}$ and γ and T_1 depending only on $\|f\|_{L^2}$, γ and $\|u_0\|_{H^s}$.

Remark 1.1. *In [12], it was proved that the KdV equations on H^s is time locally ill-posed for $s < -1/2$ under the assumption that the flow map $S(t)$ is C^2 Fréche-differentiable(see also [2]). Therefore, the condition $s \geq -1/2$ seem to be optimal.*

Remark 1.2. We ignore the nonlinear term $\frac{1}{2}\partial_x u^2$ and let $f = 0$. Then, we have

$$\|u(t)\|_{H^s} = \|u_0\|_{H^s} \exp(-\gamma t).$$

Therefore, the decay order of $\|M_2(t)u_0\|_{H^s}$ in (1.5) is optimal.

Corollary 1.2. Let $s \geq -1/2$. Then, equation (1.1)–(1.2) possess a global attractor \mathcal{A} in H^s , that is a compact subset of L^2 .

2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.2

We define $m : \mathbb{R} \mapsto \mathbb{R}$ be a smooth monotone \mathbb{R} -valued function such that

$$m(\xi) = \begin{cases} 1, & |\xi| < N \\ N^{-s}|\xi|^s, & |\xi| > 2N \end{cases}.$$

We define the operator I as following

$$\widehat{If}(\xi) = m(\xi)\widehat{f}(\xi).$$

Here, we summarize the properties of I . For any function f and $s < 0$, we have

$$\|f\|_{H^s} \leq \|If\|_{L^2} \leq N^{-s}\|f\|_{H^s}$$

Let $\widehat{g}_1 = \widehat{f}|_{|\xi| < N}$ and $\widehat{g}_2 = \widehat{f}|_{|\xi| > N}$. Then, we have

$$\|g_1\|_{L^2} \leq \|If\|_{L^2}, \quad \|g_2\|_{H^s} \leq N^s\|If\|_{L^2}.$$

We apply “ I -method” to (1.1)–(1.2). Then, we have the following a priori estimate.

Proposition 2.1. Let $T > 0$ be given and u be a solution of (1.1)–(1.2) on $t \in [0, T]$. We assume $s \geq -1/2$, $N^{3/5} > \gamma$, $N^{1/5} > C_1 T$ and

$$(2.6) \quad \|Iu_0\|_{L^2}^2 + \frac{1}{\gamma^2}\|If\|_{L^2}^2 \exp(2\gamma T) < N^{6/5}C_2,$$

then we have

$$(2.7) \quad \|Iu(T)\|_{L^2}^2 \exp(2\gamma T) < C_3 \left(\|Iu_0\|_{L^2}^2 + \frac{1}{\gamma^2}\|If\|_{L^2}^2 \exp(2\gamma T) \right).$$

We prove Theorem 1.1 by using Proposition 2.1.

Proof of Theorem 1.1. We choose $T_1 > 0$ so that

$$(2.8) \quad \exp(2\gamma T_1) > \|u_0\|_{H^s}^2 \|f\|_{L^2}^{-2} \max \left\{ -\gamma^{10s/3}, (C_1 T_1)^{-10s}, \right. \\ \left. (C_2/2)^{5s/(3+5s)} \|u_0\|_{H^s}^{-10s/(3+5s)}, (2\gamma^{-2} C_2^{-1} \|f\|_{L^2}^2 \exp(2\gamma T_1))^{-5s/3} \right\},$$

which may certainly be done because $s \geq -1/2$ and T_1 depends only on $\|f\|_{L^2}, \gamma$ and $\|u_0\|_{L^2}$. Put

$$N = \max \left\{ \gamma^{5/3}, (C_1 T_1)^5, (C_2/2)^{-5/(6+10s)} \|u_0\|_{H^s}^{5/(3+5s)}, (2\gamma^{-2} C_2^{-1} \|f\|_{L^2}^2 \exp(2\gamma T_1))^{5/6} \right\}.$$

Then, we have

$$\begin{aligned} N^{3/5} &> \gamma, & N^{1/5} &> C_1 T_1, \\ \|Iu_0\|_{L^2}^2 &\leq N^{-2s} \|u_0\|_{H^s}^2 = N^{6/5} N^{-(6+10s)/5} \|u_0\|_{H^s}^2 < C_2 N^{6/5} / 2, \\ \gamma^{-2} \|If\|_{L^2} \exp(2\gamma T_1) &< C_2 N^{6/5} / 2. \end{aligned}$$

Therefore, from Proposition 2.1 we obtain

$$(2.9) \quad \|u(T_1)\|_{H^s}^2 \leq \|Iu(T_1)\|_{L^2}^2 < C_3 (N^{-2s} \|u_0\|_{H^s}^2 \exp(-2\gamma T_1) + \gamma^{-2} \|f\|^2).$$

From (2.8), we have

$$N^{-2s} \exp(-2\gamma T_1) < \|u_0\|_{H^s}^{-2} \|f\|_{L^2}^2.$$

Therefore, we obtain

$$(2.10) \quad \|u(T_1)\|_{H^s}^2 < C_3 (1 + \gamma^{-2}) \|f\|_{L^2}^2 < K_1$$

where K_1 depends only on $\|f\|_{L^2}$ and γ . We next fix $T_2 > 0$ and solve (1.1)–(1.2) on $[T_1, T_1 + T_2]$ with initial data $u(T_1)$. Let $K_2 > 0$ be sufficiently large enough to satisfy

$$(2.11) \quad K_2 \exp(2\gamma t) > \max \left\{ \gamma^{-10s/3}, (C_1 t)^{-10s}, \right. \\ \left. (C_2^{-1} K_1)^{-5s/3}, (C_2^{-1} \gamma^{-2} \|f\|_{L^2}^2 \exp(2\gamma t))^{-5/3s} \right\}$$

for any $t > 0$, which may certainly be done because $s \geq -1/2$ and K_2 depends only on $\|f\|_{L^2}$ and γ . Put $N^{-2s} = K_2 \exp(2\gamma T_2)$. Then, from (2.11), the assumptions in Proposition 2.1 are satisfied. Therefore, we obtain

$$(2.12) \quad \|Iu(T_1 + T_2)\|_{L^2}^2 < C_3 (N^{-2s} \|u(T_1)\|_{H^s}^2 \exp(-2\gamma T_2) + \gamma^{-2} \|f\|_{L^2}^2) \\ < C_3 (K_1 K_2 + \gamma^{-2} \|f\|_{L^2}^2) < K_3$$

where K_3 depends only on $\|f\|_{L^2}$ and γ . For $t > T_1$, we define maps $M_1(t)$ and $M_2(t)$ such that

$$(2.13) \quad \widehat{M_1(t)u_0} = \widehat{S(t)u_0} \Big|_{|\xi| < 2N}, \quad \widehat{M_2(t)u_0} = \widehat{S(t)u_0} \Big|_{|\xi| > 2N}$$

where $S(t)u_0 = u(t)$ and $N = (K_2 \exp(2\gamma(t - T_1)))^{-1/2s}$. Then, for $t > T_1$, we have

$$(2.14) \quad \|M_1(t)u_0\|_{L^2}^2 < \|Iu(t)\|_{L^2}^2 < K_3,$$

$$(2.15) \quad \|M_2(t)u_0\|_{H^s}^2 < N^{2s} \|Iu(t)\|_{L^2}^2 < K_2^{-1} K_3 \exp(-2\gamma(t - T_1)).$$

Let $K = \max\{K_2^{1/2}, K_1^{-1/2} K_2^{1/2}\}$. Then, we have (1.4) and (1.5). \square

Proof of Colorally 1.2. From Theorem 1.1, $M_2(t)$ converges uniformly to 0 in H^s and $M_1(t)$ is a compact mapping for t large enough. Therefore, the semigroup $S(t)$ is asymptotically compact (see [8]). From Theorem 1.1.1 in [11], we have the existence of a global attractor \mathcal{A} for the semigroup. Moreover, \mathcal{A} is a bounded subset of \dot{L}^2 . By using a suitable version of a classical argument due to J. Ball, We obtain the compactness of \mathcal{A} . \square

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Asymptotic decay toward the rarefaction waves of solutions for viscous conservation laws in one-dimensional half space

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We consider the initial-boundary value problem for scalar viscous conservation laws in one-dimensional half space $\mathbb{R}_+ := (0, \infty)$:

$$\begin{cases} u_t + f(u)_x = u_{xx}, & x \in \mathbb{R}_+, t > 0, \\ u(0, t) = u_-, & t > 0, \\ u(x, 0) = u_0(x) = \begin{cases} = u_-, & x = 0, \\ \rightarrow u_+, & x \rightarrow \infty, \end{cases} \end{cases} \quad (1)$$

where f is a smooth function and u_{\pm} are constants. We consider this problem under the following assumptions:

$$f'' \geq \exists \alpha > 0, \quad u_* \leq u_- < u_+ \quad (f'(u_*) = 0).$$

Under these conditions, it was already shown in [5] that the solutions of (1) converge to the corresponding rarefaction waves as $t \rightarrow \infty$. The main purpose of the present research is to obtain the convergence rate. The main theorem is stated as follows:

Theorem 1 *Let $u_0 - u_+ \in (H^1 \cap L^1)(\mathbb{R}_+)$. Then the initial-boundary value problem (1) has a unique global solution $u(x, t)$. Moreover, $u(x, t)$ satisfies the following estimates:*

$$\begin{aligned} \|u(t) - r(t)\|_{L^2} &\leq C(1+t)^{-\frac{1}{4}} \log(2+t), \\ \|u(t) - r(t)\|_{L^\infty} &\leq C(1+t)^{-\frac{1}{2}} \log^3(2+t), \end{aligned}$$

where C is a positive constant depending only on u_0 .

In order to prove Theorem 1, we derive the smooth approximation $w(x, t)$ of the rarefaction wave $r(x, t)$ by employing the idea of Hattori and Nishihara [2]. We define $\tilde{w}(w, t)$ as a solution to the following Cauchy problem:

$$\begin{cases} \tilde{w}_t + \tilde{w}\tilde{w}_x = \tilde{w}_{xx}, & x \in \mathbb{R}, t > -1, \\ \tilde{w}(x, -1) = w_0^R(x), & x \in \mathbb{R}, \end{cases} \quad (2)$$

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where the initial data $w_0^R(x)$ is defined by

$$w_0^R(x) := \begin{cases} f'(u_-), & x < 0, \\ f'(u_+), & x > 0 \end{cases} \quad (f'(u_-) > 0), \quad w_0^R(x) := \begin{cases} -f'(u_+), & x < 0, \\ f'(u_+), & x > 0 \end{cases} \quad (f'(u_-) = 0).$$

Because (2) is the Burgers equation, we can get the explicit formula of $\tilde{w}(x, t)$ by using the Hopf-Cole transformation. Using $\tilde{w}(x, t)$, we define $w(x, t)$ as

$$w(x, t) := (f')^{-1}(\tilde{w}(x, t)).$$

Hattori and Nishihara show that $w(x, t)$ is a smooth approximation of rarefaction wave $r(x, t)$ in [2].

Lemma 2 For $1 \leq p \leq +\infty$ and $t \geq 0$, $w(x, t)$ satisfies the followings.

- (i) $0 \leq w(0, t) - u_- \leq Ce^{-c(1+t)}$.
- (ii) $|w_x(0, t)| \leq Ce^{-c(1+t)}$, $|w_{xx}(0, t)| \leq Ce^{-c(1+t)}$.
- (iii) $\|w(t) - r(t)\|_{L^p} \leq C(1+t)^{-\frac{1}{2} + \frac{1}{2p}}$.
- (iv) $\|w_x(t)\|_{L^p} \leq C(1+t)^{-1 + \frac{1}{p}}$, $\|w_{xx}(t)\|_{L^p} \leq C(1+t)^{-\frac{3}{2} + \frac{1}{2p}}$.
- (v) $w_x(x, t) > 0$ for $x \in \mathbb{R}$.

Successively, we define $W(x, t)$ and $\psi(x, t)$ as follows:

$$W(x, t) := w(x, t) - \psi(x, t), \quad \psi(x, t) := (w(0, t) - u_-)e^{-x},$$

where $W(x, t)$ and $\psi(x, t)$ are called “modified smooth approximation” and “modification function” respectively. Using $W(x, t)$, we define the perturbation $v(x, t)$ as

$$v(x, t) := u(x, t) - W(x, t).$$

$v(x, t)$ satisfies the following equation:

$$\begin{cases} v_t + (f(W+v) - f(W))_x = v_{xx} + R(x, t), & x \in \mathbb{R}_+, t > 0, \\ v(0, t) = 0, & t > 0, \\ v(x, 0) = v_0(x) := u_0(x) - W_0(x), & x \in \mathbb{R}_+, \end{cases} \quad (3)$$

where $R(x, t) = -\frac{f'''(w)}{f''(w)}w_x^2 - (f(W) - f(w))_x - \psi_{xx} + \psi_t$. From Lemma 2, we can see that $R(x, t)$ satisfies

$$\|R(t)\|_{L^p} \leq C(1+t)^{-2 + \frac{1}{p}}.$$

Making use of a standard iteration method, it is shown that the equation (3) has a unique solution locally in time in the space $X = C^0([0, T]; H^1(\mathbb{R}_+))$ ($T > 0$). We have a priori estimates and L^1 -estimates for $v(x, t)$ as follows.

Proposition 3 (A priori estimate) *Suppose that $v_0 \in H^1(\mathbb{R}_+)$. Then there exists a positive constants C such that the solution $v \in X$ satisfies the estimate*

$$\|v(t)\|_{H^1}^2 + \int_0^t \|\sqrt{W_x(\tau)}v(\tau)\|_{L^2}^2 + \|v_x(\tau)\|_{H^1}^2 d\tau \leq C(\|v_0\|_{H^1}^2 + 1).$$

Proposition 4 (L^1 -estimate) *Suppose that $v_0 \in (H^1 \cap L^1)(\mathbb{R}_+)$. Then the solution $v \in X$ satisfies the estimate*

$$\|v(t)\|_{L^1} \leq \|v_0\|_{L^1} + C \log(1+t).$$

Combination of H^1 -estimates and L^1 -estimates gives the decay estimates of v .

Theorem 5 (Decay estimate) *Suppose that $v_0 \in (H^1 \cap L^1)(\mathbb{R}_+)$. Then the solution $v \in X$ satisfies the following estimates for arbitrary constant $\varepsilon \in (0, \frac{1}{2})$:*

$$\begin{aligned} (i) \quad & (1+t)^{\frac{1}{2}+\varepsilon} \|v(t)\|_{L^2}^2 + \int_0^t (1+\tau)^{\frac{1}{2}+\varepsilon} \left\{ \|\sqrt{W_x}v(\tau)\|_{L^2}^2 + \|v_x(\tau)\|_{L^2}^2 \right\} d\tau \\ & \leq C(1+t)^\varepsilon \log^2(2+t), \\ (ii) \quad & (1+t)^{\frac{3}{2}+\varepsilon} \|v_x(t)\|_{L^2}^2 + \int_0^t (1+\tau)^{\frac{3}{2}+\varepsilon} \left\{ \|\sqrt{W_x}v_x(\tau)\|_{L^2}^2 + \|v_{xx}(\tau)\|_{L^2}^2 \right. \\ & \quad \left. + f'(u_-)v_x(0, \tau)^2 \right\} d\tau \leq C(1+t)^\varepsilon \log^{10}(2+t). \end{aligned}$$

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An asymptotic expansion of solutions to the Lamé system in the presence of inclusions and applications.

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Suppose that an elastic medium occupies a bounded domain Ω in \mathbb{R}^3 , with a connected Lipschitz boundary $\partial\Omega$. Let the constants (λ, μ) denote the background Lamé coefficients, that are the elastic parameters in the absence of any inhomogeneities. Suppose that one or more elastic inhomogeneities lie in Ω . We first suppose that the inhomogeneity consist of a single domain of the form

$$(0.1) \quad D = \epsilon B + z$$

where B is a bounded Lipschitz domain in \mathbb{R}^3 , ϵ denotes the order of magnitude of D and is small, and z is a center of D . We assume that there exists $d_0 > 0$ such that

$$(0.2) \quad \inf_{x \in D} \text{dist}(x, \partial\Omega) > d_0.$$

Suppose that D has the pair of Lamé constants $(\tilde{\lambda}, \tilde{\mu})$ which is different from that of the background elastic body, (λ, μ) . It is always assumed that

$$(0.3) \quad \mu > 0, \quad 3\lambda + 2\mu > 0, \quad \tilde{\mu} > 0 \quad \text{and} \quad 3\tilde{\lambda} + 2\tilde{\mu} > 0.$$

We also assume that

$$(0.4) \quad (\lambda - \tilde{\lambda})(\mu - \tilde{\mu}) \geq 0, \quad ((\lambda - \tilde{\lambda})^2 + (\mu - \tilde{\mu})^2 \neq 0).$$

We consider the following transmission problem associated to the system of elastostatics with the traction boundary condition:

$$(0.5) \quad \begin{cases} \sum_{j,k,l=1}^3 \frac{\partial}{\partial x_j} \left(C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = 0 & \text{in } \Omega, \quad i = 1, 2, 3, \\ \frac{\partial \vec{u}}{\partial \nu} |_{\partial\Omega} = \vec{g}, \end{cases}$$

where $\frac{\partial}{\partial \nu}$ denotes the conormal derivative associated with the system of elastostatics. Here the piecewise constant Lamé parameter C_{ijkl} is given by

$$(0.6) \quad C_{ijkl} := \left(\lambda \chi(\Omega \setminus D) + \tilde{\lambda} \chi(D) \right) \delta_{ij} \delta_{kl} + \left(\mu \chi(\Omega \setminus D) + \tilde{\mu} \chi(D) \right) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where $\chi(D)$ is the characteristic function of D . Let u_ϵ be the solution of (0.6). Then it is easy to see that u_ϵ converges, in an appropriate topology, to the background solution \vec{U} , which is the solution of

$$(0.7) \quad \begin{cases} \sum_{j,k,l=1}^3 \frac{\partial}{\partial x_j} \left(C_{ijkl}^0 \frac{\partial \vec{U}_k}{\partial x_l} \right) = 0 & \text{in } \Omega, \quad i = 1, 2, 3, \\ \frac{\partial \vec{U}}{\partial \nu} |_{\partial \Omega} = \vec{g}, \end{cases}$$

where

$$C_{ijkl}^0 := \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

We first derive an asymptotic expansion formula of the solution u_ϵ in terms of U as ϵ tends to 0. The asymptotic formula up to the third order terms takes the form

$$(0.8) \quad \vec{u}(x) = \vec{U}(x) + \sum_{j=1}^3 \sum_{|\alpha|=1}^3 \sum_{|\beta|=1}^{4-|\alpha|} \epsilon^{|\alpha|+|\beta|+1} (\partial^\alpha U_j)(z) \partial_z^\beta N(x, z) M_{\alpha\beta}^j + O(\epsilon^6),$$

uniformly for $x \in \partial \Omega$. Here $N(x, z)$ is the Neumann function function for the problem (0.7), and $M_{\alpha\beta}^j$ is the elastic moment tensor (or elastic polarization tensor). I want to emphasize that similar formula can be obtained even when there are multiple inhomogeneities, and when homogeneities are hard inclusions or cavities.

The elastic moment tensors, which arise naturally in the asymptotic expansion, describe the disturbance of the solution due to the discontinuity of the Lamé parameters along the boundary of D , and carry important information of D . I will explain some basic properties of these tensors such as symmetry and positive-definiteness. I will also discuss the connection of its eigenvalues to some geometric properties of D .

The formula (0.8) has many potential applications. Among them are an inverse problem to detect the location and magnitude of the inhomogeneity and computation of the effective moduli. I will explain how one can use the asymptotic expansion formula to detect the inhomogeneity, with possibly some numerical examples.

This talk is based on the joint work with H. Ammari, G. Nakamura, and K. Tanuma.

On large time behavior of solutions to the compressible Navier-Stokes equation in the half-space in \mathbf{R}^3

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In this talk I am going to talk about large time behavior of solutions to the compressible Navier-Stokes equation on the half space of \mathbf{R}^3 . The results in this talk were obtained in a *joint work* with Takayuki KOBAYASHI (Kyushu Inst. Tech.).

We consider the initial boundary value problem for the compressible Navier-Stokes equation in $\mathbf{R}_+^3 = \{x = (x', x_3); x' \in \mathbf{R}^2, x_3 > 0\}$:

$$(1) \quad \begin{aligned} & \rho_t + \operatorname{div} m = 0, \\ & m_t + \operatorname{div} \left(\frac{m \otimes m}{\rho} \right) + \nabla P(\rho) = \mu \Delta \left(\frac{m}{\rho} \right) + (\mu + \nu) \nabla \operatorname{div} \left(\frac{m}{\rho} \right), \\ & m|_{x_3=0} = 0, \quad \rho|_{t=0} = \rho_0, \quad m|_{t=0} = m_0, \end{aligned}$$

where $\rho = \rho(t, x)$ is the density; $m = (m_1(t, x), m_2(t, x), m_3(t, x))$ the momentum; and $P = P(\rho)$ the pressure; μ and ν are viscosity constants satisfying $\mu > 0$ and $\frac{2}{3}\mu + \nu \geq 0$; (ρ_0, m_0) is the initial value, which is close to a constant state $(\rho^*, 0)$. Here ρ^* is a given positive constant.

Matsumura and Nishida proved in [12] that if $\|(\rho_0 - \rho^*, m_0)\|_{H^3 \times H^3}$ is sufficiently small, then there exists a unique solution $(\rho(t), m(t))$ of (1) globally in time and $(\rho(t), m(t))$ satisfies $\|(\rho(t) - \rho^*, m(t))\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. Concerning the decay rate of the perturbation $(\rho(t) - \rho^*, m(t))$, we will show the following

Theorem 1. (i) Let $u_0 = (\rho_0 - \rho^*, m_0) \in (H^3(\mathbf{R}_+^3) \times H^3(\mathbf{R}_+^3)) \cap (L^1(\mathbf{R}_+^3) \times L^1(\mathbf{R}_+^3))$ and satisfy a suitable compatibility condition. Assume that $\partial_\rho P(\rho^*) > 0$ and that u_0 is sufficiently small in $H^3 \times H^3$. Then there exists a unique global solution $(\rho(t), m(t))$ of (1) with $U(t) = (\rho(t) - \rho^*, m(t)) \in C([0, \infty), H^3 \times H^3)$; and $U(t)$ satisfies

$$\|U(t)\|_{L^2 \times L^2} = O(t^{-3/4}) \quad \text{and} \quad \|\partial_x U(t)\|_{L^2 \times L^2} = O(t^{-9/8})$$

as $t \rightarrow \infty$.

(ii) Furthermore, we have

$$\|U(t) - \bar{U}(t)u_0\|_{L^2 \times L^2} = O(t^{-1})$$

as $t \rightarrow \infty$. Here $\bar{U}(t)u_0$ denotes the solution of the linearized problem at $(\rho^*, 0)$ with initial value u_0 .

(iii) In addition to the same assumption on $(\rho_0 - \rho^*, m_0)$, if we assume that $\int_{\mathbb{R}_+^3} (\rho_0(x) - \rho^*) dx \neq 0$, then

$$\|U(t)u_0\|_{L^2 \times L^2} \geq Ct^{-3/4}$$

as $t \rightarrow \infty$.

Decay rate $t^{-3/4}$ for $\|U(t)\|_{L^2 \times L^2}$ in Theorem 1 is the same as in the case of the Cauchy and exterior problems ([3, 8, 11]). As for $\|\partial_x U(t)\|_{L^2 \times L^2}$ we have obtained the decay rate $t^{-9/8}$ which is slower than the rate $t^{-5/4}$ for the Cauchy and exterior problems ([3, 8, 11]). This difference of decay rate is due to the analysis for the linearized problem, where we have obtained only $\|\partial_x \bar{U}(t)u_0\|_{L^2 \times L^2} = O(t^{-9/8})$, see Theorem 2 below.

The property $\|U(t) - \bar{U}(t)u_0\|_{L^2 \times L^2} = O(t^{-1})$ in Theorem 1 is also different from the one in the case of the Cauchy problem, where $\|U(t) - \bar{U}(t)u_0\|_{L^2 \times L^2} = O(t^{-5/4})$ holds ([3, 5, 6]). To prove this in the case of the Cauchy problem, the property $\|\bar{U}(t)\partial_x u_0\|_{L^2 \times L^2} = O(t^{-5/4})$ for the linearized problem is used; while in the case of the problem (1.1) on the half space we have only $\|\bar{U}(t)\partial_x u_0\|_{L^2 \times L^2} = O(t^{-1})$, which is, however, optimal (see Theorem 3 below).

We also note that Deckelnick [1, 2] showed that if $(\rho_0 - \rho^*, m_0)$ is sufficiently small in $H^3 \times H^3$ (but not necessarily belongs to $L^1 \times L^1$), then the solution of (1) satisfies

$$\begin{aligned} \|\partial_t U(t)\|_{L^2 \times L^2} &= O(t^{-1/2}), & \|\partial_x U(t)\|_{L^2 \times L^2} &= O(t^{-1/4}), \\ \|m(t)\|_{L^\infty} &= O(t^{-1/4}), & \|\rho(t) - \rho^*\|_{L^\infty} &= O(t^{-1/8}) \end{aligned}$$

as $t \rightarrow \infty$.

Theorem 1 is proved by combining the global existence results by Matsumura and Nishida ([12]) and the following decay estimates for solutions to the linearized problem at $(\rho^*, 0)$.

We write the solution $(\bar{\rho}, \bar{m})$ of the linearized problem with initial value $u_0 = (\rho_0, m_0) \in H^1 \times L^2$ as

$$\begin{aligned}\bar{U}(t)u_0 &= (\bar{\Psi}(t)u_0, \bar{V}(t)u_0), \quad \bar{\Psi}(t)u_0 = \bar{\rho}(t, \cdot), \quad \bar{V}(t)u_0 = \bar{m}(t, \cdot), \\ \bar{V}(t)u_0 &= (\bar{V}_1(t)u_0, \bar{V}_2(t)u_0, \bar{V}_3(t)u_0)\end{aligned}$$

Theorem 2. *Let $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$. Then, there exists a positive constant C such that the following estimates hold for all $t \geq 1$:*

(i)

$$\|\bar{U}(t)u_0\|_{L^2 \times L^2} \leq Ct^{-3/4}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{L^2 \times L^2}),$$

(ii)

$$\|\partial_{x'} \bar{V}(t)u_0\|_{L^2 \times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{L^2 \times L^2}),$$

$$\|\partial_x \bar{\Psi}(t)u_0\|_{L^2 \times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{H^1 \times L^2}),$$

(iii)

$$\|\partial_{x_3} \bar{V}_j(t)u_0\|_{L^2 \times L^2} \leq Ct^{-9/8}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{L^2 \times L^2}) \quad (j = 1, 2),$$

$$\|\partial_{x_3} \bar{V}_3(t)u_0\|_{L^2 \times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{L^2 \times L^2}),$$

(iv)

$$\|\bar{U}(t)\partial_{x'} u_0\|_{L^2 \times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{H^1 \times L^2}).$$

(v) *If $u_0 = (0, m_0)$, then it holds*

$$\|\bar{U}(t)\partial_{x_3} u_0\|_{L^2 \times L^2} \leq Ct^{-1}(\|m_0\|_{L^1} + \|m_0\|_{L^2}).$$

(vi) *We also have, for $u_0 = (\rho_0, m_0)$,*

$$\|\partial_t \bar{U}(t)u_0\|_{L^2 \times L^2} \leq Ct^{-5/4}(\|u_0\|_{L^1 \times L^1} + \|u_0\|_{L^2 \times L^2}).$$

The estimates in Theorem 2 (i) and (v) are optimal. In fact, we have the following lower bounds.

Theorem 3. *Let $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$.*

(i) *If $\int_{\mathbf{R}_+^3} \rho_0(x) dx \neq 0$, then*

$$\|\bar{U}(t)u_0\|_{L^2 \times L^2} \geq Ct^{-3/4}$$

as $t \rightarrow \infty$.

(ii) Assume that $u_0 = (0, m_0)$ with $m_0 = (m_{0,1}, m_{0,2}, m_{0,3}) \in H^1 \cap L^1$ and $\int_{\mathbf{R}_+^3} m_{0,j}(x) dx \neq 0$ for $j = 1$ or 2 . Then we have

$$\|\bar{U}(t)\partial_{x_3}u_0\|_{L^2 \times L^2} \geq Ct^{-1}$$

as $t \rightarrow \infty$.

Although the optimal decay rate of $\|\partial_x \bar{U}(t)u_0\|_{L^2 \times L^2}$ for general $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$ is unclear, we have the following decay rate under some additional assumption on u_0 .

Theorem 4. (i) Assume that $u_0 = (\rho_0, m_0) \in (H^1 \times L^2) \cap (L^1 \times L^1)$. Assume also that $x_3 u_0 \in L^1 \times L^1$. Then

$$\|\partial_x \bar{U}(t)u_0\|_{L^2 \times L^2} \leq Ct^{-5/4}(\|(1+x_3)u_0\|_{L^1 \times L^1} + \|u_0\|_{H^1 \times L^2}).$$

(ii) Furthermore, in addition to the assumption of (i), if $\int_{\mathbf{R}_+^3} \rho_0(x) dx \neq 0$, then

$$\|\partial_x \bar{U}(t)u_0\|_{L^2 \times L^2} \geq Ct^{-5/4}$$

as $t \rightarrow \infty$.

Finally, we should also mention that, in the case of the whole space and the exterior domains, large time behavior of solutions in L^p spaces has been studied; by Hoff and Zumbrun [3, 4], Kobayashi and Shibata [9], Liu and Wang [10], and Weike [13] for the Cauchy problem, by Kobayashi [7] and Kobayashi and Shibata [8] for the exterior problem.

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On the Half-space Problem of the Boltzmann Equation

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1 Introduction and Main Result

The Dirichlet problem of the nonlinear Boltzmann equation in the half-space arises in the analysis of the kinetic boundary layer, the condensation-evaporation problem and other problems related to the kinetic behavior of gas near the wall, [5]. The main concern is to find a solution which tends to an assigned *Maxwellian* at infinity.

An interesting feature of this problem is that not all Dirichlet data are admissible and the number of admissible conditions changes with the far Maxwellian. This has been shown for the linear case by many authors [3],[6],[7],[8], mainly in the context of the classical Milne's and Kramer's problems. Recently, a nonlinear admissible condition was derived for the discrete velocity model in [13] and the stability of steady solutions was proven in [10]. The full nonlinear problem was solved on the existence of solutions in [9] for the case of the specular reflection boundary condition, whose proof, however, does not work for the Dirichlet boundary condition, and in [2] for this case, but with the ambiguity that the far Maxwellian cannot be fixed a priori. Here, we will establish the admissible conditions for the fixed far Maxwellian. Our proof provides also a new aspect of the linear problem.

It should be mentioned that K. Aoki, Y. Sone and their group, ([1], [11], [12] and references therein), made an extensive numerical computation on the same nonlinear problem. Our result gives a partial explanation of their numerical results.

We are concerned with the steady state of a gas in the 3-dimensional half-space $D = \{(x, y, z) \in \mathbb{R}^3 | x > 0\}$, in which the mass density F of gas particles is assumed constant on each plane parallel to the boundary ∂D although the particle motion is 3-dimensional, that is, F is assumed to be a function of position x (but not of y, z) and particle velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Let ξ_1 stand for the velocity component along the x -axis. Then, F is governed by

$$(1.1) \quad \begin{cases} \xi_1 F_x = Q(F, F), & x > 0, \xi \in \mathbb{R}^3, \\ F|_{x=0} = F_0(\xi), & \xi_1 > 0, (\xi_2, \xi_3) \in \mathbb{R}^2, \\ F \rightarrow M_\infty(\xi) \quad (x \rightarrow \infty), & \xi \in \mathbb{R}^3. \end{cases}$$

Here, Q , the *collision operator*, is a quadratic integral operator in ξ . We do not use its explicit form here but need the following two classical properties, [4], [5].

(i) $Q(F) = 0$ if and only if F is a *Maxwellian*,

$$(1.2) \quad M[\rho, u, T](\xi) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|\xi - u|^2}{2T}\right),$$

which describes the distribution function of a gas in the equilibrium state with the mass density $\rho > 0$, flow velocity $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and temperature $T > 0$.

(ii) A function $\phi(\xi)$ is called a *collision invariant* of Q if

$$\langle \phi, Q(F) \rangle = 0 \quad \text{for all } F,$$

$\langle \cdot, \cdot \rangle$ being the inner product of $L^2(\mathbb{R}_\xi^3)$. Q has five collision invariants

$$(1.3) \quad \phi_0 = 1, \quad \phi_i = \xi_i \quad (i = 1, 2, 3), \quad \phi_4 = |\xi|^2,$$

which indicate the conservations of mass, momentum and energy in the course of the binary collision of particles.

The second equation in (1.1) is the Dirichlet boundary condition. The Dirichlet data $F_0(\xi)$ can be assigned only for incoming particles ($\xi_1 > 0$), but not for outgoing ones ($\xi_1 < 0$) because, then, the problem becomes ill-posed as will be seen from the a priori estimate stated in §3. This corresponds to the physical situation that one can control the incoming distribution but not the outgoing one.

It is clear that the far field $M_\infty(\xi)$ in the third equation of (1.1) cannot be assigned arbitrarily but must be a zero of Q , and hence a Maxwellian. Thus, we must take

$$M_\infty = M[\rho_\infty, u_\infty, T_\infty],$$

with some constants $\rho_\infty > 0$, $u_\infty = (u_{\infty,1}, u_{\infty,2}, u_{\infty,3}) \in \mathbb{R}^3$, and $T_\infty > 0$ which are the only quantities we can control. By a shift of the variable ξ along the boundary ∂D , we can assume without loss of generality that $u_{\infty,2} = u_{\infty,3} = 0$. Therefore, the sound speed and Mach number of this equilibrium state are given by

$$(1.4) \quad c_\infty = \sqrt{\frac{5}{3}T_\infty}, \quad \mathcal{M}^\infty = \frac{u_{\infty,1}}{c_\infty},$$

respectively, [5]. Note that the flow at infinity is incoming (resp. outgoing) if $\mathcal{M}^\infty < 0$ (resp. > 0) and supersonic (resp. subsonic) if $|\mathcal{M}^\infty| > 1$ (resp. < 1).

We will see that the Mach number \mathcal{M}^∞ provides significant changes on the solvability of (1.1). Indeed, since the boundary condition at $x = \infty$ specified by the third equation of (1.1) is imposed for all ξ , it is over-determined (ill-posed) and as a consequence, (1.1) may not be solvable unconditionally. Actually, we will show that the number of solvability conditions changes with \mathcal{M}^∞ . To state this precisely, set

$$(1.5) \quad n^+ = \begin{cases} 0, & \mathcal{M}^\infty < -1, \\ 1, & -1 < \mathcal{M}^\infty < 0, \\ 4, & 0 < \mathcal{M}^\infty < 1, \\ 5, & 1 < \mathcal{M}^\infty, \end{cases}$$

and introduce the weight function

$$(1.6) \quad W_\beta(\xi) = (1 + |\xi|)^{-\beta} \left(M[1, u_\infty, T_\infty](\xi) \right)^{1/2},$$

with $\beta \in \mathbb{R}$. Our main result is

Theorem 1.1 *Suppose $\mathcal{M}^\infty \neq 0, \pm 1$ and let $\beta > 3/2$. Then, there exist positive numbers ϵ_0, σ, C_0 , and a C^1 map*

$$(1.7) \quad \Psi : L^2(\mathbb{R}_+^3, \xi_1 d\xi) \longrightarrow \mathbb{R}^{n^+}, \quad \Psi(0) = 0,$$

such that the following holds.

(i) *For any F_0 satisfying*

$$(1.8) \quad |F_0(\xi) - M_\infty(\xi)| \leq \epsilon_0 W_\beta(\xi), \quad \xi \in \mathbb{R}_+^3,$$

and

$$(1.9) \quad \Psi(F_0 - M_\infty) = 0,$$

the problem (1.1) has a unique solution F in the class

$$(1.10) \quad |F(x, \xi) - M_\infty(\xi)| + |\xi_1 F_x(x, \xi)| \leq C_0 e^{-\sigma x} W_\beta(\xi), \quad x > 0, \xi \in \mathbb{R}^3.$$

(ii) *The set of F_0 satisfying (1.8) and (1.9) forms a (local) C^1 manifold of codimension n^+ .*

Remark 1.2 For each given M_∞ , (1.8) is a smallness condition on the deviation of F_0 from M_∞ whereas (1.9) gives restrictions on F_0 however small it may be, if $n^+ \neq 0$. Thus, our theorem says that the problem (1.1) is solvable unconditionally for any F_0 sufficiently close to M_∞ if $\mathcal{M}^\infty < -1$ but otherwise not. A physical explanation of this is that if $\mathcal{M}^\infty < -1$, any phenomena near the boundary cannot affect the far field while a part of them can propagate to infinity and affect the far field if $\mathcal{M}^\infty > -1$.

Remark 1.3 In the numerical works made in [11], [12] and references therein, the Dirichlet data F_0 is fixed to be the standard Maxwellian $M[1, 0, 1](\xi)$ (of course for $\xi_1 > 0$), and values of three parameters $(\rho_\infty, \mathcal{M}^\infty, T_\infty)$ are sought numerically which admit smooth solutions connecting F_0 and M_∞ . The conclusion is that the set of such admissible values is, in the parameter space \mathbb{R}^3 , a union of a 3-dimensional subdomain in the domain $\mathcal{M}^\infty < -1$, a 2-dimensional surface in $-1 < \mathcal{M}^\infty < 0$ and a 1-dimensional curve in $0 < \mathcal{M}^\infty < 1$ whereas no solutions are found if $\mathcal{M}^\infty > 1$. Our theorem agrees with this for the case $\mathcal{M}^\infty < 1$ in the sense that the codimension of the above mentioned regions of admissible values is just n^+ in the parameter space \mathbb{R}^3 . For the case $\mathcal{M}^\infty > 1$, $F_0 = M[1, 0, 1]$ may not satisfy the solvability condition (1.9) and hence, no solutions.

Remark 1.4 The stability of the stationary solutions obtained in Theorem 1.1 is an important issue. In the talk, we will show the exponentially asymptotic stability for the case $M^\infty < -1$.

2 A Remark on the Linearized Problem

Our proof relies on the analysis of the corresponding linearized problem at M_∞ . It provides also a new aspect of the linear problems discussed in [3],[6],[7],[8].

We shall look for the solution of (1.1) in the form

$$(2.1) \quad F(x, \xi) = M_\infty(\xi) + W_0(\xi)f(x, \xi),$$

where W_0 is the weight of (1.6) with $\beta = 0$. Then, the problem (1.1) reduces to

$$(2.2) \quad \begin{cases} \xi_1 f_x - Lf = \Gamma(f), & x > 0, \xi \in \mathbb{R}^3, \\ f|_{x=0} = a_0(\xi), & \xi \in \mathbb{R}_+^3, \\ f \rightarrow 0 (x \rightarrow \infty), & \xi \in \mathbb{R}^3, \end{cases}$$

where

$$Lf = W_0^{-1} \{ Q(M_\infty, W_0 f) + Q(W_0 f, M_\infty) \}, \quad \Gamma(f) = W_0^{-1} Q(W_0 f, W_0 f),$$

$$a_0 = W_0^{-1} (F_0 - M_\infty).$$

The operator L is linear while the remainder Γ is quadratic.

The linearized problem of (1.1) at M_∞ is just (2.2) with the term $\Gamma(f)$ dropped;

$$(2.3) \quad \begin{cases} \xi_1 f_x - Lf = 0, & x > 0, \xi \in \mathbb{R}^3, \\ f|_{x=0} = a_0(\xi), & \xi \in \mathbb{R}_+^3, \\ f \rightarrow 0 (x \rightarrow \infty), & \xi \in \mathbb{R}^3, \end{cases}$$

We can get the following linear version of Theorem 1.1.

Theorem 2.1 *Suppose $M^\infty \neq 0, \pm 1$. Then, there exist n^+ functions $r_i, 1 \leq i \leq n^+$, of $L^2(\mathbb{R}_+^3, \xi_1 d\xi)$ such that for any $a_0 \in R^\perp$ with*

$$R = \text{span}\{r_1, r_2, \dots, r_{n^+}\},$$

(2.3) has a unique L^2 solution f which tends to 0 exponentially as $x \rightarrow \infty$.

Remark 2.2 This theorem says that for the linear problem (2.3), the map Ψ of Theorem 1.1 is linear and the manifold of admissible a_0 is the hyperplane R^\perp . Actually, Ψ has the form

$$(2.4) \quad \Psi_{lin}(a) = (\langle \xi_1 r_i, a \rangle_{+})_{i=1,2,\dots,n^+},$$

where \langle, \rangle_+ denote the inner product of $L^2(\mathbb{R}_+^3)$.

In order to compare our result with the ones known so far, recall the linear operator L and put $N = \ker L$. It is classical that

$$(2.5) \quad N = \text{span}\{W_0(\xi)\phi_i(\xi)\}_{i=0,1,\dots,4}$$

where ϕ_i is as in (1.3) and thus N can be taken a 5 dimensional subspace of $L^2(\mathbb{R}^3)$. Let $P : L^2(\mathbb{R}^3) \rightarrow N$ be the orthogonal projection and define the linear operator

$$A = P\xi_1 P,$$

which is the 5-dimensional bounded self adjoint operator and has the eigenvalues

$$(2.6) \quad \lambda_1 = u_{\infty,1} - c_{\infty}, \quad \lambda_i = u_{\infty,1} \quad (i = 2, 3, 4), \quad \lambda_5 = u_{\infty,1} + c_{\infty},$$

on N . Define

$$I^+ = \{j | \lambda_j > 0\}, \quad I^- = \{j | \lambda_j < 0\}.$$

Note that n^+ of (1.5) is just $\#I^+$. Let χ_j be the eigenfunction corresponding to the eigenvalue λ_j . In [7], the following is proved.

Theorem 2.3 ([7]) *For any $a_0 \in L^2(\mathbb{R}_+^3, \xi_1 d\xi)$ and for any constants $c_j, j \in I^-$, there exists a unique L^2 solution f satisfying the first two equations of (2.3) and instead of the last one, the auxiliary condition*

$$\langle \chi_j, f(x, \cdot) \rangle = c_j, \quad x > 0, \quad j \in I^-.$$

Moreover, there exists an element $f_{\infty} \in N$ such that

$$f \rightarrow f_{\infty} \quad (x \rightarrow \infty) \quad \text{in } L^2(\mathbb{R}^3).$$

The proof in [7] does not tell us how to determine the limit f_{∞} . However, since (2.3) is linear and that $f_{\infty} \in N = \ker L$, we see that $\tilde{f} = f - f_{\infty}$ solves all of three equations in (2.3) with a_0 replaced by $a_0 - f_{\infty}$, and thereby, it follows from Theorem 2.1 that $a_0 - f_{\infty}$ should be in R^{\perp} .

3 Outline of the Proof

There are two ingredients in our proof. One is to add an artificial "damping" term and the other is to introduce the spatial weight function $e^{\sigma x}, \sigma > 0$.

To construct the damping term, decompose the operator A on N into the positive and negative parts A^+, A^- , and denote the corresponding eigenprojections by P^+, P^- . Note that if $M^{\infty} \neq 0, \pm 1$, then A has not zero eigenvalues (see (2.6)), so that

$$A = A^+ + A^-, \quad P = P^+ + P^-.$$

We modify (2.2) by adding to its right hand side the damping term defined by

$$-\gamma P^+ \xi_1 f, \quad \gamma > 0,$$

and then rewrite it by putting $f = e^{-\sigma x} g$, to deduce

$$(3.1) \quad \begin{cases} \xi_1 g_x - \sigma \xi_1 g - Lg = h - \gamma P^+ \xi_1 g, & x > 0, \xi \in \mathbb{R}^3, \\ g|_{x=0} = a_0(\xi), & \xi \in \mathbb{R}_+^3, \\ g \rightarrow 0 \quad (x \rightarrow \infty), & \xi \in \mathbb{R}^3, \end{cases}$$

with

$$(3.2) \quad h = e^{-\sigma x} \Gamma(g).$$

Note that for the case $\mathcal{M}^\infty < -1$, we have $n^+ = 0$ and $P^+ = 0$, and hence no damping term.

Take the inner product of (3.1) and g in $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and integrate by parts, to deduce

$$(3.3) \quad \langle |\xi_1| g^0, g^0 \rangle_- + (Bg, g) - (Lg, g) = \langle \xi_1 a_0, a_0 \rangle_+ + (h, g),$$

where $g^0 = g|_{x=0}$ while

$$B = -\sigma \xi_1 + \gamma P^+ \xi_1, \quad \sigma, \gamma > 0.$$

Seemingly, this has not a good sign but it *does* on N if $\gamma > \sigma > 0$ as seen from

$$PBP = -\sigma A + \gamma A^+ = -\sigma A^- + (\gamma - \sigma)A^+.$$

On the other hand, it is classical that L is negative definite on N^\perp , so that if h is assumed given and (3.1) is looked as a linear problem, then, (3.3) establishes a nice L^2 energy estimate of g .

The estimate thus obtained is enough to construct our solutions. First, the same estimate can be derived for the adjoint problem to (3.1), which then enable us, together with Riesz representation theorem, to conclude the existence of weak L^2 solution g to (3.1). Furthermore, taking suitable test functions, the "weak=strong" theorem can be established, and thus g is a unique strong solution and satisfies the above mentioned L^2 estimate. Finally, starting from this estimate and using the bootstrap argument, we can get the estimate of the L^∞ norm of g in terms of those of h and a_0 .

Now, the contraction argument applies to the nonlinear problem (3.1) with (3.2) and proves the existence of L^∞ solutions for sufficiently small a_0 .

In the case $\mathcal{M}^\infty < -1$, this gives the solutions to (2.2) and hence to the original problem (1.1). It remains to discuss the case $\mathcal{M}^\infty > -1$. Clearly, if

$$(3.4) \quad P^+ \xi_1 g = 0, \quad x > 0, \xi \in \mathbb{R}^3,$$

then g is also a solution of the original problem without the extra damping term. We can show that the condition (3.4) reduces to

$$(3.5) \quad P^+ \xi_1 g^0 = 0, \quad \xi \in \mathbb{R}^3,$$

where $g^0 = g|_{x=0}$. Clearly, g and hence g^0 are determined uniquely by the boundary data a_0 . Write

$$(3.6) \quad \Psi(a_0) = P^+ \xi_1 g^0.$$

Identifying the space P^+N with \mathbb{R}^{n^+} , we can show that (3.6) defines a C^1 map

$$(3.7) \quad \Psi : L^2(\mathbb{R}_+^3, \xi_1 d\xi) \rightarrow \mathbb{R}^{n^+},$$

and $\Psi(0) = 0$. Moreover, we can prove

Proposition 3.1 *The Fréchet derivative of Ψ at $a_0 = 0$ is given by (2.4).*

This and the implicit function theorem, then, prove that the set of a_0 's satisfying $\Psi(a_0) = 0$ forms a C^1 manifold of codimension n^+ , whence Theorem 1.1 follows.

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