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Local signature defect of fibered complex surfaces via monodromy and stable reduction

Tadashi Ashikaga

Introduction Let $\mathbf{f} : \mathbf{S} \to \mathbf{B}$ be a proper surjective holomorphic map from a compact complex surface $\mathbf{S}$ to a nonsingular curve $\mathbf{B}$ such that the general fiber of $\mathbf{f}$ is a curve of genus g. We call $\mathbf{f}$ a fibration of genus g. Let Sign $\mathbf{S}$ be the signature of the intersection form on $H^2(\mathbf{S}, \mathbf{Q})$. If there exist a finite number of fiber germs (f, F_i), $F_i = \mathbf{f}^{-1}(P_i)$ $(1 \leq i \leq s, P_i \in \mathbf{B})$ and a local invariant $\sigma(f, F_i)$ is geometrically well-defined such that

$$\text{Sign } \mathbf{S} = \sum_{i=1}^{s} \sigma(f, F_i),$$

then we say Sign $\mathbf{S}$ is localized and call $\sigma(f, F_i)$ a *local signature*.

Atiyah [At] had a deep insight which are, in some sense, the origins of several formulations of this concept. Matsumoto [Ma1], [Ma2] used the Meyer function and formulated the local signature of genus 1 and 2, and calculated them for the Lefschetz fiber germs. Endo [E] extended it for hyperelliptic fibrations of arbitrary genus. (See also [Mo].) Another local signature for hyperelliptic fibrations is defined by using the double covering method ([X], [AA1]), and these two notions are in fact coincide with each other ([Te2]).

On the other hand, Ueno [U] used the even theta constant and defined and calculated a local signature of genus 2. Iida [Ii] gave an analytic interpretation of the local signature of genus 2 by using the theta divisor and the adiabalic limit of the eta invariant. Kuno [Ku] extended the approach of the Meyer function to non-hyperelliptic fibrations of genus 3. For the survey articles, see [AK], [AE].

Now, through a series of papers, we present a certain type of a local signature including generic non-hyperelliptic and unstable fibrations of arbitrary genus. From our viewpoint, this notion should philosophically consist of two aspects of "moduli" and "monodromy" :

$$\textit{Local signature} \ = \ \text{“}\textit{Moduli}\text{” aspect} \ + \ \text{“}\textit{Monodromy}\text{” aspect}. \tag{1}$$

With respect to the moduli aspect, we refer the joint paper with K. Yoshikawa [AY] in detail. Here we only comment that, if $\mathbf{f} : \mathbf{S} \to \mathbf{B}$ is a stable fibration, then the moduli aspect of (1) purely contribute it. Namely, by developing the idea of I. Smith [Sm], we

can define an explicit "signature divisor" on the Deligne-Mumford compactification $\bar{M}_g$ and can localize Sign $\mathbf{S}$ via the pull back of it by the induced map $\mathbf{B} \to \bar{M}_g$.

The main topic of this paper is the analysis of the monodromy aspect of (1), which concerns with the direct relation between the local monodromy around the singular fibers and the local contribution of the stable reduction to the global signature. Namely, our purpose here is to write down it explicitly by the language of the Nielsen's invariants [N1] [N2] of the monodromy map.

Note that Tan [Tan] and Viehweg [V] had important studies on the contribution of the stable reduction to global invariants of $\mathbf{S}$ by certain algebro-geometric methods. On the other hand, our method is topological based on variations of the signature theorem historically started from [Hi1], and the setting is local in base.

We summarize the arguments here. Let $f : S \to \Delta$ be a degeneration of curves with the central fiber F over a closed disk Δ, and let h_S be a Riemannian metric on S such that h_S is a product metric near the boundary ∂S. We define

$$\sigma(f, F; h_S) = \text{Sign } S + \eta(\partial S, h_{\partial S}),$$

where Sign S is the signature on $H^2(S, \partial S; \mathbf{Q})$ and $\eta(\partial S, h_{\partial S})$ is the eta invariant ([APS]) with respect to the restricted metric $h_{\partial S}$ of h_S on the boundary ∂S. Let $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ be the minimal stable reduction of f with the central fiber $\widetilde{F}$ which comes from the cyclic base change $\widetilde{\Delta} \to \Delta$ of degree N. We also put $\sigma(\widetilde{f}, \widetilde{F}; h_{\partial \widetilde{S}}) = \text{Sign } \widetilde{S} + \eta(\partial \widetilde{S}, h_{\partial \widetilde{S}})$ where $h_{\partial \widetilde{S}}$ is a Riemannian metric on $\partial \widetilde{S}$ which coincides with the natural pull back of $h_{\partial \widetilde{S}}$. We define

$$\text{Lsd}(f, F; h_{\partial S}) := \sigma(f, F; h_{\partial S}) - \frac{1}{N}\sigma(\widetilde{f}, \widetilde{F}; h_{\partial \widetilde{S}}),$$

which we call the *local signature defect* of the degeneration f. This notion is analogous in some sense to the "defect part" which Hirzebruch explained in [Hi2].

Now our main theorem (Theorem 5.2.1) says that the local signature defect of f is explicitly written in terms of the Nielsen's monodromy data of f. Especially this is independent of the choice of the metric $h_{\partial S}$. Moreover the local signature defect is nothing but the local contribution of the difference of the global signatures for the global stable reduction (Lemma 5.3.6).

The idea of the proof of the main theorem is as follows: The cyclic group $G = \mathbf{Z}/(N)$ acts holomorphically on $\widetilde{S}$ so that the resolution space of the singularities of the quotient space $\widetilde{S}/G$ coincides with the original S, and the problem is reduced to the comparison between Sign $\widetilde{S}$ and Sign $\widetilde{S}/G$. Since the spaces $\widetilde{S}$ and $\widetilde{S}/G$ are also complex orbifolds, we can apply Kawasaki's orbifold signature theorem [Ka], which is a natural extension of Atiyah-Singer's G-signature theorem. The data of the integral of the equivariant L-forms

on the infinitesimal neighborhoods of the G-fixed point sets are essentially reduced to the Dedekind sum with respect to the monodromy data, which are calculated explicitly in §6.

For the preparation of this calculation, we propose the following two tools, which themselves seem to have independent meanings. The one is a new formula of Dedekind sum (Theorem 4.1.2), which we refer the joint papar with M. Ishizaka [AI2] in details. The other is a certain precise relation between the stable reduction and the Matsumoto-Montesinos' method in §§1∼3 as follows:

The study of the local monodromy has a long history, and it is settled that the local monodromy as an element in the conjugacy class of the mapping class group of genus g belongs to the class of pseudo-periodic map of negative twist. Conversely, for a given pseudo-periodic map of negative twist $\mu : \Sigma_g \to \Sigma_g$ of a Riemann surface Σ_g, Matsumoto-Montesinos ([MM1] Part I) constructed a certain topological quotient space $\Sigma_g \to \Sigma_g/\langle\mu\rangle$, which we call MM-quotient.

An algebro-geometric interpretation of $\Sigma_g/\langle\mu\rangle$ due to Takamura [Ta2] is as follows: We start from the stable reduction $\widetilde{S} \to \widetilde{\Delta}$ of f. The Galois group $G = \mathrm{Gal}(\widetilde{\Delta}/\Delta)$ acts holomorphically on $\widetilde{S}$, since it is 2-dimensional ([DM]). The G-action has natural local expressions at the nodes and the points whose isotropy groups are nontrivial as in §3.1 (cf. [Te1]). He constructed the local uniformization spaces of these actions and determined the types of singularities on $\widetilde{S}/G$. The normally minimal model of the resolution of $\widetilde{S}/G$ is nothing but the space $\Sigma_g/\langle\mu\rangle$. Therefore, to say generically, the stable reduction theorem induce the MM-quotient.

Although MM-quotient has rich informations than the usual stable reduction theorem. Indeed, we propose in §2 "a precise" stable reduction theorem based on MM-quotient as follows: Since the numerical Chorizo space of the singular fiber of f coincides with $\Sigma_g/\langle\mu\rangle$, all the irreducible components are classified into cores, tails, arcs and quasi-tails as in §1. We contract all the tails, arcs and quasi-tails to points, and put $S^\sharp$ the resulting surface. Then the natural fibration $\widetilde{S} \to \widetilde{\Delta}$ of the normalization $\widetilde{S}$ of $S^\sharp \times_\Delta \widetilde{\Delta}$ via the cyclic cover $\widetilde{\Delta} \to \Delta$ whose degree coincides with a pseudo-period of μ is nothing but the stable reduction of f (Theorem 2.2.1). This process is precise than the usual one, because the birational transformations in the process are explicit. Moreover, the description of the Milnor number of the A-type singularities on $\widetilde{S}$, the coincidence of $S^\sharp$ and $\widetilde{S}/G$, and the description of the singularities on $S^\sharp$ are all directly known.

By comparing the monodromy maps of $\widetilde{S} \to \widetilde{\Delta}$ and $\widetilde{S}/G \to \Delta = \widetilde{\Delta}/G$, we also directly obtain the coincidence of the data of the local action of G to $\widetilde{S}$ (which we call G-valency etc. in §3.1) and the Nielsen's monodromy data (Theorem 3.1.1).

In the latter half of §5, we present two examples of degeneration of curves of genus 2 and calculate their local signature defect. At the same time, we explain our method of

the stable reduction in §2 by using them.

1 Matsumoto-Montesinos quotient

In this section, we review the construction of the quotient space introduced by Matsumoto-Montesinos ([MM1] Part I) and related facts which will be used afterwards. This space is constructed by the topological quotient of a Riemann surface by an element of the subclass of the mapping class group which is called pseudo-periodic map of negative twist. Here our description is slightly modified from the original one [MM1] by using the method of Takamura [Ta2].

1.1 Let Σ_g be a Riemann surface of genus g and let $\mu : \Sigma_g \to \Sigma_g$ be a pseudo-periodic map of negative twist. By definition, μ is an orientation-preserving homeomorphism such that the following conditions are satisfied modulo isotopy: There exists a decomposition

$$\Sigma_g = \mathbf{A} \cup \mathbf{B} \tag{2}$$

into the annulus-part $\mathbf{A} = \coprod \mathcal{A}_j$ which is the disjoint union of the annular neighborhoods $\mathcal{A}_j$ of simple closed curves on Σ_g belonging to the admissible system of cut curves, and the body-part $\mathbf{B} = \coprod B_i$ which is the disjoint union of Riemann surfaces B_i with boundary, such that the boundary set $\partial\mathbf{B}$ coincides with the boundary set $\partial\mathbf{A}$. The restriction $\mu|_{\mathbf{B}}$ is periodic, i.e. the power $(\mu|_{\mathbf{B}})^N$ for some natural number N is the identity map $\mathrm{id}_{\mathbf{B}}$. The power the restriction $(\mu|_{\mathcal{A}_j})^N$ to each annulus $\mathcal{A}_j$ is a right-handed integral Dehn twist. We call such N *a pseudo-period* of μ. Note that N is a multiple of the minimal pseudo-period N_0 (i.e. the minimal natural number among all the pseudo-periods).

Now let $[\mu]$ be the equivalence class of μ of the conjugacy class $\widehat{\mathcal{M}}_g$ of the mapping class group of genus g. We review the conjugacy invariants of $[\mu]$:

Let $\vec{C}$ be an oriented simple closed curve on Σ_g. Suppose there is a natural number $m = m(\vec{C})$ such that $\mu^m(\vec{C}) = \vec{C}$ as a set, where m is assumed to be the minimal number which enjoys this property. Moreover suppose $(\mu|\vec{C})^m$ is periodic of order $\lambda = \lambda(\vec{C}) \geq 1$. Then for any point R on $\vec{C}$, there is a natural number $\sigma = \sigma(\vec{C})$ with $1 \leq \sigma \leq \lambda - 1$ such that the iteration of μ^m are situated in the order $(R,\ \mu^{m\sigma}(R),\ \mu^{2m\sigma}(R), \cdots, \mu^{(\lambda-1)m\sigma}(R))$ when viewed in the direction of $\vec{C}$. The set of triple $(m(\vec{C}), \lambda(\vec{C}), \sigma(\vec{C}))$ is called Nielen's valency at $\vec{C}$ ([N1]). Let $\delta = \delta(\vec{C})$ be the integer which satisfies $\sigma\delta \equiv 1 \pmod{\lambda}$ and $1 \leq \delta \leq \lambda - 1$. Then the map $\mu^m|_{\vec{C}}$ behaves as the rotation of angle $2\pi\delta/\lambda$ in a suitable parametrization of $\vec{C}$. Since the number δ is also important in our argument, we simply call the quardruplet $(m(\vec{C}), \lambda(\vec{C}), \sigma(\vec{C}), \delta(\vec{C}))$ *the valency* at $\vec{C}$.

Let Q be a point on the inside of a body component $B_i \backslash \partial B_i$. Let $m(B_i)$ be the minimal natural number such that $\mu^{m(B_i)}|_{B_i} = \mathrm{id}_{B_i}$. If there exists a natural number $m =$

$m(Q)$ which is strictly smaller than $m(B_i)$ such that the points $\{Q,\ \mu(Q), \cdots, \mu^{m-1}(Q)\}$ are distinct each other and $\mu^m(Q) = Q$, we call Q a *multiple point*. Then there exists a disk neighborhood D_Q of Q which is invariant under the action μ^m. The valency $(m(Q), \lambda(Q), \sigma(Q), \delta(Q))$ at Q is defined to be the valency of the boundary curve ∂D_Q of D_Q whose orientation is defined from the outside of the disk. (Note that the orientation of ∂D_Q is defined from the inside in [MM1].)

For an annulus component $\mathcal{A}_j$, we put $\partial\mathcal{A}_j = \partial\mathcal{A}_j^{(1)} \coprod \partial\mathcal{A}_j^{(2)}$ the decomposition to connected components of the boundary curve where the orientation is defined here from the outside of the annulus. The valency at $\mathcal{A}_j$ is defined to be the couple of valencies $(m(\partial\mathcal{A}_j^{(k)}), \lambda(\partial\mathcal{A}_j^{(k)}), \sigma(\partial\mathcal{A}_j^{(k)}), \delta(\partial\mathcal{A}_j^{(k)}))$ for $k = 1, 2$. If there exists a natural number β such that μ^β interchanges the boundary components of $\mathcal{A}_j$, i.e. $\mu^\beta(\partial\mathcal{A}_j^{(1)}) = \partial\mathcal{A}_j^{(2)}$, we call $\mathcal{A}_j$ an *amphidrome annulus*. Otherwise we call $\mathcal{A}_j$ a *non-amphidrome annulus*.

Let α be the smallest natural number such that $\mu^\alpha(\mathcal{A}_j) = \mathcal{A}_j$ does not interchange the boundary components. Let γ be a non-zero integer such that $\mu^\gamma|_{\partial\mathcal{A}_j}$ is the identity map. Then γ is a multiple of α, and $\mu^\gamma : \mathcal{A}_j \to \mathcal{A}_j$ is a result of e full Dehn twist, e being an integer. Then we define the *screw number* at $\mathcal{A}_j$ by $\mathbf{s}(\mathcal{A}_j) := e\alpha/\gamma$.

Then the theorem of Nielsen [N2] and Matsumoto-Montesinos [MM1] says that the conjugacy class of μ is determined by the data of
(i) valencies at multiple points $\{Q\}$ and the annuli $\{\mathcal{A}_j\}$,
(ii) screw numbers at the annuli $\{\mathcal{A}_j\}$,
(iii) the action of μ to the extended partition graph $\mathbf{\Gamma}(\mu)$, i.e. the one-dimensional oriented graph whose points correspond to $\{B_i\}$ and whose segments correspond to $\{\mathcal{A}_j\}$ in a natural way.

1.2 By a *numerical Chorizo space*, we mean a connected topological space consisting of the components which are underlining topological spaces of irreducible Riemann surfaces with nodes so that the multiplicities are attached to every components. Moreover, if two components of them intesect each other, they intersect at several points transversally.

For a given conjugacy class of pseudo-periodic map of negative twist $[\mu]$, Matsumoto-Montesinos [MM1] constructed the generalized quotient map $\pi_\mu : \Sigma_g \to \Sigma_g/\langle\mu\rangle$. Here $\Sigma_g/\langle\mu\rangle$ is a certain numerical Chorizo space, which we call *the Matsumoto-Montesinos quotient* of μ. Let $\Sigma_g/\langle\mu\rangle = \sum_i \alpha_i F_{\text{top}}^{(i)}$ be the decomposition to its components. By the construction, $F_{\text{top}}^{(i)}$ is classified into the following types (i) $\sim$ (iv):
(i) $F_{\text{top}}^{(i)}$ is a *core* component. Namely, $F_{\text{top}}^{(i)}$ is the unique component which contains the support of $\pi_\mu(B_i')$, where B_i' is the complement of disk neighborhoods of multiple points of μ on a certain body connected component B_i,
(ii) $F_{\text{top}}^{(i)}$ is a sphere component of a *tail* which comes from the quotient of a disk neighborhood of a multiple point,

(iii) $F_{\rm top}^{(i)}$ is a sphere component of an *arc* which comes from the quotient of a non-amphidrome annulus,
(iv) $F_{\rm top}^{(i)}$ is a sphere component of a *quasi-tail* which comes from the quotient of an amphidrome annulus.

We review the construction and the properties of the tails, the arcs and the quasi-tails in §§1.4, 1.5 and 1.6 respectively.

1.3 Before the construction, we prepare some terminology. Let (λ, σ) be any pair of integers with $1 \leq \sigma \leq \lambda - 1$ in this subsection. Let

$$\frac{\lambda}{\sigma} = K_1 - \frac{n_2}{\sigma} = K_1 - \cfrac{1}{K_2 - \cfrac{n_3}{n_2}} = K_1 - \cfrac{1}{K_2 - \cfrac{1}{K_3 - \cdots}} := [[K_1, K_2, \cdots, K_r]] \quad (3)$$

be the continued linear fraction. The sequence $\{n_i\}_{i=0}^{r}$ of natural numbers which satisfies $n_0 = \lambda, n_1 = \sigma$ and

$$n_i = K_{i-1} n_{i-1} - n_{i-2} \ (2 \leq i \leq r+1) \quad (4)$$

is called the *usual multiplicity sequence* of the continued linear fraction (3), or the multiplicity sequence with the initial values $n_0 = \lambda, n_1 = \sigma$. Note that $n_r = 1$ and $n_{r+1} = 0$.

We sometimes set the initial values n_0, n_1 another pair of natural numbers, and consider the multiplicity sequence of (3) with the same recursion formula (4). For instance, we consider $[[2, 2, 2, 3, 2]]$. Then the usual multiplicity sequence is $\{14, 11, 8, 5, 2, 1, 0\}$. On the other hand, the multiplicity sequence with the initial values $6, 5$ is $\{6, 5, 4, 3, 2, 3, 4\}$.

1.4 Let Q be a multiple point on $\mathbf{B}$ with the valency $(m, \lambda, \sigma, \delta)$. Let (3) be the continued linear fraction of λ/σ with respect to this valency data, and let $\{n_i\}_{i=0}^{r+1}$ be its usual multiplicity sequence.

Then we patch at a point of a disk $\mathcal{D}$ of multiplicity mn_0 to a tree of rational curves of the length r of multiplicities $mn_1, mn_2, \cdots, mn_r$ (see [MM2], p.72, Figure 1). This is the tail arizing from the multiple point Q.

1.5 Next we consider a non-amphidrome annulus $\mathcal{A} = \mathcal{A}_j$. Let $\partial\mathcal{A} = \partial\mathcal{A}^{(1)} \coprod \partial\mathcal{A}^{(2)}$ be the decomposition to the connected components of the boundary and let $(m^{(k)}, \lambda^{(k)}, \sigma^{(k)}, \delta^{(k)})$ be the valencies at $\partial\mathcal{A}^{(k)}$ $(k = 1, 2)$. Let $\mathbf{s}(\mathcal{A})$ be the screw number at $\mathcal{A}$. Then $\mathbf{s}(\mathcal{A})$ is a non-positve rational number, and is written by

$$\mathbf{s}(\mathcal{A}) = -\frac{\delta^{(1)}}{\lambda^{(1)}} - \frac{\delta^{(2)}}{\lambda^{(2)}} - \mathbf{K} \quad (5)$$

where $\mathbf{K}$ is an integer greater or equal to -1 ([MM1]). Let

$$\frac{\lambda^{(1)}}{\sigma^{(1)}} = [[K_1, K_2, \cdots, K_r]], \quad \frac{\lambda^{(2)}}{\sigma^{(2)}} = [[L_1, L_2, \cdots, L_{r'}]] \quad (6)$$

be the continued linear fractions and let $\{n_i\}_{i=0}^{r+1}$, $\{m_i\}_{i=0}^{r'+1}$ be their usual multiplicity sequences respectively.

Assume $\mathbf{K} \geq 0$. From (6), we define new continued fractions $\Phi := \Phi(\lambda^{(1)}/\sigma^{(1)}, \lambda^{(2)}/\sigma^{(2)}, \mathbf{K})$ as follows:

(i) If $\mathbf{K} \geq 1$, put $\Phi = [[K_1, K_2, \cdots, K_{r-1}, K_r + 1, \underbrace{2, \cdots, 2}_{\mathbf{K}-1}, L_{r'} + 1, L_{r'-1} \cdots, L_2, L_1]]$.

(ii) If $\mathbf{K} = 0$, $r \geq 1$ and $r' \geq 1$, put $\Phi = [[K_1, K_2, \cdots, K_{r-1}, K_r + L_{r'}, L_{r'-1} \cdots, L_2, L_1]]$.

(iii) If $\mathbf{K} = 0$, $r \geq 1$ and $r' = 0$, put $\Phi = [[K_1, K_2, \cdots, K_{r-1}]]$.

Note that Φ should be empty in the case $\mathbf{K} = 0, r = 1, r' = 0$.

LEMMA 1.5.1 *The multiplicity sequences with the initial values $\lambda^{(1)}, \sigma^{(1)}$ of the continued linear fraction Φ are as follows:*

(i) *If $\mathbf{K} \geq 1$, then $\{n_0, n_1, \cdots, n_{r-1}, n_r, \underbrace{1, \cdots, 1}_{\mathbf{K}-1}, m_{r'}, m_{r'-1}, \cdots, m_1, m_0\}$.*

(ii) *If $\mathbf{K} = 0$, $r \geq 1$ and $r' \geq 1$, then $\{n_0, n_1, \cdots, n_{r-1}, 1, m_{r'-1}, \cdots, m_1, m_0\}$.*

(iii) *If $\mathbf{K} = 0$, $r \geq 1$ and $r' = 0$, then $\{n_0, n_1, \cdots, n_{r-1}, n_r\}$.*

PROOF Easy.

Assume $\mathbf{K} = -1$. We use the following;

LEMMA 1.5.2 ([MM1] §6, [Ta2] §6.2) *Assume $\delta^{(1)}/\lambda^{(1)} + \delta^{(2)}/\lambda^{(2)} \geq 1$. Then there exists an unique pair (ω, ω') of integers with $0 \leq \omega \leq r$, $0 \leq \omega' \leq r'$ and $(\omega, \omega') \neq (0, 0)$ which satisfies*

$$n_\omega = m_{\omega'}, \quad n_{\omega+1} + m_{\omega'+1} = n_\omega.$$

We defne continued linear fractions $\Phi[-1] := \Phi(\lambda^{(1)}/\sigma^{(1)}, \lambda^{(2)}/\sigma^{(2)}, -1)$ as follows:

(i) If $\omega \geq 1, \omega' \geq 1$, put $\Phi[-1] = [[K_1, K_2, \cdots, K_{\omega-1}, K_\omega + L_{\omega'} - 1, L_{\omega'-1}, \cdots, L_2, L_1]]$.

(ii) If $\omega' = 0$, put $\Phi[-1] = [[K_1, K_2, \cdots, K_{\omega-1}]]$.

LEMMA 1.5.3 *The multiplicity sequences with the initial values $\lambda^{(1)}, \sigma^{(1)}$ of $\Phi[-1]$ are as follows;*

(i) *If $\omega \geq 1, \omega' \geq 1$, then $\{n_0, n_1, \cdots, n_{\omega-1}, n_\omega, m_{\omega'-1}, \cdots, m_1, m_0\}$.*

(ii) *If $\omega' = 0$, then $\{n_0, n_1, \cdots, n_{\omega-1}, n_\omega\}$.*

PROOF Easy.

Now we go back to the construction of the arc $\pi_\mu(\mathcal{A})$. This is related to the multiplicity sequences in Lemmata 1.5.1 and 1.5.3. Namely, this is constructed by combining two disks $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ with multiplicities mn_0 and mm_0 respectively by the following tree of spheres (see [MM2] p.73 Figure 2);

(i) If $\mathbf{K} \geq 1$, then the tree has length $r + r' + \mathbf{K} - 1$ such that the multiplicities of the components are $mn_1, mn_2, \cdots, mn_r, \underbrace{m, \cdots, m}_{\mathbf{K}-1}, mm_{r'}, \cdots, mm_2, mm_1$.

(ii) If $\mathbf{K} = 0$ and $r + r' \geq 2$, then the tree has length $r + r' - 1$ such that the multiplicities of the components are $mn_1, mn_2, \cdots, mn_{r-1}, m, mm_{r'-1}, \cdots, mm_2, mm_1$. If $\mathbf{K} = 0$ and $r + r' = 1$, then the tree is empty and we combine $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ directly at a point transversally.

(iii) If $\mathbf{K} = -1$, $\omega \geq 1$ and $\omega' \geq 1$, then the tree has length $\omega + \omega' - 1$ such that the multiplicities of the components are $mn_1, mn_2, \cdots, mn_{\omega-1}, mn_\omega, mm_{\omega'-1}, \cdots, mm_2, mm_1$. If $\mathbf{K} = -1$ and $\omega' = 0$, then the tree has length $\omega - 1$ such that the multiplicities of the components are $mn_1, mn_2, \cdots, mn_{\omega-1}$.

The above arc $\pi_\mu(\mathcal{A})$ is globally patched with the both sides of core components in a natural way.

Now the following result due to Takamura [Ta2] is useful in our discussion. Note that Tomaru [To] also proved it for the case $\mathbf{K} \geq 0$ in another motivation from ours.

PROPOSITION 1.5.4 ([Ta2], [To]) *Let $\widehat{\sigma}^{(i)}$ be the natural number which satisfies $\sigma^{(i)}\delta^{(i)} = \widehat{\sigma}^{(i)}\lambda^{(i)} + 1$ $(i = 1, 2)$, and put*

$$d = -\lambda^{(1)}\lambda^{(2)}\mathbf{s}(\mathcal{A}), \quad v = \sigma^{(1)}\delta^{(2)} + \lambda^{(2)}\widehat{\sigma}^{(1)} + \sigma^{(1)}\lambda^{(2)}\mathbf{K}.$$

Then the value of the continued linear fraction of each of Φ or $\Phi[-1]$ coincides with d/v.

1.6 Lastly we assume that the annulus $\mathcal{A}$ is amphidrome. The quasi-tail $\pi_\mu(\mathcal{A})$ is constructed as follows: Let $(2m', \lambda, \sigma, \delta)$ be the valency at both sides of boundary curves of $\mathcal{A}$, and let $\{n_i\}_{i=0}^r$ be the multiplicity sequence of the continued linear fraction of λ/σ as in (3). Then $\pi_\mu(\mathcal{A})$ is constructed by combining a disk of multiplicity $2m'm_0$ with the set of rational curves at a point transvesally. This set has the dual graph of Dynkin diagram of type D, and the multiplicities of components are $2m'n_1, 2m'n_2, \cdots, 2m'n_r, m', m'$. (the last m', m' are the multiplicities of the two tail components. See [MM2] p.73 Figure 3). The space $\pi_\mu(\mathcal{A})$ is also globally patched to a core compontent in a natural way.

2 Stable reduction in biregular sense

By a stable reduction $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ of a degeneration $f : S \to \Delta$ of curves, we mean that $\widetilde{S}$ is birationally equivalent to the fiber product $S \times_\Delta \widetilde{\Delta}$ where $\widetilde{\Delta} \to \Delta$ is a cyclic cover which is totally ramified at the origin, so that the cenrtal fiber $\widetilde{f}^{-1}(0)$ is a stable curve. The total space $\widetilde{S}$ may have singularities of type A at the nodes of the central fiber. Note

that the stable reduction is not unique for given f. Several proofs are known for the existence of the stable reduction in this simple situation ([DM], [AW] and [BPV] etc.).

Now in this section, we present another process of the stable reduction in this situation which is different from the method of above ones. Our process seems to be precise, bacause we can "catch" the stable reduction explicitly in biregular sense.

2.1 Let $f : S \to \Delta$ be a proper surjective holomorphic map from a compact complex surface S to an unit disk $\Delta = \{t \in \mathbf{C} \mid |t| < 1\}$ such that the general fiber $f^{-1}(t_0)$ $(t_0 \neq 0)$ is a nonsingular curve of genus $g \geq 1$. Let $F = f^{-1}(0)$ be the singular fiber, and let $F = \sum_{i=1}^{\gamma} \alpha_i F^{(i)}$ be its irreducible decomposition. Assume f is normally minimal, i.e. the reduced scheme $F^{\mathrm{red}} = \sum_{i=1}^{\gamma} F^{(i)}$ is normal crossing and any (-1) component in F_{red} has at least three nodes of F^{red} on it. We call f a normally minimal degeneration of curves of genus g.

Let $\mu_f : \Sigma_g \to \Sigma_g$ be a monodromy map of f, where Σ_g is considered to be a fixed general fiber of f. The conjugacy class $[\mu_f] \in \widehat{\mathcal{M}}_g$ in the maping class group is uniquely determined by f, which we call *the topological monodromy* of f. It is well-known that $[\mu_f]$ belong to the class of a pseudo-periodic map of negative twist ([Im] [MM1] etc.).

Let $F_{\mathrm{ch}} = \sum_{i=1}^{\gamma} \alpha_i F_{\mathrm{top}}^{(i)}$ be the numerical Chorizo space naturally obtained by the singular fiber $F = \sum_{i=1}^{\gamma} \alpha_i F^{(i)}$. Namely, if we only forget the analytic structure of each $F^{(i)}$, but do not forget the underlining topological structure and the multiplicity α_i and also its configuration, then we obtain F_{ch}.

The fundamental theorem of [MM1] says that F_{ch} coincides with the Matsumoto-Montesinos quotient $\Sigma_g/\langle \mu_f \rangle$ of the monodromy map μ_f. We call $F^{(i)}$ a core (resp. tail, arc, quasi-tail) component of F iff the corresponding $F_{\mathrm{top}}^{(i)}$ is a core (resp. tail, arc, quasi-tail) component in the sense of §1. By changing the order if necessary, we may assume that $\{F^{(i)}\}_{i=1}^{\gamma'}$ (for some $1 \leq \gamma' \leq \gamma$) are core components and $\{F^{(i)}\}_{i=\gamma'+1}^{\gamma}$ are non-core (i.e. tail, arc and quai-tail) components.

Since $\{F^{(i)}\}_{i=\gamma'+1}^{\gamma}$ is a proper subset (or empty set) of $F^{\mathrm{red}} = \sum_{i=1}^{\gamma} F^{(i)}$, the intersection matrix of $\{F^{(i)}\}_{i=\gamma'+1}^{\gamma}$ is negative definite. Therefore we have the bimeromorphic holomorphic map $\rho : S \to S^{\sharp}$ which contract $\{F^{(i)}\}_{i=\gamma'+1}^{\gamma}$ to points by Grauert's theorem [Gr]. Let $f^{\sharp} : S^{\sharp} \to \Delta$ be the natural holomorphic map which satisfies $f^{\sharp} \circ \rho = f$. Putting $F_i^{\sharp} = \rho(F^{(i)})$ for $1 \leq i \leq \gamma'$, the fiber $F^{\sharp} := (f^{\sharp})^{-1}(0)$ is written by $F^{\sharp} = \sum_{i=1}^{\gamma'} \alpha_i F_i^{\sharp}$.

Now the two-dimensional analytic space $S^{\sharp}$ has at most isolated singularities P so that the support of P is contained in $(F^{\sharp})^{\mathrm{red}} = \sum_{i=1}^{\gamma'} F_i^{\sharp}$. Moreover one of the following is satisfied:

(i) $(F^{\sharp})^{\mathrm{red}}$ is smooth at P, and P is the contraction image by ρ of a tail.
(ii) $(F^{\sharp})^{\mathrm{red}}$ is singular at P, and P is the contraction image by ρ of an arc.
(iii) $(F^{\sharp})^{\mathrm{red}}$ is singular at P, and P is the contraction image by ρ of a quasi-tail.

We detemine the type of singularity at P of $S^\sharp$ in the sense of Brieskorn [Br] for each case. First we consider (i). Let $(m, \lambda, \sigma, \delta)$ be the valency of the multiple point corresponds to the tail, and go back to the situation in §1.4. The point P is the contraction image of the tree $\sum_{i=1}^{r} mn_iE_i$ of rational curves. By identifying this tree with the divisor on S which a part of components of F, we have

$$0 = FE_i = m(n_{i-1} + n_iE_i^2 + n_{i+1})$$

for $1 \leq i \leq r$ (where n_{r+1} is assumed to be 0). It follows that $E_i^2 = (-n_{i-1} - n_{i+1})/n_i = -K_i$. Namely, this is the Hirzebruch-Jung string whose components have the self-intersection numbers $-K_1, \cdots, -K_r$. Therefore by the well-known argument, the germ $(S^\sharp, P)$ is a cyclic quotient singularity of type $C_{\lambda,\delta}$, i.e. the germ at the origin of the qotient $\mathbf{C}^2/(\mathbf{Z}/\lambda\mathbf{Z})$ by the action $(z_1, z_2) \to (\mathbf{e}(1/\lambda)z_1, \mathbf{e}(\delta/\lambda)z_2)$.

Second we consider (ii). We go back to §1.5, and consider the non-amphidrome annulus corresponds to the arc. The point P is the contraction image of the Hirzebruch-Jung string whose self-intersection numbers of the components coincides with the self-intersection sequence of the continued linear fraction $\Phi(\lambda^{(1)}/\sigma^{(1)}, \lambda^{(2)}/\sigma^{(2)}, \mathbf{K})$. Therefore it follows from the tautness of this type of singularity and Proposition 1.5.4 that $(S^\sharp, P)$ is a cyclic quotient singularity of type $C_{d,v}$.

We consider (iii). The point P is the contraction image of the tree of rational curves whose dual graph has Dynkin diagram of type D as in §1.6. By the argument in [Br], $(S^\sharp, P)$ is a dihedral quotient singularity of type $\langle b; 2, 1; 2, 1; \delta + \lambda(\mathbf{K} + 1), \delta + \lambda\mathbf{K}\rangle$ in p. 347 of [Br]. For more simplified notation as in [R], we call it of type $D_{\xi+\lambda,\xi}$ by putting $\xi = \delta + \lambda\mathbf{K}$. Summarizing the above argument;

LEMMA 2.1.2 *In the case* (i), $(S^\sharp, P)$ *is a cyclic quotient singularity of type* $C_{\lambda,\delta}$.

In the case (ii), $(S^\sharp, P)$ *is a cyclic quotient singularity of type* $C_{d,v}$, *where* d, v *are given in Proposition 1.5.4*

In the case (iii), $(S^\sharp, P)$ *is a dihedral quotient singularity of type* $D_{\xi+\lambda,\xi}$.

2.2 In general, let M be an analytic space and $M \to \Delta$ be a fibration of curves. Let a be a positive integer, and $h^{(a)} : \Delta^{(a)} \to \Delta$ be the covering between small disks defined by $z \to z^a$. Let $M^{(a)}$ be the normalization of the analytic space $M \times_\Delta \Delta^{(a)}$. We call the natural morphism $M^{(a)} \to \Delta^{(a)}$ *the pure a-th root fibration*[1] of $M \to \Delta$. We set $\widetilde{h}^{(a)} : M^{(a)} \to M$ the natural morphism. The following is a slightly precise version of the classical stable reduction theorem:

THEOREM 2.2.1 (i) *Let* N *be a pseudo-period of* μ_f. *Then;*

[1] This notion is slightly different from the root fibration in [BPV] p.92, because the modification after normalization is unused.

(ia) *The pure N-th root fibration $f^{(N)} : S^{(N)} \to \Delta^{(N)}$ of $f^\sharp : S^\sharp \to \Delta$ is a stable reduction of f. Moreover, $f^{(N)}$ is the unique stable reduction of f among the birational equivalence class of $S \times_\Delta \Delta^{(N)}$.*

(ib) *The cyclic group $G = \mathbf{Z}/N\mathbf{Z}$ acts holomorphically on $S^{(N)}$ so that the quotient space $S^{(N)}/G$ coincides with $S^\sharp$.*

(ic) *For a node $\widetilde{P}$ on the singular fiber $\widetilde{F} = (f^{(N)})^{-1}(0)$, the germ of the point $P = \widetilde{h}^{(N)}(\widetilde{P})$ on $S^\sharp$ is one of (ii) and (iii) in Lemma 2.1.2. Then the germ $(\widetilde{S}, \widetilde{P})$ is a rational double point of type A_{n-1} ($n \geq 1$) such that the Milnor number n is given by*

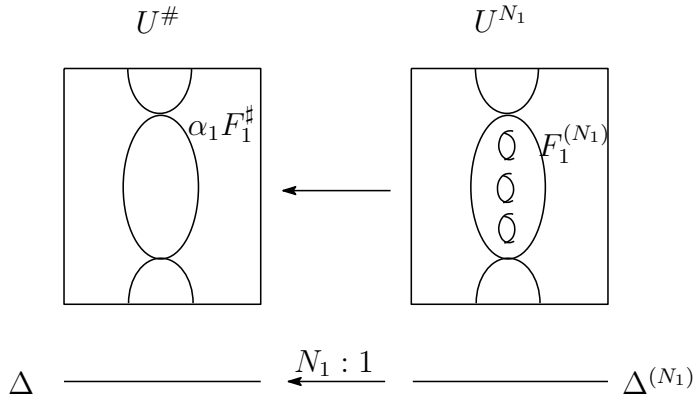
$$n = -\frac{N\mathbf{s}(\mathcal{A})}{m(\mathcal{A})}$$

where $\mathcal{A}$ is the annulus whose center curve is the vanishing cycle corresponding to P.

(ii) *Conversely, any stable reduction of f coincides with the pure N-th root fibration $f^{(N)}$ of $f^\sharp$ for some pseudo-period N of μ_f. Especially $f^{(N_0)}$ is the minimal stable reduction in the sense that the covering degree N_0 of the base change is minimal among all of them.*

PROOF (*Step 1*) We first fix a psudo-period N of μ_f. Let B be a connected component of $\mathbf{B}$ in (2). Let $\{Q_i\}_{i=1}^s$ be the set of multiple points on B, and $\{\partial B^{(i)}_{s+1 \leq i \leq s+s'}\}$ be the set of connected components of the boundary ∂B. Let $\{(m_i, \lambda_i, \sigma_i, \delta_i)\}_{1 \leq i \leq s+s'}$ be the valency at each multiple point or boundary component. Let N_1 be the minimal period of μ at B, i.e. N_1 is the minimal number such that $(\mu_B)^{N_1}$ is isotopic to the identity id_B. Then N_1 is a divisor of N and we have $N_1 = m_i\lambda_i$ for $1 \leq i \leq s + s'$.

There exists a unique component of $F^\sharp$ containing $\pi_\mu(B)$ as a set, which we may assume to be $F_1^\sharp$. Let P_i ($1 \leq i \leq s + s'$) be the point on $F_1^\sharp$ which is the image by the contraction map ρ of the tail or the arc or the quasi-tail corresponds to Q_i or $\partial B^{(i)}$ respectively. The curve $F_1^\sharp$ itself smooth or irreducible with nodes.



(Figure 1)

Now we choose an open set $U^\sharp$ of $S^\sharp$ containing the divisor $\alpha_1 F_1^\sharp$ so that the complement of $\alpha_1 F_1^\sharp$ in $U^\sharp \cap F^\sharp$ is empty or consists of small punctured disks with some multiplicities, and satisfies $f^\sharp(U^\sharp) = \Delta$ (see Figure 1).

First we consider the pure N_1-th root fibration $f_{\mathrm{loc}}^{(N_1)} : U^{(N_1)} \to \Delta^{(N_1)}$ of $f_{U^\sharp}^\sharp : U^\sharp \to \Delta$. The unique closed component $F_1^{(N_1)}$ of the central fiber $F^{(N_1)} = (f_{\mathrm{loc}}^{(N_1)})^{-1}(0)$ itself smooth or irreducible with nodes. Let $\hat{\tau}_1 : \bar{F}_1^{(N_1)} \to F_1^{(N_1)}$ be the normalization, and let $\hat{h} : \bar{F}_1^{(N_1)} \to \bar{F}_1^\sharp$ be the lift of the restriction map $\widetilde{h}_1 := \widetilde{h}^{(N_1)}|_{F_1^{(N_1)}} : F_1^{(N_1)} \to F_1^\sharp$ to the normalization. Then $\hat{h}$ is an N_1-fold cyclic covering whose branch points coincides with $\{\tau_1^{-1}(P_i)\}_{1 \le i \le s+s'}$. Moreover the total valency data of the periodic automorphism of $\bar{F}_1^{(N_1)}$ induced by this covering transformation coincides with $\{(m_i, \lambda_i, \sigma_i, \delta_i)\}_{1 \le i \le s+s'}$. (cf. [AI1] §1.) Since the natural morphism $\widetilde{h}_{\mathrm{loc}}^{(N_1)} : U^{(N_1)} \to U^\sharp$ is unramified covering over $U^\sharp \setminus \coprod_{1 \le i \le s+s'} P_i$, the singularities of $U^{(N_1)}$ are at most on $(\widetilde{h}^{(N_1)})^{-1}(P_i)$ $(1 \le i \le s+s')$.

But, over $(\widetilde{h}_{\mathrm{loc}}^{(N_1)})^{-1}(P_i)$ for $1 \le i \le s$, i.e. over the tail-contacting point, we claim that $U^{(N_1)}$ is non-singular.

Indeed, since the germ $(U^\sharp, P_i)$ is a cyclic quotient singularity of type C_{λ_i, δ_i}, the local fundamental group $\pi_{U^\sharp, P_i}$ is isomorphic to $\mathbf{Z}/\lambda_i \mathbf{Z}$ ([Br]). By looking at the local monodromy, this group is trivialized by the λ_i-fold cyclic cover whose Galois group is the stabilizer of each ramification point over $(\widetilde{h}^{(N_1)})^{-1}(P_i)$. Since the germ of the point with trivial local fundamental group is nothing but the germ of a smooth point ([Mu]), $(\widetilde{h}^{(N_1)})^{-1}(P_i)$ consists of m_i smooth points.

The total space $U^{(N)}$ of the pure N-th root fibration $f_{\mathrm{loc}}^{(N)} : U^{(N)} \to \Delta^{(N)}$ of $f_{U^\sharp}^\sharp$ is also smooth on the pull back of P_i $(1 \le i \le s)$, because the natural morphism $U^{(N)} \to U^{(N_1)}$ clealy lifts these smooth points to smooth points of $U^{(N)}$.

(*Step 2*) Next we consider the case $s+1 \le i \le s+s'$. Assume $P := P_i$ is an arc-contracting point. We choose an open set $U_P \subset U^\sharp$ of P so that the fiber $(f^\sharp|_{U_P})^{-1}(0)$ consisits of two multi-disks $nD^{(1)} + n'D^{(2)}$ $(D^{(1)} \cap D^{(2)} = P)$ and satisfies $f^\sharp(U_P) = \Delta$. We rewrite the valencies at the boundaries $\partial \mathcal{A}_i$ of the non-amphidrome annulus $\mathcal{A}_i$ by $(m, \lambda^{(k)}, \sigma^{(k)}, \delta^{(j)})$ $(k = 1, 2)$. Note that m is a divisor of both of n and n', and the general fiber of $f^\sharp|_{U_P} : U_P \to \Delta$ consists of m disjoint annuli.

First we consider the pure m-th root fibration $f_{\mathrm{loc}}^{(m)} : U^{(m)} \to \Delta^{(m)}$ of $f^\sharp|_{U_P}$. The central fiber of $f_{\mathrm{loc}}^{(m)}$ consists of two multi-disks with the multiplicities n/m and n'/m, and the general fiber consists of an annulus. Moreover the monodromy of $f_{\mathrm{loc}}^{(m)}$ is isotopic to the linear twist with the screw number $\mathbf{s}(\mathcal{A}_i)$ ([MM1] §8), i.e. there exists a parametrization $\mathcal{A}_i \simeq [0,1] \times \mathbf{R}/\mathbf{Z} \ni (t, u)$ such that the monodromy action of $f_{\mathrm{loc}}^{(m)}$ is written by $(t, u) \mapsto (t, u - \delta^{(1)}/\lambda^{(1)} + \mathbf{s}(\mathcal{A}_i)t)$.

Next we consider the pure N-th root fibraion $f_{\mathrm{loc}}^{(N)} : U^{(N)} \to \Delta^{(N)}$ of $f^\sharp|_{U_P}$. Since N/m is a multiple of both of n/m and n'/m, the central fiber $(f_{\mathrm{loc}}^{(N)})^{-1}(0)$ consists of two

(reduced) disks $\widetilde{D_1}+\widetilde{D_2}$ so that $\widetilde{D_1}$ and $\widetilde{D_2}$ meets transversally at a point $\widetilde{P}$. The general fiber $\widetilde{\mathcal{A}} := (f_{\mathrm{loc}}^{(N)})^{-1}(t_0)$ $(t_0 \neq 0)$ is an annulus, and the monodromy map $\mu_{f_{\mathrm{loc}}^{(N)}} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ is isotopic to the linear twist with the screw number $(N/m)\,\mathbf{s}(\mathcal{A}_i) = (N/m)\,(-(\delta^{(1)}/\lambda^{(1)}) - (\delta^{(2)}/\lambda^{(2)}) - \mathbf{K})$.

Since N/m is a multiple of $\mathrm{lcm}(\lambda^{(1)}, \lambda^{(2)})$, this map is an integral Dehn twist of $-(N/m)\mathbf{s}(\mathcal{A}_i)$ times to right hand direction. Therefore the numerical Chorizo space $\widetilde{\mathcal{A}}/\langle \mu_{f_{\mathrm{loc}}^{(N)}} \rangle$ consists of two disks and an $\mathbf{P}^1$-tree of length $-(N/m)\mathbf{s}(\mathcal{A}_i)-1$ whose multiplicities of all the components are one, i.e. the tree of (-2) curves of length $-(N/m)\mathbf{s}(\mathcal{A}_i)-1$. Since $(f_{\mathrm{loc}}^{(N)})^{-1}(0)$ consists of two disks, all the components of this tree should be contracted to a point. Namely, the point $\widetilde{P} \in U^{(N)} \subset \widetilde{S}$ is nothing but the contraction image of this tree, and therefore the germ $(\widetilde{S}, \widetilde{P})$ is a rational double point of type $A_{-(N/m)\mathbf{s}(\mathcal{A}_i)-1}$.

If P is a quasi-tail contracting point, i.e. $\mathcal{A}_i$ is an amphidrome annulus, the similar argument also works. (We omit it.)

Hence $f^{(N)} : S^{(N)} \to \Delta^{(N)}$ is a stable family and the assertion (ic) is verified.

If a core component of the central fiber of $(f^{(N)})$ is a (-2) curve, it has at least three nodes of the fiber. This fact is a direct consequence of Harvey's theorem [Ha]. Therefore the uniqueness of the stable family with the fixed covering degree is also clear, which induce the assertion (ia).

(*Step 3*) Let N' be any positive integer which is not a multiple of N_0, and we consider the pure N'-th root fibration $f^{(N')} : S^{(N')} \to \Delta^{(N')}$ of $f^\sharp$. The monodromy map $\mu_{f^{(N')}} : \Sigma_g \to \Sigma_g$ is isotopic to the power $(\mu_f)^{N'}$, and the decomposition to the annulus part and the body part of $\mu_{f^{(N')}}$ coincides with (2) of μ_f.

Now the assumption of N' implies that there exists a connected component B of the body part such that $\mu_{f^{(N')}}|_B$ is not isotopic to id_B. Namely the period of $\mu_{f^{(N')}}|_B$ is greater than one. Therefore the numerical Chorizo space $\Sigma_g/\langle \mu_{f^{(N')}} \rangle$ contains at least one core component with the multiplicity greater than one. If this core component is a (-1) curve in the normally minimal fiber, then it intersects at least three tail components by [Ha]. This fiber germ cannot be transposed birationally to a stable fiber germ. Hence the assertions (ii) is clear.

The remaining assertion (ib) is also clear, because the natural action of $\mathbf{Z}/N\mathbf{Z}$ to the fiber product $S^\sharp \times_\Delta \Delta^{(N)}$ lifts its normalization as a holomorphic Galois action by an easy argument. Q.E.D.

3 Group action and Monodromy

The notations are as same as §2. We fix a pseudo-period N of μ_f, and a generator g_0 of $G = \mathbf{Z}/N\mathbf{Z}$ through this section. We consider the stable reduction $\widetilde{f} = f^{(N)} : \widetilde{S} =$

$S^{(N)} \longrightarrow \widetilde{\Delta} = \Delta^{(N)}$ of $f : S \to \Delta$. In this section, we describe the direct relation of the action of G to $\widetilde{S}$ and the monodromy data of f. We also review Takamura's argument [Ta2] for later use.

3.1 (i) We call a smooth point P on $\widetilde{F} = \widetilde{f}^{-1}(0)$ *an inner multiple point* of $\widetilde{F}$ iff the stabilizer $\mathrm{Stab}_G(P)$ of P in G is strictly larger than the stabilizer $\mathrm{Stab}_G(\widetilde{F}^{(i)})$ of the irreducible component $\widetilde{F}^{(i)}$ of $\widetilde{F}$ which contains P.

For an inner multiple point P, there exists a positive integer $\widetilde{m}$ such that the local neighborhoods in $\widetilde{S}$ of the $\widetilde{m}$ mutually distinct points $P, g_0(P), \cdots, g_0^{\widetilde{m}-1}(P)$ permute cyclically and isomorphically each other by the action of g_0, and $g_0^{\widetilde{m}}$ stabilize the local neighborhood U_P of P as a set. We choose a local coordinate (x,t) on U_P so that t is a lift of a parameter on $\widetilde{\Delta}$ and x is a local parameter of $\widetilde{F}^{(i)}$ which satisfy $P = \{(x,t) = (0,0)\}$. Since the action of $g_0^{\widetilde{m}}$ is locally linearizable and of finite order, there exist relatively prime natural numbers $(\widetilde{\lambda}, \widetilde{\delta})$ with $1 \leq \widetilde{\delta} \leq \widetilde{\lambda} - 1$ so that the action of $g_0^{\widetilde{m}}$ on U_P is written by

$$(x,t) \longmapsto \left(\mathbf{e}\left(\frac{\widetilde{\delta}}{\widetilde{\lambda}}\right) x,\ \mathbf{e}\left(\frac{1}{\widetilde{\lambda}\ell}\right) t \right) \tag{7}$$

where $\ell = N/\widetilde{\lambda}$ is an integer, and $\mathbf{e}(x) = \exp(2\pi i x)$. We also put the integer $\widetilde{\sigma}$ which satisfies $\widetilde{\sigma}\widetilde{\delta} \equiv 1 \pmod{\widetilde{\lambda}}$ and $1 \leq \widetilde{\sigma} \leq \widetilde{\lambda} - 1$, and call $(\widetilde{m}, \widetilde{\lambda}, \widetilde{\sigma}, \widetilde{\delta})$ the *G-valency* at P.

(ii) Next assume P is a node of $\widetilde{F}$. There exist disks $\mathcal{D}^{(1)}, \mathcal{D}^{(2)}$ of both sides of local irreducible components of $\widetilde{F}$ at P such that $P = \mathcal{D}^{(1)} \cap \mathcal{D}^{(2)}$. We choose a suitable local coordinate neighborhood U_P on $\widetilde{S}$ at P as

$$U_P = \{(x,y,t) \in \mathbf{C}^3 \mid xy = t^n,\ |x| \leq \epsilon,\ |y| \leq \epsilon\} \tag{8}$$

where $\widetilde{F} = \{t = 0\}$, $\mathcal{D}^{(1)} = \{x = t = 0\}$, $\mathcal{D}^{(2)} = \{y = t = 0\}$ and n is a positive integer. There exists a positive integer $\widetilde{m}$ such that the local neighborhoods in $\widetilde{S}$ of the points $P, g_0(P), \cdots, g_0^{\widetilde{m}-1}(P)$ permute isomorphically each other by the action of g_0 and $g_0^{\widetilde{m}}$ stabilize U_P as a set without changing the boundaries, i.e. $g_0^{\widetilde{m}}(\mathcal{D}^{(j)}) = \mathcal{D}^{(j)}$ for $j = 1, 2$. The integer $\widetilde{m}$ is assumed to be the smallest among the positive integers which enjoys the above property.

Moreover, if $\widetilde{m}$ is even and $g_0^{\widetilde{m}/2}$ stabilize U_P exchanging $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$, we call P *an amphidrome node*. Otherwise, we call P *a non-amphidrome node*.

(iia) Assume P is a non-amphidrome node. Then, for $j = 1, 2$, there exist relatively prime natural numbers $(\widetilde{\lambda}^{(j)}, \widetilde{\delta}^{(j)})$ with $1 \leq \widetilde{\delta}^{(j)} \leq \widetilde{\lambda}^{(j)} - 1$ so that the action of $g_0^{\widetilde{m}}$ on U_P is written by

$$(x,y,t) \longmapsto \left(\mathbf{e}\left(\frac{\widetilde{\delta}^{(1)}}{\widetilde{\lambda}^{(1)}}\right) x,\ \mathbf{e}\left(\frac{\widetilde{\delta}^{(2)}}{\widetilde{\lambda}^{(2)}}\right) y,\ \mathbf{e}\left(\frac{1}{\mathrm{lcm}(\widetilde{\lambda}^{(1)}, \widetilde{\lambda}^{(2)}) \cdot \ell}\right) t \right) \tag{9}$$

where $\ell = N/\{\mathrm{lcm}(\widetilde{\lambda}^{(1)}, \widetilde{\lambda}^{(2)}) \cdot \widetilde{m}\}$ is an integer. Since the action (9) is compatible with the local equation in (8), there exists an integer $\widetilde{\mathbf{K}} \geq -1$ such that

$$n = \frac{\ell}{\gcd(\widetilde{\lambda}^{(1)}, \widetilde{\lambda}^{(2)})} \left(\widetilde{\delta}^{(1)} \widetilde{\lambda}^{(2)} + \widetilde{\delta}^{(2)} \widetilde{\lambda}^{(1)} + \widetilde{\mathbf{K}} \widetilde{\lambda}^{(1)} \widetilde{\lambda}^{(2)} \right). \tag{10}$$

We also put the integer $\widetilde{\sigma}^{(j)}$ which satisfies $\widetilde{\sigma}^{(j)} \widetilde{\delta}^{(j)} \equiv 1 \pmod{\widetilde{\lambda}^{(j)}}$ and $1 \leq \widetilde{\sigma}^{(j)} \leq \widetilde{\lambda}^{(j)} - 1$ for $j = 1, 2$, and call $(\widetilde{m}, \widetilde{\lambda}^{(j)}, \widetilde{\sigma}^{(j)}, \widetilde{\delta}^{(j)})$ the G-valency at P. Moreover we set

$$\widetilde{\mathbf{s}} = -\frac{\widetilde{\delta}^{(1)}}{\widetilde{\lambda}^{(1)}} - \frac{\widetilde{\delta}^{(2)}}{\widetilde{\lambda}^{(2)}} - \widetilde{\mathbf{K}}, \tag{11}$$

which we call the *G-screw number* at P.

(iib) Assume P is an amphidrome node. There exist relatively prime natural numbers $(\widetilde{\lambda}, \widetilde{\delta})$ with $1 \leq \widetilde{\delta} \leq \widetilde{\lambda} - 1$ such that $g_0^{\widetilde{m}/2}$ acts on U_P by

$$(x, y, t) \longmapsto \left(\mathbf{e}\left(\frac{\widetilde{\delta}}{2\widetilde{\lambda}}\right) y,\ \mathbf{e}\left(\frac{\widetilde{\delta}}{2\widetilde{\lambda}}\right) x,\ \mathbf{e}\left(\frac{1}{2\widetilde{\lambda}\ell}\right) t \right) \tag{12}$$

where $\ell = N/(\widetilde{\lambda}\widetilde{m})$ is an integer. Moreover there exists a non-negative integer $\widetilde{\mathbf{K}}$ with

$$n = 2\ell \left(\widetilde{\delta} + \widetilde{\mathbf{K}} \widetilde{\lambda} \right). \tag{13}$$

We put the integer $\widetilde{\sigma}$ in the same way, and call $(\widetilde{m}, \widetilde{\lambda}, \widetilde{\sigma}, \widetilde{\delta})$ the G-valency at P. We set

$$\widetilde{\mathbf{s}} = -\frac{2\widetilde{\delta}}{\widetilde{\lambda}} - 2\widetilde{\mathbf{K}} \tag{14}$$

which we call the G-screw number at P.

(iii) Let $\mathbf{\Gamma}(\widetilde{F})$ be the dual graph of $\widetilde{F}$. This is the one-dimensional oriented graph so that the vertices correspond to the irreducible components of $\widetilde{F}$ and the oriented segments corresponds to the nodes of $\widetilde{F}$ in a canonical way. The action of G to $\widetilde{F}$ naturally induces an action to $\mathbf{\Gamma}(\widetilde{F})$.

Now we claim that the data (i) $\sim$ (iii) in this section are essentially identified with the data (i) $\sim$ (iii) in §1.1. One way to obtain this result is to compare the types of singularities on $S^\sharp$ and the quotient singularities by the G-action in this section (see Takamura [Ta2] and §3.2). Another way is to describe directly the monodromy map of the quotient family of the action as follows:

THEOREM 3.1.1 (i) *There exists a natural isomorphism $\mathbf{\Gamma}(\widetilde{F}) \to \mathbf{\Gamma}(\mu_f)$ so that the action of G to $\mathbf{\Gamma}(\widetilde{F})$ coincides with the action of μ_f to $\mathbf{\Gamma}(\widetilde{F})$. Especially, there exists a one-to-one correspondence between the set $\{F^{(i)}\}$ of irreducible components of $\widetilde{F}$ and*

the set $\{B^{(i)}\}$ of body components for μ_f, between the set $\{P_k\}$ of non-amphidrome nodes of $\widetilde{F}$ and the set $\{\mathcal{A}_k\}$ of non-amphidrome annulus for μ_f, and between the set $\{P_\ell\}$ of amphidrome nodes of $\widetilde{F}$ and the set $\{\mathcal{A}_\ell\}$ of amphidrome annulus for μ_f, respectively.

(ii) *Let $\{P_j\}$ be the set of inner multiple points on a irreducible component $F^{(i)}$, and let $\{Q_j\}$ be the set of multiple points on the corresponding body component $B^{(i)}$. Then there exists a one-to-one correspondence between $\{P_j\}$ and $\{Q_j\}$ such that the G-valency $(\widetilde{m}(P_j), \widetilde{\lambda}(P_j), \widetilde{\sigma}(P_j), \widetilde{\delta}(P_j))$ coincides with the valency $(m(Q_j), \lambda(Q_j), \sigma(Q_j), \delta(Q_j))$.*

(iii) *The G-valency $(\widetilde{m}(P_k), \widetilde{\lambda}^{(j)}(P_k), \widetilde{\sigma}^{(j)}(P_k), \widetilde{\delta}^{(j)}(P_k))$ $(j = 1, 2)$ at a non-amphidrome node P_k coincides with the valency $(m(\mathcal{A}_k), \lambda^{(j)}(\mathcal{A}_k), \sigma^{(j)}(\mathcal{A}_k), \delta^{(j)}(\mathcal{A}_k))$ at the corresponding non-amphidrome annulus $\mathcal{A}_k$. Moreover the G-screw number $\widetilde{\mathbf{s}}(P_k)$ coincides with the screw number $\mathbf{s}(\mathcal{A}_k)$.*

(iv) *The G-valency $(\widetilde{m}(P_\ell), \widetilde{\lambda}(P_\ell), \widetilde{\sigma}(P_\ell), \widetilde{\delta}(P_\ell))$ at an amphidrome node P_ℓ coincides with the valency $(m(\mathcal{A}_\ell), \lambda(\mathcal{A}_\ell), \sigma(\mathcal{A}_\ell), \delta(\mathcal{A}_\ell))$ at the corresponding amphidrome annulus $\mathcal{A}_\ell$. Moreover the G-screw number $\widetilde{\mathbf{s}}(P_\ell)$ coincides with the screw number $\mathbf{s}(\mathcal{A}_\ell)$.*

Proof *Step 1* First, according to the well-known argument ([C] [MM1] etc.), we decribe the monodromy map of $\widetilde{f}$.

We consider a general fiber $\widetilde{F}_{t_0} = \widetilde{f}^{-1}(t_0)$ $(t_0 \neq 0)$ around the stable fiber $\widetilde{F}$. For each node P and its neighborhood U_P by (8), we put $\mathcal{A}_{P,t_0} := U_P \cap \widetilde{F}_{t_0}$. We have the decomposition

$$\widetilde{F}_{t_0} = (\coprod_P \mathcal{A}_{P,t_0}) \cup (\coprod_i \mathcal{B}^{(i)}_{t_0}) \tag{15}$$

where $\coprod_i \mathcal{B}^{(i)}_{t_0}$ is the decomposition to connected components of $\widetilde{F}_{t_0} \setminus \coprod_P \mathcal{A}_{P,t_0}$. Clearly there exists a natural one-to-one correspondence between the set $\{\mathcal{B}^{(i)}_{t_0}\}$ and the set $\{\widetilde{F}^{(i)}\}$ of irreducible components of $\widetilde{F}$.

Now we specially consider the fiber $\widetilde{F}_\delta$ for a sufficiently small positive real number δ. Set $\epsilon' = \delta^n/\epsilon$. We define a homeomorphism $\varphi_{P,\delta} : [0,1] \times S^1 \to \mathcal{A}_{P,\delta}$ as follows; We first define a map $\mathbf{u} : [0,\epsilon] \times [0,\epsilon] \to [0,1]$ by

$$\mathbf{u}(|x|,|y|) = \begin{cases} (\epsilon - |y|)/(2\epsilon - 2|x|) & \text{if } |x| \leq |y| \leq \epsilon, \\ 1 - (\epsilon - |x|)/(2\epsilon - 2|y|) & \text{if } |y| \leq |x| \leq \epsilon. \end{cases}$$

For each $u \in [0,1]$, there uniquely exists a pair of real numbers $(r(u), s(u))$ such that $r(u)s(u) = \delta^n$, $0 \leq r(u) \leq \epsilon$ and $0 \leq s(u) \leq \epsilon$. Then $(r(u), s(u))$ defines a curve connecting from the point $(r(0), s(0)) = (\epsilon', \epsilon)$ on $\partial\mathcal{A}^{(2)}_{P,\delta} = \{y = \epsilon\}$ to the point $(r(1), s(1)) = (\epsilon, \epsilon')$ on $\partial\mathcal{A}^{(1)}_{P,\delta} = \{x = \epsilon\}$.
Then we define $\varphi_{P,\delta} : [0,1] \times S^1 \to \mathcal{A}_{P,\delta}$ by

$$\varphi_{P,\delta}(u, \alpha) := (\mathbf{e}(-\alpha)r(u), \mathbf{e}(\alpha)s(u), \delta^n) \tag{16}$$

where $\alpha \in \mathbf{R}/\mathbf{Z} \simeq S^1$.

Now we define a homeomorhism $h_\theta : \widetilde{F}_\delta \to \widetilde{F}_{\mathbf{e}(\theta)\delta}$ for $0 \leq \theta \leq 1$ as follows; For an annulus part $\mathcal{A}_{P,\delta}$, by using the parametrization $\varphi_{P,\delta}$, we define $h_\theta|_{\mathcal{A}_{P,\delta}} : \mathcal{A}_{P,\delta} \to \mathcal{A}_{P,\mathbf{e}(\theta)\delta}$ by

$$(x, y, \delta) \longrightarrow \Big(\mathbf{e}\Big((1 - \mathbf{u}(|x|, |y|))\theta n\Big)x,\ \mathbf{e}\Big(\mathbf{u}(|x|, |y|)\theta n\Big)y,\ \mathbf{e}(\theta)\delta \Big). \tag{17}$$

For a body part $\mathcal{B}^{(i)}_\delta$ which is locally defined by $t = \delta$ in a natural coordinate (z, t) of $\widetilde{S}$, we define $h_\theta|_{\mathcal{B}^{(i)}_\delta} : \mathcal{B}^{(i)}_\delta \to \mathcal{B}^{(i)}_{\mathbf{e}(\theta)\delta}$ by $(z, \delta) \to (\mathbf{e}(\theta n)z, \mathbf{e}(\theta)\delta)$. These are globally well-patched and define a homeomorhism $h_\theta : \widetilde{F}_\delta \to \widetilde{F}_{\mathbf{e}(\theta)\delta}$.

Then the monodromy homeomorphism is nothing but $h_1 : \widetilde{F}_\delta \to \widetilde{F}_\delta$ by putting $\theta = 1$. Since

$$(h_1|_{\mathcal{A}_{P,\delta}} \circ \varphi_{P,\delta})(u, \alpha) = \varphi_{P,\delta}(u, \alpha - nu)$$

by (16) and (17), $h_1|_{\mathcal{A}_{P,\delta}}$ is a result of n-full Dehn twist. Therefore h_1 induce integral Dehn twists on the annulus part and the identity map on the body part .

Step 2 We put $\delta^* = \delta^N$, and consider the smooth fiber $F_{\delta^*} = f^{-1}(\delta^*)$. Note that $g_0(\widetilde{F}_\delta) = \widetilde{F}_{\mathbf{e}(1/N)\delta}$. Let $\pi : \widetilde{S} \to \widetilde{S}/G$ be the projection, and let $\pi_1 = \pi|_{\widetilde{F}_\delta} : \widetilde{F}_\delta \to F_{\delta^*}$ and $\pi_2 = \pi|_{\widetilde{F}_{\mathbf{e}(1/n(P))\delta}} : \widetilde{F}_{\mathbf{e}(1/N)\delta} \to F_{\delta^*}$ be the two isomorphisms. Since f coincedes with $\widetilde{f}_G : \widetilde{S}/G \to \Delta$ over the non-critical locus on Δ, the homeomorhism

$$\mathbf{h} := \pi_2 \circ h_{1/N} \circ (\pi_1)^{-1} : F_{\delta^*} \longrightarrow F_{\delta^*} \tag{18}$$

is nothing but the monodromy homeomorphism of f.

We set $\mathcal{A}_{P^*,\delta^*} = \pi_1(\mathcal{A}_{P,\delta})$ and $\mathcal{B}^{(i)}_{\delta^*} = \pi_1(\mathcal{B}^{(i)}_\delta)$, and consider the decomposition

$$F_{\delta^*} = (\coprod_{P^*} \mathcal{A}_{P^*,\delta^*}) \cup (\coprod_i \mathcal{B}^{(i)}_{\delta^*}). \tag{19}$$

Since the restriction of $\mathbf{h}^N$ to $\coprod \mathcal{B}^{(i)}_{\delta^*}$ is the identity map and the restriction to $\mathcal{A}_{P^*,\delta^*}$ is an integral Dehn twist on itself, $\mathbf{h}$ is a pseudo-periodic map so that (19) coincides with the decomposition of the annular neighborhood of the admissible system of cut curves and the body parts of $\mathbf{h}$. Moreover the minimal pseudo-period of $\mathbf{h}$ is a divisor of N.

Since (19) is the decomposition with respect to the monodromy $\mathbf{h}$, the partition graph $\mathbf{\Gamma}(\mu_f)$ is isomorphic to $\mathbf{\Gamma}(\mu_{\widetilde{f}})$. Since the dual graph $\mathbf{\Gamma}(\widetilde{F})$ is isomorphic to $\mathbf{\Gamma}(\mu_{\widetilde{f}})$, this is isomorphic to $\mathbf{\Gamma}(\mu_f)$. These isomorphisms are canonical and clearly compatible with the G-action and the monodromy action. Therefore we have the assertion (i).

Now by using (16), we define a paramerization on $\mathcal{A}_{P^*,\delta^*}$ by

$$\varphi_{P^*,\delta^*} = \pi_1 \circ \varphi_{P,\delta} : [0, 1] \times S^1 \longrightarrow \mathcal{A}_{P^*,\delta^*}.$$

Assume P is a non-amphidrome node. It follows from (9) (10) (11) (16) (17) (18) that

$$\left((\mathbf{h}^m|_{\mathcal{A}_{P^*,\delta^*}})\circ\varphi_{P^*,\delta^*}\right)(u,\alpha)=\varphi_{P^*,\delta^*}\left(u,\ \alpha-\frac{n\widetilde{m}u}{N}+\frac{\widetilde{\delta}^{(2)}}{\widetilde{\lambda}^{(2)}}\right)=\varphi_{P^*,\delta^*}\left(u,\ \alpha+\widetilde{\mathbf{s}}u+\frac{\widetilde{\delta}^{(2)}}{\lambda^{(2)}}\right).$$

Namely, this is the linear twist with the screw number $\widetilde{\mathbf{s}}$ and the valency $\widetilde{\delta}^{(2)}/\widetilde{\lambda}^{(2)}$ at the one sided boundary. The valency at the another side boundary coincides with $\widetilde{\delta}^{(1)}/\widetilde{\lambda}^{(1)}$ by (11). Therefore the assertion (iii) holds.

Assume P is an amphidrome node. It follows from (11) (12) (13) (16) (17) (18) that

$$\begin{aligned}\left((\mathbf{h}^m|_{\mathcal{A}_{P^*,\delta^*}})\circ\varphi_{P^*,\delta^*}\right)(u,\alpha)&=\varphi_{P^*,\delta^*}\left(1-u,\ -\alpha+\frac{\widetilde{m}nu}{N}+\frac{\widetilde{\delta}}{2\widetilde{\lambda}}\right)\\&=\varphi_{P^*,\delta^*}\left(1-u,\ -\alpha-\frac{\widetilde{\mathbf{s}}u}{2}+\frac{\widetilde{\delta}}{2\widetilde{\lambda}}\right).\end{aligned}$$

We also have $\left((\mathbf{h}^{2m}|_{\mathcal{A}_{P^*,\delta^*}})\circ\varphi_{P^*,\delta^*}\right)(u,\alpha)=\varphi_{P^*,\delta^*}(u,\alpha+\widetilde{\mathbf{s}}u+\widetilde{\delta}/\widetilde{\lambda})$. Therefore the assertion (iv) follows.

It remains to prove (ii). For the irreducible component $\widetilde{F}^{(i)}$, we put $\widetilde{F}^{(i)*}=\widetilde{F}^{(i)}\setminus\coprod_P(U_P\cap\widetilde{F}^{(i)})$ where P is a node of $\widetilde{F}$ on $\widetilde{F}^{(i)}$. Then there exists a canonical analytic isomorphism $\varphi^{(i)}_{\mathbf{e}(\theta)\delta}:\mathcal{B}^{(i)}_{\mathbf{e}(\theta)\delta}\to\widetilde{F}^{(i)*}$ as Riemann surfaces with boundary.

Now let $g_0^{m(i)}$ be the generator of $Stab_G(\widetilde{F}^{(i)})$. Then the restriction map $h_{m(i)/N}|_{\mathcal{B}^{(i)}_\delta}:\mathcal{B}^{(i)}_\delta\to\mathcal{B}^{(i)}_{\mathbf{e}(m(k)/N)\delta}$ is also an analytic isomorphism so that the composition map

$$\varphi^{(i)}_{\mathbf{e}(m(i)/N)\delta}\circ h_{m(i)/N}|_{\mathcal{B}^{(i)}_\delta}\circ(\varphi^{(i)}_\delta)^{-1}:\widetilde{F}^{(i)*}\longrightarrow\widetilde{F}^{(i)*}$$

coincides with the analytic automorphism $g_0^{m(i)}|_{\widetilde{F}^{(i)*}}:\widetilde{F}^{(i)*}\to\widetilde{F}^{(i)*}$. Therefore the body part $\mathcal{B}^{(i)}_{\delta^*}=\pi_1(\mathcal{B}^{(i)}_\delta)$ of $F_{\delta^*}$is analytically isomorphic to $\widetilde{F}^{(i)*}$ and the restriction of the power of the monodromy map $\mathbf{h}^{m(i)}|_{\mathcal{B}^{(i)}_{\delta^*}}:\mathcal{B}^{(i)}_{\delta^*}\longrightarrow\mathcal{B}^{(i)}_{\delta^*}$ is nothing but $g_0^{m(i)}|_{\widetilde{F}^{(i)*}}:\widetilde{F}^{(i)*}\to\widetilde{F}^{(i)*}$ via this isomorphism. Hence the assertion (ii) is clear by (7) and the definition of the valency. Q.E.D.

3.2 Since Theorem 3.1.1 is established, we use the simplified notations $\widetilde{m}=m,\widetilde{\lambda}=\lambda,\cdots$ by omitting the "tilde". For instance, we write $(x,t)\to(\mathbf{e}(\delta/\lambda)x,\mathbf{e}(1/\lambda\ell)t)$ for (7), etc.

We already described the singularities on $S^\sharp=\widetilde{S}/G$ in Theorem 2.2.1 and Lemma 2.1.2. Here, we also expain why these types of singularities appear on $S^\sharp$ by the quotient of the action (7), (9) and (12). The argument of this subsection is due to Takamura [Ta2].

(i) We consider the neighborhood U_P of an inner multiple point P. Since the action to U_P of the cyclic group $\langle g_0^m\rangle$ generated by g_0^m of (7) is not small (i.e. has a reflection), we descend it as follows; Let $U_P\to V_{P'}=\{(z,u)\in\mathcal{D}\times\mathcal{D}\}$ ($\mathcal{D}$ is a small disk) be the

map defined by $(x,t) \to (z,u) = (x,t^\ell)$. As the descend map of (7), we define the action $\widehat{g_0}$ to $V_{P'}$ by

$$(z,u) \longmapsto \left(\mathbf{e}\left(\frac{\delta}{\lambda}\right) z, \mathbf{e}\left(\frac{1}{\lambda}\right) u\right). \tag{20}$$

The action to $V_{P'}$ of the cyclic group $\langle\widehat{g_0}\rangle \simeq \mathbf{Z}/\lambda\mathbf{Z}$ is small so that the quotient space $U_P/\langle g_0^m\rangle$ is isomorphic to $V_{P'}/\langle\widehat{g_0}\rangle$, whose singularity at the origin is of type $C_{\lambda,\sigma}$ by (20). Namely, this is the case (i) of Lemma 2.1.2.

(ii) We consider U_P in (8) for a node P. Let $\widetilde{U}_P = \{(X,Y) \in \mathcal{D}\times\mathcal{D}\} \longrightarrow U_P$ be the minimal local uniformization map defined by

$$(X,Y) \longmapsto (x,y,t) = (X^n, Y^n, XY). \tag{21}$$

(iia) Assume P is a non-amphidrome node. The number n is determined by (10). We lift the action (9) to $\widetilde{U}_P$ as follows: An elementary number theoretic consideration imply that there exists a pair of integers (a_1,a_2) such that

$$a_1 + a_2 = \mathbf{K},\ \ (\delta^{(1)} + a_1\lambda^{(1)}, d) = (\delta^{(2)} + a_2\lambda^{(2)}, d) = 1, \tag{22}$$

where $d = \lambda^{(1)}\delta^{(2)} + \lambda^{(2)}\delta^{(1)} + \mathbf{K}\lambda^{(1)}\lambda^{(2)}$ is defined in Proposition 1.5.4. As a lift of (9) by (21), we define the action on $\widetilde{U}_P$ by

$$(X,Y) \longmapsto \left(\mathbf{e}\left(\frac{\delta^{(1)} + a_1\lambda^{(1)}}{\lambda^{(1)}n}\right) X,\ \mathbf{e}\left(\frac{\delta^{(2)} + a_2\lambda^{(2)}}{\lambda^{(2)}n}\right) Y\right). \tag{23}$$

The cyclic action generated by (23) is not small. We descend it as follows: By putting $\tilde{\lambda}^{(i)} = \lambda^{(i)}/\gcd(\lambda^{(1)},\lambda^{(2)})$ $(i=1,2)$, let $\widetilde{U}_P \longrightarrow V_{P'} = \{(\bar{X},\bar{Y}) \in \mathcal{D}\times\mathcal{D}\}$ be the map defined by $X \longmapsto \bar{X} = X^{\ell\tilde{\lambda}^{(1)}}$, $Y \longmapsto \bar{Y} = Y^{\ell\tilde{\lambda}^{(2)}}$. Then the descent map of (23) is written as

$$\widehat{g}_0 : (\bar{X},\bar{Y}) \longmapsto \left(\mathbf{e}\left(\frac{\delta^{(1)} + a_1\lambda^{(1)}}{d}\right)\bar{X},\ \mathbf{e}\left(\frac{\delta^{(2)} + a_2\lambda^{(2)}}{d}\right)\bar{Y}\right), \tag{24}$$

which generates the small action. By (22), $\xi = \mathbf{e}\left((\delta^{(1)} + a_1\lambda^{(1)})/d\right)$ is a primitive d-th root of unity and the number $\mathbf{e}\left((\delta^{(2)} + a_2\lambda^{(2)})/d\right)$ coincides with ξ^v, where v is defined in Proposition 1.5.4. The quotient space $U_P/\langle g_0^m\rangle$ is isomorphic to $V_{P'}/\langle\widehat{g_0}\rangle$, whose singularity at the origin is of type $C_{d,v}$. Namely this is the case (ii) of Lemma 2.1.2.

(iib) Assume P is an amphidrome node. The action which is the lift by (21) of the cyclic group generated by (12) is generated by two maps

$$\begin{aligned} h_0 &: (X,Y) \longmapsto \left(\mathbf{e}\left(\frac{\delta + a_1\lambda}{\lambda n}\right) X,\ \mathbf{e}\left(\frac{\delta + a_2\lambda}{\lambda n}\right) Y\right), \\ h_1 &: (X,Y) \longmapsto \left(\mathbf{e}\left(\frac{1}{2\lambda\ell}\right) Y,\ \mathbf{e}\left(\frac{1}{2\lambda\ell}\right) X\right), \end{aligned} \tag{25}$$

where $n = \ell(2\delta + 2\lambda\mathbf{K})$ and (a_1, a_2) is a pair of integers with $a_1 + a_2 = 2\mathbf{K}$, $(\delta + a_1\lambda, \lambda n) = (\delta + a_2\lambda, \lambda n) = 1$. Note that h_1^2 is contained in the subgroup gerenated by h_0. This action is not small. Let $\widetilde{U}_P \longrightarrow V_{P'} = \{(\bar{X}, \bar{Y}) \in \mathcal{D} \times \mathcal{D}\}$ be the map defined by $(X, Y) \longmapsto (\bar{X}, \bar{Y}) = \left(X^\ell,\ Y^\ell\right)$. By putting $\xi = \delta + \lambda\mathbf{K}$, the descent action is written by

$$(\bar{X}, \bar{Y}) \longmapsto \left(\mathbf{e}\left(\frac{\delta + a_1\lambda}{2\lambda\xi}\right)\bar{X},\ \mathbf{e}\left(\frac{\delta + a_2\lambda}{2\lambda\xi}\right)\bar{Y}\right),\ \ (\bar{X}, \bar{Y}) \longmapsto \left(\mathbf{e}\left(\frac{1}{2\lambda}\right)\bar{Y},\ \mathbf{e}\left(\frac{1}{2\lambda}\right)\bar{X}\right),$$

which generate the small action, and the germ of singularity at the origin of the quotient space is nothing but $D_{\xi+\lambda,\xi}$-singularity in (iii) of Lemma 2.1.2.

4 Dedekind sum

In this section, we prepare some number-theoretic arguments which relates to the Dedekind sum.

4.1 Let (λ, σ) be any pair of mutually prime natural numbers with $1 \leq \sigma \leq \lambda - 1$. Let

$$s(\sigma, \lambda) := \sum_{k=1}^{\lambda-1} \left(\left(\frac{k}{\lambda}\right)\right)\left(\left(\frac{k\sigma}{\lambda}\right)\right)$$

be the Dedekind sum with respect to (λ, σ), where $((x)) = x - [x] - 1/2$ for a non-integer x and $((x)) = 0$ for an integer x. ($[x]$ is the greatest integer not exceeding x.) The following formula is classical.

PROPOSITION 4.1.1 (Rademacher) *Let k_1, k_2 be any integers. Then*

$$\sum_j \cot\frac{\pi j}{k_1\lambda}\cot\frac{\pi\sigma j}{k_2\lambda} = 4k_1k_2 \cdot s(\sigma, \lambda)$$

where j move integers satisfying $1 \leq j \leq k_1k_2\lambda - 1$, $j \not\equiv 0 \pmod{k_1\lambda}$ and $j \not\equiv 0 \pmod{k_2\lambda}$.

When $k_1 = k_2 = 1$, it is the classical Rademacher's formula. The above formula is written in Hirzebruch-Zagier [HZ] p.179, 180.

Let $\lambda/\sigma = [[K_1, K_2, \cdots, K_r]]$ be the continued linear fraction, and set $n_0 = \lambda, n_1 = \sigma$ and $n_i = K_{i-1}n_{i-1} - n_{i-2}$ $(2 \leq i \leq r)$. Let δ be the integer with $\sigma\delta \equiv 1 (\mathrm{mod}\ \lambda)$ and $1 \leq \delta \leq \lambda - 1$. We express $s(\sigma, \lambda)$ via these data of the continued linear fraction. For the proof of the following theorem, see [AI2].

THEOREM 4.1.2 ([AI2])

$$s(\sigma, \lambda) = -\frac{r}{4} + \frac{1}{12}\left(\frac{\sigma + \delta}{\lambda} + \sum_{i=1}^{r} K_i\right).$$

4.2 We consider the Dedekind sum with respect to the monodromy data at a non-amphidrome annulus $\mathcal{A}$. We follow the notations in §1.5.

LEMMA 4.2.1 ([Ta2] §6.3) *Let d, v be as in Proposition 1.5.4 and put $v^* = \sigma^{(2)}\delta^{(1)} + \lambda^{(1)}\widehat{\sigma}^{(2)} + \sigma^{(2)}\lambda^{(1)}\mathbf{K}$. Then we have $vv^* \equiv 1 \pmod{d}$. Moreover;*
(i) *If $\mathbf{K} \geq 0$, then $1 \leq v^* \leq d-1$.*
(ii) *If $\mathbf{K} = -1$, $\omega \geq 1$ and $\omega' \geq 1$ (in the notation of Lemma 1.5.2), then $1 \leq v^* \leq d-1$.*
(iii) *If $\mathbf{K} = -1$, $\omega \geq 1$ and $\omega' = 0$, then $v^* < 0$.*

LEMMA 4.2.2

$$-4s(v,d) + 4s(\sigma^{(1)}, \lambda^{(1)}) + 4s(\sigma^{(2)}, \lambda^{(2)}) - \frac{1}{3}\left(\frac{\delta^{(1)}}{\lambda^{(1)}} + \frac{\delta^{(2)}}{\lambda^{(2)}}\right) + \frac{2}{3}\mathbf{K} - \frac{(\lambda^{(1)})^2 + (\lambda^{(2)})^2}{3d\lambda^{(1)}\lambda^{(2)}}$$

$$= \begin{cases} \mathbf{K} - 1 & \text{if } \mathbf{K} \geq 0 \\ \omega + \omega' - r - r' - 1 + \dfrac{1}{3}\left(\displaystyle\sum_{i=\omega+1}^{r} K_i + \sum_{i=\omega'+1}^{r'} L_i + 2\right) & \text{if } \mathbf{K} = -1, \omega \geq 1, \omega' \geq 1 \\ 2\omega - 2r - 1 + \dfrac{1}{3}\left(-\alpha + 1 + K_\omega + 2\displaystyle\sum_{i=\omega+1}^{r} K_i\right) & \text{if } \mathbf{K} = -1, \omega \geq 1, \omega' = 0 \end{cases}$$

where α is the integer which satisfies $1 \leq v^* + \alpha d \leq d-1$.

PROOF It follows from Proposition 1.5.4 and Theorem 4.1.2 that

$$s(v,d) = \begin{cases} -\dfrac{r + r' + \mathbf{K} - 1}{4} + \dfrac{1}{12}\left(\dfrac{v + v^*}{d} + \displaystyle\sum_{i=1}^{r} K_i + \sum_{i=1}^{r'} L_i + 2\mathbf{K}\right) & \mathbf{K} \geq 0 \\ -\dfrac{\omega + \omega' - 1}{4} + \dfrac{1}{12}\left(\dfrac{v + v^*}{d} + \displaystyle\sum_{i=1}^{\omega} K_i + \sum_{i=1}^{\omega'} L_i + 1\right) & \mathbf{K} = -1, \omega \geq 1, \omega' \geq 1 \\ -\dfrac{\omega - 1}{4} + \dfrac{1}{12}\left(\dfrac{v + v^*}{d} + \alpha + \displaystyle\sum_{i=1}^{\omega-1} K_i\right) & \mathbf{K} = -1, \omega \geq 1, \omega' = 0 \end{cases}$$

On the other hand, a calculation shows that

$$\frac{v + v^*}{d} - \frac{\sigma^{(1)} + \delta^{(1)}}{\lambda^{(1)}} - \frac{\sigma^{(2)} + \delta^{(2)}}{\lambda^{(2)}} = -\left(\frac{\delta^{(1)}}{\lambda^{(1)}} + \frac{\delta^{(2)}}{\lambda^{(2)}}\right) - \frac{(\lambda^{(1)})^2 + (\lambda^{(2)})^2}{d\lambda^{(1)}\lambda^{(2)}}.$$

Therefore, by applying Theorem 4.1.2 again to $s(\sigma^{(1)}, \lambda^{(1)})$ and $s(\sigma^{(2)}, \lambda^{(2)})$, we easily have the assertion. Q.E.D.

5 Local signature defect and Examples

5.1 Let $0 < \epsilon < 1$ be a sufficiently small real number. In this section, let $f : S \to \Delta$ be a normally minimal degeneration of curves of genus g over a closed ϵ-disk $\Delta = \{t \in \mathbf{C} \mid |t| \leq \epsilon\}$ with the central fiber $F = f^{-1}(0)$. (By a little bit abuse of the notation, one may consider f the restriction over a small closed disk of the degeneration defined in §2.1.) We denote the restriction of f to its boundary by $\partial f : \partial S \longrightarrow \partial\Delta \simeq S^1$.

Since the restricted family over the punctured disk $f^{-1}(\Delta^*) \to \Delta^* = \Delta \setminus \{0\}$ is differentiably locally trivial, there exists a Riemannian metric $h_S = \{h_{ij}\}$ on S such that the restriction over a tubular neighborhhod of ∂S is a product metric. We put $h_{\partial S}$ the restriction of h_S to ∂S. The choice of h_S is not unique, and we fix one of them. Let Sign$S$ be the signature of the intersection form on $H^2(S, \partial S; \mathbf{Q})$, and $\eta(\partial S, h_{\partial S})$ be the eta-invariant of Atiyah-Patodi-Singer [APS]. Inspired by Furuta's discussion [Fu][2] , we put

$$\sigma(f, F; h_{\partial S}) = \mathrm{Sign}S + \eta(\partial S, h_{\partial S}).$$

Let $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ be the stable reduction of degree N of f, and let $\rho : \widetilde{\Delta} \to \Delta$, $\widetilde{\rho} : \widetilde{S} \to S$ be the natural maps. Let $h_{\widetilde{S}}$ be a Riemannian metric on $\widetilde{S}$ so that $h_{\widetilde{S}}$ is an extension of the natural pull back of the restricted metric of h_S to a tubular neighborhood of ∂S. Since the map $\widetilde{\rho}$ is umramified near the boundary, it is well-defined. The metric $h_{\widetilde{S}}$ is a product metric near the boundary $\partial\widetilde{S}$, and the eta-invariant $\eta(\partial\widetilde{S}, h_{\partial\widetilde{S}})$ is well-defined. We also put $\sigma(\widetilde{f}, \widetilde{F}; h_{\partial\widetilde{S}}) = \mathrm{Sign}(\widetilde{S}) + \eta(\partial\widetilde{S}, h_{\partial\widetilde{S}})$. We define

$$\mathrm{Lsd}(f, F; h_{\partial S}; N) := \sigma(f, F; h_{\partial S}) - \frac{1}{N}\sigma(\widetilde{f}, \widetilde{F}; h_{\partial\widetilde{S}}),$$

and call it the *local signature defect* of $(f, F; h_{\partial S})$ of order N.

5.2 We summarize the notations of the monodromy data of f. Let $\Sigma_g = \mathbf{B} \cup \mathbf{A}' \cup \mathbf{A}''$ be the decomposition to the set of body connected components $\mathbf{B} = \coprod B_i$, the set of non-amphidrome annuli $\mathbf{A}' = \coprod \mathcal{A}_j$ and the set of amphidrome annuli $\mathbf{A}'' = \coprod \widetilde{\mathcal{A}}_k$ with respect to (the representative of) the monodromy map $\mu_f : \Sigma_g \to \Sigma_g$. Let $\mathbf{B}/\sim = \coprod [B_i]$ be the orbit decomposition with respect to the cyclic action generated by μ_f. Namely two components B_{i_1} and B_{i_2} are equivalent iff $B_{i_2} = (\mu_f)^n(B_{i_1})$ for some integer n, and $\mathbf{B}/\sim$ is its equivalence class. We set $\mathbf{A}'/\sim = \coprod [\mathcal{A}_j]$, $\mathbf{A}''/\sim = \coprod [\widetilde{\mathcal{A}}_k]$ similarly.

Let $\{m_\alpha, \lambda_\alpha^{(i)}, \sigma_\alpha^{(i)}, \delta_\alpha^{(i)}\}_{1 \leq \alpha \leq \varphi(i)}$ be the set of all the valencies which are attached to multiple points and boundary curves of B_i. Set $\lambda_\alpha^{(i)}/\sigma_\alpha^{(i)} = [[K_1(\alpha, i), K_2(\alpha, i), \cdots, K_{r(\alpha,i)}(\alpha, i)]]$.

Let $(m(\mathcal{A}_j), \lambda^{(1)}(\mathcal{A}_j), \sigma^{(1)}(\mathcal{A}_j), \delta^{(1)}(\mathcal{A}_j))$ and $(m(\mathcal{A}_j), \lambda^{(2)}(\mathcal{A}_j), \sigma^{(2)}(\mathcal{A}_j), \delta^{(2)}(\mathcal{A}_j))$ be the valencies of both boundary curves of the non-amphidrome annulus $\mathcal{A}_j$, and $\mathbf{s}(\mathcal{A}_j) =$

[2] Furuta [Fu] discussed this type of invariants under a more general setting of the linear connection.

$-\delta^{(1)}(\mathcal{A}_j)/\lambda^{(1)}(\mathcal{A}_j)-\delta^{(2)}(\mathcal{A}_j)/\lambda^{(2)}(\mathcal{A}_j)-\mathbf{K}(\mathcal{A}_j)$ be the screw number. Set $\lambda^{(1)}(\mathcal{A}_j)/\sigma^{(1)}(\mathcal{A}_j) = [[K_1^{(j)}, K_2^{(j)}, \cdots, K_{r(j)}^{(j)}]]$ and $\lambda^{(2)}(\mathcal{A}_j)/\sigma^{(2)}(\mathcal{A}_j) = [[L_1^{(j)}, L_2^{(j)}, \cdots, L_{r'(j)}^{(j)}]]$. We define a rational number $\epsilon(\mathcal{A}_j)$ by

$$\epsilon(\mathcal{A}_j) = \begin{cases} 0, & \mathbf{K}(\mathcal{A}_j) \geq 0 \\ \dfrac{1}{3}\left(\displaystyle\sum_{i=\omega(j)+1}^{r(j)} K_i^{(j)} + \sum_{i=\omega'(j)+1}^{r'(j)} L_i^{(j)} + 2\right) - 1, & \mathbf{K}(\mathcal{A}_j) = -1, \omega(j) \geq 1, \omega'(j) \geq 1 \\ \dfrac{1}{3}\left(-\alpha(j) + 1 + K_{\omega(j)}^{(j)} + 2\displaystyle\sum_{i=\omega(j)+1}^{r(j)} K_i^{(j)}\right) - 1, & \mathbf{K}(\mathcal{A}_j) = -1, \omega(j) \geq 1, \omega'(j) = 0 \end{cases}$$

where $\omega(j), \omega'(j)$ are defined in Lemma 1.5.2 and $\alpha(j)$ is defined in Lemma 4.2.2. Moreover we put $\ell(\mathcal{A}_j) = N/\{\mathrm{lcm}(\lambda^{(1)}(\mathcal{A}_j), \lambda^{(2)}(\mathcal{A}_j)) \cdot m(\mathcal{A}_j)\}$.

Let $(m(\widetilde{\mathcal{A}_k}), \lambda(\widetilde{\mathcal{A}_k}),\ \sigma(\widetilde{\mathcal{A}_k}), \delta(\widetilde{\mathcal{A}_k}))$ be the valency of a boundary curve of the amphidrome annulus $\widetilde{\mathcal{A}_k}$, and $\mathbf{s}(\widetilde{\mathcal{A}_k}) = -2\delta(\widetilde{\mathcal{A}_k})/\lambda(\widetilde{\mathcal{A}_k}) - 2\mathbf{K}(\widetilde{\mathcal{A}_k})$ be the screw number. Put $\ell(\widetilde{\mathcal{A}_k}) = N/\{\lambda(\widetilde{\mathcal{A}_k})m(\widetilde{\mathcal{A}_k})\}$.

Note that the above data are independent of the choice of a representative of $[B_i]$, $[\mathcal{A}_j]$ and $[\widetilde{\mathcal{A}}_k]$ respectively.

Now the proof of the following theorem is postponed to §6.

Theorem 5.2.1

$$\mathrm{Lsd}(f, F; h_{\partial S}; N) = -\frac{1}{3}\sum_{[B_i]}\sum_{\alpha=1}^{\varphi(i)}\left(\frac{\sigma_\alpha^{(i)} + \delta_\alpha^{(i)}}{\lambda_\alpha^{(i)}} + \sum_{j=1}^{r(\alpha,i)} K_j(\alpha, i)\right)$$

$$+\sum_{[\mathcal{A}_j]}\left(\frac{\gcd(\lambda^{(1)}(\mathcal{A}_j), \lambda^{(2)}(\mathcal{A}_j))}{\ell(\mathcal{A}_j)\lambda^{(1)}(\mathcal{A}_j)\lambda^{(2)}(\mathcal{A}_j)} - \mathbf{K}(\mathcal{A}_j) + \epsilon(\mathcal{A}_j)\right) + \sum_{[\widetilde{\mathcal{A}_k}]}\left(\frac{1}{2\ell(\widetilde{\mathcal{A}_k})\lambda(\widetilde{\mathcal{A}_k})} - \mathbf{K}(\widetilde{\mathcal{A}_k}) - 2\right)$$

where the summations move all the classes of the orbit decompositions.

Since $\mathrm{Lsd}(f, F; h_{\partial S}; N)$ is independent of $h_{\partial S}$, we write it by $\mathrm{Lsd}(f, F; N)$ from now on. Moreover, if we choose N minimal, then we simply write it by $\mathrm{Lsd}(f, F)$ and call it the *local signaure defect* of the degeneration f. This invariant only depends on the fiber germ at F.

Now let $\widehat{f} : \widehat{S} \to \Delta$ be the semi-stable reduction of f. The surface $\widehat{S}$ is obtained from $\widetilde{S}$ by the resolution $\widehat{S} \to \widetilde{S}$ of rational double points of type A at the double points of $\widetilde{F}$. The boundary $\partial\widehat{S}$ of $\widehat{S}$ coincides with $\partial\widetilde{S}$, and has the natural Riemannian metric $h_{\partial\widehat{S}}$. For the fiber germ $(\widehat{f}, \widehat{F})$, we put $\sigma(\widehat{f}, \widehat{F}; h_{\partial\widehat{S}}) = \mathrm{Sign}(\widehat{S}) + \eta(\partial\widehat{S}, h_{\partial\widehat{S}})$. Similarly we define

$$\mathbf{Lsd}(f, F; h_{\partial S}; N) := \sigma(f, F; h_{\partial S}) - \frac{1}{N}\sigma(\widehat{f}, \widehat{F}; h_{\partial\widehat{S}}).$$

Corollary 5.2.2

$$\mathbf{Lsd}(f,F;h_{\partial S};N) = -\frac{1}{3}\sum_{[B_i]}\sum_{\alpha=1}^{\varphi(i)}\left(\frac{\sigma_\alpha^{(i)}+\delta_\alpha^{(i)}}{\lambda_\alpha^{(i)}} + \sum_{j=1}^{r(\alpha,i)} K_j(\alpha,i)\right)$$
$$+\sum_{[\mathcal{A}_j]}\left(\frac{\sigma^{(1)}(\mathcal{A}_j)}{\lambda^{(1)}(\mathcal{A}_j)} + \frac{\sigma^{(2)}(\mathcal{A}_j)}{\lambda^{(2)}(\mathcal{A}_j)} + \epsilon(\mathcal{A}_j)\right) + \sum_{[\widetilde{\mathcal{A}_k}]}\left(\frac{\delta(\widetilde{\mathcal{A}_k})}{\lambda(\widetilde{\mathcal{A}_k})} - 2\right).$$

Proof Let $n(\mathcal{A}_j)$ be the Milnor number of the singularity of type A corresponding to the non-amphidrome annulus $\mathcal{A}_j$ via Theorem 3.1.1. By (10), $n(\mathcal{A}_j)$ coincides with

$$\frac{\ell(\mathcal{A}_j)}{\gcd(\lambda^{(1)}(\mathcal{A}_j),\lambda^{(2)}(\mathcal{A}_j))}\left(\delta^{(1)}(\mathcal{A}_j)\lambda^{(2)}(\mathcal{A}_j) + \delta^{(2)}(\mathcal{A}_j)\lambda^{(1)}(\mathcal{A}_j) + \mathbf{K}(\mathcal{A}_j)\lambda^{(1)}(\mathcal{A}_j)\lambda^{(2)}(\mathcal{A}_j)\right).$$

Moreover the number N is written by $N = \ell(\mathcal{A}_j)\cdot \mathrm{lcm}(\lambda^{(1)}(\mathcal{A}_j),\lambda^{(2)}(\mathcal{A}_j))$.

Let $n(\widetilde{\mathcal{A}_k})$ be the Milnor number of the singularity of type A corresponding to the amphidrome annulus $\widetilde{\mathcal{A}_k}$. By (13), we have $n(\widetilde{\mathcal{A}_k}) = 2\ell(\widetilde{\mathcal{A}_k})\left(\delta(\widetilde{\mathcal{A}_k}) + \mathbf{K}(\widetilde{\mathcal{A}_k})\lambda(\widetilde{\mathcal{A}_k})\right)$ and $N = 2\ell(\widetilde{\mathcal{A}_k})\lambda(\widetilde{\mathcal{A}_k})$. Therefore we have

$$\frac{1}{N}\left(\mathrm{Sign}(\widetilde{S}) - \mathrm{Sign}(\widehat{S})\right) = \sum_{[\mathcal{A}_j]}\frac{n(\mathcal{A}_j)-1}{N} - \sum_{[\widetilde{\mathcal{A}_k}]}\frac{n(\widetilde{\mathcal{A}_k})-1}{N}$$
$$= \sum_{[\mathcal{A}_j]}\left(\frac{\sigma^{(1)}(\mathcal{A}_j)}{\lambda^{(1)}(\mathcal{A}_j)} + \frac{\sigma^{(2)}(\mathcal{A}_j)}{\lambda^{(2)}(\mathcal{A}_j)} + \mathbf{K}(\mathcal{A}_j) - \frac{1}{\ell(\mathcal{A}_j)\cdot\mathrm{lcm}(\lambda^{(1)}(\mathcal{A}_j),\lambda^{(2)}(\mathcal{A}_j))}\right)$$
$$+\sum_{[\widetilde{\mathcal{A}_k}]}\left(\frac{\sigma(\widetilde{\mathcal{A}_k})}{\lambda(\widetilde{\mathcal{A}_k})} + \mathbf{K}(\widetilde{\mathcal{A}_k}) - \frac{1}{2\ell(\widetilde{\mathcal{A}_k})\lambda(\widetilde{\mathcal{A}_k})}\right).$$

On the other hand, it follows from $\eta(\partial\widetilde{S},h_{\partial\widetilde{S}}) = \eta(\partial\widehat{S},h_{\partial\widehat{S}})$ that

$$\mathbf{Lsd}(f,F;h_{\partial S};N) = \mathrm{Lsd}(f,F;h_{\partial S};N) - \frac{1}{N}\left(\mathrm{Sign}(\widetilde{S}) - \mathrm{Sign}(\widehat{S})\right).$$

Hence the assertion follows from Theorem 5.2.1.

5.3 We consider the global stable reduction $\widetilde{\mathbf{f}}:\widetilde{\mathcal{S}}\to\widetilde{\mathcal{C}}$ of $\mathbf{f}:\mathcal{S}\to\mathcal{C}$ of order N. The description of $\widetilde{\mathbf{f}}$ is as follows:

Let N_i be a pseudo-period of the local monodromy map around the singular fiber $F_i = \mathbf{f}^{-1}(P_i)$ $(1\le i\le r)$. Let N be a common multiple of all of N_i. By adding a non-critical point P_0 ("a dummy point") if necesarry, we can construct a cyclic branched covering $\varphi:\widetilde{\mathcal{C}}\to\mathcal{C}$ of order N whose branch points coincide with $P_0,P_1,\cdots,P_r$. Moreover, for $1\le i\le r$, the fiber $\varphi^{-1}(P_i)$ consists of N/N_i points so that the ramification indices at them are N_i. The analytic surface $\widetilde{\mathcal{S}}$ has at most A-type singularities and is birationally

equivalent to $\mathcal{S} \times_{\mathcal{C}} \widetilde{\mathcal{C}}$. Explicit construction is nothing but the natural patching of N/N_i-copies of the local stable reduction of the fiber germ of F_i $(1 \leq i \leq r)$ discussed in §2. Note that $\mathbf{f}^{-1}(\varphi^{-1}(P_0))$ consists of multiple fibers whose reduced schemes are nonsingular and $\widetilde{\mathcal{S}}$ itselfis smooth around the fiber.

The local signature defect of the germ of the singular fiber is nothing but the local contribution of the difference of the global signature under the global stabe reduction, i.e. we have the following Lemma whose proof is also postponed to §6.

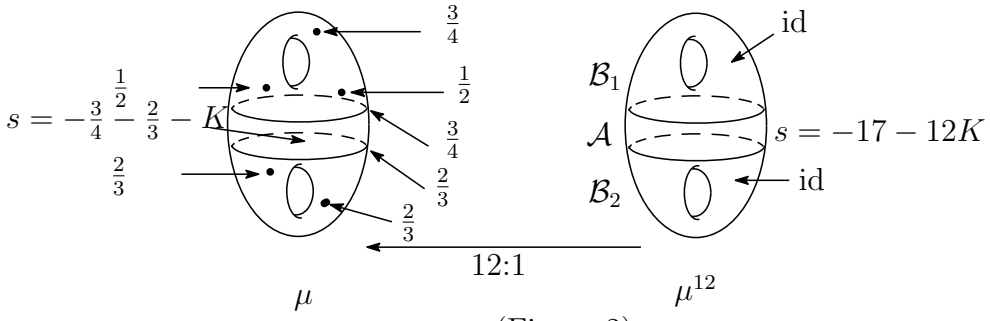
LEMMA 5.3.1

$$\text{Sign } \mathcal{S} - \frac{\text{Sign } \widetilde{\mathcal{S}}}{N} = \sum_{i=1}^{r} \text{Lsd}(f_i, F_i; N_i)$$

where f_i is the restricted fibration of $\widetilde{\mathbf{f}}$ to a tubular neighborhood of F_i. and $\widetilde{F}_i$ is its stable fiber of the local stable reduction $\widetilde{f}_i$ of f_i of order N_i.

5.4 We present two examples of singular fiber germs of genus 2 and calculate their local signature defect. At the same time, we explain our method of the stable reduction in §2 using them.

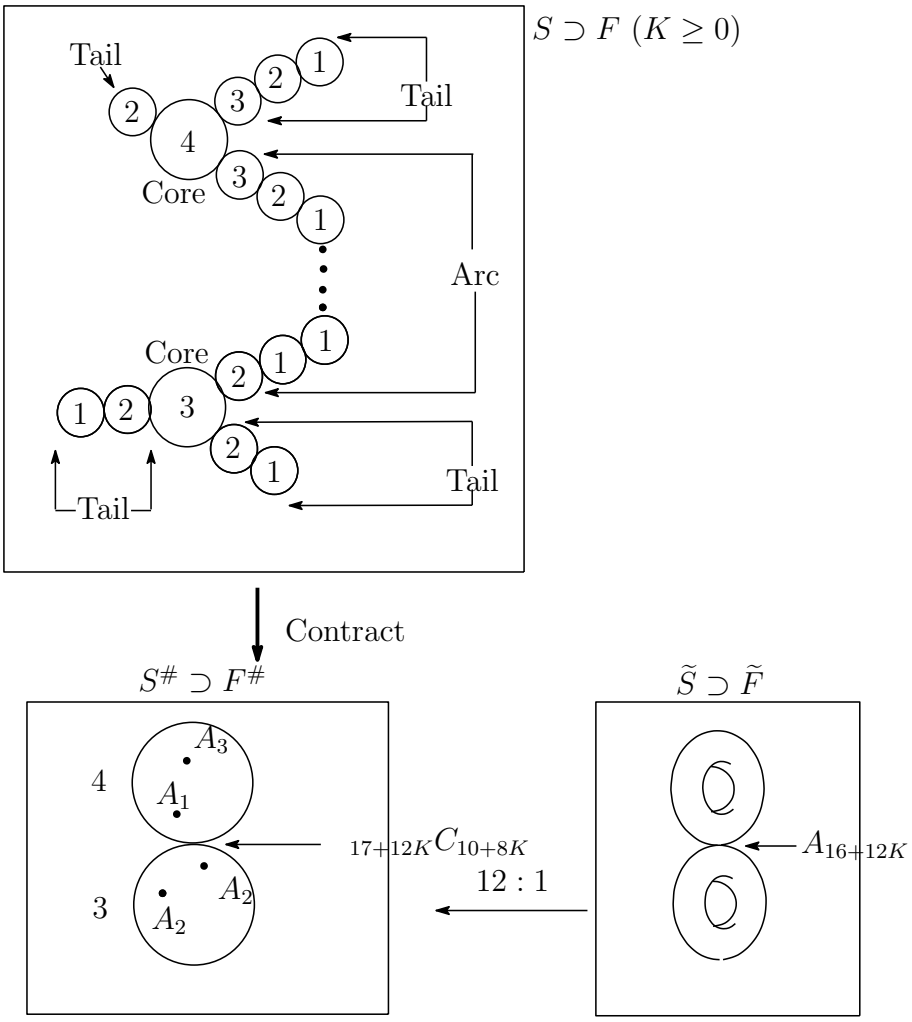
EXAMPLE 5.4.1 Let $\mu : \Sigma_2 \to \Sigma_2$ be the pseudo-periodic map of negative twist of genus 2 as follows: The decomposition (2) for μ is written by $\Sigma_2 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{A}$, where each $\mathcal{B}_i$ $(i = 1, 2)$ is a body connected component consisting of a one-punctured torus and $\mathcal{A}$ is a non-amphidrome annulus. The valencies on $\mathcal{B}_1$ are $(1, 4, 3, 3)$, $(1, 4, 3, 3)$ and $(1, 2, 1, 1)$, where $(1, 4, 3, 3)$ is attached to the boundary curve $\partial\mathcal{B}_1 = \partial\mathcal{A}^{(1)}$ and the others are attached to the multiple points. According to [AI1], we simply say that the total valency on $\mathcal{B}_1$ is $\mathbf{3/4} + 3/4 + 1/2$. By the same way, the total valency on $\mathcal{B}_2$ is $\mathbf{2/3} + 2/3 + 2/3$. The screw number $\mathbf{s}(\mathcal{A})$ is $-3/4 - 2/3 - \mathbf{K}$ $(\mathbf{K} \geq -1)$. The minimal pseudo-period of μ is 12. (See the left half of Figure 2.)



(Figure 2)

Let $f : S \to \Delta$ be the degeneration whose topological monodromy coincides with μ.

Note that f is listed as the types $[IV^* - III^* - m]$, $[IV^* - III^* - \alpha]$ in Namikawa-Ueno's table [NU]. The irreducible decomposition of the singular fiber $F = f^{-1}(0)$ is written by $F = 4F^{(1)} + \sum_{k=1}^{2}\sum_{j=1}^{3}(4-j)F_{kj}^{(1)} + 2F_3^{(1)} + \sum_{j=1}^{\mathbf{K}-1}F_j^{(0)} + 3F^{(2)} + \sum_{k=1}^{3}\sum_{j=1}^{2}(3-j)F_{kj}^{(2)}$ for $\mathbf{K} \geq 0$,[3] and by $F = 4F^{(1)} + \sum_{j=1}^{3}(4-j)F_{2j}^{(1)} + 2F_3^{(1)} + 3F_{11}^{(1)} + 2F_{12\cdot 11}^{(1\cdot 2)} + 3F^{(2)} + \sum_{k=2}^{3}\sum_{j=1}^{2}(3-j)F_{kj}^{(2)}$ for $\mathbf{K} = -1$. The configuration of F is as in Figure 3.



(Figure 3)

The fiber F has two core components $F^{(1)}$ and $F^{(2)}$, four connected components of the tail $\sum_{j=1}^{3}(4-j)F_{2j}^{(1)}$, $2F_3^{(1)}$, $\sum_{j=1}^{2}(3-j)F_{2j}^{(2)}$ and $\sum_{j=1}^{2}(3-j)F_{3j}^{(2)}$, and one connected component of the arc $\sum_{j=1}^{3}(4-j)F_{1j}^{(1)} + \sum_{j=1}^{\mathbf{K}-1}F_j^{(0)} + \sum_{j=1}^{2}jF_{1j}^{(2)}$ (resp. $3F_{11}^{(1)} + 2F_{12\cdot 11}^{(1\cdot 2)}$) for $\mathbf{K} \geq 0$ (resp. $\mathbf{K} = -1$). All the irreducible components are smooth rational. The self-intersection number of each of $F_{13}^{(1)}, F_{12}^{(2)}, F_{12\cdot 11}^{(1\cdot 2)}$ is -3, and other components are all (-2)-curves.

[3] If $\mathbf{K} = 0$, then $F_{13}^{(1)} = F_{12}^{(2)}$ and $\sum_{j=1}^{\mathbf{K}-1}F_j^{(0)}$ is empty.

Let $S \to S^\sharp$ be the contraction in §2. The irreducible decomposition of the central fiber $F^\sharp$ of $S^\sharp \to \Delta$ is written by $F^\sharp = 4F_1^\sharp + 3F_2^\sharp$. The normal analytic surface $S^\sharp$ has five isolated singularities $P_1^{(1)}, P_2^{(1)}, P_{12}, P_1^{(2)}$ and $P_2^{(2)}$. The points $P_1^{(1)}, P_2^{(1)}, P_1^{(2)}, P_2^{(2)}$ are tail contracting points, and are rational double points of types A_3, A_1, A_2, A_2 respectively. The point $P_{12} = F_1^\sharp \cap F_2^\sharp$ is an arc contraction point, and is the cyclic quotient singularity of type $C_{17+12\mathbf{K},12+9\mathbf{K}}$ (resp. type $C_{5,3}$) for $\mathbf{K} \geq 0$ (resp. $\mathbf{K} = -1$), because of $[[2,2,3,\underbrace{2,\cdots,2}_{\mathbf{K}-1},3,2]] = (17+12\mathbf{K})/(12+9\mathbf{K})$ (resp. $[[2,3]] = 5/3$). (See Figure 3.)

Let $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ be the stable family obtained by the composition of the $12:1$ base change and its normalization as in §2. The central fiber $\widetilde{F}$ consists of two components $\widetilde{F} = \widetilde{F}_1 + \widetilde{F}_2$ so that $\widetilde{F}_1$ (resp. $\widetilde{F}_2$) is a smooth elliptic curve, since it is a 4-fold (resp. 3-fold) cyclic cover of $\mathbf{P}^1$ branched at 3 points whose branch indices are $4,4,2$ (resp. 3,3,3). The topological monodromy around $\widetilde{F}$ coincides with μ^{12}, and therefore the decomposition to the bodies and the annuli is as same as μ and the total valencies at $\mathcal{B}_i$ $(i = 1,2)$ are trivial and the screw number at $\mathcal{A}$ is $-17-12\mathbf{K}$, i.e. it is the right-handed Dehn twist of $17+12\mathbf{K}$ times. (See the right half of Figure 2.) The surface $\widetilde{S}$ has a rational double point of type $A_{16+12\mathbf{K}}$ at the node $P = \widetilde{F}_1 \cap \widetilde{F}_2$. By Theorem 5.2.1, we have

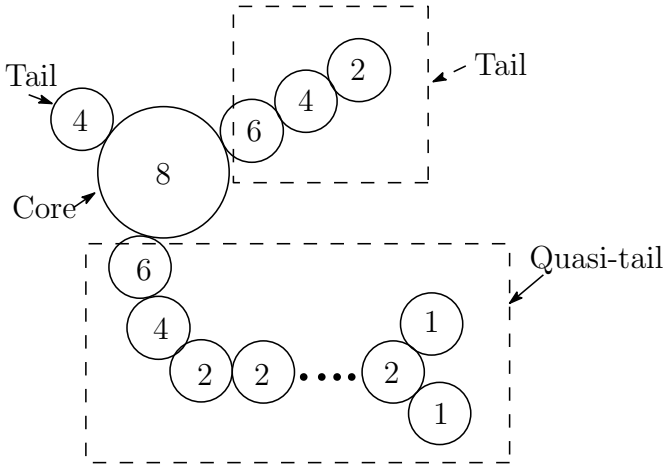
$$\mathrm{Lsd}(f,F) = \mathrm{Lsd}(f,F;12) = -\frac{34}{3} + \left(\frac{1}{12} - \mathbf{K}\right) = -\frac{45}{4} - \mathbf{K}.$$

Example 5.4.2 Let $\mu : \Sigma_2 \to \Sigma_2$ be the pseudo-periodic map with the same decomposition $\Sigma_2 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{A}$ as in Example 7.1 such that the total valency of each of $\mathcal{B}_i$ $(i = 1,2)$ is $\mathbf{3/4} + 3/4 + 1/2$ and $\mathcal{A}$ is an amphidrome annulus with the screw number $-3/4 - 3/4 - 2\mathbf{K}$ $(\mathbf{K} \geq 0)$. The minimal pseudo-period is 8. The configuration of the singular fiber F of the degeneration $f : S \to \Delta$ whose topological monodromy coincides with μ is as in Figure 4 (f is of the type $[2III^* - m]$ in [NU]).

The fiber F has one core component, two connected components of the tails and one connected component of the quasi-tails. After the contraction $S \to S^\sharp$, the surface $S^\sharp$ has three isolated singularities on its central fiber $F^\sharp$ of $S^\sharp \to \Delta$. Two of them are rational double points of types A_3 and A_1 which are the tail contracting points, and one of them is a dihedral singularity of type $D_{7+4\mathbf{K},3+4\mathbf{K}}$ which is the quasi-tail contracting point.

Let $\widetilde{f} : \widetilde{S} \to \widetilde{\Delta}$ be the stable family obtained by $8:1$ base change of f and its normalization. The central fiber $\widetilde{F}$ consists of two smooth elliptic curves with one node so that $\widetilde{S}$ has a rational double point of type $A_{5+8\mathbf{K}}$ at the node. By Theorem 5.2.1, we have

$$\mathrm{Lsd}(f,F) = \mathrm{Lsd}(f,F;8) = -6 + \left(\frac{1}{8} - \mathbf{K} - 2\right) = -\frac{63}{8} - \mathbf{K}.$$



(Figure 4)

6 Proof of Main Theorem

We prove Theorem 5.2.1 and Lemma 5.3.1. Our basic tool is a complex 2-dimensional version of the orbifold signature theorem with smooth boundary. We review the notations and the result of it in §6.1, and then applied it to our proof in §6.2.

6.1 In general, let M be a complex 2-dimensional V-manifold in the sense of Satake [Sa] with a real 3-dimensional manifold boundary ∂M. Let $M = \bigcup U$ be a V-coordinate covering, namely, the local uniformization maps $\widetilde{U} \to U$ and the finite Galois groups $G_U = \mathrm{Gal}(\widetilde{U}/U)$ are defined with the natural compatibility conditions, i.e. if $U_1 \subset U_2$, then there exist a holomorphic open embedding $\widetilde{U}_1 \hookrightarrow \widetilde{U}_2$ and an injective group homomorphism $G_{U_1} \hookrightarrow G_{U_2}$ which induce $\widetilde{U}_1/G_{U_1} = U_1 \hookrightarrow U_2 = \widetilde{U}_2/G_{U_2}$. Since M is a rational homology manifold, the signature $\mathrm{Sign}\, M$ over $H^2(M, \partial M, \mathbf{Q})$ is well-defined.

Let $\{h^U_{ij}\}$ be a V-Riemannian metric on M, i.e. each h^U_{ij} is a G_U-invariant Riemannian metric on $\widetilde{U}$ with the natural compatibility conditions. We assume $\{h^U_{ij}\}$ induces a product metric near ∂M. The V-Levi-Civita connection $\{\bigtriangledown_U\}$ with respect to $\{h^U_{ij}\}$, and the V-curvature matrix $\{R_U\}$ with respect to $\{\bigtriangledown_U\}$ are defined naturally ([Sa]). The V-Pontrjagin form $\{p_1(U)\}$ is defined by $p_1(U) = (2\pi)^{-2}\,\mathrm{Tr}\,(R^2_U)$.

For each V-coordinate U and an element $g \in G_U$, we define the equivariant L-form $L^g(U)$ as follows (cf. [AS] §6) : If $g = \mathrm{id}$, put $L^g(U) = (1/3)\,p_1(U)$, i.e the Hirzebruch L_1 form. Assume $g \neq \mathrm{id}$, and let $\widetilde{U}^g$ be the g-fixed locus on $\widetilde{U}$, which is an isolated fixed point or a fixed complex curve.

Suppose $\widetilde{U}^g := P$ is an isolated fixed point. Then g acts on the tangent space $T_P(\widetilde{U})$ linearly with its eigen values $e^{i\alpha_g}$ and $e^{i\beta_g}$ $(0 < \alpha_g, \beta_g < 2\pi)$. Then we put $L^g(U) = -\cot(\alpha_g/2)\cot(\beta_g/2)$.

Suppose $\widetilde{U}^g := C$ is a fixed complex curve. Then g acts on each fiber of the normal bundle $(N_{C/\widetilde{U}})_Q$ of a generic point $Q \in C$ linearly with its eigen value $e^{i\theta_g}$ $(0 < \theta_g < 2\pi)$. Put $L^g(U) = \mathrm{cosec}^2(\theta_g/2) \cdot c_1(N_{C/\widetilde{U}})$, where $c_1(N_{C/\widetilde{U}})$ is the first Chern form of $N_{C/\widetilde{U}}$.

Then the orbifold signature theorem ([Ka]) says that

$$\mathrm{Sign}\ M + \eta(\partial M) = \sum_U \frac{1}{|G_U|} \sum_{g \in G_U} \int_{\widetilde{U}^g} \rho_U L^g(U) \tag{26}$$

where ρ_U is a cut function on U so that the system $\{\rho_U\}$ defines a partition of unity on M and $\eta(\partial M)$ is the eta invariant with respect to the above V-connection. Note that (26) is the "primitive version" in some sense, and Kawasaki reformulates it by describing the correction term from the L_1 class as the equivariant L class on the orbifolds which are the preimages of the V-singular locus of M by the natural immersion. (See Theorem and Corollary in [Ka] p.78,79.)

6.2 We prove Theorem 5.2.1 by several steps.

Step 1 First we consider the surface $\widetilde{S}$, which has a simple V-manifold structure. The support of the V-singular locus (i.e. the set of points whose isotropy groups in the Galois groups are non-trivial) of $\widetilde{S}$ coincides with the set of nodes $\{P\}$ on the central fiber $\widetilde{F}$ at which $\widetilde{S}$ has A_n-singularities $(n \geq 1)$.

We may assume that the V-chart U_P containing P is unique, and the local uniformization map $\widetilde{U}_P \to U_P$ is given by (21). Since the generator of the Galois group $G_{U_P} \simeq \mathbf{Z}/(n)$ acts on $\widetilde{U}_P$ as $(\bar{X}, \bar{Y}) \mapsto (\mathbf{e}(1/n)\bar{X}, \mathbf{e}((n-1)/n)\bar{Y})$, it follows from Proposition 4.1.1 that

$$\begin{aligned} \frac{1}{|G_{U_P}|} \sum_{g \in G_{U_P} \setminus \{\mathrm{id}\}} L^g(U_P) &= \frac{1}{n} \sum_{j=1}^{n-1} \cot \frac{\pi j}{n} \cot \frac{\pi(n-1)j}{n} \\ &= -4s(n-1, n) = \frac{n}{3} - 1 + \frac{2}{3n}. \end{aligned} \tag{27}$$

If $P = P_{\mathrm{noam}}$ is a non-amphidrome node, by putting $d = \delta^{(1)}\lambda^{(2)} + \delta^{(2)}\lambda^{(1)} + \mathbf{K}\lambda^{(1)}\lambda^{(2)}$ and $c = \gcd(\lambda^{(1)}, \lambda^{(2)})$, we have $n = (\ell d)/c$ and $N = (\ell\lambda^{(1)}\lambda^{(2)})/c$. If $P = P_{\mathrm{am}}$ is an amphidrome node, we have $n = 2\ell\xi$ and $N = 2\ell\lambda$. Therefore it follows from (26) and (27) that

$$\begin{aligned} \frac{1}{N}\left(\mathrm{Sign}\ \widetilde{S} + \eta(\partial\widetilde{S})\right) &= \frac{1}{3N}\int_{\widetilde{S}} p_1(\widetilde{S}) + \sum_{P_{\mathrm{noam}}} \left(\frac{d}{3\lambda^{(1)}\lambda^{(2)}} - \frac{c}{\ell\lambda^{(1)}\lambda^{(2)}} + \frac{2c^2}{3\ell^2 d\lambda^{(1)}\lambda^{(2)}}\right) \\ &\quad + \sum_{P_{\mathrm{am}}} \left(\frac{\xi}{3\lambda} - \frac{1}{2\ell\lambda} + \frac{1}{3\ell\lambda n}\right) \end{aligned} \tag{28}$$

where the first summation moves all the data of non-amphidrome nodes and the second summation moves all the data of amphidrome nodes.

STEP 2 Next we consider the surface $S^\sharp$ defined in §2. The cyclic group $G = \mathbf{Z}/(N)$ acts holomorphically on $\widetilde{S}$ so that $\widetilde{S}/G$ coincides with $S^\sharp$. The surface $S^\sharp$ has the V-manifold structure via the composition of the restriction of the global covering map $\widetilde{S} \to S^\sharp$ and the local uniformization map of the V-chart of $\widetilde{S}$.

First we calculate the contribution of the integral of the equvariant L-form of the isolated fixed points P of the Galois groups. We may assume the V-chart U_P containing P is unique. Let $G^\star_{U_P}$ be the subset of elements $g \in G_{U_P}$ such that $\widetilde{U}^g_P = \{P\}$, i.e. g has the isolated fixed point at P. We set

$$L\langle P\rangle = \frac{1}{|G_{U_P}|} \sum_{g \in G^\star_{U_P}} L^g(U_P).$$

Note that P is a tail contracting point, or an arc contracting point, or a quasi-tail contracting point defined in §2.1.

(i) Assume P is a tail contracting point. Then $\widetilde{U}_P$ is a neighborhood of an inner multiple point with the valency $(m, \lambda, \sigma, \delta)$ in §3.2 (i). The group G_{U_P} is isomorphic to $\mathbf{Z}/(\ell\lambda)$, whose generator g_1 acts on $\widetilde{U}_P$ as the map (7). We put $J^\star = \{j \in \mathbf{Z} \mid 1 \le j \le \ell\lambda - 1,\ j \not\equiv 0 \pmod{\ell}\}$. Then $G^\star_{U_P} = \{g_1^j \mid j \in J^\star\}$. Therefore it follows from Proposition 4.1.1 that

$$L\langle P\rangle = -\frac{1}{\ell\lambda} \sum_{j \in J^\star} \cot\frac{\pi j}{\ell\lambda} \cot\frac{\pi\delta j}{\lambda} = -4s(\delta, \lambda). \tag{29}$$

(ii) Assume P is an arc contracting point. Then $\widetilde{U}_P$ coincides with the local uniformization space of the neighborhood of the non-amphidrome node in §3.2 (iia). The group G_{U_P} is isomorphic to $\mathbf{Z}/\left(\ell^2\lambda^{(1)}\lambda^{(2)}d/c^2\right)$, whose generator g_1 acts on $\widetilde{U}_P$ as the map (22). We put $J^\star = \{j \in \mathbf{Z} \mid 1 \le j \le \ell^2\lambda^{(1)}\lambda^{(2)}d/c^2 - 1,\ j \not\equiv 0 \pmod{\ell\lambda^{(1)}d/c},\ j \not\equiv 0 \pmod{\ell\lambda^{(2)}d/c}\}$. Then $G^\star_{U_P} = \{g_1^j \mid j \in J^\star\}$. By changing the primitive d-th root of unity as in the argument in §3.2 (iia), it follows from Proposition 4.1.1 that

$$L\langle P\rangle = -\frac{c^2}{\ell^2\lambda^{(1)}\lambda^{(2)}d} \sum_{j \in J^\star} \cot\frac{\pi cj}{\ell\lambda^{(1)}d} \cot\frac{\pi cj}{\ell\lambda^{(2)}d} = -4s(v, d). \tag{30}$$

(iii) Assume P is a quasi-tail contracting point. Then $\widetilde{U}_P$ coincides with the local uniformization space of the neighborhood of the amphidrome node in §3.2 (iib). The group G_{U_P} is isomorphic to the dihedral group of order $2\ell\lambda n$ whose generatores h_0 and h_1 act on $\widetilde{U}_P$ as the maps (25), and is written by $G_{U_P} = \{h_0^j,\ h_0^j h_1 \mid 0 \le j \le \ell\lambda n - 1\}$.

The map h_0^j has an isolated fixed point at P iff $j \not\equiv 0 \pmod{\ell n}$. Moreover, by putting $\sigma\delta = \lambda\hat{\sigma} + 1$ and $v = \sigma\delta + \lambda\hat{\sigma} + 2\sigma\lambda\mathbf{K}$, the action of the subgroup generated by h_0 equivalently descends to the action generated by $(z, u) \mapsto (\mathbf{e}(v/2\lambda\xi)z, \mathbf{e}(1/2\lambda\xi)u)$ by Proposition 1.5.4.

On the other hand, since $h_0^j h_1$ acts as $X \longmapsto \mathbf{e}\,(1/(2\ell\lambda) + (\delta + a_2\lambda)j/(\lambda n))\,Y$, $Y \longmapsto \mathbf{e}\,(1/(2\ell\lambda) + (\delta + a_1\lambda)j/(\lambda n))\,X$, the eigen-values of this action are $\mathbf{e}((1+j)/(2\ell\lambda))$ and $\mathbf{e}((1+j)/(2\ell\lambda)+1/2)$. The map $h_0^j h_1$ has an isolated fixed point at P iff $j \not\equiv -1\,(\mathrm{mod}\,\ell\lambda)$.

Since $\cot\frac{\pi(1+j)}{4\ell\lambda}\cot\left(\frac{\pi(1+j)}{4\ell\lambda}+\frac{\pi}{2}\right) = -1$, the similar argument implies that

$$L\langle P\rangle = -2s(v, 2\lambda\xi) + \frac{\ell\lambda - 1}{2\ell\lambda}. \tag{31}$$

Step 3 Next we calculate the contribution of the fixed complex curves C on $S^\sharp$ of the Galois groups. Namely, let $C \subset \bigcup U_i$ be a covering of V-charts in $S^\sharp$, and let $G^\star_{U_i}(C)$ be the point-wise stabilizer of $C|_{U_i}$ in G_{U_i}. Put

$$L\langle C\rangle = \sum_{U_i} \frac{1}{|G_{U_i}|} \sum_{g \in G^\star_{U_i}(C)} \int_{C\cap U_i} \rho_{U_i} L^g(U_i).$$

Note that C is an irreducible component of the central fiber $F^\sharp = \sum \alpha_j F_j^\sharp$, or a horizontal hyperplane around a quasi-tail contracting point. We calculate it for each case.

(i) We consider the former case. Since the normal bundle of $F_j^\sharp$ in $S^\sharp$ is locally trivial on a neighborhood of a generic point of $F_j^\sharp$, the data in order to determine $L\langle F_j^\sharp\rangle$ are concentrated on the neighborhoods of the nodes. Let $P_{j,k}$ be a point written locally as an intersection point of two components $F_j^\sharp$ and $F_k^\sharp$, and $U_{P_{j,k}}$ be the chart containing $P_{j,k}$.

Choose the coordinate (X, Y) of the uniformization space $\widetilde{U}_{P_{j,k}}$ so that $Y = 0$ (resp. $X = 0$) defines the local lift $\widetilde{F}_j^\sharp$ (resp. $\widetilde{F}_k^\sharp$) of $F_j^\sharp$ (resp. $F_k^\sharp$). We take another chart U_i with $F_j^\sharp \cap U_i \neq \emptyset$, $U_i \cap U_{P_{j,k}} \neq \emptyset$ and $U_i \not\ni P_{j,k}$ such that the coordinate (X', t) of $\widetilde{U}_i$ is chosen so that $t = 0$ defines the lift $\widetilde{F}_j^\sharp$, and the coordinate transformation on $\widetilde{U}_i \cap \widetilde{U}_{P_{j,k}}$ is given by $t = XY$, $X' = X$. Since the transition function $f_{i,P_{j,k}}$ of $N_{\widetilde{E}_j/\widetilde{U}_{P_{j,k}}} \simeq [\widetilde{E}_j]|_{\widetilde{E}_j}$ on $\widetilde{U}_i \cap \widetilde{U}_{P_{j,k}}$ is given by $f_{i,P_{j,k}} = Y$, the system $\{\phi|_{\widetilde{U}_{P_{j,k}}} = 1/Y,\ \phi|_{\widetilde{U}_i} = 1\}$ defines a meromorphic section of the normal bundle of $\widetilde{E}_j$ over $\widetilde{U}_i \cup \widetilde{U}_{P_{j,k}}$. Namely, a divisor corresponding to the normal bundle has a simple pole at $\widetilde{P}_{j,k} = \{(X, Y) = (0, 0)\}$. Therefore, it follows from the well-known local calculation (cf. [GH] p. 141) that

$$\int_{\widetilde{E}_j\cap\widetilde{U}_{P_{j,k}}} \rho_{\widetilde{U}_{P_{j,k}}} c_1(N_{\widetilde{E}_j/\widetilde{U}_{P_{j,k}}}) = -1, \quad \int_{\widetilde{E}_j\cap\widetilde{U}_i} \rho_{\widetilde{U}_i} c_1(N_{\widetilde{E}_j/\widetilde{U}_i}) = 0.$$

If $P_{j,k}$ is an arc contacting point, then $G^\star_{P_{j,k}}(F_j^\sharp)$ is a cyclic group of order $\ell\lambda^{(1)}/c$ generated by $n\lambda^{(2)}$-th power of the map (23). If $P_{j,k}$ is a quasi-tail contacting point, then $G^\star_{P_{j,k}}(F_j^\sharp)$ is a cyclic subgroup of order ℓ generated by $h_0^{\lambda n}$ of (25).

On the other hand, we have

$$\sum_{k=1}^{r-1} \mathrm{cosec}^2 \frac{\pi k}{r} = \frac{r^2 - 1}{3}$$

for a natural number r (cf. [HZ] p. 178). From these, we have

$$\sum_{F_j^\sharp} L\langle F_j^\sharp\rangle = -\sum_{P_{noam}} \frac{c^2}{\ell^2 d\lambda^{(1)}\lambda^{(2)}} \sum_{i=1}^{2} \frac{1}{3}\left\{\left(\frac{\ell\lambda^{(i)}}{c}\right)^2 - 1\right\} - \sum_{P_{am}} \frac{1}{2\ell\lambda n}\cdot\frac{2(\ell^2-1)}{3}\,. \tag{32}$$

(ii) We consider the latter case. Let P be a quasi-tail contracting point, and we follow the notations in Step 2 (iii). If $j = k\ell\lambda$ $(1 \le k \le n)$, the map $h_0^j h_1$ has a fixed curve component

$$H_j := \left\{(X,Y) \in \widetilde{U}_p \;\middle|\; \mathbf{e}\left(\frac{1}{2\ell\lambda} + \frac{(\delta + a_1\lambda)j}{\lambda n}\right) X - Y = 0\right\},$$

so that the action of $h_0^j h_1$ to the normal bundle of H_j in $\widetilde{U}_p$ is the reflection, i.e. with eigenvalue -1. Since H_j is a horizontal hyperplane with respect to the global fibration, the self-intersection H_j^2 on $\widetilde{U}_p$ (cf. [AS] p. 583) is nothing but a single point with multiplicity 1. Therefore we have

$$\sum_{k=1}^{n} L\langle H_j\rangle = \frac{1}{2\ell\lambda n}\sum_{k=1}^{n} \operatorname{cosec}^2\frac{\pi}{2}\cdot H_j^2 = \frac{1}{2\ell\lambda}. \tag{33}$$

Step 4 We first sum up (29) $\sim$ (33), and write $\operatorname{Sign} S^\sharp + \eta(\partial S^\sharp)$ as the summation of the contribution of the monodromy data of inner multiple points, non-amphidrome annuli and amphidrome annuli by using (26) and Theorem 3.1.1. Second, we substitute the contribution of the valency data of both sides of boundary curves of the non-amphidrome and amphidrome annuli from them, and add them as the contribution of the body parts by using Lemma 4.2.2 and Theorem 4.1.2. This calculation implies that

$$\begin{aligned}\operatorname{Sign} S^\sharp \;+\; \eta(\partial S^\sharp) \;&=\; \frac{1}{3}\int_{S^\sharp} p_1(S^\sharp) \;+\; \sum_{[B_i]}\sum_{\alpha=1}^{\varphi(i)}\left\{r(\alpha,i) - \frac{1}{3}\left(\frac{\sigma_\alpha^{(i)} + \delta_\alpha^{(i)}}{\lambda_\alpha^{(i)}} + \sum_{j=1}^{r(\alpha,i)} K_j(\alpha,i)\right)\right\} \\ &+\; \sum_{[\mathcal{A}_j]}\left(\frac{d}{3\lambda^{(1)}\lambda^{(2)}} - 1 + \epsilon'(\mathcal{A}_j) + \frac{2c^2}{3\ell^2 d\lambda^{(1)}\lambda^{(2)}}\right) \;+\; \sum_{[\widetilde{\mathcal{A}}_k]}\left(\frac{\xi}{3\lambda} + \frac{1}{3\ell\lambda n}\right)\end{aligned} \tag{34}$$

where $\epsilon'(\mathcal{A}_j) = 0$ for $\mathbf{K} \ge 0$, $\epsilon'(\mathcal{A}_j) = \omega + \omega' - r - r' + 1 + \epsilon(\mathcal{A}_j)$ for $\mathbf{K} = -1, \omega \ge 1, \omega' \ge 1$, and $\epsilon'(\mathcal{A}_j) = 2\omega - 2r + 1 + \epsilon(\mathcal{A}_j)$ for $\mathbf{K} = -1, \omega \ge 1, \omega' = 0$ in the notations of §5.2.

On the other hand, since the intersection matrix of the exceptional set is negative definite, the contraction $S \to S^\sharp$ induces the decrease of the dimension of the negative eigen space of the intersection form as the same number of the exceptinal curves. Therefore

$$\operatorname{Sign} S - \operatorname{Sign} S^\sharp = -\sum_{[B_i]}\sum_{\alpha=1}^{\varphi(i)} r(\alpha,i) - \sum_{[\mathcal{A}_j]}(\mathbf{K} - 1 + \epsilon'(\mathcal{A}_j) - \epsilon(\mathcal{A}_j)) - \sum_{[\widetilde{\mathcal{A}}_k]}(\mathbf{K}+2). \tag{35}$$

By the definition of the orbifold Pontrjagin form, we have $\int_{S^\sharp} p_1(S^\sharp) = (1/N)\int_{\widetilde{S}} p_1(\widetilde{S})$. Moreover, since G acts on $\partial\widetilde{S}$ freely, $\eta(\partial S) = \eta(\partial S^\sharp)$ coincides with $(1/N)\eta(\partial\widetilde{S})$ ([APS] II). Hence the assertion of Theorem 5.2.1 follows from (28) (34) (35). Q.E.D.

We prove Lemma 5.3.1. The Galois group $\mathbf{G} = \mathrm{Gal}(\widetilde{\mathcal{C}}/\mathcal{C})$ of the base extension $\widetilde{\mathcal{C}} \to \mathcal{C}$ naturally acts on $\widetilde{\mathcal{S}}$ so that the restriction of this action to a neighborhood of a singular fiber of $\mathbf{f}$ is nothing but the action which we already discussed here. Since the action of $\mathbf{G}$ is free outside the singular fibers and the dummy fiber $\mathbf{f}^{-1}(P_0)$, it suffices to show that any data of $\mathbf{f}^{-1}(P_0)$ does not contribute $\mathrm{Sign}\,\mathcal{S} - (1/N)\,\mathrm{Sign}\,\widetilde{\mathcal{S}}$.

Although $\mathbf{f}^{-1}(P_0)$ is a component of fixed curves of some element of $\mathbf{G}$, it does not contribute the integral of the equivariant L-form because the normal bundle of $\mathbf{f}^{-1}(P_0)$ in $\mathcal{S}$ is trivial. Therefore the assertion is clear. Q.E.D.

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Tadashi Ashikaga
Faculty of Engineering, Tohoku-Gakuin University, Tagajo, Miyagi, 985-8537, Japan.
E-mail:tashikaga@tjcc.tohoku-gakuin.ac.jp